Robust Kalman Filtering under Model Perturbations

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Abstract

We consider a family of divergence-based minimax approaches to perform robust filtering. The mismodeling budget, or tolerance, is specified at each time increment of the model. More precisely, all possible model increments belong to a ball which is formed by placing a bound on the Tau-divergence family between the actual and the nominal model increment. Then, the robust filter is obtained by minimizing the mean square error according to the least favorable model in that ball. It turns out that the solution is a family of Kalman like filters. Their gain matrix is updated according to a risk sensitive like iteration where the risk sensitivity parameter is now time varying. As a consequence, we also extend the risk sensitive filter to a family of risk sensitive like filters according to the Tau-divergence family.

Index Terms

Robust Kalman filtering, Tau-divergence family, minimax problem, risk sensitive filtering.

I. INTRODUCTION

Kalman filter is ubiquitous in many applications. The main reason is due by its iterative structure, allowing its implementation very simple. On the other hand, this filter is designed with respect to a linear state space model. The latter is often inadequate to describe phenomena, accordingly the resulting Kalman filter does not perform well in the practice. Since the beginning, therefore, it was clear the importance to develop robust versions of the standard Kalman filter.

Robust filtering can be performed according to the risk sensitive approach, [14], [13], [1], [7], [12], [10]. Here, the robust estimator is designed according to the nominal model but in

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such a way to avoid large errors. The sensitivity to large errors is tuned by the so called risk sensitivity parameter. It is worth noting this approach has been interpreted as a minimax problem [2], [15], [4], [5], [6]. The appealing aspect of the risk sensitive approach is that the solution is a Kaman like filter. On the other hand, the risk sensitive parameter is not explicitly connected to the discrepancy between the actual and the nominal model. Recently, a divergence-based minimax approach has been proposed in [8], [9], [20]. More precisely, in [8] the robust static estimation problem of a signal given noisy observations has been considered. Here, all possible models belong to a ball which is formed by placing a bound on the Kullback-Leibler divergence between the actual and the nominal model. This bound, say tolerance, represents the mismodeling budged. Then, the robust filter is obtained by minimizing the mean square error according to the least favorable model in this ball. It turns out that the Bayes estimator is robust under model uncertainty characterized by this ball. In [9], a dynamic extension to this problem (i.e. a robust filtering problem) has been considered. More precisely, drawing inspiration from [3], [11], the mismodeling budged is specified to each time increment of the model, that is the model uncertainty is expressed in an incremental way. Roughly speaking, the idea is to iterate the Bayes estimator with the least favorable statistics found in [8]. It turns out the robust estimator has a Kalman like structure. More precisely, it is a risk sensitivity like filter, where the risk sensitivity parameter is now time varying.

In [19], the robust static estimation problem proposed in [8] has been extended, in the Gaussian case, to a family of uncertainty classes. The latter is formed by placing a bound on a set of divergences (called τ -divergence family, [17]) between the actual and the nominal model. This particular divergence family is chosen because, in contrast to the alpha and the beta family [18], [16], [21], it allows to characterize uncertainty balls for which the Bayes estimator is still robust.

The contribution of this paper is to extend the robust Kalman filter in [9] to a family of robust Kalman filters parametrized by the τ -divergence family using the results in [19]. This family of filters is characterized by a time varying risk sensitive parameter. Therefore, by adopting the perspective given in [2], we also extend the risk sensitive filter to a family of risk sensitive like filters parametrized by the τ -divergence family, say τ -risk sensitive filters. Finally, we present a simulation study which shows that parameter τ tunes how conservative the robust filter is.

In the paper we will use the following notation. ||x|| denotes the Euclidean norm of $x \in \mathbb{R}^n$. $||x||_A$ denotes the weighted Euclidean norm with A symmetric and positive definite.

II. ROBUST STATIC ESTIMATION

We review the robust static estimation problem under model uncertainty characterized by the τ -divergence family introduced in [19]. Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$ be two jointly Gaussian random vectors. Let $z := \begin{bmatrix} x^T & y^T \end{bmatrix}^T$. Its joint nominal probability density f is

$$f(z) = \frac{1}{\sqrt{(2\pi)^{p+n} \det K_z}} \exp\left(-\frac{1}{2}(z-m_z)^T K_z^{-1}(z-m_z)\right)$$

where the mean vector $m_z \in \mathbb{R}^{n+p}$ and the covariance matrix $K_z \in \mathcal{Q}^{n+p}_+$ are known. We conformably partition the mean vector and the covariance matrix of z according to x and y:

$$m_z = \begin{bmatrix} m_x \\ m_y \end{bmatrix}, \quad K_z = \begin{bmatrix} K_x & K_{xy} \\ K_{yx} & K_y \end{bmatrix}$$

Let \tilde{f} denote the actual joint probability density of z

$$\tilde{f}(z) = \frac{1}{\sqrt{(2\pi)^{p+n} \det \tilde{K}_z}} \exp\left(-\frac{1}{2}(z-m_z)^T \tilde{K}_z^{-1}(z-m_z)\right)$$

where the mean vector $\tilde{m}_z \in \mathbb{R}^{n+p}$ and the covariance matrix $\tilde{K}_z \in \mathcal{Q}^{n+p}_+$ are unknown. Since both f and \tilde{f} are Gaussian, the deviation between f and \tilde{f} may be directly measured by the deviation between (m_z, K_z) and $(\tilde{m}_z, \tilde{K}_z)$ through the τ -divergence, [19]:

$$\mathcal{D}_{\tau}(\tilde{f}||f) = \begin{cases} \|\Delta m_{z}\|_{K_{z}^{-1}}^{2} + \operatorname{tr}\left(-\log(\tilde{K}_{z}K_{z}^{-1})\right) \\ +\tilde{K}_{z}K_{z}^{-1} - I_{n+p}\right), & \tau = 0 \\ \|\Delta m_{z}\|_{\frac{1-\tau}{1-\tau}K_{z}^{-1}}^{2} + \operatorname{tr}\left(-\frac{1}{\tau(1-\tau)}(L_{z}^{-1}\tilde{K}_{z}L_{z}^{-T})^{\tau} \\ +\frac{1}{1-\tau}\tilde{K}_{z}K_{z}^{-1} + \frac{1}{\tau}I_{n+p}\right), & 0 < \tau < 1 \\ \delta_{\infty}(\Delta m_{z}) + \operatorname{tr}\left(L_{z}^{-1}\tilde{K}_{z}L_{z}^{-T}\log(L_{z}^{-1}\tilde{K}_{z}L_{z}^{-T}) \\ -\tilde{K}_{z}K_{z}^{-1} + I_{n+p}\right), & \tau = 1 \end{cases}$$

$$(1)$$

where L_z is such that $K_z = L_z L_z^T$, $\Delta m_z = \tilde{m}_z - m_z$, and

$$\delta_{\infty}(v) := \begin{cases} 0 & \text{if } v = 0\\ \infty & \text{otherwise.} \end{cases}$$

Note that, $\mathcal{D}_{\tau}(\tilde{f}||f) \geq 0$ and equality holds if and only if $\tilde{f} = f$. This divergence takes root in the prediction theory. Let $e^N = L_z^{-1}(z - m_z)$ with $z \sim \tilde{f}$. e^N can be understood as a normalized

prediction error, where m_z represents the minimum variance prediction of z based on f. If $\tilde{f} = f$, then e^N has zero mean and covariance matrix I. Hence, this divergence measures the discrepancy between e^N and the Gaussian random vector with zero mean and covariance I. We consider the closed ball centered on f:

$$\mathcal{B}_{\tau} := \{ f \text{ s.t. } \mathcal{D}_{\tau}(f \| f) \le c \}$$

$$\tag{2}$$

where $c \in \mathbb{R}_+$ is a fixed tolerance. Accordingly, \mathcal{B}_{τ} represents the set of all possible probability densities of z consistent with the allowed mismodelling budget.

The robust estimator of x given y is designed according to the minimax point of view [8], [6]. More precisely, whenever we seek to design an estimator minimizing a suitable loss function, an hostile player, say "nature", conspires to select the worst possible probability density in \mathcal{B}_{τ} . Let g(y) denote an estimator of x based on the observation vector y. The optimal robust filter is solution to the following minimax problem

$$\min_{g \in \mathcal{G}} \max_{\tilde{f} \in \mathcal{B}_{\tau}} J(\tilde{f}, g)$$
(3)

where

$$J(\tilde{f},g) = \mathbb{E}_{\tilde{f}}[\|x - g(y)\|^2] = \int_{\mathbb{R}^{n+p}} \|x - g(y)\|^2 \tilde{f}(z) dz$$

denotes the mean square error and \mathcal{G} denotes the set of all estimators g(y) such that $\mathbb{E}_{\tilde{f}}[\|g(y)\|^2]$ is finite for any $\tilde{f} \in \mathcal{B}_{\tau}$.

Theorem 2.1: Let $0 \le \tau \le 1$. The least favorable probability density \tilde{f}° has mean vector $\tilde{m}_z^{\circ} = m_z$ and covariance matrix with the following structure

$$\tilde{K}_{z}^{\circ} = \begin{bmatrix} \tilde{K}_{x} & K_{xy} \\ K_{yx} & K_{y} \end{bmatrix}$$

$$\tag{4}$$

wherein only the covariance of x is perturbed with respect to the nominal covariance matrix. Let

$$P = K_x - K_{xy} K_y^{-1} K_{yx}$$
$$V = \tilde{K}_x - K_{xy} K_y^{-1} K_{yx}$$
(5)

denote the nominal and the perturbed *a posteriori* covariance matrix of x given y. Then,

$$V = \begin{cases} L_P \left(I_n - \theta (1 - \tau) L_P^T L_P \right)^{\frac{1}{\tau - 1}} L_P^T, & 0 \le \tau < 1 \\ L_P \exp \left(\theta L_P^T L_P \right) L_P^T, & \tau = 1 \end{cases}$$
(6)

where L_P is such that $P = L_P L_P^T$. Here θ^{-1} , with $\theta^{-1} > (1 - \tau) ||P||$, is the unique Lagrange multiplier satisfying the relation $c = \gamma_{\tau}(P, \theta)$ where

$$\gamma_{\tau}(P,\theta) = \begin{cases} -\log \det(I_n - \theta P)^{-1} + \operatorname{tr}((I_n - \theta P)^{-1} - I_n), \quad \tau = 0\\ \operatorname{tr}(-\frac{1}{\tau(1-\tau)}(I_n - \theta(1-\tau)L_P^T L_P)^{\frac{\tau}{\tau-1}} \\ +\frac{1}{1-\tau}(I_n - \theta(1-\tau)L_P^T L_P)^{\frac{1}{\tau-1}} + \frac{1}{\tau}I_n), \quad 0 < \tau < 1\\ \operatorname{tr}(\exp(\theta L_P^T L_P)(\theta L_P^T L_P - I_n) + I_n), \quad \tau = 1 \end{cases}$$
(7)

The optimal robust estimator is the Bayes estimator

$$g^{\circ}(y) = G^{\circ}(y - m_y) + m_x \tag{8}$$

with $G^{\circ} = K_{xy}K_y^{-1}$.

Theorem 2.1 shows that the Bayes estimator is robust with respect to the uncertainty class, parametrized by τ , in (2). Clearly, this optimality holds in the Gaussian case. Without this assumption, the least favorable probability density could be more different than the one in Theorem 2.1.

Corollary 2.1: Let $\theta > 0$ be a priori fixed and such that $\theta^{-1} > (1 - \tau) ||P||$. Consider the minimax problem

$$\min_{g \in \mathcal{G}} \max_{\tilde{f} \in \mathcal{B}_{\tau}} \mathbb{E}_{\tilde{f}}[\|x - g(y)\|^2] - \theta^{-1} \mathcal{D}_{\tau}(\tilde{f}\|f)$$

where $\mathcal{B}_{\tau} = \{\tilde{f} \text{ s.t. } \mathcal{D}_{\tau}(\tilde{f}||f) < \infty\}$ and \mathcal{G} is the set of all estimators such that $\mathbb{E}_{\tilde{f}}[||g(y)||^2]$ is finite for any $\tilde{f} \in \mathcal{B}_{\tau}$. Then, the least favorable probability density \tilde{f}_t° has mean vector $\tilde{m}_z^{\circ} = m_z$ and covariance matrix \tilde{K}_z° as in (4). The perturbed *a posteriori* covariance matrix V of x given y is in (5). Moreover, its relation with P is given by (6) where θ now has been *a priori* chosen. The optimal estimator is the Bayes estimator (8).

III. ROBUST FILTERING PROBLEM

We consider a nominal Gauss-Markov state space model of the form

$$x_{t+1} = A_t x_t + B_t v_t$$

$$y_t = C_t x_t + D_t v_t$$
(9)

where $x_t \in \mathbb{R}^n$ is the state process, $y_t \in \mathbb{R}^p$ is the observation process, and $v_t \in \mathbb{R}^m$ is WGN with unit variance, i.e. $\mathbb{E}[v_t v_s^T] = I_m \delta_{t-s}$ where δ_t denotes the Kronecker delta function. We assume that the noise v_t is independent of the initial state, whose nominal distribution is given by $f_0(x_0) \sim \mathcal{N}(\hat{x}_0, V_0)$. Let $z_t = [x_{t+1}^T y_t^T]^T$. Model (9) is characterized by the nominal transition probability density of z_t given x_t :

$$\phi_t(z_t|x_t) \sim \mathcal{N}\left(\left[\begin{array}{c}A_t\\C_t\end{array}\right]x_t, \left[\begin{array}{c}B_t\\D_t\end{array}\right]\left[\begin{array}{c}B_t^T\\D_t\end{array}\right]\left[\begin{array}{c}B_t^T\\D_t\end{array}\right]\right).$$
(10)

As noticed in [9], when entropy-like indexes are used to measure the proximity of statistical models, all the relations between dynamic variables or observations should be uncertain, otherwise those indexes take infinite value. To avoid such a situation, we assume that the noise v_t affects all the components of the dynamics and observations in (9), possibly with a very small variance for relations which are viewed as almost certain. Therefore, the covariance matrix

$$K_{z_t|x_t} = \begin{bmatrix} B_t \\ D_t \end{bmatrix} \begin{bmatrix} B_t^T & D_t^T \end{bmatrix}$$

is positive definite. Moreover, the matrix $\Gamma_t = \begin{bmatrix} B_t^T D_t^T \end{bmatrix}^T$ has full column rank, and without loss of generality we can assume Γ_t is square and invertible, so that m = n + p. Otherwise, we can compress the column space of Γ_t and remove noise components which do not affect model (9).

We adopt the minimax approach proposed [9, Section III] to characterize the robust filter. Let $\tilde{\phi}_s(z_t|x_t)$ be the least favorable transition probability density of z_t given x_t . Let $\tilde{f}_t(x_t|Y_{t-1})$ be the *a priori* probability density of x_t conditioned on the observations $Y_{t-1} = \{y_s, 0 \le s \le t-1\}$ and based on the least favorable model. We introduce the marginal probability densities

$$\bar{f}_t(z_t|Y_{t-1}) = \int \phi_t(z_t|x_t) \tilde{f}_t(x_t|Y_{t-1}) \mathrm{d}x_t$$
(11)

$$\tilde{f}_t(z_t|Y_{t-1}) = \int \tilde{\phi}_t(z_t|x_t) \tilde{f}_t(x_t|Y_{t-1}) dx_t.$$
(12)

Note that, $\overline{f}_t(z_t|Y_{t-1})$ can be viewed as the pseudo-nominal density of z_t conditioned on Y_{t-1} computed from the conditional least favorable density $\tilde{f}_t(x_t|Y_{t-1})$ and the nominal transition probability density $\phi_t(x_t|z_t)$. As in [9], we assume that

$$\tilde{f}_t(x_t|Y_{t-1}) \sim \mathcal{N}(\hat{x}_t, V_t).$$
(13)

In this way the conditional probability density $\bar{f}_t(z_t|Y_{t-1})$ is Gaussian. We make the additional assumption that $\tilde{\phi}(z_t|x_t)$ is such that $\tilde{f}_t(z_t|Y_{t-1})$ is Gaussian. In [9], the latter assumption was not made. However, it is worth noting that the least favorable solution found is such that (12) is Gaussian, see Remark 3.2. Therefore, we can measure the deviance between $\tilde{\phi}_t$ and ϕ_t as deviance between $\tilde{f}_t(z_t|Y_{t-1})$ and $\bar{f}_t(z_t|Y_{t-1})$ using the τ -divergence (1). Then, we assume that $\tilde{\phi}_t$ belongs to the closed ball about ϕ_t : $\mathcal{B}_{t,\tau} = \{\tilde{\phi}_t(z_t|x_t) \text{ s.t. } \mathcal{D}_{\tau}(\tilde{f}_t||\bar{f}_t) \leq c_t\}$ where $c_t \in \mathbb{R}_+$ is the tolerance specified at each time step. Let \mathcal{G}_t denote the class of estimators with finite second-order moments with respect all densities $\tilde{\phi}_t(z_t|x_t)\tilde{f}_t(x_t|Y_{t-1})$ such that $\tilde{\phi}_t(z_t|x_t) \in \mathcal{B}_{t,\tau}$. The the robust filter is characterized by the following minimax problem

$$(g_t^{\circ}, \tilde{\phi}_t^{\circ}) = \arg\min_{g_t \in \mathcal{G}_t} \max_{\tilde{\phi}_t \in \mathcal{B}_{t,\tau}} J_t(\tilde{\phi}_t, g_t)$$
(14)

where

$$J_t(\tilde{\phi}_t, g_t) = \mathbb{E}_{\tilde{f}_t}[\|x_{t+1} - g_t(y_t)\|^2 |Y_{t-1}]$$

= $\int \int \|x_{t+1} - g_t(y_t)\|^2 \tilde{\phi}_t(z_t|x_t) \tilde{f}_t(x_t|Y_{t-1}) dx_t dz_t$

denotes the mean square error of the estimator $\hat{x}_{t+1} = g_t^{\circ}(y_t)$ of x_{t+1} evaluated with respect to the transition density $\tilde{\phi}_t$ in $\mathcal{B}_{t,\tau}$. It is worth noting that \hat{x}_{t+1} depends on Y_t , and not only on y_t , but this dependency is suppressed to simplify the notations.

Remark 3.1: In the minimax problem (14) we require that $\tilde{f}_t(z_t|Y_{t-1})$ defined in (12) is a conditional probability density, that is

$$\int \int \tilde{\phi}_t(z_t|x_t) \tilde{f}_t(x_t|Y_{t-1}) \mathrm{d}z_t \mathrm{d}x_t = 1,$$
(15)

but we do not require that $\tilde{\phi}_t(z_t|x_t)$ is a transition probability density for each x_t . Therefore, the *a priori* conditional probability density $\tilde{f}_t(x_t|Y_{t-1})$ is not required to coincide with the *a* posteriori one computed from $\tilde{\phi}_t(z_t|x_t)\tilde{f}_t(x_t|Y_{t-1})$.

Remark 3.2: In [9], $\tilde{\phi}_t(z_t|x_t)$ is not required to be such that (12) is Gaussian. The constraint on $\tilde{\phi}_t(z_t|x_t)$ is that $\mathbb{D}_{KL}(\tilde{f}_t||\bar{f}_t) \leq c_t$ where \mathbb{D}_{KL} is the *Kullback-Leibler* divergence among probability densities. On the other hand, the solution $\tilde{\phi}_t^{\circ}(z_t|x_t)$ to the corresponding minimax problem is such that (12) is Gaussian, see [9, Formula (16)]. Hence, the corresponding $\tilde{f}_t^{\circ}(z_t|Y_{t-1})$ is Gaussian. Note that, $\mathbb{D}_{KL}(\tilde{f}_t||\bar{f}_t) = \mathcal{D}_0(\tilde{f}_t||\bar{f}_t)$ when \tilde{f}_t , \bar{f}_t are Gaussian. We conclude that, for $\tau = 0$, the solution to (14) coincides with the one in [9].

IV. ROBUST KALMAN FILTERS

We show that the optimal robust estimator solution to the minimax problem (14) is a Kalman like filter parametrized by τ . In this way, we obtain a parametric family of robust Kalman filters. First, Problem (14) can be reformulated as the static minimax problem (3). Consider the ball $\bar{\mathcal{B}}_{t,\tau} = \{\tilde{f}_t(z_t|Y_{t-1}) \text{ s.t. } \mathcal{D}_{\tau}(\tilde{f}_t || \bar{f}_t) \leq c_t\}$ which is the set of all probability densities having structure (12) with $\tilde{\phi}_t \in \mathcal{B}_{t,\tau}$. The equivalent minimax problem is

$$(\tilde{f}_t^\circ, g_t^\circ) = \arg\min_{g_t \in \mathcal{G}_t} \max_{\tilde{f}_t \in \bar{\mathcal{B}}_{t,\tau}} \bar{J}_t(\tilde{f}_t, g_t)$$

where

$$\bar{J}_t(\tilde{f}_t, g_t) = \int \|x_{t+1} - g_t(y_t)\|^2 \tilde{f}_t(z_t | Y_{t-1}) \mathrm{d}z_t.$$

In view of (10) and (13), the pseudo-nominal density is Gaussian

$$\bar{f}_t(z_t|Y_{t-1}) \sim \mathcal{N}\left(\left[\begin{array}{c}A_t\\C_t\end{array}\right]\hat{x}_t, K_{z_t}\right)$$
(16)

where the conditional covariance matrix K_{z_t} is given by

$$K_{z_{t}} = \begin{bmatrix} K_{x_{t+1}} & K_{x_{t+1},y_{t}} \\ K_{y_{t},x_{t+1}} & K_{y_{t}} \end{bmatrix}$$
$$= \begin{bmatrix} A_{t} \\ C_{t} \end{bmatrix} V_{t} \begin{bmatrix} A_{t}^{T} & C_{t}^{T} \end{bmatrix} + \begin{bmatrix} B_{t} \\ D_{t} \end{bmatrix} \begin{bmatrix} B_{t}^{T} & D_{t}^{T} \end{bmatrix}.$$
(17)

Applying Theorem 2.1 with $f \mapsto \bar{f}_t$, $\tilde{f} \mapsto \tilde{f}_t$ and $g \mapsto g_t$, the least favorable conditional density $\tilde{f}_t^{\circ}(z_t|Y_{t-1})$ is such that

$$\tilde{f}_t^{\circ}(z_t|Y_{t-1}) \sim \mathcal{N}\left(\left[\begin{array}{c}A_t\\C_t\end{array}\right]\hat{x}_t, \tilde{K}_{z_t}^{\circ}\right)$$
(18)

where the least favorable conditional covariance matrix is

$$\tilde{K}_{z_t}^{\circ} = \begin{bmatrix} \tilde{K}_{x_{t+1}} & K_{x_{x+1}y_t} \\ K_{y_t x_{t+1}} & K_{y_t} \end{bmatrix}$$

Let

$$P_{t+1} = K_{x_{t+1}} - K_{x_{t+1},y_t} K_{y_t}^{-1} K_{y_t,x_{t+1}}$$
$$V_{t+1} = \tilde{K}_{x_{t+1}} - K_{x_{t+1},y_t} K_{y_t}^{-1} K_{y_t,x_{t+1}}$$

denote the nominal and the least favorable conditional covariance of x_{x+1} given Y_t . Then,

$$V_{t+1} = \begin{cases} L_{P_{t+1}} \left(I_n - \theta_t (1-\tau) L_{P_{t+1}}^T L_{P_{t+1}} \right)^{\frac{1}{\tau-1}} L_{P_{t+1}}^T, & 0 < \tau < 1 \\ L_{P_{t+1}} \exp \left(\theta_t L_{P_{t+1}}^T L_{P_{t+1}} \right) L_{P_{t+1}}^T, & \tau = 1 \end{cases}$$

where $L_{P_{t+1}}$ is such that $P_{t+1} = L_{P_{t+1}}L_{P_{t+1}}^T$ and $\theta_t^{-1} > (1-\tau)||P_{t+1}||$ is the unique solution to $c_t = \gamma_\tau(P_{t+1}, \theta_t)$ where γ_τ has been defined in (7). The optimal robust estimator takes the form,

$$\hat{x}_{t+1} = g_t^{\circ}(y_t) = A_t \hat{x}_t + G_t(y_t - C_t \hat{x}_t)$$
(19)

with gain matrix $G_t = K_{x_{t+1},y_t} K_{y_t}^{-1}$. From (17), we obtain

$$G_{t} = (A_{t}V_{t}C_{t}^{T} + B_{t}D_{t}^{T})(C_{t}V_{t}C_{t}^{T} + D_{t}D_{t}^{T})^{-1}$$

$$P_{t+1} = A_{t}V_{t}A_{t}^{T} - G_{t}(C_{t}V_{t}C_{t}^{T} + D_{t}D_{t}^{T})G_{t}^{T} + B_{t}B_{t}^{T}.$$
(20)

Algorithm 1 shows the iterative scheme of the optimal robust estimator we found for the case $0 \le \tau < 1$. The algorithm for the limit case $\tau = 1$ is the same with the exception that the updating of V_{t+1} is different. It is clear that the robust filter has the same iterative structure of the Kalman filter with the exception that P_t is applied a distortion through matrix V_t . In particular, G_t is governed by a Riccati-like equation.

Algorithm 1: Robust Kalman filter at time t Input : c_t , \hat{x}_t , V_t , y_t Output: \hat{x}_{t+1} , V_{t+1} 1 $G_t = (A_t V_t C_t + B_t D_t^T)^T (C_t V_t C_t^T + D_t D_t^T)^{-1}$ 2 $\hat{x}_{t+1} = A_t \hat{x}_t + G_t (y_t - C_t \hat{x}_t)$ 3 $P_{t+1} = A_t V_t A_t^T - G_t (C_t V_t C_t^T + D_t D_t^T) G_t^T + B_t B_t^T$ 4 Find θ_t such that $c_t = \gamma_\tau (P_{t+1}, \theta_t)$ 5 Compute $V_{t+1} = L_{P_{t+1}} (I_n - \theta_t (1 - \tau) L_{P_{t+1}}^T L_{P_{t+1}})^{\frac{1}{\tau-1}} L_{P_{t+1}}^T$

It remains to characterize the least favorable transition density $\tilde{\phi}_t^{\circ}(z_t|x_t)$. It is not difficult to

prove that, [19, Theorem 2.1],

$$K_{z_t}^{-1} = \begin{bmatrix} I_n & 0\\ -G_t^T & I_p \end{bmatrix} \begin{bmatrix} P_{t+1}^{-1} & 0\\ 0 & K_{y_t}^{-1} \end{bmatrix} \begin{bmatrix} I_n & -G_t\\ 0 & I_p \end{bmatrix}$$
$$(\tilde{K}_{z_t}^{\circ})^{-1} = \begin{bmatrix} I_n & 0\\ -G_t^T & I_p \end{bmatrix} \begin{bmatrix} V_{t+1}^{-1} & 0\\ 0 & K_{y_t}^{-1} \end{bmatrix} \begin{bmatrix} I_n & -G_t\\ 0 & I_p \end{bmatrix}$$

Accordingly

$$K_{z_t}^{-1} - (\tilde{K}_{z_t}^{\circ})^{-1} = \begin{bmatrix} I_n \\ -G_t^T \end{bmatrix} \Phi_t \begin{bmatrix} I_n & -G_t \end{bmatrix}$$

where $\Phi_t = P_{t+1}^{-1} - V_{t+1}^{-1}$ which is positive definite. Let $e_t = x_t - \hat{x}_t$ denote the estimation error. Define

$$\hat{m}_{z_t} = \mathbb{E}_{f_t}[z_t|Y_{t-1}] = \mathbb{E}_{\tilde{f}_t^{\circ}}[z_t|Y_{t-1}] = \begin{bmatrix} A_t^T & C_t^T \end{bmatrix}^T \hat{x}_t$$

Therefore,

$$(z_t - \hat{m}_{z_t})^T (K_{z_t}^{-1} - (\tilde{K}_{z_t}^{\circ})^{-1})(z_t - \hat{m}_{z_t})$$

$$= (z_t - \hat{m}_{z_t})^T \begin{bmatrix} I_n \\ -G_t^T \end{bmatrix} \Phi_t \begin{bmatrix} I_n & -G_t \end{bmatrix} (z_t - \hat{m}_{z_t})$$

$$= \|x_{t+1} - (A_t \hat{x}_t + G_t (y_t - C_t \hat{x}_t))\|_{\Phi_t}^2$$

$$= \|x_{t+1} - \hat{x}_{t+1}\|_{\Phi_t}^2 = \|e_{t+1}\|_{\Phi_t}^2.$$

By (18) and (16), we have

$$\tilde{f}_t^{\circ}(z_t|Y_{t-1}) \sim \exp\left(\frac{1}{2} \|e_{t+1}\|_{\Phi_t}^2\right) \bar{f}_t(z_t|Y_{t-1}).$$

By (11), we obtain

$$\tilde{f}_t^{\circ}(z_t|Y_{t-1}) \sim \int \exp\left(\frac{1}{2} \|e_{t+1}\|_{\Phi_t}^2\right) \phi_t(z_t|x_t) \tilde{f}_t(x_t|Y_{t-1}) \mathrm{d}x_t$$

and by (12) we conclude that

$$\tilde{\phi}_t^{\circ}(z_t|x_t) = \frac{1}{M_t(\Phi_t)} \exp\left(\frac{1}{2} \|e_{t+1}\|_{\Phi_t}^2\right) \phi_t(z_t|x_t)$$
(21)

where the normalizing constant $M_t(\Phi_t)$ is such that (15) holds. It is worth noting that in the case $\tau = 0$, i.e. the case considered in [9], the distortion is a radial function of the estimation error e_{t+1} , because $\Phi_t = \theta_t^{-1} I_n$ for $\tau = 0$. On the contrary, in the case $\tau \neq 0$ such distortion is nonradial.

V. LEAST-FAVORABLE MODEL

For simulation and performance evaluation purposes, in particular for choosing parameters c_t and τ , it is important to characterize the least favorable model which is the solution to (14). The idea is to characterize it through (21). Note that, there is a one to one correspondence between z_t and v_t , given x_t , through the relation

$$v_t = \Gamma_t^{-1} \left(z_t - \left[\begin{array}{c} A_t \\ C_t \end{array} \right] x_t \right)$$

because matrix Γ_t is invertible. Accordingly, we can characterize the least favorable model with model (9) where the distortion has been moved now in noise v_t . Applying the same arguments used in [9], see also [6, Section 17.7], it is not difficult to prove that the least favorable probability density of v_t depends on e_t and is distributed as $\tilde{\psi}_t(v_t|e_t) \sim \mathcal{N}(H_te_t, \tilde{K}_{v_t})$ where $\tilde{K}_{v_t} = (I_{n+p} - (B_t - G_t D_t)^T (\Omega_{t+1}^{-1} + \Phi_t)(B_t - G_t D_t))^{-1}$ and $H_t = \tilde{K}_{v_t}(B_t - G_t D_t)^T (\Omega_{t+1}^{-1} + \Phi_t)(A_t - G_t C_t)$. Matrix Ω_{t+1}^{-1} is computed from the backward recursion

$$\Omega_t^{-1} = (A_t - G_t C_t)^T (\Omega_{t+1}^{-1} + \Phi_t) (A_t - G_t C_t) + H_t^T \tilde{K}_{v_t}^{-1} H_t$$
(22)

where the final point can be initialized with $\Omega_{T+1}^{-1} = 0$ and T is the simulation horizon. The backward recursion is due by the fact that integrating $\tilde{\phi}_t^{\circ}(z_t|x_t)$ over z_t we obtain a positive function of e_t , therefore the "nature" has the opportunity to change retroactively the least favorable density of x_t . It is not difficult to see that the least favorable model admits a state space representation with matrices

$$\tilde{A}_{t} = \begin{bmatrix} A_{t} & B_{t}H_{t} \\ 0 & A_{t} - G_{t}C_{t} + (B_{t} - G_{t}D_{t})H_{t} \end{bmatrix}$$

$$\tilde{B}_{t} = \begin{bmatrix} B_{t} \\ B_{t} - G_{t}D_{t} \end{bmatrix} L_{t}, \quad \tilde{C}_{t} = \begin{bmatrix} C_{t} & D_{t}H_{t} \end{bmatrix}, \quad \tilde{D}_{t} = D_{t}L_{t}$$
(23)

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and the input is WGN with unit variance. Note that, to construct the least favorable model, first we generate the gains G_t performing a forward sweep of the robust filter (19)-(20) over interval [0, T], then we generate the matrices Ω_t through a backward sweep over interval [0, T]. Therefore, increasing the simulation interval beyond [0, T] requires performing a new backward sweep of recursion (22). Then, we can evaluate the performance of an arbitrary estimator

$$\hat{x}_{t+1}' = A_t \hat{x}_t' + G_t' (y_t - C_t \hat{x}_t') \tag{24}$$

applied to the the least favorable model. Let

$$\Pi_t = \mathbb{E}\left[\left[\begin{array}{c} e_t' \\ e_t \end{array} \right] \left[\begin{array}{c} (e_t')^T & e_t^T \end{array} \right] \right]$$

where e_t is the estimation error of the optimal filter (19) and e'_t is the estimation error of filter (24). Then, it can be proven that Π_t obeys to the *Lyapunov* equation, [9],

$$\Pi_{t+1} = \left(\tilde{A}_t - \begin{bmatrix} G'_t \\ 0 \end{bmatrix} \tilde{C}_t\right) \Pi_t \left(\tilde{A}_t - \begin{bmatrix} G'_t \\ 0 \end{bmatrix} \tilde{C}_t\right)^T + \left(\tilde{B}_t - \begin{bmatrix} G'_t \\ 0 \end{bmatrix} \tilde{D}_t\right) \left(\tilde{B}_t - \begin{bmatrix} G'_t \\ 0 \end{bmatrix} \tilde{D}_t\right)^T$$
(25)

where $\Pi_0 = I_2 \otimes V_0$.

VI. RISK SENSITIVE FILTERING

Consider the robust Kalman filter we presented in Section IV with $\tau = 0$. If we replace θ_t with a constant value θ we recognize immediately that we obtain the risk sensitive filter, [13], [1], [14]. This suggest us that the risk sensitive filter can extended using the τ -divergence family. Consider the *Markov-Gauss* state space model (9). Let $\bar{f}_t(z_t|Y_{t-1})$ be the conditional density of z_t given Y_{t-1} based on the model (9) and defined in (16). The classic risk sensitive estimator g_t° at time t is defined as

$$g_t^{\circ} = \underset{g_t \in \mathcal{G}_t}{\operatorname{argmin}} \mathbb{E}_{\bar{f}_t} [\exp(\theta \| x_{t+1} - g_t(y_t) \|^2) \, | \, Y_{t-1}]$$
(26)

where \mathcal{G}_t is the set of estimators for which the objective function in (26) is finite. $\theta > 0$ is the risk sensitivity parameter. More precisely, the larger θ is the more the objective function in (26) penalizes estimators with large errors. In [2], it has been shown that the risk sensitive estimator is solution to the following minimax problem

$$g_t^{\circ} = \underset{g_t \in \mathcal{G}_t}{\operatorname{argminmax}} \mathbb{E}_{\tilde{f}_t} [\|x_{t+1} - g_t(y_t)\|^2 | Y_{t-1}] \\ - \theta^{-1} \mathbb{D}_{KL}(\tilde{f}_t \| f_t)$$

$$(27)$$

where $\mathcal{B}_t = {\tilde{f}_t \text{ s.t. }} \mathbb{D}_{KL}(\tilde{f}_t || f_t) < \infty}$. The second term in the objective function in (27) is always nonpositive because $\mathbb{D}_{KL}(\tilde{f}_t || f_t) \ge 0$. For small values of θ , it takes large negative values for conditional densities not close to the nominal one. Therefore, the maximizer is obliged

to choose a conditional density close to the nominal one. On the contrary, for large values of θ , it takes (negative) values close to zero for some conditional densities not close to the nominal one. In such a situation, the maximizer has the possibility to choose those conditional densities. Note that, this behaviour does not change if we replace \mathbb{D}_{KL} with another divergence measure.

In our setting $\overline{f}_t(z_t|Y_{t-1})$ is Gaussian by assumption. In addition, if we assume that $\widetilde{f}_t(z_t|Y_{t-1})$ is Gaussian, then $\mathbb{D}_{KL}(\widetilde{f}_t||f_t) = \mathcal{D}_0(\widetilde{f}_t||f_t)$ where \mathcal{D}_0 has been defined in (1). It is then natural to extend the minimax problem (27) to the τ -divergence family:

$$g_t^{\circ} = \underset{g_t \in \mathcal{G}_t}{\operatorname{argmin}} \underset{\tilde{f}_t \in \mathcal{B}_t}{\operatorname{argmin}} \mathbb{E}_{\tilde{f}_t} [\|x_{t+1} - g_t(y_t)\|^2 \,|\, Y_{t-1}] \\ - \theta^{-1} \mathcal{D}_{\tau}(\tilde{f}_t \| f_t)$$

By applying Corollary 2.1, the optimal τ -risk sensitive estimator takes the form of (19)-(20) where

$$V_{t+1} = \begin{cases} L_{P_{t+1}} \left(I_n - \theta(1-\tau) L_{P_{t+1}}^T L_{P_{t+1}} \right)^{\frac{1}{\tau-1}} L_{P_{t+1}}^T, & 0 < \tau < 1 \\ L_{P_{t+1}} \exp\left(\theta L_{P_{t+1}}^T L_{P_{t+1}}\right) L_{P_{t+1}}^T, & \tau = 1. \end{cases}$$
(28)

It is worth noting that, for the case $0 \le \tau < 1$, V_{t+1} is defined provided that $0 < P_{t+1} < (\theta(1-\tau))^{-1}I_n$, while for the case $\tau = 1$, it is well defined whenever P_{t+1} is positive definite. The algorithmic scheme is similar to Algorithm 1: the unique difference is that Step 4 is now removed. Finally, while the risk sensitivity parameter of the robust filter of Section IV is time varying, and its evolution is governed by c_t , now it is constant.

VII. SIMULATION RESULTS

We consider the time-invariant model (9) with

$$A = \begin{bmatrix} 0.1 & 1 \\ 0 & 1.2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 0.1 \end{bmatrix}$$

and $x_0 \sim \mathcal{N}(0, V_0)$ with $V_0 = 0.01 \cdot I_2$. We consider the following three filters: KF is the standard Kalman filter; RKF₀ is the robust Kalman filter of Section IV with $\tau = 0$; RKF₁ is the robust Kalman filter of Section IV with $\tau = 1$. In Figure 1 we show the evolution of the risk sensitivity

parameter of RKF₀ and RKF₁ for $c = 10^{-1}$. We notice the θ_t is constant in the steady state, that is RKF₀ and RKF₁ coincide with the risk sensitive filters of Section VI with $\tau = 0$, $\theta \approx 0.19$ and $\tau = 1$, $\theta \approx 0.23$, respectively, in the steady state.



Fig. 1. Evolution of the risk sensitive parameter θ_t with $c = 10^{-1}$.

In what follows we evaluate the performance of RK, RKF₀ and RKF₁, which have structure (24), applied to the least favorable model (23). More precisely, for each filter, applied to (23), we consider the estimation error $e'_t = [e^1_t e^2_t]^T$. Then, we compute the variance of e^1_t and e^2_t through (25). We consider two situations: c large, i.e. nominal and least favorable model are very different; c small, i.e. nominal and least favorable model are similar.

A. Large tolerance

Here RKF₀ and RKF₁ have tolerance $c = 10^{-1}$. In the first experiment, we apply these filters to the nominal model (9). The variance of e_t^1 and e_t^2 are depicted in the first row of Figure 2. As expected, KF performs better than the others. Moreover, the variances of RKF₀ are slightly larger than the ones of RKF₁. In the second experiment, we apply these filters to the least favorable model (23) with $\tau = 0$ and $c = 10^{-1}$. The variances of e_t^1 and e_t^2 are depicted in the second row of Figure 2. Obviously, RKF₀ is the best estimator because it has been designed with respect to this model. Although RKF₁ has been designed with respect to another model, it performs better than KF. In the third experiment, we apply these filters to the least favorable model (23) with



Fig. 2. Variances of e_t^1 and e_t^2 when the filters are applied to the nominal model (first row); to (23) with $\tau = 0$ and $c = 10^{-1}$ (second row); to (23) with $\tau = 1$ and $c = 10^{-1}$ (third row). Here RKF₀ and RKF₁ have $c = 10^{-1}$.

 $\tau = 1$ and $c = 10^{-1}$. The variances of e_t^1 and e_t^2 are depicted in the third row of Figure 2. In this case RKF₁ is the best estimator because it is optimal with respect to the underlying model. Also in this case, the worst estimator is KF. From these experiments we can conclude that:

- the smaller τ is, the more conservative the filter is, that is, the smaller τ is, the more the uncertainty class contains models with larger mean square error. This property has been noticed also for the static estimation problem in [19].
- the family of robust Kalman filters provide better performances than the standard Kalman filter, even in the case that the least favorable model belongs to an uncertainty class parametrized by a different τ .

B. Small tolerance

We perform the same experiment three experiments of before where the unique difference is the tolerance which now is $c = 5 \cdot 10^{-3}$, see Figure 3. RKF₀ and RKF₁ provides the same performance which is comparable with the one of KF. Therefore, as long as the discrepancy between the nominal and the least favorable model is not too large, then the performance of KF applied to (23) does not deteriorate too much.

VIII. CONCLUSIONS

In this paper, we have considered a robust filtering problem under incremental model perturbations characterized by the τ -divergence family. The family of robust estimators we proposed is the solution to a minimax problem. These robust estimators have an iterative structure similar to the one of the Kalman filter. We have derived the corresponding least favorable models. Moreover, we have extended the risk sensitive filter to a family of risk sensitive like filters. Finally, a simulation study shows that parameter τ tunes how conservative the robust filter is.

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Fig. 3. Variances of e_t^1 and e_t^2 when the filters are applied to the nominal model (first row); to (23) with $\tau = 0$ and $c = 5 \cdot 10^{-3}$ (second row); to (23) with $\tau = 1$ and $c = 5 \cdot 10^{-3}$ (third row). Here RKF₀ and RKF₁ have $c = 5 \cdot 10^{-3}$.

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