

Optimal estimation and control for lossy network: stability, convergence, and performance

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Abstract—In this paper, we study the problems of optimal estimation and control, i.e., the linear quadratic Gaussian (LQG) control, for systems with packet losses but without acknowledgment. Such acknowledgment is a signal sent by the actuator to inform the estimator of the incidence of control packet losses. For such system, which is usually called as a User Datagram Protocol (UDP)-like system, the optimal estimation is nonlinear and its calculation is time-consuming, making its corresponding optimal LQG problem complicated. We first propose two conditions: 1) the sensor has some computation abilities; and 2) the control command, exerted to the plant, is known to the sensor. For a UDP-like system satisfying these two conditions, we derive the optimal estimation. By constructing the finite and infinite product probability measure spaces for the estimation error covariances (EEC), we give the stability condition for the expected EEC, and show the existence of a measurable function to which the EEC converges in distribution, and propose some practical methods to evaluate the estimation performance. Finally, the LQG controllers are derived, and the conditions for the mean square stability of the closed-loop system are established.

Index Terms—networked control systems, optimal estimation and control, LQG, packet loss, UDP-like system, smart sensor

I. INTRODUCTION

A. Background and Motivations

In recent years, A great deal of attention has been devoted to networked control systems (NCSs) in which control loops are closed over network. The introduction of network does bring numerous advantages, but in the meantime, it also causes some network-induced constraints, such as limited bandwidth, quantization errors, which may result in transmission packet

losses [1–5]. For NCSs with packet losses, there are two fundamental protocols in network communication, which are the transmission control protocol (TCP) and the user datagram protocol (UDP). The TCP features the successful data transmission by retransmitting lost data until the sending node receives a acknowledgment (ACK) signal from the receiving node. However, such retransmission mechanism in turn leads to several drawbacks, e.g., network jitter, transmission delay, additional consumption of energy and bandwidth [6]. In unreliable networks, sometimes, it would be difficult to implement the TCP by sending the ACK in time [7–9]. Thus, the TCP is commonly employed in NCSs where high security and data integrity are required and the communication is relatively reliable, such as smart grids and smart transportation. For the UDP, no ACK scheme is used and thus no retransmission of the lost data is required. Therefore, the UDP, with relatively less transmission reliability, avoids unnecessary energy consumption, causes lower latency, and allows more consistent communication, making it a preferable choice for real-time NCSs, e.g. robot control and remote teleoperations [10]. The system, in which there is no ACK signal sent from the actuator to notice the estimator the occurrence of control packet losses, is commonly called a UDP-like system (see Fig. 1-A or 1-B), and the one with ACK signals is called a TCP-like system, and the one with the ACK randomly lost is called a Quasi-TCP-like system [8, 11].

It is shown in recent work [12, 13] that for a UDP-like system the optimal estimator is nonlinear and consisted of exponentially increasing terms, and thus its design is time-consuming. By replacing the optimal estimator with the linear minimum-mean-square-error (LMMSE) estimator, the LQG problem, actually a sub-optimal LQG control, was studied in [14]. It is reported that for general UDP-like systems, the separation principle does not hold, and the LQG controller is a nonlinear function of the estimated state and its solution involves solving non-convex optimization problems. Various linear sub-optimal LQG controllers were developed in [7, 15, 16]. Therefore, we may conclude that if the conventional UDP-like structure (Fig. 1-A) remains unchanged, the optimal estimation and control are not only technically involved to be obtained, but also useless in practical use due to unaffordable time-consuming computations. It motivates us to consider whether there are some feasible or reasonable conditions or structures for UDP-like systems under which the optimal estimation and LQG control can be obtained and useful properties can be further analyzed.

Based on this motivation, we propose two mild and feasible

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conditions as follows. **Condition 1:** The sensor has some computation abilities, enough to run a standard Kalman filter, namely, a smart sensor; and **Condition 2:** The control command actually exerted to the plant (i.e., u_k^a defined later) is available for the sensor. These two conditions are not so stringent that can be satisfied in some scenarios. For Condition 1, smart sensors have a wide range of applications in many fields [17–19]. Practical techniques with complex computations—including speech/handwriting recognition and discrete Fourier transform—have been implemented in smart sensors. In particular, the smart sensors being able to perform Kalman filtering have been widely used to improve state estimation [20–22]. Moreover, thanks to the micro-electromechanical technology, these smart sensors can be designed to be smaller in size and lower in cost [18]. Condition 2 may be achieved in some scenarios: 1) The sensor and actuator are implemented in one device, such as the smart sensor node used in [17], in which the actuator is able to communicate and even control the sensor; 2) The actuator possesses communication ability to transfer data, such as the wireless smart actuator which can share information with the sensor [19].

In the following, we call the UDP-like system satisfying these two conditions a smart-sensor-based UDP (SS-UDP) system. Under Conditions 1 and 2 the sensor is able to obtain the information of control packet losses, that is, the ACKs. However, the way these ACKs are processed in a SS-UDP system is different from that in a Quasi-TCP-like system. In a Quasi-TCP-like system, the raw ACKs sent to the estimator over an unreliable channel is randomly lost, and thus it is required to estimate the value of ACK, making the optimal estimator [23] and the LQG controller [8] nonlinear. In a SS-UDP system, the ACK is fused into a local estimate at the sensor side. Although the local estimate is also randomly lost over the sensor-estimator channel, the resulting optimal estimator is linear, as shown later in Sec. III-A.

It can be seen from [1, 24] that among the topics on optimal estimation and control for the system with packet loss, there are some closely related fundamental issues: the stability of the estimator [25, 26], the distribution and convergence of the estimation error covariance [27, 28], the estimation performance evaluation [29, 30], and the stability of the closed-loop system [14, 31]. These four issues have been fully investigated for TCP-like systems, but they are seldom studied for SS-UDP systems. Therefore, in this paper, we are concerned with the optimal estimation and control problems for SS-UDP systems, especially the aforementioned four fundamental issues.

B. Related work and contributions

To our best knowledge, there is few work on optimal estimation and control for the proposed SS-UDP system. Recent advancements on UDP-like systems have been summarized above. In the sequel, we briefly review the state of art of these four fundamental issues for TCP-like systems.

For a conventional TCP-like system, that is, the system without a smart sensor, it is well known that its optimal estimator is the time-varying Kalman filter. Its stability was studied in the pioneering work [25] where it is pointed out that

there exists a critical value which determines the boundedness of the expected estimation error covariance (EEC), i.e., $\mathbb{E}[\mathbf{P}_k]$. Following [25], various aspects have been further researched, including the bound for the critical value [32], the distribution for EEC [27, 28], the Markov packet losses case [26, 33–35]. In [29, 30], the authors pointed out that $\mathbb{P}(\{\mathbf{P}_k \leq M\})$ is a better evaluation for the estimation performance than the quantity $\mathbb{E}[\mathbf{P}_k]$, and then obtained the lower and upper bounds for $\mathbb{P}(\{\mathbf{P}_k \leq M\})$. The LQG problem for TCP-like systems has been comprehensively investigated in [14, 31, 36, 37]. However, these results fail to apply to UDP-like or SS-UDP systems, as the structure of the optimal estimator for TCP-like systems is different from that for UDP-like or SS-UDP systems. For TCP-like systems with smart sensor, the EEC is convergent under some condition [38]. In [39], the authors proposed a LQG controller by using a smart sensor to design an encoder/decoder to compensate observation packet losses.

The introduction of smart sensors facilitates the design of the optimal estimator, however the estimation-related issues are still challenging. The resulting EEC \mathbf{P}_k is a random quantity. To better formulate its distribution, convergent, and the limit involves an infinite product probability space. The constructions of the σ -field and the probability measure on it are complicated. Moreover, the \mathbf{P}_k in the SS-UDP system contains u_k , and thus its properties relies on u_k , which is different from the cases in which there is no control input [20, 29, 30]. Therefore, the existing results on TCP-like systems cannot directly apply to SS-UDP systems.

In this paper, we study the optimal estimation and LQG control issues for SS-UDP systems. The main contributions are summarized as follows:

- 1) We construct finite and infinite product probability spaces for the estimation error covariance \mathbf{P}_k , which is the core in solving these aforementioned estimation issues.
- 2) We show the condition for the convergence of \mathbf{P}_k in distribution. Moreover, we explicitly describe the limit, a measurable function on an infinite product probability space. We establish the stability condition for $\mathbb{E}[\mathbf{P}_k]$, and propose some practical methods to evaluate the estimation performance. For TCP-like systems these estimation issues are respectively addressed by different approaches in [25, 27, 28, 30]. For SS-UDP systems we solve them in an unified way by the probability space based method.
- 3) For SS-UDP systems, we obtain the finite horizon LQG controller, establish the condition for the existence of the infinite horizon LQG controller, and then prove that the corresponding closed-loop system is mean square stable. It is worth noting that the optimal LQG controller for UDP-like systems has not yet been obtained; and there is no solution to the infinite horizon LQG problem for TCP-like systems as the Kalman gain is not convergent.

The rest of the paper is organized as follows: The system setup and problems are formulated in Section II. In Section III, finite and infinite product probability spaces are constructed, and then the stability, convergence, and performance evaluation of the optimal estimator are studied. The LQG problem is solved in Section IV. Conditions 1 and 2 are further discussed

in Section V. Numerical examples are presented in Section VI to illustrate our main results. The conclusions are given in Section VII. The proofs of all lemmas and propositions are given in Appendix.

Notations:

- $\mathbb{P}(\cdot)$ denotes the probability measure.
- $\mathbb{E}[\cdot]$ denotes the probability expectation.
- D^c denotes the complement of a set D .
- $(\cdot)'$ denotes the transpose of a matrix or vector.
- Let M be a matrix. $[\cdot]_M^2$ denotes $(\cdot)'M(\cdot)$.
- λ_M and $\underline{\lambda}_M$ stand for the maximum and the minimum singular values of M , respectively.
- $\|\cdot\|$ denotes the norm. Specifically, for a vector x , $\|x\| \triangleq \sqrt{x'x}$; for a matrix M , $\|M\| \triangleq \lambda_M$.
- \uplus stands for the union of sets and the symbol $+$ is used to emphasize that these sets are mutually disjoint.
- \mathbb{R} , \mathbb{N} , and \mathbb{Z}_+ denote the set of real numbers, natural numbers, and nonnegative integers, respectively.
- $\overline{\mathbb{Z}}_+$ is defined as the set of the extend nonnegative integers, that is, $\overline{\mathbb{Z}}_+ \triangleq \mathbb{Z}_+ \cup \{\infty\}$.
- $\psi_1 \circ \psi_2$: Let ψ_1 and ψ_2 be two functions. $\psi_1 \circ \psi_2(X)$ is defined as $\psi_1(\psi_2(X))$.
- In a binary sequence, e.g., $(0, 1, \times, \dots)$, the symbol \times means either 0 or 1.

II. SYSTEM SETUP AND PROBLEM FORMULATION

We propose one of the possible implementations of the SS-UDP system. That is the one illustrated in Fig. 1-B, in which the actuator and the smart sensor are located together in one device, like the smart sensor node used in [17], so that the actuator can provide the information u_k^a to the smart sensor.

A. System setup

Consider the following discrete-time linear SS-UDP system with the framework illustrated in Fig. 1-B:

- Plant:

$$x_{k+1} = Ax_k + Bu_k^a + \omega_k \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times q}$ are constant matrices, $x_k \in \mathbb{R}^n$ is the system state, $u_k^a \in \mathbb{R}^q$ is the control input actually exerted to the plant by the actuator, and ω_k is a zero mean Gaussian noise with covariance $Q \geq 0$. In this paper, we consider the unstable system, i.e., $\lambda_A > 1$.

- Actuator:

$$u_k^a = \nu_k u_k \quad (2)$$

where $u_k \in \mathbb{R}^q$ is the control command sent from the controller over the controller-actuator (C/A) channel. $\{\nu_k\}$, a sequence of i.i.d. Bernoulli random variables with $\mathbb{P}(\nu_k = 1) = \nu$, models the control packet loss. That is, $\nu_k = 1$ means that the control command u_k has been successfully transmitted to the actuator, otherwise $\nu_k = 0$.

- Sensor:

$$y_k = Cx_k + v_k, \quad (3)$$

where $C \in \mathbb{R}^{p \times n}$ is a constant matrix, $y_k \in \mathbb{R}^p$ is the observation, and v_k is a zero mean Gaussian noises

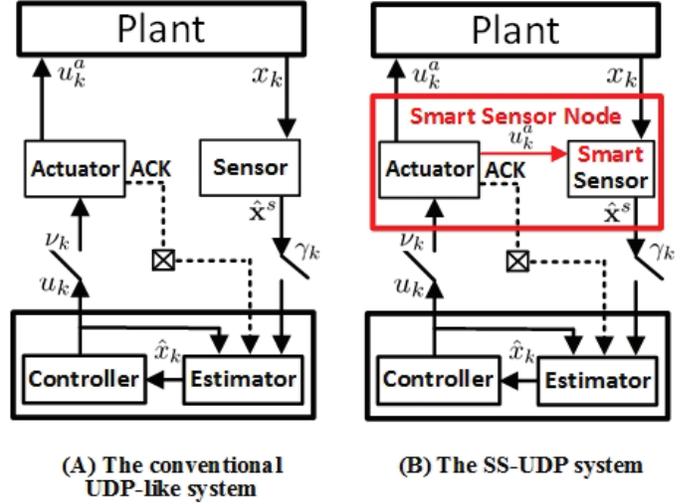


Fig. 1. The conventional UDP-like system and the SS-UDP system. The dash line with \boxtimes indicates that there is no communication channel from the actuator to the estimator for sending acknowledgments.

with covariance $R > 0$. As assumed in Condition 1, the smart sensor is able to run a Kalman filter to obtain a local optimal state estimate $\hat{x}_k^s = \mathbb{E}[x_k|Y_k]$, where $Y_k = \{y_k, \dots, y_1\}$ (The detailed derivations are given in Section III). Then it sends \hat{x}_k^s instead of the raw observation y_k to the estimator over the sensor-estimator (S/E) channel. $\{\gamma_k\}$, a sequence of i.i.d. Bernoulli random variables with $\mathbb{P}(\gamma_k = 1) = \gamma$, describes the observation packet loss. Namely, $\gamma_k = 1$ means that the local estimate \hat{x}_k^s has been successfully transmitted to the estimator; otherwise $\gamma_k = 0$.

- Estimator and controller unit: Denote the information set received at the estimator side by $\mathcal{I}_k = \{\gamma_k \hat{x}_k^s, \dots, \gamma_1 \hat{x}_1^s\}$. When $\gamma_k = 0$, $\gamma_k \hat{x}_k^s$ stands for an empty set. The task of the estimator/controller unit is to determine the optimal state estimation \hat{x}_k based on \mathcal{I}_k , and then provide the optimal control input u_k . The implementation of this unit is presented in Lemma 1 and Theorem 4, respectively.

For the system described in (1)-(3), some assumptions are given in the following.

Assumption 1. *The initial state x_0 is a Gaussian random variable with mean \bar{x}_0 and covariance P_0 . x_0 , ω_k , v_k , ν_k , and γ_k are mutually independent. The pair $(A, Q^{1/2})$ is controllable, and the pair (A, C) is observable.*

B. Problems formulation

All the problems listed below are assumed to be proposed for SS-UDP systems. The aforementioned four fundamental issues are formulated in the following five problems.

1) Estimation problems:

Definition 1. *An estimation of x_k , denoted by \hat{x}_k , is said to be optimal in the minimum mean square error (MMSE) sense, if \hat{x}_k minimizes $\mathbb{E}[\|x_k - \hat{x}_k|\mathcal{I}_k\|^2]$.*

Denote by \mathbf{P}_k the estimation error covariance of $\hat{\mathbf{x}}_k$. It is shown later that \mathbf{P}_k is a random variable. Although there are many convergence criteria for a random variable, a suitable one to describe \mathbf{P}_k is the convergence in distribution. Since the probability space of \mathbf{P}_k changes with k , most of the convergence criteria are not fit for this case.

Definition 2 (Convergence in distribution [40]). *Let X_n and X be random variables with distributions \mathbb{P}_n and \mathbb{P} , respectively. X_n is said to converge in distribution to X , denoted by $X_n \xrightarrow{d} X$, if $\lim_{n \rightarrow \infty} \mathbb{P}_n(\{X_n < x\}) = \mathbb{P}(\{X < x\})$ for every x such that $\mathbb{P}(\{X = x\}) = 0$.*

Problem 1. *Determine the condition under which \mathbf{P}_k converges in distribution, and find out the limit.*

As \mathbf{P}_k is a random quantity, the stability of the estimator is usually examined by the boundedness of $\mathbb{E}[\mathbf{P}_k]$, as in [22, 25, 26].

Definition 3 (Stability). *The estimation error covariance \mathbf{P}_k is said to be stable in the mean sense, i.e., $\mathbb{E}[\mathbf{P}_k]$ is stable, if $\mathbb{E}[\mathbf{P}_k]$ is bounded, that is, $\sup_k \mathbb{E}[\mathbf{P}_k] < +\infty$.*

Problem 2. *Determine the stability condition for $\mathbb{E}[\mathbf{P}_k]$.*

As pointed out in [29, 30], $\mathbb{P}_k(\{\mathbf{P}_k \leq \alpha I\})$ is a better evaluation of the estimation performance than $\mathbb{E}[\mathbf{P}_k]$, as it gives a complete characterization of the estimation performance.

Problem 3. *Analyze the estimation performance in term of $\mathbb{P}_k(\{\mathbf{P}_k \leq \alpha I\})$.*

2) *LQG problems:* Given an integer N , and let W , Λ , W_N , $\{W_k\}$ and $\{\Lambda_k\}$ for $1 \leq k \leq N-1$ be positive definite matrices. Define the finite horizon cost function:

$$J_N(\pi_{N-1}, \bar{x}_0, P_0) = \mathbb{E}[x'_N W_N x_N + \sum_{k=0}^{N-1} x'_k W_k x_k + \nu_k u'_k \Lambda_k u_k | \pi_{N-1}, \bar{x}_0, P_0]$$

and the infinite horizon cost function:

$$J_\infty(\pi_\infty, \bar{x}_0, P_0) = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E}[\sum_{k=0}^{N-1} x'_k W_k x_k + \nu_k u'_k \Lambda_k u_k | \pi_{N-1}, \bar{x}_0, P_0],$$

where $\pi_{N-1} = \{u_0, \dots, u_{N-1}\}$ and $\pi_\infty = \{u_0, \dots, u_k, \dots\}$ stand for sequences of control inputs. Each u_k is a function of \mathcal{I}_k , i.e., $u_k = f_k(\mathcal{I}_k)$. For the optimal control, some assumptions are given as follows:

Assumption 2. *The pair (A, B) is controllable, and the pair $(A, W^{1/2})$ is observable.*

Problem 4. *Determine the optimal control sequence, denoted by π_{N-1}^* (or π_∞^*), minimizing the cost function J_N (or J_∞),*

$$J_N^* \triangleq J_N(\pi_{N-1}^*, \bar{x}_0, P_0) = \min_{\pi_{N-1}} J_N(\pi_{N-1}, \bar{x}_0, P_0)$$

$$J_\infty^* \triangleq J_\infty(\pi_\infty^*, \bar{x}_0, P_0) = \min_{\pi_\infty} J_\infty(\pi_\infty, \bar{x}_0, P_0).$$

The closed-loop system, whenever mentioned in the following, refers to the SS-UDP system with the feedback control

$u_k = L_\infty \hat{\mathbf{x}}_k$, where L_∞ is the infinite horizon LQG control gain. The closed-loop system is consisted of three dynamic subsystems, whose states are the plant x_k , the smart sensor $\hat{\mathbf{x}}_k^s$, and the estimator $\hat{\mathbf{x}}_k$, respectively. It has been pointed out in [31] that even when the infinite LQG control exists, the controller does not necessarily stabilize the closed-loop system. Hence, we consider the following problem.

Definition 4 (Mean square stability (MMS)). *The closed-loop system is said to be mean square (m.s.) stable, if $\mathbb{E}[\|x_k\|^2] < +\infty$, $\mathbb{E}[\|\hat{\mathbf{x}}_k\|^2] < +\infty$, and $\mathbb{E}[\|\hat{\mathbf{x}}_k^s\|^2] < +\infty$ for all $k \in \mathbb{N}$.*

Problem 5. *Determine the condition under which the closed-loop system is m.s. stable.*

Our objectives are to solve these five problems above. The following table provides a reference for their answers. The column of main topics explains in which problem the terms “stability, convergence, and performance” in the title of this paper are involved.

Problems	Answers	Main topics
1	Theorem 1	Convergence
2	Theorem 2	Stability
3	Theorem 3	Estimation Performance
4	Theorems 4 and 5(i)	LQG control
5	Theorem 5(ii)	Stability

III. OPTIMAL ESTIMATOR AND ITS PROPERTIES

A. Smart sensor based optimal estimator

As assumed in Condition 1, the smart sensor is able to run standard Kalman filter to obtain the local optimal estimate $\hat{\mathbf{x}}_k^s$ and its corresponding estimation error covariance S_k as follows:

$$\bar{\mathbf{x}}_{k+1}^s = A\hat{\mathbf{x}}_k^s + \nu_k B u_k = A\hat{\mathbf{x}}_k^s + B u_k^a \quad (4)$$

$$\bar{S}_{k+1} = A S_k A' + Q \quad (5)$$

$$K_{k+1} = \bar{S}_{k+1} C' (C \bar{S}_{k+1} C' + R)^{-1}$$

$$\hat{\mathbf{x}}_{k+1}^s = \bar{\mathbf{x}}_{k+1}^s + K_{k+1} (y_{k+1} - C \bar{\mathbf{x}}_{k+1}^s)$$

$$S_{k+1} = (I - K_{k+1} C) \bar{S}_{k+1}, \quad (6)$$

with $\hat{\mathbf{x}}_0^s = \bar{x}_0$ and $S_0 = P_0$.

Lemma 1. *For the SS-UDP system, the optimal estimator is the following.*

$$\hat{\mathbf{x}}_k = \gamma_k \hat{\mathbf{x}}_k^s + (1 - \gamma_k) (A \hat{\mathbf{x}}_{k-1} + \nu B u_{k-1}) \quad (7)$$

$$\mathbf{P}_k = \gamma_k S_k + (1 - \gamma_k) (A \mathbf{P}_{k-1} A' + Q + U_{k-1}), \quad (8)$$

with $\hat{\mathbf{x}}_0 = \bar{x}_0$ and $\mathbf{P}_0 = P_0$, where $U_{k-1} \triangleq \bar{\nu} \nu B u_{k-1} u'_{k-1} B'$ and $\bar{\nu} = 1 - \nu$.

B. Construction of probability spaces

Problems 1, 2, and 3 in Section II are closely related to \mathbf{P}_k . From (8), it is clear that \mathbf{P}_k is a random variable. The probability space of \mathbf{P}_k and its limit play an important role in solving these problems. Therefore, in this section, we first construct the finite product probability space $(\Omega_k, \mathcal{F}_k, \mathbb{P}_k)$ of \mathbf{P}_k . Then we construct an infinite product probability space $(\Omega_\infty, \mathcal{F}_\infty, \mathbb{P}_\infty)$ —which characterizes the behavior of the limit

of \mathbb{P}_k —and a measurable function (i.e., a random variable) on it. Based on them, these problems are solved in Section III-C, -D, and -E, respectively.

We start with some definitions relevant to the probability space.

Definition 5 ([41]). *Some definitions relevant to **field** are given as follows.*

- (Field and σ -field) Let \mathcal{F} be a collection of subsets of a set Ω . Then \mathcal{F} is called a field if (i) $\Omega \in \mathcal{F}$; (ii) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$; (iii) If $A_1, \dots, A_n \in \mathcal{F}$, $\bigcup_{i=1}^n A_i \in \mathcal{F}$. \mathcal{F} is called a σ -field, if (iii) is replaced by “If $A_1, A_2, \dots \in \mathcal{F}$, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.”
- (σ -field generated by sets) Let \mathcal{L} be a collection of subsets of Ω , a σ -field \mathcal{F}_0 is called the σ -field generated by \mathcal{L} , denoted by $\sigma(\mathcal{L})$, if \mathcal{F}_0 is included in any σ -field containing \mathcal{L} .

Definition 6 ([41]). *Some definitions relevant to **measure** are given as follows.*

- (Finitely and countably additive) Let \mathcal{F} be a σ -field of a set Ω . A set function $\mu : \mathcal{F} \rightarrow \mathbb{R}$ is called countably additive on \mathcal{F} , if whenever A_1, A_2, \dots form a finite or countable collection of disjoint sets in \mathcal{F} whose union also belong to \mathcal{F} , we have $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$. If this equity holds only for finite collections of disjoint sets in \mathcal{F} , then μ is called finitely additive.
- (Measure) A measure on a σ -field (or field) is a non-negative real value countably additive set function μ . If $\mu(\Omega) = 1$, μ is called a probability measure, usually denoted by \mathbb{P} .
- (Probability space) A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a set and is usually called the sample space, \mathcal{F} is a σ -field on Ω , and \mathbb{P} is probability measure on \mathcal{F} .
- (Measurable function) Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be measurable spaces. A function $h : \Omega_1 \rightarrow \Omega_2$ is said to be $\mathcal{F}_1 - \mathcal{F}_2$ measurable (or measurable for short) if $h^{-1}(A) \in \mathcal{F}_1$ for each $A \in \mathcal{F}_2$.
- (σ -finite) A nonnegative, finitely additive set function μ on the field \mathcal{F}_1 is called σ -finite if Ω can be written as $\bigcup_{i=1}^{\infty} A_i$ where $A_i \in \mathcal{F}_1$ and $\mu(A_i) < \infty$.

1) *Probability space $(\Omega_k, \mathcal{F}_k, \mathbb{P}_k)$: $\{\gamma_k\}$ is a sequence of i.i.d. Bernoulli random variables. According to the definition above, we denote the probability space for each γ_k by $(\Omega_\gamma, \mathcal{F}_\gamma, \mathbb{P}_\gamma)$ where $\Omega_\gamma = \{0, 1\}$, \mathcal{F}_γ is the conventional σ -field for Ω_γ , i.e., $\mathcal{F}_\gamma = \{\emptyset, \Omega_\gamma, \{0\}, \{1\}\}$. \mathbb{P}_γ is the probability measure on \mathcal{F}_γ with $\mathbb{P}_\gamma(\gamma_k = 1) = \gamma$, where the subscript γ is used to distinguish those \mathbb{P}_k and \mathbb{P}_∞ defined later. In the sequel, we construct the probability space of \mathbf{P}_k .*

Ω_k : It can be seen from (8) that \mathbf{P}_k depends on $(\gamma_k, \dots, \gamma_1)$, which may take 2^k different values. The sample space Ω_k is the set consisting of all these 2^k points. The point in Ω_k is denoted by $\theta \triangleq (\gamma_k, \dots, \gamma_1)$.

\mathcal{F}_k : We take the σ -field \mathcal{F}_k to consist of all subsets of Ω_k , i.e., $\mathcal{F}_k \triangleq \{A \subseteq \Omega_k\}$.

\mathbb{P}_k : The probability measure \mathbb{P}_k is a set function: $\mathcal{F}_k \rightarrow$

$[0, 1]$, which can be defined in the following way. For each single point set $\{\theta\} = \{(\gamma_k, \dots, \gamma_1)\} \subseteq \Omega_k$, due to the mutual independence of $(\gamma_k, \dots, \gamma_1)$, $\mathbb{P}_k(\{\theta\}) = \prod_{i=1}^k \mathbb{P}_\gamma(\gamma_i)$. Then we can assign each set $A \in \mathcal{F}_k$ a probability measure in such a way that $\mathbb{P}_k(A) \triangleq \sum_{\theta \in A} \mathbb{P}_k(\{\theta\})$. It is easy to check that \mathbb{P}_k is indeed a probability measure.

2) *Random variable \mathbf{P}_k* : As a random variable, \mathbf{P}_k is in fact a measurable function mapping Ω_k to $\mathbb{R}^{n \times n}$. To better describe \mathbf{P}_k , we will define some subsets of Ω_k . On each subset, \mathbf{P}_k takes a constant value.

Recall that Ω_k consists of points of the form $\theta = (\gamma_k, \dots, \gamma_1)$. We define $F_k^{[k]} \triangleq \{(0, \dots, 0)\}$, and for $0 \leq i \leq k-1$,

$$F_k^{[i]} \triangleq \left\{ \begin{pmatrix} 0 & \dots & 0 & 1 & \times & \dots & \times \\ 1 & \dots & i & i+1 & i+2 & \dots & k \end{pmatrix} \right\}$$

where from left to right the first “1” occurs in the $i+1$ th position. That is, for each $\theta \in F_k^{[j]}$, we have

$$\theta = \{\gamma_k = 0, \dots, \gamma_{k-j+1} = 0, \gamma_{k-j} = 1, \times, \dots, \times\}.$$

Take $k = 3$ for example:

$$F_3^{[0]} = \{(1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}, \quad F_3^{[1]} = \{(0, 1, 0), (0, 1, 1)\}, \quad F_3^{[2]} = \{(0, 0, 1)\}, \quad F_3^{[3]} = \{(0, 0, 0)\}.$$

Proposition 1. *For the subsets $F_k^{[i]}$, the following facts hold.*

- (i) $F_k^{[i]} \cap F_k^{[j]} = \emptyset$ for $0 \leq i \neq j \leq k$ (\emptyset means an empty set).
- (ii) $\Omega_k = \bigcup_{i=0}^k F_k^{[i]}$.
- (iii) $\mathbb{P}_k(F_k^{[j]}) = \gamma \bar{\gamma}^j$ for $0 \leq j \leq k-1$, and $\mathbb{P}_k(F_k^{[k]}) = \bar{\gamma}^k$.

Define two functions $\psi_k(P)$ and $\Psi_k^{[i]}(P)$ as follows:

$$\psi_k(P) = APA' + Q + U_k \quad (9)$$

$$\Psi_k^{[i]}(P) = \begin{cases} P, & \text{for } i = 0 \\ \psi_{k-1} \circ \dots \circ \psi_{k-i}(P), & \text{for } 1 \leq i \leq k \end{cases} \quad (10)$$

For $0 \leq i \leq k$, we define

$$P_k^{[i]} \triangleq \Psi_k^{[i]}(S_{k-i}), \quad (11)$$

where S_k is computed by (5) and (6).

Lemma 2. *For $\forall k$ and $\forall \{U_k\}$, $P_k^{[n]} \leq P_k^{[n+1]}$, $0 \leq n \leq k-1$.*

Proposition 2. *As a function, \mathbf{P}_k can be described as follows:*

$$\mathbf{P}_k(\theta) = P_k^{[j]}, \text{ for } \theta \in F_k^{[j]} \subseteq \Omega_k.$$

3) *Probability space $(\Omega_\infty, \mathcal{F}_\infty, \mathbb{P}_\infty)$* : As $k \rightarrow \infty$, Ω_k becomes an infinite product space $\Omega_\infty \triangleq \Omega_\gamma \times \Omega_\gamma \times \dots$. To study the behavior of the limit of \mathbf{P}_k , we construct the probability space $(\Omega_\infty, \mathcal{F}_\infty, \mathbb{P}_\infty)$ as follows.

Ω_∞ : Since $\Omega_\infty = \Omega_\gamma \times \Omega_\gamma \times \dots$, it consists of infinitely many points and each point $\theta \in \Omega_\infty$ takes the form $\theta = (\theta_1, \dots, \theta_i, \dots)$ with $\theta_i = 0$ or 1 for $i \in \mathbb{N}$.

\mathcal{F}_∞ : Clearly, Ω_∞ is an uncountable set. A well known result in the measure theory is that if the σ -field consists of all the subsets of an uncountable set Ω_∞ , then there is no measure on the σ -field [40].

Therefore, we have to equip Ω_∞ with a suitable σ -field and then construct a probability measure on it. Define some countable subsets of Ω_∞ as follows: Let $A = \{(0, 0, \dots)\}$, and let

$$A^{[i]} = \left\{ \begin{pmatrix} 0 & \dots & 0 & 1 & \times & \dots \\ & & 1 & \dots & i & i+1 & i+2 & \dots \end{pmatrix} \right\}.$$

For the convenience of indexing the sets A and $A^{[i]}$, we define $A^{[\infty]} \triangleq A$. Let \mathcal{A}_0 be a collection of sets $\{\emptyset, \Omega_\infty, A^{[0]}, A^{[1]}, \dots, A^{[\infty]}\}$, and let \mathcal{A} be a collection of all the finite and countable unions of subsets of \mathcal{A}_0 . That is, if $B \in \mathcal{A}$, there exists a finite or countable set

$$\mathcal{N}_B = \{n_1, n_2, \dots\} \subseteq \overline{\mathbb{Z}}_+ \quad (12)$$

such that $B = \bigcup_{i \in \mathcal{N}_B} A^{[i]}$.

Define $\mathcal{F}_\infty = \sigma(\mathcal{A})$, the σ -field generated by \mathcal{A} .

\mathbb{P}_∞ : Generally, it is difficult to explicitly define a measure \mathbb{P} by assigning a value for every set in \mathcal{F}_∞ . Conventionally, one can define a desired measure \mathbb{P}^* on some subsets of interest, such as \mathcal{A} , and then extend \mathbb{P}^* to a probability measure \mathbb{P} on \mathcal{F}_∞ . Such \mathbb{P} is identical to \mathbb{P}^* when acting on \mathcal{A} . Following this idea, we define a set function $\mathbb{P}^* : \mathcal{A} \rightarrow [0, 1]$ in such a way that

$$\mathbb{P}^*(A^{[\infty]}) = 0, \text{ and } \mathbb{P}^*(A^{[i]}) = \gamma\bar{\gamma}^i \triangleq p_i, \quad (13)$$

and

$$\mathbb{P}^*(B) \triangleq \sum_{i \in \mathcal{N}_B} p_i, \text{ for } B \in \mathcal{A}, \quad (14)$$

where $B = \bigcup_{i \in \mathcal{N}_B} A^{[i]}$. The existence of \mathbb{P} on \mathcal{F}_∞ is formulated later in Proposition 5.

The following proposition lists some properties of the structure of \mathcal{A} .

Proposition 3. *The following facts hold.*

- (i) $A^{[i]} \cap A^{[j]} = \emptyset$ for $i \neq j \in \overline{\mathbb{Z}}_+$.
- (ii) $\Omega_\infty = \bigcup_{i \in \overline{\mathbb{Z}}_+} A^{[i]}$, i.e., $\Omega_\infty = A \bigcup (\bigcup_{i=0}^\infty A^{[i]})$.
- (iii) Given a set $B \in \mathcal{A}$, the set \mathcal{N}_B mentioned in (12) is unique. If B_i and B_j are disjoint, so are \mathcal{N}_{B_i} and \mathcal{N}_{B_j} . Moreover, $B_i \bigcup B_j = \bigcup_{k \in \mathcal{N}_{B_i} \cup \mathcal{N}_{B_j}} A^{[k]}$.
- (iv) Let $\{B_i\}$ be a sequence of mutually disjoint sets. Then $\bigcup_{i=1}^\infty B_i = \bigcup_{k \in \bigcup_{i=1}^\infty \mathcal{N}_{B_i}} A^{[k]}$.

To extend \mathbb{P}^* to a probability measure on \mathcal{F}_∞ , we require the famous measure extension theorem as follows.

Lemma 3 (Carathéodory measure extension theorem [41]). *Let Ω^* be a nonempty set, and let μ^* be a measure on the field \mathcal{F}_0 of subsets of Ω^* , and assume that μ^* is σ -finite on \mathcal{F}_0 , then μ^* has a unique extension to a measure μ on $\sigma(\mathcal{F}_0)$ such that $\mu(A) = \mu^*(A)$ for $A \in \mathcal{F}_0$.*

The following proposition shows that \mathcal{A} and \mathbb{P}^* meet the requirements in the preceding measure extension theorem.

Proposition 4. *The following facts hold. (i) \mathcal{A} is a field; (ii) \mathbb{P}^* is a measure on the field \mathcal{A} ; and (iii) \mathbb{P}^* is σ -finite on \mathcal{A} .*

Based on Propositions 3, 4 and the measure extension theorem, the existence of the probability measure \mathbb{P} on \mathcal{F}_∞ is shown as follows.

Proposition 5. *There exists a probability measure \mathbb{P}_∞ on \mathcal{F}_∞ such that $\mathbb{P}_\infty(A) = \mathbb{P}^*(A)$ for $\forall A \in \mathcal{A}_0$, that is, $\mathbb{P}_\infty(\emptyset) = 0$, $\mathbb{P}_\infty(\Omega_\infty) = 1$, and $\mathbb{P}_\infty(A^{[i]}) = \mathbb{P}^*(A^{[i]})$ for $\forall i \in \overline{\mathbb{Z}}_+$.*

4) *Construction of the random variable \mathbf{P} on Ω_∞ :*

In the following, we construct a random variable \mathbf{P} on $(\Omega_\infty, \mathcal{F}_\infty, \mathbb{P}_\infty)$, which will be the limit of \mathbf{P}_k . For a probability space, a random variable is in fact a measurable function. Hence, to construct \mathbf{P} , we only need to designate its domain and the values it takes. Define two functions

$$\psi(P) = APA' + Q + U \quad (15)$$

$$\Psi^{[m]}(P) = \begin{cases} P, & \text{for } m = 0 \\ \underbrace{\psi \circ \dots \circ \psi(P)}_{m \text{ times}}, & \text{for } m \geq 1. \end{cases} \quad (16)$$

It follows from [42, Theorem 17.53] that under Assumption 1 \bar{S}_k and S_k in (5) and (6) converge, and their limits, denoted by \bar{S} and S respectively, are positive definite. Define

$$S^{[m]} \triangleq \Psi^{[m]}(S), \text{ for } m \geq 0, \quad (17)$$

and then define a random variable $\mathbf{P} : \Omega_\infty \mapsto \mathbb{R}^{n \times n}$ as

$$\mathbf{P}(\theta) = \begin{cases} 0, & \text{for } \theta \in A^{[\infty]} \\ S^{[m]}, & \text{for } \theta \in A^{[m]}, m \in \mathbb{Z}_+. \end{cases} \quad (18)$$

From the definition above, the range of \mathbf{P} is countable. For any subset $M \subseteq \mathbb{R}^{n \times n}$, the inverse image $\mathbf{P}^{-1}(M)$ consists of finite or countable subsets $A^{[i]}$, i.e., the union of these subsets $A^{[i]}$ ($\mathbf{P}^{-1}(M) = \bigcup A^{[i]}$). From the construction of \mathcal{A} , it is clear that $\bigcup A^{[i]} \in \mathcal{A}$, which indicates that \mathbf{P} is a measurable function, that is, a random variable. To formulate the distribution of \mathbf{P} , we give a lemma as follows.

Denote the maximum singular values of $S^{[m]}$ and $P_k^{[m]}$ by $\lambda_S^{[m]}$ and $\lambda_{P_k}^{[m]}$, respectively.

Lemma 4. *The following facts hold.*

- (i) $\{\lambda_S^{[n]}\}$ for $n \in \mathbb{Z}_+$ is a strictly increasing sequence.
- (ii) $\lambda_S^{[n]}$ is an increasing sequence, and $\lim_{n \rightarrow \infty} \lambda_S^{[n]} = \infty$.
- (iii) For given $\alpha \geq \lambda_S$ and $U \geq 0$, there exists a unique integer $n_\alpha \in \mathbb{N}$ such that $\lambda_S^{[n_\alpha]} \leq \alpha < \lambda_S^{[n_\alpha+1]}$. The following function is well-defined

$$n_\alpha = n(\alpha, U). \quad (19)$$

For brevity, denote the subset $\{\theta \in \Omega_k | \mathbf{P}_k(\theta) \leq \alpha I\}$ by $\{\mathbf{P}_k \leq \alpha I\}$ and $\{\theta \in \Omega_\infty | \mathbf{P}(\theta) \leq \alpha I\}$ by $\{\mathbf{P} \leq \alpha I\}$.

Proposition 6. *The distribution of \mathbf{P} is the following.*

- (i) $\mathbb{P}_\infty(\{\mathbf{P} = S^{[m]}\}) = \gamma\bar{\gamma}^m$ for $m \in \overline{\mathbb{Z}}_+$. ($\gamma\bar{\gamma}^\infty \triangleq 0$)
- (ii) $\mathbb{P}_\infty(\{\mathbf{P} \leq \alpha I\}) = 1 - \bar{\gamma}^{n_\alpha+1}$, where $n_\alpha = n(\alpha, U)$ and $\alpha \geq \lambda_S$.

C. Convergence of \mathbf{P}_k

Based on the probability spaces constructed above, we deal with Problem 1 in this section, and the result is formulated in the following theorem.

Theorem 1. *If U_k is convergent, then \mathbf{P}_k converges to \mathbf{P} in distribution, i.e., $\mathbf{P}_k \xrightarrow{d} \mathbf{P}$, where \mathbf{P} is constructed by (15)-(18).*

To prove Theorem 1, some lemmas are given as follows.

Lemma 5. *Let X, Y , and S be matrices where $S > 0$, and let $\alpha > 0$ and $\varepsilon > 0$ be real numbers. The following facts hold.*

- (i) *If $\|X - Y\| < \varepsilon$, then $-\varepsilon I < X - Y < \varepsilon I$.*
- (ii) *Let $d = \alpha - \lambda_S$. If $d > 0$, then $\alpha I - S \geq dI$.*

Lemma 6. *Suppose $\lim_{k \rightarrow \infty} S_k = S$ and $\lim_{k \rightarrow \infty} U_k = U$. Given an integer $n > 0$ and a real number $\alpha > \lambda_S$, if $\lambda_S^{[n]} < (>) \alpha$, there is an integer $N_n > 0$ such that $\lambda_{P_k}^{[n]} < (>) \alpha$ for $k > N_n$.*

Lemma 7. *Suppose that $\{U_k\}$ is convergent. Let $\alpha > \lambda_S$ but $\alpha \notin \{\lambda_S^{[1]}, \lambda_S^{[2]}, \dots\}$, and let n_α be the unique integer satisfying $\lambda_S^{[n_\alpha]} < \alpha < \lambda_S^{[n_\alpha+1]}$. Then there exists an integer $N > 0$ such that for $\forall k > N$,*

- (i) $\lambda_{P_k}^{[n_\alpha]} < \alpha < \lambda_{P_k}^{[n_\alpha+1]}$.
- (ii) $\mathbb{P}_k(\{\mathbf{P}_k \leq \alpha I\}) = 1 - \bar{\gamma}^{n_\alpha+1}$.

Proof of Theorem 1: From Proposition 6 (i), it follows that: 1) For the multi-variable case, when $X = S^{[m]}$ with $m \in \mathbb{Z}_+$, $\mathbb{P}_\infty(\{\mathbf{P} = X\}) \neq 0$; 2) For the scalar case, $S^{[m]} = \lambda_S^{[m]}$, and thus when $x = \lambda_S^{[m]}$ with $m \in \mathbb{Z}_+$, $\mathbb{P}_\infty(\{\mathbf{P} = x\}) \neq 0$. Therefore, according to the definition of convergence in distribution, to avoid the possibility that $\mathbb{P}_\infty(\{\mathbf{P} = \alpha I\}) \neq 0$, we consider $\alpha \in (\lambda_S, \infty)$ but $\alpha \notin \{\lambda_S^{[1]}, \lambda_S^{[2]}, \dots\}$.

It follows from Lemma 4(iii) and Proposition 6(ii) that for a given $\alpha > \lambda_S$, there exists an integer n_α such that $\lambda_S^{[n_\alpha]} < \alpha < \lambda_S^{[n_\alpha+1]}$ and $\mathbb{P}_\infty(\{\mathbf{P} \leq \alpha I\}) = 1 - \bar{\gamma}^{n_\alpha+1}$. From Lemma 7, it follows that there exists an integer N such that for $k > N$, $\mathbb{P}_k(\{\mathbf{P}_k \leq \alpha I\}) = 1 - \bar{\gamma}^{n_\alpha+1} = \mathbb{P}_\infty(\{\mathbf{P} \leq \alpha I\})$, which implies $\mathbb{P}_k(\{\mathbf{P}_k \leq \alpha I\})$ converges to $\mathbb{P}_\infty(\{\mathbf{P} \leq \alpha I\})$. Therefore, according to Definition 2, we have $\mathbf{P}_k \xrightarrow{d} \mathbf{P}$. ■

D. Stability of $\mathbb{E}[\mathbf{P}_k]$

In this section, we deal with Problem 2. Before formulating the result, we give a lemma as follows.

Lemma 8. $\mathbb{E}[\mathbf{P}_k] = P_k^{[k]} \bar{\gamma}^k + \sum_{n=0}^{k-1} P_k^{[n]} \gamma \bar{\gamma}^n$.

Theorem 2. *For the SS-UDP system in (1) with bounded inputs, under Assumption 1, if $\bar{\gamma} < \lambda_A^{-2}$, $\mathbb{E}[\mathbf{P}_k]$ is stable for $\forall P_0 \geq 0$; if $\bar{\gamma} > \lambda_A^{-2}$, there exists a P_0 such that $\mathbb{E}[\mathbf{P}_k] \rightarrow \infty$.*

Proof: Under Assumption 1, S_k is convergent. Then it is bounded. Denote its bound by S_b . Let \bar{U} be the bound for U_k . \bar{U} exists by hypothesis. From (37), we have that for $0 \leq n \leq k$

$$\begin{aligned} P_k^{[n]} &\leq (\lambda_A^2)^n S_b + (\bar{U} + Q) \sum_{i=1}^n (\lambda_A^2)^{n-i} \\ &\leq (\lambda_A^2)^n (S_b + (\bar{U} + Q)/(\lambda_A^2 - 1)) = (\lambda_A^2)^n S^*, \end{aligned}$$

where $S^* \triangleq S_b + (\bar{U} + Q)/(\lambda_A^2 - 1)$. By Lemma 8,

$$\begin{aligned} \mathbb{E}[\mathbf{P}_k] &= \sum_{n=0}^{k-1} P_k^{[n]} \gamma \bar{\gamma}^n + P_k^{[k]} \bar{\gamma}^k \\ &\leq \gamma S^* \sum_{n=0}^{k-1} (\bar{\gamma} \lambda_A^2)^n + S^* (\bar{\gamma} \lambda_A^2)^k \\ &\leq S^* \sum_{n=0}^k (\bar{\gamma} \lambda_A^2)^n \leq S^*/(1 - \bar{\gamma} \lambda_A^2). \end{aligned} \quad (20)$$

Therefore, for $\forall P_0 \geq 0$, $\mathbb{E}[\mathbf{P}_k]$ is bounded.

From (37), we have $P_k^{[n]} \geq A^n S_{k-n} (A^n)'$ for $0 \leq n \leq k$. Then $P_k^{[k]} > A^k S_0 (A^k)'$. From (20), it follows that $\mathbb{E}[\mathbf{P}_k] > \bar{\gamma}^k P_k^{[k]} > \bar{\gamma}^k A^k P_0 (A^k)'$ where $S_0 = P_0$. When $\lambda_A > 1$, there is an eigenvalue λ of A such that $\lambda_A = |\lambda|$. Denote the corresponding eigenvector of λ by $p \neq 0$. By letting $P_0 = pp'$, $\mathbb{E}[\mathbf{P}_k] \geq \bar{\gamma}^k A^k pp' (A^k)' = \bar{\gamma}^k \lambda^k p (\lambda^k p)' = (\bar{\gamma} \lambda_A^2)^k P_0$. Hence, $\mathbb{E}[\mathbf{P}_k] \rightarrow \infty$ as $k \rightarrow \infty$. The proof is completed. ■

Remark 1. *For the SS-UDP system, the stability of $\mathbb{E}[\mathbf{P}_k]$ depends on γ but is independent of ν . Similar phenomenon for the UDP-like system has been proved in [13].*

E. Estimation Performance Evaluation

In this section, we focus on the estimation performance evaluation, but does not consider the design of controller. We adopt the metric $\mathbb{P}_k(\{\mathbf{P}_k \leq \alpha I\})$ to evaluate the performance, as it gives a more complete characterization of the estimation performance [29, 30].

As previously shown, \mathbf{P}_k depends on $\{u_k\}$, so does the computation of $\mathbb{P}_k(\{\mathbf{P}_k \leq \alpha I\})$. However, in some cases, all the values of $\{u_k\}$ are not always known. For example, at the stage of the system design or before the running of system, designers sometimes do not know all the values of $\{u_k\}$ in advance, especially when u_k is computed on-line, and only some partial information of the control sequence maybe available, such as its bound or limit. Therefore, we consider the following cases, in which $\{u_k\}$ is assumed to be bounded.

- Case 1: All the values of $\{u_k\}$ are known;
- Case 2: Only the upper bound \bar{u} of $\{u_k\}$ is known;
- Case 3: $\{u_k\}$ is convergent and its limit u_∞ is known.

Clearly, for case 2 U_k is bounded by $\bar{U} = \nu \bar{\nu} \bar{u}^2 B B'$, and for case 3 U_k converges to $U_\infty = \nu \bar{\nu} B u_\infty u_\infty' B'$.

Let $\Gamma_k(\alpha) \triangleq \mathbb{P}_k(\{\mathbf{P}_k \leq \alpha I\})$.

When $\{u_k\}$ is convergent, Lemma 7 (ii) provides a way to compute $\Gamma_k(\alpha)$. However, it applies to the case that $\alpha \notin M_S$, where $M_S \triangleq \{\lambda_S^{[1]}, \lambda_S^{[2]}, \dots\}$. When $\alpha \in M_S$, we can use $\Gamma_k(\alpha + \epsilon)$ to approximately calculate $\Gamma_k(\alpha)$, where we can choose ϵ to be a sufficiently small quantity, e.g., 10^{-5} . In the following theorem, when Lemma 7 (ii) is used to compute $\Gamma_k(\alpha)$ and $\alpha \in M_S$, we assume that $\Gamma_k(\alpha) = \Gamma_k(\alpha + \epsilon)$.

Theorem 3. *Given $\alpha > 0$, for different cases, we have the following results for $\Gamma_k(\alpha)$:*

- For case 1, $\Gamma_k(\alpha)$ can be computed by Algorithm 1.
- For case 2, there is an integer N such that $\Gamma_k(\alpha) \geq 1 - \bar{\gamma}^{n_b+1} \triangleq p_{bd}$, for $k > N$,

- For case 3, there is an integer N such that $\Gamma_k(\alpha) = 1 - \bar{\gamma}^{n_c+1} \triangleq p_{conv}$, for $k > N$,

where $n_b = n(\alpha, \bar{U})$ and $n_c = n(\alpha, U_\infty)$.

Algorithm 1 Computation of the distribution of \mathbf{P}_k

Initialization: compute $P_k^{[0]}, \dots, P_k^{[k]}$ by (9-11). $\Gamma_k(\alpha) = 0$.
if $P_k^{[k]} \leq \alpha I$ **then**
 $\Gamma_k(\alpha) = 1$
else
 $i = 0$
while $P_k^{[i]} \leq \alpha I$ **do**
 $\Gamma_k(\alpha) = \Gamma_k(\alpha) + \gamma \bar{\gamma}^i$; $i = i + 1$
end while
end if

Proof: For case 1: If $0 < \alpha I < P_k^{[0]}$, then $\alpha I < P_k^{[i]}$ for $0 \leq i \leq k$ due to the monotonicity of $P_k^{[i]}$. Consequently, $\Gamma_k(\alpha) = 0$, which is equal to the result Algorithm 1 yields.

In the following, we consider the case $P_k^{[0]} \leq \alpha I$.

If $P_k^{[k]} \leq \alpha I$, then $\mathbf{P}_k(\theta) \leq \alpha I$ with $\theta \in F_k^{[i]}$ for $0 \leq i \leq k$. From Proposition 1 (ii), we have $\{\mathbf{P}_k < \alpha I\} = \Omega_k$. Then $\Gamma_k(\alpha) = \mathbb{P}_k(\{\mathbf{P}_k < \alpha I\}) = \mathbb{P}_k(\Omega_k) = 1$.

Then we consider the case that $P_k^{[0]} \leq \alpha I < P_k^{[k]}$. Due to the monotonicity of $P_k^{[i]}$, it is clear that there exists an integer $0 \leq n_\alpha < k$ such that $P_k^{[n_\alpha]} \leq \alpha I$ and $P_k^{[n_\alpha+1]} \not\leq \alpha I$. This integer n_α in fact is the largest integer n such that $P_k^{[n]} \leq \alpha I$ holds. From Proposition 2, $\mathbf{P}_k(\theta) = P_k^{[i]}$ for $\theta \in F_k^{[i]}$. Therefore, $\mathbf{P}_k(\theta) \leq \alpha I$ if and only if $\theta \in \bigcup_{i=0}^{n_\alpha} F_k^{[i]}$, which implies $\{\mathbf{P}_k \leq \alpha I\} = \bigcup_{i=0}^{n_\alpha} F_k^{[i]}$. $\Gamma_k(\alpha) = \mathbb{P}_k(\{\mathbf{P}_k \leq \alpha I\}) = \mathbb{P}_k(\bigcup_{i=0}^{n_\alpha} F_k^{[i]}) = \sum_{i=0}^{n_\alpha} \mathbb{P}_k(F_k^{[i]}) = \sum_{i=0}^{n_\alpha} \gamma \bar{\gamma}^i$. Clearly, the routine in the while loop is to find out the n_α and then sum up $\gamma \bar{\gamma}^i$ to obtain $\sum_{i=0}^{n_\alpha} \gamma \bar{\gamma}^i$.

For case 2: Define a random variable $\bar{\mathbf{P}}_k$ on Ω_k by letting U_k in ψ_k be a constant quantity, i.e., $U_k = \bar{U}$. Then define $\bar{\psi}(P) = APA' + Q + \bar{U}$, and

$$\bar{\Psi}^{[m]}(P) = \begin{cases} P, & \text{for } m = 0 \\ \underbrace{\bar{\psi} \circ \dots \circ \bar{\psi}}_m(P), & \text{for } 1 \leq m \leq k. \end{cases}$$

Define $\bar{P}_k^{[i]} \triangleq \bar{\Psi}^{[i]}(S_{k-i})$ and $\bar{\mathbf{P}}_k(\theta) = \bar{P}_k^{[i]}$ for $\theta \in F_k^{[i]}$.

Note that $\bar{\psi}^{[m]}(P) \geq \psi_k^{[m]}(P)$ for $\forall P \geq 0$. By comparing the definitions of $\bar{\Psi}^{[m]}(P)$ and $\Psi_k^{[m]}(P)$, it is easy to obtain that $\bar{\Psi}^{[m]}(P) \geq \Psi_k^{[m]}(P)$ for $\forall P \geq 0$. Thus, $\bar{P}_k^{[i]} = \bar{\Psi}^{[i]}(S_{k-i}) \geq \Psi_k^{[i]}(S_{k-i}) = P_k^{[i]}$. Then we have $\bar{\mathbf{P}}_k(\theta) \geq \mathbf{P}_k(\theta)$ for $\forall \theta \in \Omega_k$.

Let $u_k \equiv \bar{u}$. Obviously, $\{u_k\}$ is a convergent sequence. It follows from Lemma 7 (ii) that there is an integer N such that for $k > N$, $\mathbb{P}_k(\{\bar{\mathbf{P}}_k \leq \alpha I\}) = 1 - \bar{\gamma}^{n_b+1} = p_{bd}$ where $n_b = n(\alpha, \bar{U})$. Since $\bar{\mathbf{P}}_k(\theta) \geq \mathbf{P}_k(\theta)$, we have $\{\bar{\mathbf{P}}_k \leq \alpha I\} \subset \{\mathbf{P}_k \leq \alpha I\}$. Then $\mathbb{P}_k(\{\mathbf{P}_k \leq \alpha I\}) \geq \mathbb{P}_k(\{\bar{\mathbf{P}}_k \leq \alpha I\}) = p_{bd}$.

For case 3: The case 3 is the result of Lemma 7 (ii). \blacksquare

Estimation performance evaluation: For different cases, the results on the computation of $\Gamma_k(\alpha)$ are formulated in Theorem 3. For cases 2 and 3, without full knowledge of $\{u_k\}$,

Algorithm 1 does not work and the exact value of cannot be obtained. When k is sufficiently large, p_{bd} is a lower bound for $\Gamma_k(\alpha)$ and p_{conv} is identical to $\Gamma_k(\alpha)$. They can be viewed as long term evaluations for estimation performance.

Remark 2. In Algorithm 1, at the time k , it requires to compute $k + 1$ quantities, i.e., $P_k^{[0]}, \dots, P_k^{[k]}$. The amounts of computation and PC memory increase linearly with time, which is not suitable for the system running over a long time. Hence, even when the full knowledge of $\{u_k\}$ is known, to avoid the time-consuming computation, one can use the results in cases 2 and 3 to obtain an approximation.

IV. OPTIMAL CONTROL

In this section, we start with some useful lemmas, and then give answers to Problems 4 and 5 in Theorems 4 and 5.

A. Finite Horizon LQG control

From (8), we have

$$\mathbb{E}[\mathbf{P}_{k+1}|\mathcal{I}_k] = \bar{\gamma}(A\mathbf{P}_k A' + Q + \bar{\nu}\nu B u_k u_k' B') + \gamma S_{k+1}. \quad (21)$$

Define the optimal value function

$$V_N(x_N) = \mathbb{E}[x_N' W_N x_N | \mathcal{I}_N] \quad (22)$$

$$V_k(x_k) = \min_{u_k} \mathbb{E}[x_k' W_k x_k + \nu_k u_k' \Lambda_k u_k + V_{k+1}(x_{k+1}) | \mathcal{I}_k]. \quad (23)$$

Lemma 9. $V_k(x_k)$ can be calculated as follows.

$$V_k(x_k) = \mathbb{E}[x_k' M_k x_k | \mathcal{I}_k] + \text{tr}(\Delta_k) + \text{tr}(H_k \mathbf{P}_k) \quad (24)$$

where $H_k = T_k - M_k$,

$$M_k = A' M_{k+1} A + W_k - \nu A' M_{k+1} B \times (\Lambda_k + B'(\alpha M_{k+1} + \bar{\alpha} T_{k+1}) B)^{-1} B' M_{k+1} A \quad (25a)$$

$$T_k = \gamma A' M_{k+1} A + \bar{\gamma} A' T_{k+1} A + W_k \quad (25b)$$

$$\Delta_k = \Delta_{k+1} + (\bar{\gamma} T_{k+1} + \gamma M_{k+1}) Q + \gamma H_{k+1} S_{k+1}, \quad (25c)$$

with $M_N = T_N = W_N$ and $\Delta_N = 0$.

Theorem 4. For the SS-UDP system, the optimal control minimizing J_N is $u_k = L_k \hat{\mathbf{x}}_k$, where

$$L_k = -(\Lambda_k + B'(\alpha M_{k+1} + \bar{\alpha} T_{k+1}) B)^{-1} B' M_{k+1} A \quad (26)$$

and the corresponding cost function

$$J_N^* = \mathbb{E}[x_0' M_0 x_0] + \text{tr}(T_0 - M_0) P_0 + \sum_{j=1}^N \text{tr}((\bar{\gamma} T_j + \gamma M_j) Q + \gamma (T_j - M_j) S_j) \quad (27)$$

where M_k and T_k are computed by (25).

Proof: From the derivation of $V_k(x_k)$ in Lemma 9, it is clear that $u_k = L_k \hat{\mathbf{x}}_k$ is the optimal control law minimizing the cost function. It follows from the dynamic programming theory that $J_N^* = V_0(x_0)$. Then (27) can be easily obtained by computing $V_0(x_0)$ via (24)(25c). The proof is completed. \blacksquare

Remark 3. It is known that the separation principle holds for the TCP-like system but fails for the UDP-like one [14]. For

the SS-UDP system, the feedback gain L_k can be computed offline by (25a)(25b)(26), but $\hat{\mathbf{x}}_k$ and \mathbf{P}_k in (7) and (8) depend on u_{k-1} . Thus, the separation principle still does not hold.

B. Infinite Horizon LQG control

Let $W_N = W_k = W$ and $\Lambda_k = \Lambda$. Define two functions

$$\begin{aligned}\phi_M(M, T) &= A'MA + W \\ &\quad - \nu A'MB(\Lambda + B'(\alpha M + \bar{\alpha}T)B)^{-1}B'MA \\ \phi_T(M, T) &= \bar{\gamma}A'TA + \gamma A'MA + W\end{aligned}$$

where $\bar{\alpha} = \bar{\gamma}\bar{\nu}$ and $\alpha = 1 - \bar{\alpha}$.

If $\lim_{N \rightarrow \infty} \frac{1}{N} J_N^*$ exists, so does the optimal control for the infinite horizon case. The existence of this limit depends on the convergences of $M_{k+1} = \phi_M(M_k, T_k)$ and $T_{k+1} = \phi_T(M_k, T_k)$, which has been studied in Lemmas 8.2 and 8.3 of [14]. The result is formulated as follows.

Lemma 10 ([14]). *There are $M_\infty > 0$ and $T_\infty > 0$ such that*

$$M_\infty = \phi_M(M_\infty, T_\infty) \text{ and } T_\infty = \phi_T(M_\infty, T_\infty), \quad (28)$$

if and only if $M_{k+1} = \phi_M(M_k, T_k)$ and $T_{k+1} = \phi_T(M_k, T_k)$ converge to M_∞ and T_∞ , respectively.

Theorem 5. *For the SS-UDP system, if there exist solutions $M_\infty > 0$ and $T_\infty > 0$ for (28), then*

(i) *the LQG control for infinite horizon case exists, i.e., $u_k = L_\infty \hat{\mathbf{x}}_k$,*

$$L_\infty = (\Lambda + B'(\alpha M_\infty + \bar{\alpha}T_\infty)B)^{-1}B'M_\infty A,$$

and

$$J_\infty = \gamma \text{tr}(T_\infty - M_\infty)S + \text{tr}(\bar{\gamma}T_\infty + \gamma M_\infty)Q. \quad (29)$$

(ii) *The closed-loop system is mean square stable.*

Proof of part (i): It follows from Lemma 10 that M_k and T_k converge, which implies the existence of the optimal control. L_∞ and J_∞ can be obtained by taking the limits of L_k in (26) and $\frac{1}{N} J_N^*$ in (27).

Proof of part (ii): Let $e_k^s = x_k - \hat{\mathbf{x}}_k^s$. Note that under Assumption 1, $\mathbb{E}[\|e_k^s\|^2] = \mathbb{E}[(e_k^s)'e_k^s] = \text{tr}(S_k)$ is convergent and then is bounded. Thus, if $\mathbb{E}[\|x_k\|^2]$ is bounded, so is $\mathbb{E}[\|\hat{\mathbf{x}}_k^s\|^2]$. Similarly, due to $x_k = \hat{\mathbf{x}}_k + e_k$, if $\mathbb{E}[\|\hat{\mathbf{x}}_k\|^2]$ and $\mathbb{E}[\|e_k\|^2]$ are bounded, then $\mathbb{E}[\|x_k\|^2]$ is bounded. Therefore, the MSS of the closed-loop system is studied by examining the boundedness of $\mathbb{E}[\|\hat{\mathbf{x}}_k\|^2]$ and $\mathbb{E}[\|e_k\|^2]$ instead of $\mathbb{E}[\|x_k\|^2]$, $\mathbb{E}[\|\hat{\mathbf{x}}_k^s\|^2]$, and $\mathbb{E}[\|e_k^s\|^2]$.

From (1) and (2), $x_{k+1} = Ax_k + \nu_k Bu_k + \omega_k$. Then,

$$\begin{aligned}\hat{\mathbf{x}}_{k+1} &= \bar{\gamma}_{k+1}(A\hat{\mathbf{x}}_k + \nu Bu_k) + \gamma_{k+1}\hat{\mathbf{x}}_{k+1}^s \\ &= (A + (\bar{\gamma}_{k+1}\nu + \gamma_{k+1}\nu_k)BL)\hat{\mathbf{x}}_k \\ &\quad + \gamma_{k+1}Ae_k + \gamma_{k+1}\omega_k - \gamma_{k+1}e_{k+1}^s.\end{aligned} \quad (30)$$

From (7) and (30), by some algebraic derivations, we have

$$\begin{aligned}e_{k+1} &= \bar{\gamma}_{k+1}(Ae_k + (\nu_k - \nu)BL\hat{\mathbf{x}}_k) \\ &\quad + \bar{\gamma}_{k+1}\omega_k + \gamma_{k+1}e_{k+1}^s.\end{aligned} \quad (31)$$

Since $\mathbb{E}[\|e_{k+1}^s\|^2]$ and $\mathbb{E}[\|\omega_k\|^2]$ are bounded, it is pointed out in [31] that $\hat{\mathbf{x}}_{k+1}$ and e_{k+1} in (30) and (31) are m.s. stable if

and only if they are m.s. stable in the homogeneous part of (30) and (31) as follows.

$$\hat{\mathbf{x}}_{k+1} = (A + (\bar{\gamma}_{k+1}\nu + \gamma_{k+1}\nu_k)BL)\hat{\mathbf{x}}_k + \gamma_{k+1}Ae_k \quad (32)$$

$$e_{k+1} = \bar{\gamma}_{k+1}(\nu_k - \nu)BL\hat{\mathbf{x}}_k + \bar{\gamma}_{k+1}Ae_k. \quad (33)$$

For brevity, the subscripts ∞ of M_∞ and T_∞ are removed. Denote the optimal control by u_k^* . Let $W_k = W$ and $\Lambda_k = \Lambda$ in (23). It follows from (23) that $V_k(x_k) = \mathbb{E}[x_k' W x_k + \nu_k (u_k^*)' \Lambda u_k^* + V_{k+1}(x_{k+1}) | \mathcal{I}_k]$. By using the established property in [14] that

$$\mathbb{E}[x_k' M x_k | \mathcal{I}_k] = \hat{\mathbf{x}}_k' M \hat{\mathbf{x}}_k + \text{tr}(M \mathbf{P}_k), \quad (34)$$

Eq. (24) can be rewritten as $V_k(x_k) = \hat{\mathbf{x}}_k' M_k \hat{\mathbf{x}}_k + \text{tr}(\Delta_k) + \mathbb{E}[e_k' T_k e_k | \mathcal{I}_k]$. By taking mathematical expectation over information \mathcal{I}_k , we have

$$\begin{aligned}\mathbb{E}[V_{k+1}(x_{k+1}) - V_k(x_k)] &= -\mathbb{E}[x_k' W x_k + \nu (u_k^*)' \Lambda u_k^*] \\ \mathcal{V}_k &\triangleq \mathbb{E}[\hat{\mathbf{x}}_k' M_k \hat{\mathbf{x}}_k + e_k' T_k e_k] = \mathbb{E}[V_k(x_k)] - \mathbb{E}[\text{tr}(\Delta_k)].\end{aligned}$$

From (25c), we have

$$\begin{aligned}\mathcal{V}_{k+1} - \mathcal{V}_k &= -\mathbb{E}[x_k' W x_k + \nu (u_k^*)' \Lambda u_k^*] \\ &\quad + (\bar{\gamma}T_{k+1} + \gamma M_{k+1})Q + \gamma H_{k+1} S_{k+1}.\end{aligned}$$

Observe that in Lemma 9, x_k and e_k are determined by (30) and (31). Here, what we consider is their homogeneous parts (32) and (33). Clearly, in the homogeneous parts there is no process noise ω_k and the estimation error e_k^s is set to be zero as well, which is equivalent to setting their covariances to be zeros, that is, $Q = S_k = 0$. Consequently, for the homogeneous parts,

$$\begin{aligned}\mathcal{V}_{k+1} - \mathcal{V}_k &= -\mathbb{E}[x_k' W x_k + \nu (u_k^*)' \Lambda u_k^*] \\ &= -\mathbb{E}[x_k' W x_k + \nu \hat{\mathbf{x}}_k' L' \Lambda L \hat{\mathbf{x}}_k] \\ &= -\mathbb{E}[\hat{\mathbf{x}}_k' (W + \nu L' \Lambda L) \hat{\mathbf{x}}_k + e_k' W e_k].\end{aligned}$$

By repeatedly using the preceding equation, we have $\mathcal{V}_{k+1} - \mathcal{V}_0 = -\sum_{i=0}^k \mathbb{E}[\hat{\mathbf{x}}_i' (W + \nu L' \Lambda L) \hat{\mathbf{x}}_i + e_i' W e_i]$. Since $\mathcal{V}_{k+1} \geq 0$,

$$\mathcal{V}_0 \geq \sum_{i=0}^k \mathbb{E}[\hat{\mathbf{x}}_i' (W + \nu L' \Lambda L) \hat{\mathbf{x}}_i + e_i' W e_i]. \quad (35)$$

Note that $\mathcal{V}_0 = \mathbb{E}[\hat{\mathbf{x}}_0' M_0 \hat{\mathbf{x}}_0 + e_0' T_0 e_0]$ is bounded and that $W + \nu L' \Lambda L$ and W are positive definite. The series in the right hand side of (35) is monotonically increasing and bounded so that it is convergent, which implies that $\mathbb{E}[\|\hat{\mathbf{x}}_i\|^2] \rightarrow 0$ and $\mathbb{E}[\|e_i\|^2] \rightarrow 0$ as $i \rightarrow \infty$. Then $\mathbb{E}[\|\hat{\mathbf{x}}_k\|^2]$ and $\mathbb{E}[\|e_k\|^2]$ are bounded. Thus, $\hat{\mathbf{x}}_{k+1}$ and e_{k+1} are m.s. stable in the homogeneous part of (30) and (31), which implies the MMS of the closed-loop system. ■

V. FURTHER DISCUSSIONS ON CONDITIONS 1 AND 2

As mentioned in Section I, for UDP-like systems, the theoretical analysis and practical use of the optimal estimation and control encounter intractable difficulties. In this section, we show that these difficulties remain if one of the Conditions 1 and 2 is not satisfied. Consider the following two cases.

Case I: Condition 1 holds, but Condition 2 is not satisfied. Even when the sensor has computational abilities, $\hat{\mathbf{x}}_k^s$ cannot

be computed by the smart sensor, since its computation requires u_k^a , which can be seen from (4). What the smart sensor can send to the estimator is the raw measurement y_k . Hence, in this case, the system works like the conventional UDP-like system.

Case II: Condition 2 holds, but Condition 1 fails. Without computational abilities, the useful information the smart sensor can provide the estimator via the S/E channel is u_k^a and y_k . Then the estimator can recover the ν_k from u_k^a . However, $\{\nu_k\}$ is partially and stochastically recovered due to the losses of u_k^a in the S/E channel. Therefore, this case can be viewed as the systems with ACK signals randomly lost. It is revealed in [11, 23] that it is computationally prohibitive to obtain the optimal estimator and LQG controller if ACK signals are randomly lost.

It can be concluded that if Conditions 1 and 2 are not satisfied simultaneously, solving the optimal estimation and control problems will confront with intractable difficulties.

VI. NUMERICAL EXAMPLES

In this section, the main results we obtained are verified by some numerical examples. Consider the double inverted pendulum model in [43] with following parameters:

$A =$

$$\begin{bmatrix} 1 & -0.0004 & 0 & 0.0093 & 0 & 0 \\ 0 & 1.0034 & -0.0010 & 0.0016 & 0.0090 & 0.0003 \\ 0 & -0.0038 & 1.0032 & -0.0004 & 0.0008 & 0.0094 \\ 0 & -0.0786 & 0.0063 & 0.8730 & 0.0083 & -0.0048 \\ 0 & 0.6544 & -0.2380 & 0.3101 & 0.9034 & 0.0664 \\ 0 & -0.7149 & 0.6137 & -0.0751 & 0.1579 & 0.8770 \end{bmatrix}$$

$B = [0.0001; -0.0003; 0.0001; 0.0274; -0.0668; 0.0162]'$

$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. Let $Q = 0.01I_6$ and $R =$

$0.01I_3$, where I_3 and I_6 are identity matrices with dimensions 3 and 6, respectively. $W = I_6$ and $\Lambda = 2$.

1) *Convergence of \mathbf{P}_k* : Let $\gamma = \nu = 0.5$. Here, we do not draw the graphs of all $\Gamma_k(\alpha)$ to show $\Gamma_k(\alpha) \rightarrow \Gamma_\infty(\alpha)$, as their trajectories seriously overlap. Hence, we draw $\Gamma_k(\alpha)$ for some specific values $k = 20, 40, 60, 80$, and 100. As shown in Fig. 2 (a semi-log graph), $\Gamma_k(\alpha)$ converges to $\Gamma_\infty(\alpha)$, which implies the convergence of \mathbf{P}_k to \mathbf{P} in distribution.

2) *Stability of $\mathbb{E}[\mathbf{P}_k]$* : The relationship between $\mathbb{E}[\mathbf{P}_k]$ and γ is shown in Fig. 3. It can be seen that under different control inputs and packet arrival rate ν , the stability condition for $\mathbb{E}[\mathbf{P}_k]$ is the same and is determined by the observation packet arrival rates. As claimed in Theorem 2, for $\mathbb{E}[\mathbf{P}_k]$ to be stable, it requires $\bar{\gamma} < \lambda_A^{-2}$, i.e., $\gamma > 0.0763$.

3) *Estimation Performance Evaluation*: Let $\gamma = \nu = 0.5$. The exact values of $\Gamma_k(1)$ and $\Gamma_k(1.5)$ are calculated by Algorithm 1. Fig. 4(a) shows that if $\{u_k\}$ is bounded but not convergent, so is $\Gamma_k(1.5)$. When its bound $\bar{u} = 15$ is known, by Theorem 3, the lower bound p_{bd} for $\Gamma_k(1.5)$ can be obtained. It can be viewed as a lower bound for estimation performance. It can also be seen in Fig. 4(b) that if $\{u_k\}$ is convergent, so is $\Gamma_k(1)$, as claimed in Theorem 1. Based

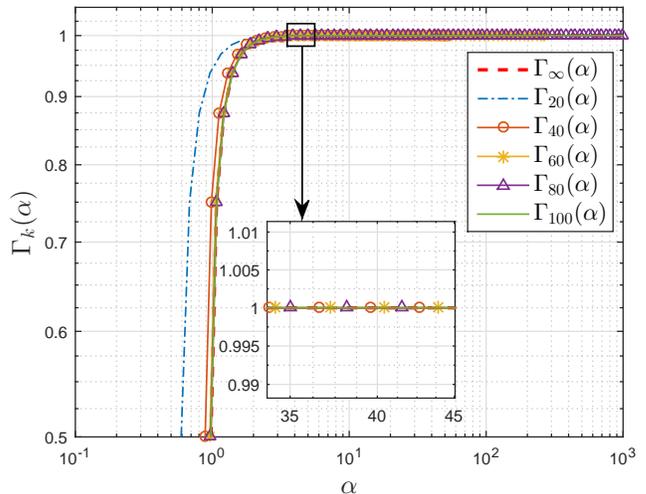


Fig. 2. Convergence of $\Gamma_k(\alpha) = \mathbb{P}_k(\{\mathbf{P}_k \leq \alpha\})$

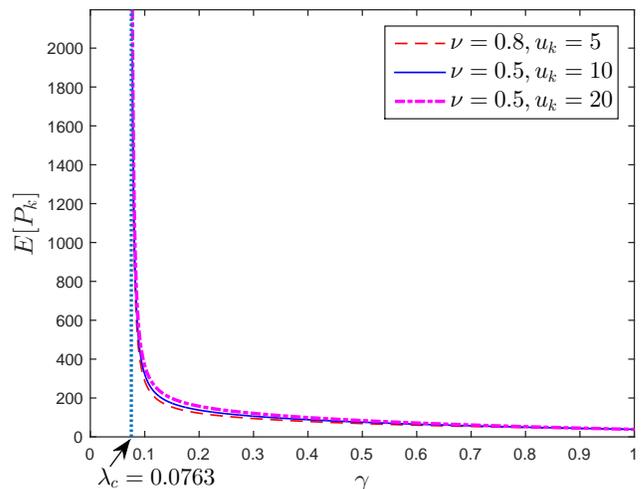
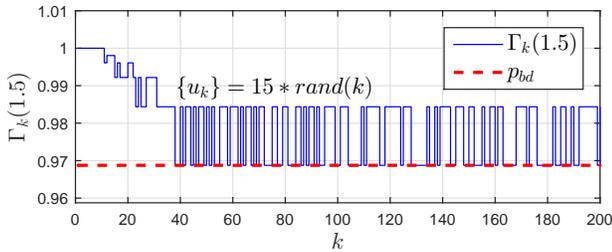


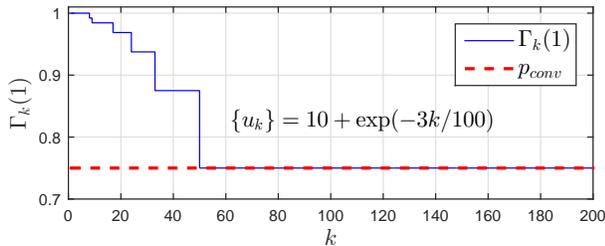
Fig. 3. Relationship between $\mathbb{E}[\mathbf{P}_k]$ and γ

on the limit of $\{u_k\}$, the quantity p_{conv} can be computed by Theorem 3. When k is sufficiently large, the exact value of $\Gamma_k(1) = p_{conv}$ is obtainable even when full knowledge of $\{u_k\}$ is unknown.

4) *LQG control and MSS of the closed-loop system*: Let $\gamma = \nu = 0.8$. As shown in Fig. 5, it is easy to check the convergences of M_k and T_k by simulation. Then, it follows from Lemma 10 that there exist solutions $M_\infty > 0$ and $T_\infty > 0$ for (28), which guarantees the existence of the infinite LQG control. For the closed-loop SS-UDP system, we carry out 1000 times Monte Carlo simulations and then compute the mean values $\mathbb{E}[\|\hat{x}_k\|]$ and $\mathbb{E}[\|e_k\|]$. As shown in Fig. 5, $\mathbb{E}[\|\hat{x}_k\|]$ and $\mathbb{E}[\|e_k\|]$ are bounded, which also implies the boundedness of $\mathbb{E}[\|\hat{x}_k\|^2]$ and $\mathbb{E}[\|e_k\|^2]$. Note that $\mathbb{E}[\|e_k^s\|^2] = \text{tr}(S_k)$ is bounded under Assumption 1. Thus, the closed-loop SS-UDP system is mean square stable, as claimed in Theorem 5.



(a) The bounded but not convergent input case.



(b) The convergent input case.

Fig. 4. $\mathbb{P}_k(\{\mathbf{P}_k < I\})$ and $\mathbb{P}_k(\{\mathbf{P}_k < 1.5I\})$. The $\text{rand}(k)$ stands for a sequence of i.i.d. Bernoulli random variables taking value in 0 or 1.

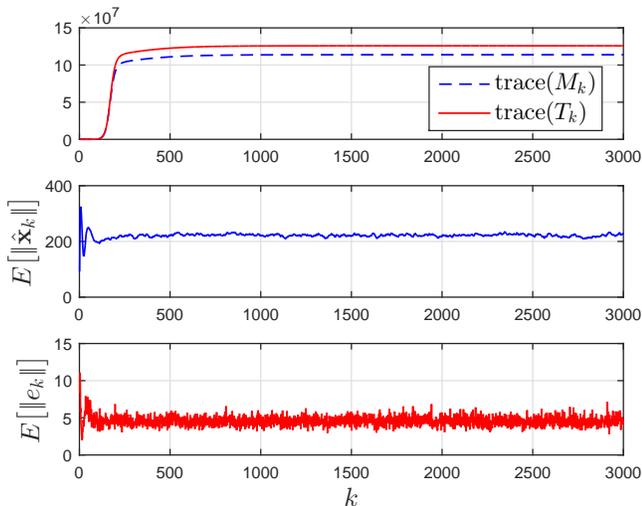


Fig. 5. Mean square stability of the closed-loop SS-UDP system

VII. CONCLUSION

In this paper, we have shown that for conventional UDP-like systems there are some intrinsic difficulties to design the optimal estimation and LQG control. To overcome these difficulties, we have proposed two feasible conditions for the system structure. For UDP-like systems satisfying these two conditions, the optimal estimation has been derived. By constructing the probability spaces of the estimation error covariances, the stability condition for $\mathbb{E}[\mathbf{P}_k]$, the convergence of \mathbf{P}_k in distribution, and some practical ways to evaluate the estimation performance have been established. The LQG controllers for both finite and infinite horizon cases have been developed. Examples are given to demonstrate the potential and effectiveness of the proposed new design techniques.

There are two possible extensions. One is to design event-triggering conditions on the smart sensor side and then develop the corresponding event-based remote estimator, whose advantages [44] may help extend the results in this paper to UDP-like networks suffering from the limitation of communication and the restriction on the actuator/sensor power consumption. The other is to extend the obtained results for the single smart sensor node to the multiple-nodes case, and consider the distributed estimation and control problems [45] over UDP-like networks.

APPENDIX

Proof of Lemma 1: For the estimator, if $\gamma_k = 1$, then the estimator receives $\hat{\mathbf{x}}_k^s$. Thus, $\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^s$ and $\mathbf{P}_k = S_k$.

If $\gamma_k = 0$, without $\hat{\mathbf{x}}_k^s$, the optimal estimate is in fact the prediction $\hat{\mathbf{x}}_k = A\hat{\mathbf{x}}_{k-1} + \nu Bu_{k-1}$. Then, the prediction error covariance

$$\begin{aligned} \mathbf{P}_k &= \mathbb{E}[(x_k - \hat{\mathbf{x}}_k)(x_k - \hat{\mathbf{x}}_k)' | \mathcal{I}_{k-1}] \\ &= \mathbb{E}[A(x_{k-1} - \hat{\mathbf{x}}_{k-1})(x_{k-1} - \hat{\mathbf{x}}_{k-1})' A' | \mathcal{I}_{k-1}] \\ &\quad + \mathbb{E}[\omega_{k-1}\omega_{k-1}'] + \mathbb{E}[(\nu_k - \nu)^2] B u_{k-1} u_{k-1}' B' \\ &= A\mathbf{P}_{k-1}A' + Q + U_{k-1}. \end{aligned}$$

Proof of Proposition 1: Proof of part (i): Without loss of generality, we assume that $i < j$. By definition, for each $\theta_a \in F_k^{[i]}$, the element in the $i + 1$ th position of θ_a is 1. According to the definition of $F_k^{[j]}$, for each $\theta_b \in F_k^{[j]}$, from left to right, the first 1 occurs in the $j + 1$ th position. It means that the element in the $i + 1$ th position of θ_b must be 0 due to $i + 1 < j + 1$. Therefore, $\theta_a \notin F_k^{[j]}$, vice versa. Part (i) is proved.

Proof of part (ii): It is clear that $F_k^{[i]} \subseteq \Omega_k$. Thus, we have $\bigcup_{i=0}^k F_k^{[i]} \subseteq \Omega_k$. In the following, we show that $\bigcup_{i=0}^k F_k^{[i]} \supseteq \Omega_k$. For each $\theta \in \Omega_k$, if all the elements in θ are 0, then by definition $\theta \in F_k^{[k]}$. If not, there must be at least one 1 in θ . From left to right, denote the position where the first 1 occurs by $i + 1$. According to the definition of $F_k^{[i]}$, $\theta \in F_k^{[i]}$. Thus, we have $\theta \in F_k^{[i]}$ for $\forall \theta \in \Omega_k$, which means $\bigcup_{i=0}^k F_k^{[i]} \supseteq \Omega_k$. Part (ii) is proved.

Proof of part (iii): By definition, $F_k^{[j]}$ consists of points of the form $\theta = (0, \dots, 0, \gamma_{k-j} = 1, \times, \dots, \times)$. For each $\theta \in F_k^{[j]}$, $\mathbb{P}_k(\{\theta\}) = \bar{\gamma}^j \gamma \mathbb{P}_\gamma(\gamma_{k-j-1}) \cdots \mathbb{P}_\gamma(\gamma_1)$.

According to the definition of $\mathbb{P}_k(A)$,

$$\begin{aligned} \mathbb{P}_k(F_k^{[j]}) &= \sum_{\theta \in F_k^{[j]}} \mathbb{P}_k(\{\theta\}) = \sum_{\gamma_{k-j-1}=0}^1 \cdots \sum_{\gamma_1=0}^1 \mathbb{P}_k(\{\theta\}) \\ &= \sum_{\gamma_{k-j-1}=0}^1 \cdots \sum_{\gamma_1=0}^1 \gamma \bar{\gamma}^j \mathbb{P}_\gamma(\gamma_{k-j-1}) \cdots \mathbb{P}_\gamma(\gamma_1) \\ &= \gamma \bar{\gamma}^j \left(\sum_{\gamma_{k-j-1}=0}^1 \mathbb{P}_\gamma(\gamma_{k-j-1}) \right) \cdots \left(\sum_{\gamma_1=0}^1 \mathbb{P}_\gamma(\gamma_1) \right) = \gamma \bar{\gamma}^j \end{aligned}$$

where the last line is obtained by noting that $\sum_{\gamma_i=0}^1 \mathbb{P}_\gamma(\gamma_i) = 1$, for $1 \leq i \leq k$. For $F_k^{[k]}$, $\mathbb{P}_k(F_k^{[k]}) = \mathbb{P}_k(\{(0, \dots, 0)\}) = \bar{\gamma}^k$, which completes the proof. \blacksquare

Proof of Lemma 2: By (5) and (6),

$$\begin{aligned} S_{k-n} &= \bar{S}_{k-n} - K_{k-n} C' \bar{S}_{k-n} \\ &\leq \bar{S}_{k-n} = A S_{k-n-1} A' + Q \\ &\leq A S_{k-n-1} A' + Q + U_{k-n-1} \\ &= \psi_{k-n-1}(S_{k-n-1}). \end{aligned}$$

From the definition of $\Phi_k^{[n]}$, it is clear that if $P_1 \leq P_2$, then $\Phi_k^{[n]}(P_1) \leq \Phi_k^{[n]}(P_2)$. $P_k^{[n+1]} = \Phi_k^{[n+1]}(S_{k-n-1}) = \Phi_k^{[n]}(\psi_{k-n-1}(S_{k-n-1})) \geq \Phi_k^{[n]}(S_{k-n}) = P_k^{[n]}$. ■

Proof of Proposition 2: For each $\theta \in F_k^{[j]}$, it takes the form that $\theta = \{\gamma_k = 0, \dots, \gamma_{k-j+1} = 0, \gamma_{k-j} = 1, \times, \dots, \times\}$. From (8), it follows that if $\gamma_{k-j} = 1$, then $P_{k-j} = S_{k-j}$ no matter what values $\{\gamma_{k-j-1}, \dots, \gamma_1\}$ take. Due to $\{\gamma_k = 0, \dots, \gamma_{k-j+1} = 0\}$, it follows from (8) that $P_k = \psi_{k-1} \circ \dots \circ \psi_{k-j}(S_{k-j}) = \Psi_k^{[j]}(S_{k-j}) = P_k^{[j]}$. ■

Proof of Proposition 3: Parts (i) and (ii) can be easily proved by following the similar line of argument in the proofs of parts (i) and (ii) of Proposition 1. Thus, the proofs of (i) and (ii) is not presented here for saving space.

We prove part (iii) by contradiction. It is no loss of generality to consider B is a union of countable sets. Suppose that the representation of B is not unique, then $B = \biguplus_{i \in \mathcal{N}_{B_1}} A^{[i]} = \biguplus_{i \in \mathcal{N}_{B_2}} A^{[i]}$, where $\mathcal{N}_{B_1} = \{n_1, n_2, \dots\} \uplus \{n_1^a, n_2^a, \dots\}$, $\mathcal{N}_{B_2} = \{n_1, n_2, \dots\} \uplus \{n_1^b, n_2^b, \dots\}$, $\{n_1, n_2, \dots\}$ is the common set in \mathcal{N}_{B_1} and \mathcal{N}_{B_2} , and $\{n_1^a, n_2^a, \dots\} \cap \{n_1^b, n_2^b, \dots\} = \emptyset$. For $\forall n_i^a$ with $i \geq 1$, let $\theta \in A^{[n_i^a]}$. From part (i), it follows that $\theta \notin A^{[n_j]}$ due to $n_j \neq n_i^a$ for $j \geq 1$, and that $\theta \notin A^{[n_j^b]}$ due to $n_i^a \neq n_j^b$ for $\forall i$ and j . Therefore, $\theta \notin (\biguplus_{j \in \{n_1, n_2, \dots\}} A^{[j]}) \uplus (\biguplus_{j \in \{n_1^b, n_2^b, \dots\}} A^{[j]}) = \biguplus_{i \in \mathcal{N}_{B_2}} A^{[i]} = B$, which is a contradiction to $\theta \in n_i^a \subseteq B$. Thus, the presentation of B is unique.

Let $B_i = \biguplus_{k \in \mathcal{N}_{B_i}} A^{[k]}$ and $B_j = \biguplus_{k \in \mathcal{N}_{B_j}} A^{[k]}$. For any $n_i \in \mathcal{N}_{B_i}$, $A^{[n_i]} \subseteq B_i$ but $A^{[n_i]} \not\subseteq B_j$ due to the hypothesis that B_i and B_j are disjoint, which implies $n_i \notin \mathcal{N}_{B_j}$. Similarly, we can obtain that for any $n_j \in \mathcal{N}_{B_j}$, $n_j \notin \mathcal{N}_{B_i}$, which implies $\mathcal{N}_{B_i} \cap \mathcal{N}_{B_j} = \emptyset$. Then, it is straightforward to have $B_i \uplus B_j = (\biguplus_{k \in \mathcal{N}_{B_i}} A^{[k]}) \uplus (\biguplus_{k \in \mathcal{N}_{B_j}} A^{[k]}) = \biguplus_{k \in \mathcal{N}_{B_i} \uplus \mathcal{N}_{B_j}} A^{[k]}$.

Proof of part (iv): For $\forall \theta \in \biguplus_{i=1}^{\infty} B_i$, there exists a unique integer i such that $\theta \in B_i$, and then there exists an unique integer $j \in \mathcal{N}_{B_i}$ such that $\theta \in A^{[j]}$. Thus, $\theta \in A^{[j]} \subseteq \biguplus_{k \in \biguplus_{i=1}^{\infty} \mathcal{N}_{B_i}} A^{[k]}$, due to $j \in \biguplus_{i=1}^{\infty} \mathcal{N}_{B_i}$. Then we have $\biguplus_{i=1}^{\infty} B_i \subseteq \biguplus_{k \in \biguplus_{i=1}^{\infty} \mathcal{N}_{B_i}} A^{[k]}$. It is clear that for any $A^{[j]}$ in $\biguplus_{k \in \biguplus_{i=1}^{\infty} \mathcal{N}_{B_i}} A^{[k]}$, it must come from some B_i . Thus, $\biguplus_{k \in \biguplus_{i=1}^{\infty} \mathcal{N}_{B_i}} A^{[k]} \subseteq \biguplus_{i=1}^{\infty} B_i$, which proves part (iv). ■

Proof of Proposition 4: *Proof of part (i):* The three conditions (i-iii) in Definition 5 for a collection of subsets to be a field are checked for \mathcal{A} . (i) By definition of \mathcal{A} , $\Omega_{\infty} \in \mathcal{A}$. (ii) If $B \in \mathcal{A}$, then there exists a finite or countable subset $\mathcal{N}_B \subseteq \mathbb{Z}_+$ such that $B = \biguplus_{i \in \mathcal{N}_B} A^{[i]}$. Note $\mathcal{N}_B^c \subseteq \mathbb{Z}_+$ is a finite or countable subset as well. It follows from Proposition 3(ii) that $B^c = \biguplus_{i \in \mathcal{N}_B^c} A^{[i]} \in \mathcal{A}$. (iii) Let $B_1, \dots, B_n \in \mathcal{A}$. Then $\bigcup_{i=1}^n B_i = \bigcup_{i=1}^n \biguplus_{j \in \mathcal{N}_{B_i}} A^{[j]} = \biguplus_{j \in \mathcal{N}_B} A^{[j]}$, where

$\mathcal{N}_B = \bigcup_{i=1}^n \mathcal{N}_{B_i}$. $\mathcal{N}_B \subseteq \mathbb{Z}_+$ is a finite or countable set. Thus, $\bigcup_{i=1}^n B_i \in \mathcal{A}$. Therefore, \mathcal{A} is a field.

Proof of part (ii): Let $\{B_i\}$ be a countable collection of mutually disjoint subsets of \mathcal{A} . Denote $B = \biguplus_{i=1}^{\infty} B_i$. From Proposition 3(iv), $B = \biguplus_{k \in \biguplus_{i=1}^{\infty} \mathcal{N}_{B_i}} A^{[k]}$. Let $\mathcal{N}_B = \biguplus_{i=1}^{\infty} \mathcal{N}_{B_i}$. By (14),

$$\begin{aligned} \mathbb{P}^*(B) &= \sum_{k \in \mathcal{N}_B} p_k = \sum_{k \in \biguplus_{i=1}^{\infty} \mathcal{N}_{B_i}} p_k \\ &= \sum_{k \in \{\mathcal{N}_{B_1}, \mathcal{N}_{B_2}, \dots\}} p_k \\ &= \sum_{i=1}^{\infty} \left(\sum_{k \in \mathcal{N}_{B_i}} p_k \right) = \sum_{i=1}^{\infty} \mathbb{P}^*(B_i). \end{aligned}$$

By definition, \mathbb{P}^* is a nonnegative countably additive set function on the field \mathcal{A} , and thus it is a measure on the field \mathcal{A} , which proves part (ii).

Proof of part (iii): As \mathbb{P}^* is countably additive, it naturally is finitely additive. According to the definition of σ -finite, part (iii) can be readily proved by noting the facts that $\Omega_{\infty} = \biguplus_{i \in \mathbb{Z}_+} A^{[i]}$ (Proposition 3), and that $\mathbb{P}^*(A^{[i]}) < 1$, $i \in \mathbb{Z}_+$. ■

Proof of Proposition 5: It has been proved that \mathbb{P}^* and \mathcal{A} satisfy the conditions required in Lemma 3. From Lemma 3, it follows that there exists a probability measure \mathbb{P}_{∞} on \mathcal{F}_{∞} such that $\mathbb{P}_{\infty}(A) = \mathbb{P}^*(A)$ for $\forall A \in \mathcal{A}$. By noting that $\mathcal{A}_0 \subseteq \mathcal{A}$, Proposition 5 is proved. ■

Proof of Lemma 4: *Proof of part (i):* From (15), (16), and (17), by some algebraic computations, we have

$$S^{[n]} = A^n S (A^n)' + \sum_{i=1}^n A^{n-i} (U + Q) (A^{n-i})'. \quad (36)$$

From the definitions of ψ and ψ_k , it follows that 1) if $P_1 > P_2$, then $\psi(P_1) > \psi(P_2)$ and $\psi_k(P_1) > \psi_k(P_2)$; 2) $S^{[k+1]} = \Psi^{[k+1]}(S) = \psi(\Psi^{[k]}(S)) = \psi(S^{[k]})$. Note that $S^{[0]} = S = \bar{S} - \bar{S} C' (C \bar{S} C' + R)^{-1} C \bar{S} \leq \bar{S} = A S A' + Q \leq A S A' + Q + U = \psi(S) = S^{[1]}$. Then $\lambda_S^{[0]} \leq \lambda_S^{[1]}$. Part (i) holds for $n = 0$. Suppose that part (i) holds for $n = 1, \dots, k$. We check the case $n = k + 1$. $S^{[k+1]} = \psi(S^{[k]}) \geq \psi(S^{[k-1]}) = S^{[k]}$. Hence, $\{S^{[n]}\}$ is an increasing sequence.

To prove that $\{S^{[n]}\}$ strictly increases, we show that $S^{[n+1]} \neq S^{[n]}$ for $\forall n \in \mathbb{Z}_+$. We have $S^{[n]} > 0$ for $\forall n \in \mathbb{Z}_+$ due to $S > 0$. If there is an integer $k \in \mathbb{Z}_+$ such that $S^{[k+1]} = S^{[k]}$, then from (15)-(17) we have $S^{[k]} = S^{[k+1]} = A S^{[k]} A' + Q + U$. However, it is well known that for $\lambda_A > 1$ there is no positive definite solution P for $P = A P A' + Q + U$. Hence, $S^{[k+1]} \neq S^{[k]}$ for $\forall n \in \mathbb{Z}_+$. Part (i) is proved.

Proof of part (ii): $\lambda_S^{[k]} \leq \lambda_S^{[k+1]}$ is an immediate result of part (i). From (36), $S^{[n]} \geq A^n S (A^n)' \geq \lambda_S (A A')^n$. Thus, $\lambda_S^{[n]} \geq \lambda_S (\lambda_A)^{2n}$. Due to the assumption that $\lambda_A > 1$, we have $\lim_{n \rightarrow \infty} \lambda_S^{[n]} = \infty$.

Proof of part (iii): The existence and uniqueness of such n_{α} are evident, due to the monotonicity of $\lambda_S^{[n]}$ and the property that $\lambda_S^{[n]} \rightarrow \infty$ as $n \rightarrow \infty$.

When A and Q is fixed, n_{α} only depends on α and the U in $\psi(\cdot)$, and is also uniquely determined by α and U . Thus, the function $n_{\alpha} = n(\alpha, U)$ is well-defined. ■

Proof of Proposition 6: Proof of part (i): From (18), it follows that $\mathbf{P}(\theta) = 0$ only when $\theta \in A^{[\infty]}$. Thus, $\{\mathbf{P}(\theta) = 0\} = A^{[\infty]}$. By Proposition 5 and (13), $\mathbb{P}_\infty(A^{[\infty]}) = \mathbb{P}^*(A^{[\infty]}) = 0$. By Lemma 4(i) and the definition of $\mathbf{P}(\theta)$ in (18), it is clear that $\mathbf{P}(\theta) = S^{[m]}$ only when $\theta \in A^{[m]}$. Thus, $\{\mathbf{P}(\theta) = S^{[m]}\} = A^{[m]}$. Then it follows from Proposition 5 and (13) that $\mathbb{P}_\infty(\{\mathbf{P}(\theta) = S^{[m]}\}) = \mathbb{P}_\infty(A^{[m]}) = \mathbb{P}^*(A^{[m]}) = \gamma\bar{\gamma}^m$.

Proof of part (ii): According to the definition of \mathbf{P} , it may take the values $\{0, S, S^{[1]}, \dots\}$. Note that $\lambda_S^{[n]} \leq \alpha$ is equivalent to $S^{[n]} \leq \alpha I$. It follows from Lemma 4(i) and (iii) that for a given α , $S^{[i]} \leq \alpha I$ for $0 \leq i \leq n_\alpha$, where $n_\alpha = n(\alpha, U)$. By the definition of \mathbf{P} , $\{\mathbf{P} \leq \alpha I\}$ consists of $\{A^{[\infty]}, A^{[0]}, \dots, A^{[n_\alpha]}\}$. That is, $\{\mathbf{P} \leq \alpha I\} = A^{[\infty]} \uplus (\uplus_{i=0}^{n_\alpha} A^{[i]})$. Then $\mathbb{P}_\infty(A^{[\infty]} \uplus (\uplus_{i=0}^{n_\alpha} A^{[i]})) = \mathbb{P}_\infty(A^{[\infty]}) + \sum_{i=0}^{n_\alpha} \mathbb{P}_\infty(A^{[i]}) = \sum_{i=0}^{n_\alpha} \gamma\bar{\gamma}^i = 1 - \bar{\gamma}^{n_\alpha+1}$. ■

Proof of Lemma 5: Part (i) can be proved by noting the fact if $\|M\| < \varepsilon$ then $-\varepsilon < M < \varepsilon$, and part (ii) is proved by noting that $(\alpha - d)I = \lambda_S I \geq S$. ■

Proof of Lemma 6: From (9), (10), and (11), we have

$$\begin{aligned} P_k^{[n]} &= \Psi_k^{[n]}(S_{k-n}) \\ &= A^n S_{k-n} (A^n)' + A^{n-1} (U_{k-1} + Q) (A^{n-1})' \\ &\quad + A^{n-2} (U_{k-2} + Q) (A^{n-2})' + \dots + (U_{k-n} + Q) \\ &= A^n S_{k-n} (A^n)' + \sum_{i=1}^n A^{n-i} (U_{k-i} + Q) (A^{n-i})'. \end{aligned} \quad (37)$$

Due to the convergence of $\{S_k\}$ and $\{U_k\}$, for $\forall \varepsilon_S > 0$ and $\forall \varepsilon_U > 0$, there exists an integer $N_n \in \mathbb{N}$ such that $\|S_{k-n} - S\| < \varepsilon_S$ and $\|U_{k-n} - U\| < \varepsilon_U$, for $k > N_n$. By (36),

$$\begin{aligned} &\|P_k^{[n]} - S^{[n]}\| \\ &\leq \|A^n (S_{k-n} - S) (A^n)' + \sum_{i=1}^n A^{n-i} (U_{k-i} - U) (A^{n-i})'\| \\ &\leq \varepsilon_S \|A^n (A^n)'\| + \varepsilon_U \sum_{i=0}^{n-1} \|A^i (A^i)'\|. \end{aligned}$$

Let $d = |\alpha - \lambda_S^{[n]}|$. For a given ε with $0 < \varepsilon < d$, we can obtain $\|P_k^{[n]} - S^{[n]}\| < \varepsilon$ for $k > N_n$ by choosing sufficiently small ε_S and ε_U . By Lemma 5 (i), we have $-\varepsilon I < S^{[n]} - P_k^{[n]} < \varepsilon I$, $\forall k > N_n$.

Proof of $(\lambda_S^{[n]} < \alpha) \Rightarrow (\lambda_{P_k}^{[n]} < \alpha)$: By using Lemma 5 (ii), $\alpha I - P_k^{[n]} = (\alpha I - S^{[n]}) + (S^{[n]} - P_k^{[n]}) > dI - \varepsilon I > 0$. Hence, $P_k^{[n]} < \alpha I$, which implies $\lambda_{P_k}^{[n]} < \alpha$.

Proof of $(\lambda_S^{[n]} > \alpha) \Rightarrow (\lambda_{P_k}^{[n]} > \alpha)$: $S^{[n]} - P_k^{[n]} < \varepsilon I \Rightarrow S^{[n]} < \varepsilon I + P_k^{[n]} \Rightarrow \lambda_S^{[n]} < \varepsilon + \lambda_{P_k}^{[n]} \Rightarrow \lambda_{P_k}^{[n]} - \lambda_S^{[n]} > -\varepsilon$. Then $\lambda_{P_k}^{[n]} - \alpha = (\lambda_{P_k}^{[n]} - \lambda_S^{[n]}) + (\lambda_S^{[n]} - \alpha) > -\varepsilon + d > 0$. ■

Proof of Lemma 7: Proof of part (i): From Lemma 6, we have $\lambda_{P_k}^{[n_\alpha]} < \alpha$ for $k > N_{n_\alpha}$ and $\alpha < \lambda_{P_k}^{[n_\alpha+1]}$ for $k > N_{n_\alpha+1}$. By letting $N = \max(N_{n_\alpha}, N_{n_\alpha+1})$, part (i) is proved.

Proof of part (ii): From part (i) and the monotonicity of $P_k^{[n]}$ (Lemma 2), it follows that for $k > N$, $\mathbf{P}_k \leq \alpha I$ if and only if \mathbf{P}_k takes the value within $\{P_k^{[0]}, \dots, P_k^{[n_\alpha]}\}$. It is shown in Proposition 2 that $\mathbf{P}_k(\theta) = P_k^{[i]}$ for $\theta \in F_k^{[i]}$. Therefore, $\{\mathbf{P}_k \leq \alpha I\}$ consists of $\{F_k^{[0]}, \dots, F_k^{[n_\alpha]}\}$. Hence, $\{\mathbf{P}_k \leq \alpha I\} = \uplus_{i=0}^{n_\alpha} F_k^{[i]}$. It follows from Proposition 1

that $\mathbb{P}_k(\{\mathbf{P}_k \leq \alpha I\}) = \mathbb{P}_k(\uplus_{i=0}^{n_\alpha} F_k^{[i]}) = \sum_{i=0}^{n_\alpha} \mathbb{P}_k(F_k^{[i]}) = \sum_{i=0}^{n_\alpha} \gamma\bar{\gamma}^i = 1 - \bar{\gamma}^{n_\alpha+1}$. ■

Proof of Lemma 8: Taking mathematical expectation of (8) yields

$$\mathbb{E}[\mathbf{P}_{k+1}] = \bar{\gamma}(A\mathbb{E}[\mathbf{P}_k]A' + Q + U_k) + \gamma S_{k+1}. \quad (38)$$

From (9), (10), and (11),

$$\begin{aligned} P_{k+1}^{[i+1]} &= \Psi_{k+1}^{[i+1]}(S_{k-i}) \\ &= \psi_{(k+1)-1} \circ \psi_{(k+1)-2} \circ \dots \circ \psi_{(k+1)-(i+1)}(S_{k-i}) \\ &= \psi_k(\Psi_k^{[i]}(S_{k-i})) \\ &= \psi_k(P_k^{[i]}) = AP_k^{[i]}A' + Q + U_k. \end{aligned} \quad (39)$$

In the following, we prove Lemma 8 by mathematical induction. For $k = 1$, from (8), $\mathbf{P}_1 = \gamma_1 S_1 + (1 - \gamma_1)(AP_0A' + Q + U_0)$. \mathbf{P}_1 takes the value $S_1 = \Psi_1^{[0]}(S_1) = P_1^{[0]}$ at $\gamma_1 = 1$, and $AP_0A' + Q + U_0 = \psi_0(S_0) = \Psi_1^{[1]}(S_0) = P_1^{[1]}$ at $\gamma_1 = 0$. Thus, $\mathbb{E}[\mathbf{P}_1] = P_1^{[0]}\mathbb{P}_\gamma(\{\gamma_1 = 1\}) + P_1^{[1]}\mathbb{P}_\gamma(\{\gamma_1 = 0\}) = P_1^{[0]}\gamma + P_1^{[1]}\bar{\gamma}$. Clearly, Lemma 8 holds at $k = 1$.

Suppose that Lemma 8 is true for $1, \dots, k$. From (38),

$$\begin{aligned} \mathbb{E}[\mathbf{P}_{k+1}] &= \bar{\gamma}(A\mathbb{E}[\mathbf{P}_k]A' + Q + U_k) + \gamma S_{k+1} \\ &\stackrel{(a)}{=} \bar{\gamma}(A(\sum_{n=0}^{k-1} P_k^{[n]}\gamma\bar{\gamma}^n + P_k^{[k]}\bar{\gamma}^k)A' \\ &\quad + (\sum_{n=0}^{k-1} \gamma\bar{\gamma}^n + \bar{\gamma}^k)(Q + U_k)) + \gamma S_{k+1} \\ &\stackrel{(b)}{=} \sum_{n=0}^{k-1} \gamma\bar{\gamma}^{n+1} P_{k+1}^{[n+1]} + P_{k+1}^{[k+1]}\bar{\gamma}^{k+1} + \gamma P_{k+1}^{[0]} \\ &= \sum_{n=0}^k \gamma\bar{\gamma}^n P_{k+1}^{[n]} + P_{k+1}^{[k+1]}\bar{\gamma}^{k+1}, \end{aligned}$$

where (a) is obtained by using the hypothesis and noting that $\sum_{n=0}^{k-1} \gamma\bar{\gamma}^n + \bar{\gamma}^k = 1$, and (b) is obtained by using (39) and noting that $P_{k+1}^{[0]} = \Psi_{k+1}^{[0]}(S_{k+1}) = S_{k+1}$.

Hence, (38) is true for $k + 1$, which completes the proof. ■

Proof of Lemma 9: This lemma is proved by mathematical induction. Clearly, (24) is true for $k = N$. Suppose that (24) is true for $k + 1$. Then we check V_k as follows.

$$\begin{aligned} V_k(x_k) &= \min_{u_k} \mathbb{E}[x_k' W_k x_k + \nu_k u_k' \Lambda_k u_k + V_{k+1}(x_{k+1}) | \mathcal{I}_k] \\ &\stackrel{(a)}{=} \min_{u_k} \mathbb{E}[x_k' (W_k + A' M_{k+1} A) x_k | \mathcal{I}_k] \\ &\quad + \bar{\gamma} \text{tr}(H_{k+1}(A\mathbf{P}_k A' + Q + \bar{\nu}\nu B u_k u_k' B')) \\ &\quad + \nu u_k' (\Lambda_k + B' M_{k+1} B) u_k + 2\nu u_k' B' M_{k+1} A \hat{\mathbf{x}}_k \\ &\quad + \text{tr}(\Delta_{k+1} + M_{k+1} Q + \gamma H_{k+1} S_{k+1}) \\ &= \min_{u_k} \mathbb{E}[x_k' (W_k + A' M_{k+1} A) x_k | \mathcal{I}_k] + 2\nu u_k' B' M_{k+1} A \hat{\mathbf{x}}_k \\ &\quad + \nu u_k' (\Lambda_k + B' (M_{k+1} + \bar{\gamma}\bar{\nu} H_{k+1}) B) u_k + \text{tr}(\Delta_{k+1} \\ &\quad + M_{k+1} Q + \bar{\gamma} H_{k+1} (A\mathbf{P}_k A' + Q) + \gamma H_{k+1} S_{k+1}) \end{aligned} \quad (40)$$

where (a) is obtained by using (21), (24), and some algebraic manipulations. Differentiating $V_k(x_k)$ with respect to u_k and

solving $\frac{\partial V_k}{\partial u_k} = 0$ yield the minimizer $u_k^* = L_k \hat{x}_k$ where $L_k = -(\Lambda_k + B'(\alpha M_{k+1} + \bar{\alpha} T_{k+1})B)^{-1} B' M_{k+1} A$. By substituting u_k^* back to (40) and then using (34), we have

$$\begin{aligned} V_k(x_k) &\stackrel{(a)}{=} \mathbb{E}[x_k' M_k x_k | \mathcal{I}_k] \\ &\quad + \text{tr}((A' M_{k+1} A + W_k - M_k) \mathbf{P}_k) \\ &\quad + \bar{\gamma} \text{tr}(A' H_{k+1} A \mathbf{P}_k) + \text{tr}(\gamma H_{k+1} S_{k+1}) \\ &\quad + \text{tr}(\Delta_{k+1}) + \text{tr}(M_{k+1} Q) + \bar{\gamma} \text{tr}(H_{k+1} Q) \\ &\stackrel{(b)}{=} \mathbb{E}[x_k' M_k x_k | \mathcal{I}_k] + \text{tr}(H_k \mathbf{P}_k) + \text{tr}(\Delta_k) \end{aligned}$$

where (a) is obtained by (25a)(25b), and (b) is obtained by (25c). Thus, (24) holds for V_k . The proof is completed. ■

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