Lyapunov functions for persistently-excited cascaded time-varying systems: application to consensus

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***Abstract*—We present some results on stability of linear time-varying systems with particular structures. Such systems appear in diverse problems, which include the analysis of adaptive systems, persistently-excited observers and consensus of systems interconnected through time-varying links. The originality of our statements rely in the fact that we provide smooth strict Lyapunov functions hence, our proofs are constructive and direct. Moreover, we establish uniform global exponential stability with explicit stability and decay estimates. For illustration we address a brief but representative case-study of consensus of Lagrangian systems interconnected through unreliable links.**

I. Introduction

The problem of establishing uniform global exponential stability for linear time-varying systems under conditions of persistency of excitation was initially motivated by the analysis of adaptive control systems. A considerable bulk of literature is available, some of which is nicely presented, *e.g.*, in [1]. Beyond the pure question of stability and convergence, lays that of performance. Specifically, to determine explicit exponential estimates that relate the property of persistency of excitation to the overshoot and convergence rates. For so-called "gradient" systems explicit bounds were independently provided in [2] and [3]. For more complex cases, such as that of model-reference adaptive control systems see [4]. It is to be noted, however, that the methods of proof in these references is rather intricate since they do not rely on the construction of strict Lyapunov functions.

As far as we know, the first Lyapunov functions for systems with a structure reminiscent of model-reference adaptive control appeared in [5], [6]. The method consists in constructing a strict Lyapunov function starting from a non strict one that satisfies $\dot{V}(t,x) \leq -q(t)V(t,x)$ where $q(t)$ is a positive persistently exciting signal. Our study in this note starts with this inequation.

Persistency of excitation also plays a fundamental role in control design, as for instance, in systems in which the control input is multiplied by a time-varying function –see [7]. Such is the case of certain systems in aerospace engineering applications –see *e.g.*, [8], [9], and [10]. In [7] and [11] uniform global asymptotic stability is established via the construction of a non-strict control Lyapunov function.

Another interesting case-study in which stability analysis tools for linear time-varying systems are useful is that of consensus under the assumption that communication links are time-varying and the graph has a spanning tree. In this scenario, stating conditions of persistency of excitation on the communication channels is particularly useful [12]. The aim is to guarantee the so-called semistability property [13], which covers that of set asymptotic stability [14] for systems having a continuum of equilibria E, in which each solution initially close to some equilibrium point remains close to one of the equilibria in E. Although Lyapunov analysis of semistability does not lead to an intuitive construction of Lyapunov functions, these methods are useful to address consensus problems where the communication topology switches between a finite number of time invariant graphs with certain connectivity property, and over a certain dwell time –see *e.g.*, [15], [16]. In [17] the consensus problem for networks whose topologies switch among *time-varying* graphs was addressed, that is, over *two* time scales.

In much of the existing literature, however, the study of consensus under time-varying communication links makes use of trajectory based approaches by means of a *non differentiable* Lyapunov functions to establish the contraction of trajectories. See for instance the seminal work of Moreau [18] in which the communication signals take arbitrarily positive values. Similar problems are treated, for example, in [19] and [20] under relatively relaxed conditions on communication signals and on the graph topologies.

In this paper we present several constructions of strict Lyapunov functions for linear time-varying systems under persistency of excitation conditions that apply in different contexts. In Section II, we present a preliminary statement for a scalar positive system that serves as basis for our main results: these hold for cascaded non-autonomous systems and are presented in Section III. Together with the comparison lemma, our results may be used to make straightforward statements for systems that appear in applications ranging from state-estimation via Luenberger-type observers to consensus under time-varying interconnections and with a directed spanning-tree topology –see Section IV.

From a theoretical viewpoint our consensus result *per se* is covered in the literature. However, as far as we know, we provide for the first time a strict smooth Lyapunov function. The importance of this can hardly be overestimated; strict Lyapunov functions are a fundamental step for analysis and design of robust control under realistic conditions, such as delays and sampling [21]. From a technical viewpoint, our constructions are inspired by [6], but we also use the results in [22] and [23], mainly for the strictification of Lyapunov functions with a non-positive persistently-exciting bounds on the time derivatives. In Section IV we also provide a concise but representative example of mutual synchronization of Lagrangian systems [24] interconnected through a spanning-tree topology. Finally, some concluding remarks are provided in Section V.

II. A positive comparison system

We start with a simple preliminary statement that, in addition to setting the basis for our main results, is interesting in its own right. Consider the differential equation

$$\dot{v} = -q(t)v, \qquad v \in \mathbb{R} \tag{1}$$

where $q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Invoking standard results on adaptive control –see *e.g.*, [1], one may conclude that the origin is uniformly globally exponentially stable if q is continuous and persistently exciting that is, if there exist T, $\mu > 0$ such that

$$\int_t^{t+T} q(s)ds > \mu \quad \forall\, t \geq 0. \tag{2}$$

Remark 1: The requirement that $q(t) \geq 0$ is not necessary –see [25, Lemma 1]. •

The following statement presents a strict Lyapunov function which establishes this, otherwise well-known, result –*cf.* [25].

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Lemma 1: Let $q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be essentially bounded and let inequality (2) hold. Under these conditions, for the system (1), under condition (2), the function $W : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$, defined by

$$\begin{aligned} W(t,v) &= \frac{1}{2}\big[1 + 2\bar{q}T + \frac{2}{T}p(t)\big]v^2 && \text{(3a)} \\ p(t) &:= -\int_t^{t+T}\int_t^m q(s)ds\, dm && \text{(3b)} \end{aligned}$$

is a strict Lyapunov function hence, $\{v = 0\}$ is uniformly globally exponentially stable. □

Remark 2: The function $t \mapsto p$ in (3b) was first introduced in [6] under the equivalent form

$$p(t) = \int_t^{t+T} (s - t - T)\, q(s)ds, \tag{4}$$

which is obtained by changing the order of integration. •

Proof of Lemma 1: Let $\bar{q}$ be such that $|q(t)| \leq \bar{q}$ for all $t \geq t_0$ and define $p_M := \bar{q}T^2$. Since, moreover, $q(t) \geq 0$, we have $-p_M \leq p(t) \leq 0$, $|p(t)| \leq p_M$ for all $t \geq 0$ hence,

$$\frac{1}{2}v^2 \leq W(t,v) \leq \big[\frac{1}{2} + \bar{q}T\big]v^2. \tag{5}$$

The derivative of W along the trajectories of (1) yields

$$\dot{W}(t,v) = -\Big[q(t)\big[1 + 2\bar{q}T + \frac{2}{T}p(t)\big] - \frac{\dot{p}}{T}\Big]v^2$$

where, by definition, $\bar{q}T + \frac{1}{T}p(t) \geq 0$ and, after the fundamental theorem of calculus, the derivative of p in (3b) yields

$$\dot{p}(t) = Tq(t) - \int_t^{t+T} q(s)ds, \quad \forall\, t \geq 0. \tag{6}$$

Hence,

$$\dot{W} \leq -\left[\frac{1}{T}\int_t^{t+T} q(s)ds\right]v^2 \qquad \forall t \geq 0 \tag{7}$$

and, in view of (2), we obtain

$$\dot{W}(t,v) \leq -\frac{\mu}{T}v^2 \tag{8}$$

for all $t \geq t_0$ and $v \in \mathbb{R}$. Now, in view of (5), we obtain

$$\dot{W}(t,v) \leq -\frac{2\mu}{(1 + 2\bar{q}T)T}W(t,v) \tag{9}$$

which, by integrating along the trajectories, yields

$$|v(t)| \leq \sqrt{1 + 2\bar{q}T}\,|v(t_\circ)|\exp\Big[-\frac{\mu(t - t_\circ)}{(1 + 2\bar{q}T)T}\Big] \quad \forall t \geq t_0. \tag{10}$$

■

The simplicity of Lemma 1 should not eclipse its utility in stability analysis. For instance, along with the comparison theorem, it may be used to establish uniform global asymptotic stability, with guaranteed convergence rates, for certain nonlinear time-varying systems. To see this, consider the equation

$$\dot{z} = f(t,z) \tag{11}$$

and let $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be positive definite, proper and decrescent, that is, assume that there exist α_1, $\alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|z|) \leq V(t,z) \leq \alpha_2(|z|). \tag{12}$$

Assume, further, that there exists a globally Lipschitz continuous function $q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, satisfying (2),

$$\dot{V}(t,z) \leq -q(t)V(t,z). \tag{13}$$

Then, let us define $v(t) := V(t, z(t))$, so that $\dot{v}(t) \leq -q(t)v(t)$ for all $t \geq 0$. In view of the monotonicity properties of V and the comparison theorem, Lemma 1 directly establishes uniform global asymptotic stability of the origin, $\{z = 0\}$, with an explicit decay estimate. Indeed, from (10), (12) and the comparison Lemma, we obtain

$$\begin{aligned} |z(t)| &\leq \alpha_1^{-1}\Big(k_v\alpha_2(|z_\circ|)e^{-\lambda_v(t-t_\circ)}\Big) && \text{(14a)} \\ \lambda_v &:= \frac{\mu}{k_v^2 T}, \qquad k_v := \sqrt{1 + 2\bar{q}T}. && \text{(14b)} \end{aligned}$$

A. Example: nonlinear observer design

To illustrate further the utility of Lemma 1, consider the problem of designing an observer for a bilinear system

$$\begin{aligned} \dot{x} &= A(u,y)x + B(u,y) && \text{(15a)} \\ y &= Cx. && \text{(15b)} \end{aligned}$$

Since the system is linear in the unmeasured variable, we may proceed with a "Luenberger-like" design –see, *e.g.*, [26] and references therein. To that end, let $\hat{x}$ denote the state estimate and let us define its dynamics through the equation

$$\dot{\hat{x}} = A(u,y)\hat{x} + B(u,y) - L(u,y)C(u,y)[\hat{x} - x] \tag{16}$$

where the observer gain, L, is to be designed in order to ensure that the origin of the estimation-errors system is uniformly globally exponentially stable. This may be accomplished by imposing a condition of persistency of excitation along the trajectories [27], [28].

Proposition 1: Consider the system (15) and the observer (16). Let L be continuous, and let u, y be such that there exist a continuously-differentiable function $P : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a continuous function $q_m : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and positive constants p_m, p_M, μ and T such that:

(i) defining $\mathcal{A}(t) := A(u(t), y(t)) - L(u(t), y(t))C(u(t), y(t))$ and $Q(t) := -\dot{P}(t) - P(t)\mathcal{A}(t) - \mathcal{A}(t)^\top P(t)$, we have
$$Q(t) \geq q_m(t)I \geq 0 \quad \forall\, t \geq 0;$$
(ii) q_m is persistently exciting uniformly in $y(t)$ and $u(t)$ *i.e.*, it satisfies (2) with μ and T independent of the initial conditions;
(iii) the matrix $P(t)$ is uniformly positive definite and bounded, *i.e.*,
$$p_m I \leq P(t) \leq p_M I.$$

Then, the estimation errors $z(t)$ satisfy the bound

$$|z(t)| \leq k_v\sqrt{\frac{p_M}{p_m}}|z_\circ|e^{-\lambda_v(t-t_\circ)} \tag{17}$$

where k_v and λ_v are defined in (14b). □

Proof: Let the estimation errors be defined as $z := \hat{x} - x$ hence,

$$\dot{z} = \mathcal{A}(t)z. \tag{18}$$

Then, consider the function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by $V(t,z) := z^\top P(t)z$. This function satisfies (12) with $\alpha_1(s) := p_m s^2$ and $\alpha_2(s) := p_M s^2$. Moreover, defining $q(t) := \frac{q_m(t)}{p_M}$, a direct computation shows that the time derivative of V along the trajectories of (18) satisfies (13). Therefore, by Lemma 1, we see that

$$\mathcal{W}(t,z) := \frac{1}{2}\big[1 + 2\bar{q}T + \frac{2}{T}p(t)\big]\big[z^\top P(t)z\big]^2$$

is a Lyapunov function for the estimation error dynamics (18) and (14a) holds which, in this case, is equivalent to (17). ■

The statement of Proposition 1 generalizes some results that rely on a uniform complete *observability* condition, *e.g.*, the choice:

$$\begin{aligned} \dot{P} &= -\varepsilon P - \big[A(u,y)^\top P + PA(u,y)\big] + 2C^\top C && \text{(19a)} \\ L &:= P^{-1}C^\top, \qquad\qquad P(t_\circ) \geq p_m I, && \text{(19b)} \end{aligned}$$

commonly used in observer design for bilinear systems –*cf.* [26], guarantees that $P(t)$, hence $Q(t) := \varepsilon P(t)$, is positive definite and bounded, for all $t \geq T$. The persistency of excitation condition on Q, imposed in Proposition 1, is less restrictive than positivity; moreover, the gain $L(t)$ as defined in (19b) may reach very high values [26]. Yet, the advantage of this choice is that it leads directly to an exponential-convergence estimate and provides a strict Lyapunov function for the estimation error-system. That is, this construction naturally lends itself for output-feedback high-gain designs, notably for systems with Lipschitz non-linearities –see *e.g.*, [29]. On the other hand, for such systems, notably chaotic oscillators, the main result in [28] provides an observer of the type of (16), under the less restrictive persistency of excitation condition on $Q(t)$. Thus, the statement of Proposition 1 covers all the previously mentioned results by providing an explicit stability bound under the weaker condition of persistency of excitation.

III. Cascaded systems

In this section, using Lemma 1, we establish a more general result which applies to cascades of persistently-excited systems. To start with, consider the 2nd-order system:

$$\dot{x}_1 = -a_1(t)x_1 + a_2(t)x_2 \tag{20a}$$
$$\dot{x}_2 = -a_2(t)x_2 \tag{20b}$$

under the assumption that a_1 and a_2 are continuous, uniformly bounded, and persistently exciting, functions taking non-negative values.

For this system, exponential stability of the origin $\{x_1 = x_2 = 0\}$ may be assessed following a direct cascades argument. Indeed, this follows, *e.g.*, from the results in [30] observing that, by Lemma 1, the respective origins of

$$\dot{x}_1 = -a_1(t)x_1 \qquad \dot{x}_2 = -a_2(t)x_2 \tag{21}$$

are uniformly globally exponentially stable and $a_2(t)$ is bounded hence, the solutions $x_1(t)$ of equation (20a) are uniformly globally bounded. The statement also follows from the fact that (20a) is input-to-state stable with Lyapunov function $W(t, x_1)$ defined by (3) and input x_2. However, even though the cascades argument is straightforward for the case of two interconnected systems, the argument is hard to extend to cascades of $n > 2$ time-varying systems,

$$\Sigma_n' : \begin{cases} \dot{x}_1 = - a_1(t)x_1 + a_2(t)x_2 \\ \dot{x}_2 = - a_2(t)x_2 + a_3(t)x_3 \\ \qquad \vdots \\ \dot{x}_{n-1} = - a_{n-1}(t)x_{n-1} + a_n(t)x_n \\ \dot{x}_n = - a_n(t)x_n, \end{cases} \tag{22}$$

relying purely on converse Lyapunov theorems. Our next statement removes this difficulty by providing a strict Lyapunov function.

Theorem 1: Consider the system (22) under the following hypotheses:

A1 *(Non-negativity):* $a_i(t) \geq 0$ for all $i \leq n$ and all $t \geq 0$.
A2 *(Boundedness):* There exists $\bar{a} > 0$ such that $|a_i(t)| \leq \bar{a}$ for all $t \geq 0$ and all $i \leq n$.
A3 *(Persistency of Excitation):* There exist μ, $T > 0$ such that

$$\int_t^{t+T} a_i(s)ds > \mu \quad \forall i \leq n, \ \forall\, t \geq 0. \tag{23}$$

Then, defining $\beta_1 = 0$ and, for each $i \leq n$,

$$\beta_i \geq \frac{T}{2\mu}\big[1 + \bar{a}T\big]^2 + \frac{T\bar{a}^2}{2\mu}\beta_{i-1}, \quad \forall i \geq 2,$$
$$p_i(t) := -\int_t^{t+T}\int_t^m a_i(s)ds\, dm, \tag{24}$$

the function $V_n : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, defined as

$$V_n(t, x) := x^\top P(t)x \tag{25}$$

with

$$P(t) := \frac{1}{2}\text{diag}\big[1 + 2\bar{a}T + \frac{2}{T}\, p_i(t) + \beta_i\bar{a}\big],$$

is a strict Lyapunov function. Consequently, the origin is uniformly globally exponentially stable. □

Proof: The proof is constructed based upon that of Lemma 1. We show that the Lyapunov function candidate V_n is positive definite, proper and its total derivative satisfies

$$\dot{V}_n(t, x) \leq -\frac{\mu}{2T}\sum_{i=1}^n x_i^2. \tag{26}$$

Firstly, note that

$$-\bar{a}T^2 \leq p_i(t) \leq 0, \quad \forall\, i \leq n, \ t \geq 0 \tag{27}$$

therefore,

$$\frac{1}{2}\text{diag}\big[1 + \beta_i\bar{a}\big] \leq P(t) \leq \frac{1}{2}\text{diag}\big[1 + 2\bar{a}T + \beta_i\bar{a}\big].$$

Next, we proceed by induction and using Lemma 1. For $n = 1$ the system (22) corresponds to

$$\Sigma_1' : \dot{x}_1 = -a_1(t)x_1$$

and

$$V_1(t, x_1) = \frac{1}{2}\big[1 + 2\bar{a}T + \frac{2}{T}p_1(t)\big]x_1^2 \tag{28}$$

is a strict Lyapunov function for Σ_1'. The latter follows by mimicking the proof of Lemma 1 to obtain

$$\dot{V}_1(t, x_1) \leq -\frac{\mu}{T}x_1^2 \tag{29}$$

–*cf.* Eq. (8). Actually, later we shall use the fact that, for any index $i \geq 1$, the derivative of the right-hand side of (28) along the trajectories of $\dot{x}_i = -a_i(t)x_i$ satisfies an inequality similar to (29), *i.e.*, $\dot{V}_i \leq -(\mu/T)x_i^2$.

For $n = 2$, the cascaded system Σ_2' corresponds to (20), for which we define the function $V_2 : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ as

$$V_2(t, \bar{x}_{12}) = V_1(t, x_1) + \frac{1}{2}\big[1 + 2\bar{a}T + \frac{2}{T}p_2(t)\big]x_2^2 + \frac{1}{2}\beta_2\bar{a}x_2^2 \tag{30}$$

with $\bar{x}_{1j} := [x_1 \cdots x_j]^\top$ and, according to (24),

$$\beta_2 \geq \frac{T}{2\mu}\big[1 + 2\bar{a}T\big]^2. \tag{31}$$

Furthermore, using the bound $\bar{a} \geq a_i(t) \geq 0$, following the proof-lines of Lemma 1, we see that the time derivative of V_2 satisfies

$$\dot{V}_2(t, \bar{x}_{12}) \leq \dot{V}_1(t, x_1) - \beta_2 a_2(t)^2 x_2^2 - \frac{\mu}{T}x_2^2 \tag{32}$$

and, along the trajectories of (20a), $\dot{V}_1$ satisfies

$$\dot{V}_1(t, x_1) \leq -\frac{\mu}{T}x_1^2 + \big[1 + 2\bar{a}T + \frac{2}{T}p_1(t)\big]x_1 a_2(t)x_2.$$

In turn, this implies that

$$\dot{V}_2(t, x) \leq -\frac{\mu}{2T}(x_1^2 + 2x_2^2) + \phi_2(t, \bar{x}_{12}, \beta_2)$$
$$\phi_2(t, \bar{x}_{12}, \beta_2) := -\frac{\mu}{2T}x_1^2 - \beta_2 a_2(t)^2 x_2^2 + \big[1 + 2\bar{a}T + \frac{2}{T}p_1(t)\big]x_1 x_2 a_2(t).$$

Now, notice that $\phi_2 \leq 0$ if β_2 satisfies (31). To show this, we introduce

$$\epsilon^2 := \frac{\mu}{T\big[1 + 2\bar{a}T\big]} \tag{33}$$

and we use the triangle inequality

$$x_1 a_2(t) x_2 \le \frac{1}{2}\epsilon^2 x_1^2 + \frac{1}{2\epsilon^2} a_2(t)^2 x_2^2 \quad \forall\, \epsilon \neq 0, \tag{34}$$

as well as the fact that $p_i(t) \le 0$, to obtain

$$\begin{aligned}\phi_2(t,\bar{x}_{12},\beta_2) \le &- \frac{x_1^2}{2}\left[\frac{\mu}{T} - \epsilon^2\big[1+2\bar{a}T\big]\right]\\ &- x_2^2 a_2(t)^2\left[\beta_2 - \frac{1}{2\epsilon^2}\big[1+2\bar{q}T\big]\right] \quad \forall \epsilon \neq 0.\end{aligned}$$

From (33) and (31) it follows that $\phi_2 \le 0$ hence, we conclude that

$$\dot{V}_2(t,\bar{x}_{12}) \le -\frac{\mu}{2T}\big(x_1^2 + x_2^2\big) - \frac{\mu}{2T}x_2^2. \tag{35}$$

Next, we proceed by induction. For any $j \in (2,n]$, let V_j be a strict Lyapunov function for Σ'_j *–cf.* (22), and let it be defined as

$$V_j(t,\bar{x}_{1j}) = V_{j-1}(t,\bar{x}_{1j-1}) + \frac{1}{2}\big[1+2\bar{a}T + \frac{2}{T}p_j(t)\big]x_j^2 + \frac{1}{2}\beta_j \bar{a} x_j^2. \tag{36}$$

To evaluate its total time-derivative along the trajectories of Σ'_j we first see that

$$\dot{V}_{j-1}(t,\bar{x}_{1j-1}) \le -\frac{\mu}{2T}\sum_{i=1}^{j-1} x_i^2 - \frac{\mu}{2T}x_{j-1}^2 + \frac{\partial V_{j-1}}{\partial x_{j-1}} a_j x_j$$

and, in view of (36),

$$\frac{\partial V_{j-1}}{\partial x_{j-1}} = \big[1 + 2\bar{a}T + \frac{2}{T}p_{j-1}(t) + \beta_{j-1}\bar{a}\big]x_{j-1}. \tag{37}$$

Hence, it follows that

$$\dot{V}_j(t,\bar{x}_{1j}) \le -\frac{\mu}{2T}\sum_{i=1}^{j} x_i^2 - \frac{\mu}{2T}x_j^2 + \phi_j(t,\bar{x}_{1j},\beta_j,\beta_{j-1})$$

where

$$\begin{aligned}\phi_j(t,\bar{x}_{1j},\beta_j,\beta_{j-1}) = &-\frac{\mu}{2T}x_{j-1}^2 - \beta_j a_j(t)^2 x_j^2\\ &+ \left[1 + 2\bar{a}T + \frac{2}{T}p_{j-1}(t)\right] a_j(t) x_j x_{j-1}\\ &+ \beta_{j-1}\bar{a} a_j(t) x_j x_{j-1}.\end{aligned} \tag{38}$$

Now, in view of (27), the factor of $a_j(t)x_j x_{j-1}$ is non-negative hence, applying the triangle inequality to the last two terms on the right-hand side of (38), we obtain that, for any ϵ, $\sigma \neq 0$,

$$\begin{aligned}\phi_j(t,\bar{x}_{1j},\beta_j,\beta_{j-1}) \le &-\frac{\mu}{2T}x_{j-1}^2 - \beta_j a_j(t)^2 x_j^2\\ &+\Big[1+2\bar{a}T + \frac{2}{T}p_{j-1}\Big]\Big[\frac{a_j(t)^2}{2\epsilon^2}x_j^2 + \frac{1}{2}\epsilon^2 x_{j-1}^2\Big]\\ &+ \beta_{j-1}\Big[\frac{1}{2}\sigma^2 a_j(t)^2 x_j^2 + \frac{\bar{a}^2}{2\sigma^2}x_{j-1}^2\Big]\end{aligned}$$

which, in turn, since $p_i(t) \le 0$, implies that

$$\begin{aligned}\phi_j(t,\bar{x}_{1j},\beta_j,\beta_{j-1}) \le &-\frac{x_{j-1}^2}{2}\left[\frac{\mu}{T} - \frac{\bar{a}^2}{\sigma^2} - \epsilon^2\Big[1+2\bar{a}T\Big]\right]\\ &- x_j^2 a_j(t)^2\left[\beta_j - \frac{1}{2\epsilon^2}\Big[1+2\bar{a}T\Big] - \frac{\sigma^2}{2}\beta_{j-1}\right]\end{aligned}$$

for all $\sigma \neq 0$, $\epsilon \neq 0$. To render non-positive the factors of x_j^2 and x_{j-1}^2 above, we choose

$$\sigma^2 = \frac{2T\bar{a}^2}{\mu}.$$

Then, the factor of x_{j-1}^2 equals to zero if (33) holds, while the factor of $-x_j^2 a_j(t)^2$ is non-negative if

$$\beta_j \ge \frac{T}{2\mu}\left[1+2\bar{a}T\right]^2 + \frac{T\bar{a}^2}{\mu}\beta_{j-1}$$

for all $j \in (2,n]$ *–cf.* (24). It follows that $\phi_j \le 0$ and, consequently,

$$\dot{V}_j(t,\bar{x}_{1j}) \le -\frac{\mu}{2T}\sum_{i=1}^{j} x_i^2 - \frac{\mu}{2T}x_j^2. \tag{39}$$

The latter holds for any integer $j \in [3,n]$ hence, together with (29) and (35), the inequality (26) follows. ■

Remark 3: From the previous proof it also follows that the trajectories of (22) satisfy

$$|x(t)|^2 \le \alpha_M |x_\circ|^2 \mathrm{e}^{-(\mu/2T\alpha_M)(t-t_\circ)} \qquad \forall t \ge t_\circ$$

where $\alpha_M := 1 + (2T + \beta_n)\bar{a}$. To see this, we observe that the Lyapunov function V_n satisfies (since $\beta_n > \beta_{n-1} > \cdots \beta_1 = 0$)

$$(1/2)\alpha_M |x|^2 \ge V_n(t,x) \ge (1/2)|x|^2.$$

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An interesting extension of Theorem 1 relies on the use of the comparison theorem, applied this time in a manner reminiscent of vector Lyapunov functions, to obtain the following statement for cascaded linear-time-varying persistently-excited systems

$$\begin{aligned}\dot{x}_1 &= A_1(t)x_1 + B_1(t)x_2\\ &\vdots\\ \dot{x}_{n-1} &= A_{n-1}(t)x_{n-1} + B_{n-1}(t)x_n\\ \dot{x}_n &= A_n(t)x_n, \qquad x_i \in \mathbb{R}^m,\end{aligned} \tag{40}$$

under the following hypotheses:

A4 *(Boundedness)* There exists $\bar{B} > 0$ such that[1] $\|B_i\|_\infty \le \bar{B}$.

A5 *(Lyapunov Equation)* There exist positive definite matrices $P_i(t)$ and positive semi-definite matrices $Q_i(t)$, verifying:

$$\dot{P}_i + A_i^\top P_i + P_i A_i = -Q_i \tag{41}$$

A6 *(Persistency of excitation)* There exists a positive constants P_{iM}, P_{im}, μ, T and a function $q_{im} : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ such that:

$$0 < P_{im} I_n \le P_i(t) \le P_{iM} I_n \tag{42}$$

$$0 \le q_{im}(t) I_n \le Q_i(t) \tag{43}$$

$$\int_t^{t+T} q_{im}(s)ds > \mu \qquad \forall t \ge 0. \tag{44}$$

Theorem 2: Under assumptions **A4**, **A5** and **A6** there exists a quadratic strict differentiable Lyapunov function for (40). □

Sketch of Proof. For each $i \le n$, let us define $V_i(t,x) = x_i^\top P_i(t) x_i$. The derivative of each V_i along the trajectories of (40), satisfies

$$\begin{aligned}\dot{V}_1 \le &- x_1^\top Q_1(t) x_1 + 2x_1^\top P_1 B_1(t) x_2\\ &\vdots\\ \dot{V}_{n-1} \le &- x_{n-1}^\top Q_{n-1}(t)x_{n-1} + 2x_{n-1}^\top P_{n-1}B_{n-1}(t)x_n\\ \dot{V}_n \le &- x_n^\top Q_n(t) x_n.\end{aligned} \tag{45}$$

Then, consider the modified Lyapunov function $W_i : \mathbb{R}_{\ge 0} \times \mathbb{R}^{nm} \to \mathbb{R}_{\ge 0}$ defined by $W_i(t,x) = \phi_i(t)V_i(t,x)$ with

$$\phi_i(t) = \alpha_i - \frac{1}{T}\int_t^{t+T}\int_t^m q_{im}(s)ds\, dm$$

where α_i are constants such that $\phi_i \ge P_{iM} + 1$. Then, defining $\nu_i : \mathbb{R}_{\ge 0} \to \mathbb{R}^{n\times n}$ such that $P_i(t) = \nu_i(t)^\top \nu_i(t)$ and $M_i(t) = \phi_i(t)\nu_i(t)B_i\nu_{i+1}(t)^{-1}$, $\beta_1 = 1$, $\beta_{i+1} = \frac{6T^2}{\mu^2}\|M_i\|_\infty^2 \beta_i$, we see that

$$\mathcal{W}(t,x) = \sum_{i=1}^{n} \beta_i W_i(t,x_i)$$

constitutes a strict Lyapunov function for (40). ■

[1] We use $\|B_i\|_\infty := \sup_{t\ge 0}\|B_i(t)\|$ where $\|B_i(t)\|$ denotes the induced L_2 norm of $B_i(t)$ or any other congruent matrix norm.

IV. ISS Consensus under spanning tree

To illustrate the utility of our main results we consider now a classical tracking consensus problem concerning n agents interconnected in a spanning-tree topology with time-varying interconnection gains. That is, each agent communicates only with two neighbors. Even though here we consider that each agent communicates always with the same neighbours, in general, this does not need to be the case –*cf.* [17]. We limit our case-study to this topology because in concrete cases of formation control, or follow-the-leader tracking control for that matter, using such communication topology excludes communication redundancy. From a strictly theoretical viewpoint, however, our main stability statement *per se* in this section is covered by, *e.g.*, [14]. On the other hand, as far as we know, we provide for the first time a *strict smooth* Lyapunov function which, in turn, leads to establish input-to-state stability (ISS).

Thus, let us consider n dynamical systems defined by

$$\dot{z}_i = f_i(t, z_i) + u_i, \qquad z_i \in \mathbb{R}^m,\ i \leq n \tag{46}$$

which are required to follow a reference trajectory $z^* : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ generated by an exogenous system $\dot{z}^* := f^*(t, z^*)$. We assume that only the controller for the nth agent has access to the reference trajectory. Then, the ith agent receives information from the $i+1$st, thereby establishing a spanning-tree topology, albeit through unreliable channels.

To recast this consensus-tracking problem into a stabilization one we introduce the error system with state variables $x_i := z_i - z_{i+1}$ for all $i \leq n$, with $z_{n+1} := z^*$ and $f_{n+1} := f^*$. That is,

$$\dot{x}_i = f_i(t, x_i + z_{i+1}(t)) - f_{i+1}(t, z_{i+1}(t)) + u_i - u_{i+1} \tag{47a}$$
$$\dot{x}_n = f_n(t, x_n + z^*(t)) - f^*(t, z^*(t)) + u_n. \tag{47b}$$

The consensus problem boils down to stabilizing the origin $\{x = 0\}$, with $x := [x_1, \cdots, x_n]^\top$, for the system non-autonomous system (47). For this, we use the control inputs

$$u_i := -\gamma a_i(t)[z_i - z_{i+1}] + w_i, \qquad a_i(t) \geq 0, \quad \forall\, t \geq 0 \tag{48}$$

where the functions a_i are assumed to be bounded and persistently exciting, $\gamma > 0$ is the interconnection strength, and w_i denote "additional" inputs to be defined. Then, the closed-loop system is

$$\dot{x}_i = -\gamma a_i(t) x_i + \gamma a_{i+1}(t) x_{i+1} + \psi_i(t, x_i) + v_i \tag{49a}$$
$$\dot{x}_n = -\gamma a_n(t) x_n + \psi_n(t, x_n) + v_n \tag{49b}$$

with $v_i := w_i - w_{i+1}$ and

$$\psi_i(t, x_i) := f_i(t, x_i + z_{i+1}(t)) - f_{i+1}(t, z_{i+1}(t)), \quad i \leq n. \tag{50}$$

Note that the system (49) may be regarded as a "perturbed" version of (22) hence, the following statement, which implies robust consensus-tracking of (46), follows as a useful corollary of Theorem 1.

Lemma 2: Consider the system (49) under assumptions **A1**–**A3**. For each $i \leq n$, let v_i be measurable functions, let $\psi_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}^m$ be such that there exist once-continuously-differentiable class $\mathcal{K}_\infty$ functions L_i such that

$$|\psi_i(t, x_i)| \leq L_i(|x_i|). \tag{51}$$

Let R_i be such that for all $x_i \in B_{R_i}$, $B_{R_i} := \{x_i \in \mathbb{R} : |x_i| \leq R_i\}$,

$$\Big| \frac{\partial L_i}{\partial s}\big(|x_i|\big) \Big| \leq \ell_i$$

and the interconnection strength γ is such that

$$\frac{\mu\gamma}{2T} > 2\ell_i \big[1 + \bar{a}(2T + \beta_i)\big].$$

Then, the system (49) is input-to-state-stable from the input $v := [v_1, \cdots, v_n]^\top$, for all initial conditions $t_\circ \geq 0$ and $x_{i\circ} \in \mathbb{R}^n$ which produce complete trajectories satisfying $x_i(t, t_\circ, x_{i\circ}) \in B_{R_i}$. □

Sketch of proof: Following the proof of Theorem 1 the Lyapunov function V_n defined in (25) is found to satisfy

$$\dot{V}_n(t, x) \leq -\sum_{i=1}^{n} \Big[\frac{\mu\gamma}{2T} - \ell_i \big[1 + \bar{a}(2T + \beta_i)\big]\Big] x_i^2 + \big[1 + \bar{a}(2T + \beta_i)\big] x_i v_i \tag{52}$$

for all $x_i \in B_{R_i}$. Then, we see that $|v_i| \leq \ell_i |x_i|$ implies that

$$\dot{V}_n(t, x) \leq -\sum_{i=1}^{n} \Big[\frac{\mu\gamma}{2T} - 2\ell_i \big[1 + \bar{a}(2T + \beta_i)\big]\Big] x_i^2. \tag{53}$$

It follows that V_n is an input-to-stable Lyapunov function for all $x_i \in B_{R_i}$ and each $i \leq n$. Hence, the system is input-to-state stable for all initial conditions $t_\circ \geq 0$, $x_{i\circ} \in B_{R_i}$ generating complete trajectories that satisfy $|x_i(t, t_\circ, x_{i\circ})| \leq R_i$ for all $t \geq t_\circ \geq 0$ and all $i \leq n$. ■

A. Example

For the sake of illustration let us consider the following case-study of consensus-tracking control of Lagrangian systems,

$$D_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + g_i(q_i) = \tau_i, \qquad \tau_i,\, q_i \in \mathbb{R}^p. \tag{54}$$

The functions D_i, C_i and g_i are, respectively, the inertia matrix, the Coriolis matrix and the potential forces vector. The control torques are denoted by τ_i.

We consider the problem of tracking and mutual synchronisation —see [24] in which all systems are required to follow a common exogenous trajectory $t \mapsto q^*$. Now, we assume that the systems are interconnected in a spanning-tree topology through unreliable links hence, on intervals of time the nodes may be isolated.

To each system we apply the preliminary linearizing feedback (this is possible because D is full rank) $\tau_i = D_i(q_i)u_i + C_i(q_i, \dot{q}_i)\dot{q}_i + g_i(q_i)$ so that the equation of each node becomes $\ddot{q}_i = u_i$. Then, emulating the unreliability of the communication channel by a square-pulse function $a : \mathbb{R}_{\geq 0} \to \{0, \bar{a}\}$ the control input becomes

$$u_i = a(t)[-k_1(q_i - q_{i+1}) - k_2(\dot{q}_i - \dot{q}_{i+1}) + \ddot{q}_{i+1}]$$

that is, the control is active only when $a(t) = \bar{a} > 0$.

Now, for each $i \leq n$, define $x_i := [q_i^\top\ \dot{q}_i^\top]^\top - [q_{i+1}^\top\ \dot{q}_{i+1}^\top]^\top$. We see that the error dynamics, in closed loop, takes the form

$$\dot{x}_i = A_i(t)x_i + B_i(t)x_{i+1} + v_i(t), \quad i \leq n-1$$

where the perturbation v_i, which stems from the fact the "feedforward" term $\ddot{q}_{i+1}$ in u_i is not available all the time, is defined as $v_i(t) := [a(t) - 1][\ddot{q}_{i+1}(t) - \ddot{q}_{i+2}(t)]$. Furthermore,

$$A_i(t) := \begin{bmatrix} 0 & 1 \\ -a(t)k_1 & -a(t)k_2 \end{bmatrix}, \quad B_i(t) = \begin{bmatrix} 0 \\ a(t) \end{bmatrix}$$

and, for $i = n$ we have $\dot{x}_n = A_n(t)x_n + v_n$ with $v_n(t) := [a(t) - 1]\ddot{q}^*(t)$. By Theorem 2, for $v_i \equiv 0$, the origin is uniformly exponentially stable and admits a strict smooth Lyapunov function provided that **A4**–**A6** hold. To verify these assumptions, we follow the second construction in [7] for double integrators with time-varying peristently-exciting input gain, $\ddot{x} = \alpha(t)u$, and define

$$a(t) := \frac{\alpha(t)}{\alpha(t) + \varepsilon}, \quad \varepsilon \in (0, 1).$$

In the current example we used $k_1 = k_2 = 1$ for all agents but different arbitrary gains may be used. We chose $\alpha(t)$ as a periodic pulse function of period $T = 40$s, with a duty cycle of 70% and $\varepsilon = 0.01$. Hence, $a(t) \approx \alpha(t)$ is persistently exciting –see the bottom plot in Figure 1, and the conditions **A1**–**A3** hold. The "nominal" dynamics $\dot{x}_i = A_i(t)x_i$ is studied in [7, Proposition 2]. After the proof of the latter and some numeric computations we see that Assumptions

A5 and **A6** hold with $q_{im}(t) \approx a(t)$, for the particular choice of $Q_i := 0.16255\,I$. From Theorem 2, with $v_i(t) \equiv 0$, we conclude uniform global exponential stability hence, formation tracking control of (54). Input-to-state stability with respect to the disturnace v_i also may be concluded.

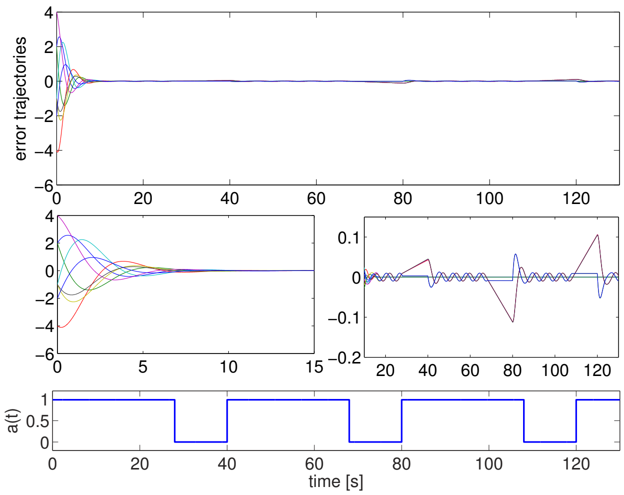

Fig. 1. Mutual synchronization of four Lagrangian systems

Some numerical results are illustrated in Figure 1, for the case in which all systems follow the reference $q^*(t)$ =sin(t). The steady-state error depicted in the zoomed portion of the figure illustrates the ISS statement. It may be diminished at will by increasing k_1 and k_2.

V. Conclusions

The novelty of our work lies in the provided *strict* Lyapunov functions; indeed, stability statements for more general classes of systems have been established before by other means. We believe, however, that our statements may be used as *off-the-shelf* results in a variety of problems appearing in adaptive control systems, state estimation of bilinear systems, and consensus with persistently-exciting interconnections. This claim is supported through a concise but representative example concerning consensus of Lagrangian systems which, we believe, may serve as basis for future work in the same direction. For instance, more general graphs, beyond spanning-tree topologies should be considered. Also, while we adopted here (due to page constraints) the use of an *ad hoc* preliminary feedback-linearizing control loop, applying other control methods is desirable.

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