Phase limitations of Zames-Falb multipliers

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Abstract—Phase limitations of both continuous-time and discrete-time Zames-Falb multipliers and their relation with the Kalman conjecture are analysed. A phase limitation for continuous-time multipliers given by Megretski is generalised and its applicability is clarified; its relation to the Kalman conjecture is illustrated with a classical example from the literature. It is demonstrated that there exist fourth-order plants where the existence of a suitable Zames-Falb multiplier can be discarded and for which simulations show unstable behavior. A novel phase-limitation for discrete-time Zames-Falb multipliers is developed. Its application is demonstrated with a second-order counterexample to the Kalman conjecture. Finally, the discrete-time limitation is used to show that there can be no direct counterpart of the off-axis circle criterion in the discrete-time domain

I. INTRODUCTION

The absolute stability of a negative feedback interconnection between an LTI system G and a nonlinearity ϕ with a slope restriction k has aroused the interests of many researchers. The stability tests include the circle criterion, Popov criterion [1], [2], and off-axis circle criterion [3], [4] in continuous time and the circle criterion [5], Tsypkin criterion [6] and Jury-Lee criterion [7], [8] in discrete time. For a recent discussion, see [9] and [10]. Apart from these, loop transformation and multiplier theory are both important tools to establish the stability of feedback interconnections. The Zames-Falb multipliers are a class of multipliers with the property of preserving the positivity of monotone and bounded nonlinearities, and hence of slope-restricted nonlinearities after loop transformation. The class of Zames-Falb multipliers can be defined in either continuous time [11], [12] or discrete time [13], [14]. Specifically, after loop transformation, the stability of the negative interconnection between an LTI system G and a nonlinearity ϕ with a slope restriction k is guaranteed if there exists a Zames-Falb multiplier M such that

$$Re\{M(1+kG)\} > 0,$$
 (1)

with M and G evaluated over all frequencies. That is to say, at $j\omega$, $\omega \in \mathbb{R}$ for continuous-time systems and at $e^{j\omega}$, $\omega \in [0, 2\pi]$ for discrete-time systems.

The Zames-Falb multipliers may be considered a classical tool [15]. Nevertheless, there has been considerable recent interest, largely sparked by the availability of numerical searches

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\hat{k}_{ZF}	Maximum slope for which a Zames-Falb multiplier is known					
k_{ZF}	Maximum slope for which there exists a Zames-Falb multiplier					
k_S	Maximum slope for which the Lur'e system is absolutely stable					
k_{PL}	Minimum slope for which phase limitation implies there is no					
	Zames-Falb multiplier					
\hat{k}_C	Minimum slope for which a counterexample to absolute					
	stability is known					
k_O	Slope for direct discrete-time counterpart off-axis circle criterion					
	(which is false)					
k_{RO}	Slope for Reduced Off-axis circle criterion in [42]					
k_N	Nyquist value					

([16], [17], [18], [19], [20], [21], [22], for continuous time; [23], [24] for discrete time) and their encapsulation within an IQC (integral quadratic constraint) framework [25], [26], [27], [28]. There has also been interest in generalising the class, both to MIMO (multi-input, multi-output) nonlinearities [29], [30], [31], [32], [33] and to nonlinearities outside the original classes considered by Zames and Falb [34], [35], [28], [36]. In addition to determining stability conditions, they can be used to analyse performance [37], [38]; further, they can be used to obtain tighter versions of the Popov criterion [39]. Applications of Zames-Falb multipliers range from input-constrained model predictive control [40] to first order numerical optimisation algorithms [41].

Although both continuous-time and discrete-time Zames-Falb multipliers are defined with similar conditions, there are clear distinctions between their properties. In discrete time the Zames-Falb multipliers are the full set of multipliers preserving the positivity of monotone and bounded nonlinearities, besides direct phase substitutions [14], [43]. In continuous time matters are more nuanced, but the class of Zames-Falb multipliers remains the widest known class

Fig. 1. Relations between slope restrictions discussed in the text. Conjecture I.2 is that $k_{ZF} = k_S$ and hence $k_S < k_{PL}$.

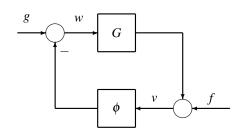


Fig. 2. Lur'e problem

of multipliers preserving the positivity of monotone and bounded nonlinearities, up to phase equivalence [39], [44]. For a tutorial introduction to the phase properties of continuous-time Zames-Falb multipliers, phase-equivalence results and the issues associated with causality, see [45]. Phase properties are essential to our understanding of Zames-Falb multipliers.

For example, if $k < k_N$ (see Table I for various slope restrictions discussed in this paper), then the phase of (1+kG) lies between -180° and 180° . Meanwhile multipliers must be positive so are restricted to lie between -90° and 90° . But as the Kalman conjecture is false, any set of suitable multipliers must be restricted by some further fundamental limitations. This follows from the obvious but important fact:

Fact I.1 If the system is not absolutely stable, there can be no appropriate Zames-Falb multiplier.

However, only a few papers discuss such limitations. Megretski [46] shows that there exists a phase limitation for continuous-time Zames-Falb multipliers. Another phase limitation of Zames-Falb multipliers is given by Jönsson and Laiou [47], [48]. Such limitations are often ignored when new searches for multipliers are presented (see for example [45] and references therein). Often only k_N is provided as an upper limit for the slope restriction.

We discuss Megretski's phase restriction [46] with respect to a fourth-order continuous-time plant whose phase drops from $+180^{\circ}$ to -180° ; similarly with k sufficiently big the phase of 1+kG drops from above $+90^{\circ}$ degrees to below -90° . The limitation cannot be applied to first, second or third-order plants whose phase is in the range (-180,90) degrees or (-90,180) degrees; this agrees with the well-known result that the Kalman conjecture is true for such plants [49].

To the best of authors' knowledge, no similar limitation has been developed in the discrete-time domain. Since there exist second-order discrete-time counterexamples to the Kalman conjecture whose phase is in the range (-180,0) degrees [50], [51] one might expect a *simpler* limitation for discrete-time multipliers; this turns out to be indeed the case.

The contribution of this paper is for both continuous-time and discrete-time multipliers. We generalise Megretski's limitation [46] for continuous-time multipliers to a wider choice of frequency intervals. Further, we show that Megrestki's limitation [46] only applies for the class of Zames-Falb multipliers which do not require the odd condition on the nonlinearity; we provide the corresponding result when the nonliearity is odd. We discuss the limitation's numerical calculation and demonstrate its application in the context of a classical example due to O'Shea [11], [45]. In particular we demonstrate a fourthorder counterexample of the Kalman conjecture for which the constraint is active. A further contribution of the paper is the development of a phase limitation for discrete-time Zames-Falb multipliers. This limitation is fundamentally different to Megrestki's limitation as it only requires the phase of (1+kG)to be either in the interval (90, 180) degrees or in the interval (-180, -90) degrees. The limitation is easy to compute, and is active for a second-order discrete-time counterexample of the Kalman conjecture. This close link between the preclusion of a Zames-Falb multiplier and unstable behaviour leads us to the following conjecture as the counterpart to Fact I.1; however no proof (or counterexample) is offered in this paper:

Conjecture I.2 If there is no appropriate Zames-Falb multiplier, the system is not absolutely stable.

One direct application of the phase limitation is to show there can be no direct discrete-time counterpart of the off-axis circle criterion. The continuous-time off-axis circle criterion is a useful graphical stability test and is shown to be a less conservative criterion compared to the circle criterion [3], [4]. The derivation is based on the phase properties of RL/RC multipliers. The direct discrete-time counterpart of the offaxis circle criterion is sometimes assumed to be true in the literature (e.g. [52], [53]). However only a highly restrictive discrete-time version is proposed in [42], without discussion as to whether the direct discrete-time counterpart off-axis circle criterion is true or false. In this paper, we show that in some cases there are no Zames-Falb multipliers with the requisite phase properties for its derivation - i.e. the direct counterpart off-axis circle criterion cannot be derived using multiplier theory. The invalidation is completed by counterexample.

Some preliminary results related with Theorem IV.3 part (i) were presented in [54].

II. NOTATION AND PRELIMINARY RESULTS

A. Signal spaces

For continuous-time signals let $\mathcal{L}_2[0,\infty)$ be the Hilbert space of square integrable and Lebesgue measurable functions $f:[0,\infty)\to\mathbb{R}$ and let \mathcal{L}_2 be defined similarly for $f:\mathbb{R}\to\mathbb{R}$. Let $\mathcal{L}_{2e}[0,\infty)$ be the extended space of $\mathcal{L}_2[0,\infty)$ [43].

For discrete-time signals let \mathbb{Z} and \mathbb{Z}^+ be the set of integer numbers and positive integer numbers including 0, respectively. Let ℓ be the space of all real-valued sequences, $h: \mathbb{Z}^+ \to \mathbb{R}$ and let ℓ_2 denote the Hilbert space of all square-summable and measurable real sequences $f: \mathbb{Z}^+ \to \mathbb{R}$ (ℓ is the extended space of ℓ_2). Similarly, we can define the Hilbert space $\ell_2(\mathbb{Z})$ by considering real sequences $f: \mathbb{Z} \to \mathbb{R}$.

B. Lur'e problem and the Kalman conjecture

The feedback interconnection system is a Lur'e system represented in Fig. 2 with both G and ϕ mapping $\mathcal{L}_{2e}[0,\infty) \to \mathcal{L}_{2e}[0,\infty)$ (continuous time) or $\ell \to \ell$ (discrete time). The object G is assumed LTI stable and the object ϕ memoryless and slope-restricted (see below). The interconnection relationship is

$$\begin{cases} v = f + Gw, \\ w = -(\phi v) + g. \end{cases}$$
 (2)

The system (2) is well-posed if the map $(v,w) \mapsto (g,f)$ has a causal inverse on $\ell \times \ell$, and this feedback interconnection is ℓ_2 -stable if for any $f,g \in \ell_2$, both $w,v \in \ell_2$.

Definition II.1 (Memoryless slope-restricted nonlinearity) The nonlinearity $\phi: \mathcal{L}_{2e}[0,\infty) \to \mathcal{L}_{2e}[0,\infty)$ or $\phi: \ell \to \ell$ is said to be memoryless and slope-restricted in S[0,k], if there

is a function $N: \mathbb{R} \to \mathbb{R}$ such that $(\phi u)(t) = N(u(t))$ or $(\phi u)(k) = N(u(k)), \ N(0) = 0, \ and$

$$0 \le \frac{N(x_1) - N(x_2)}{x_1 - x_2} \le k, \quad \forall x_1, x_2 \in \mathbb{R}, x_1 \ne x_2.$$
 (3)

In addition, ϕ is said to be odd if N is odd, i.e. N(x) = -N(-x), for all $x \in \mathbb{R}$.

We define the Nyquist value and state the Kalman conjecture for both continuous-time and discrete-time systems.

Definition II.2 (Nyquist value) Given a stable LTI system G, the Nyquist value k_N is the supremum of all the positive real numbers k such that τkG satisfies the Nyquist Criterion for all $\tau \in [0,1]$. It can also be expressed as:

$$k_N = \sup\{k > 0 : \inf_{\Omega}\{|1 + \tau kG|\} > 0\}, \forall \tau \in [0, 1]\},$$
 (4)

with G evaluated over all frequencies (i.e. $\omega \in \mathbb{R}$ for continuous-time systems and $\omega \in [0,2\pi]$ for discrete-time systems).

Conjecture II.3 (Kalman Conjecture, [55]) *Let* ϕ *be a memoryless slope-restricted nonlinearity such that there exists a continuously differentiable* $N : \mathbb{R} \to \mathbb{R}$ *and* S > 0 *such that* $\phi(v)(t) = N(v(t))$ *(or* $\phi(v)(k) = N(v(k))$ *and*

$$0 \le \frac{dN(x)}{dx} \le S, \quad \forall x \in \mathbb{R}. \tag{5}$$

Then the negative feedback interconnection of the continuoustime (or discrete-time) LTI systems $G \sim [A,B,C,0]$ and ϕ (Fig 2) is globally asymptotically stable if A-BCk is Hurwitz (Schur) for all $k \in [0,S]$.

There exist fourth-order continuous-time counterexamples to the Kalman conjecture [56], [49], [57] and second-order discrete-time counterexamples [50], [51].

C. Zames-Falb multipliers

The characteristics of continuous-time Zames-Falb multipliers is given in the following theorem that defines two different classes of multipliers.

Theorem II.4 (Continuous-time Zames-Falb multipliers, [12].) Consider the continuous-time feedback system in Fig. 2 with G a stable LTI system and ϕ memoryless and sloperestricted in S[0,k]. Suppose that there exists an LTI multiplier $M: \mathcal{L}_2 \to \mathcal{L}_2$ whose transfer function has the form

$$M(s) = 1 - H(s) \tag{6}$$

such that the impulse response h of H satisfies

$$\int_{-\infty}^{\infty} |h(t)| \, dt < 1. \tag{7}$$

Moreover, let us assume that either ϕ is odd or h(t) > 0. Suppose further there is some $\delta > 0$ such that

$$Re\{M(j\omega)(1+kG(j\omega))\} \ge \delta \text{ for all } \omega \in \mathbb{R}.$$
 (8)

Then the feedback interconnection (2) is \mathcal{L}_2 -stable.

Remark II.5 With some abuse of notation, we denote h(t) as the addition of a real-valued function $h_a(t)$ and impulses at different instants, i.e.

$$h(t) = h_a(t) + \sum_{i=1}^{\infty} h_i \delta(t_i). \tag{9}$$

Definition II.6 The class of continuous-time Zames-Falb multipliers \mathcal{M}^c is defined as the LTI systems $M: \mathcal{L}_2 \to \mathcal{L}_2$ whose transfer function has the form

$$M(s) = 1 - H(s) \tag{10}$$

such that the impulse response h of H satisfies that $h(t) \ge 0$ for all t and

$$\int_{-\infty}^{\infty} h(t) \, dt < 1. \tag{11}$$

Definition II.7 The class of continuous-time "odd" Zames-Falb multipliers \mathcal{M}_{odd}^c is defined as the LTI systems $M: \mathcal{L}_2 \to \mathcal{L}_2$ whose transfer function has the form

$$M(s) = 1 - H(s) \tag{12}$$

such that the impulse response h of H satisfies

$$\int_{-\infty}^{\infty} |h(t)| \, dt < 1. \tag{13}$$

By definition, $\mathcal{M}^c \subset \mathcal{M}^c_{\text{odd}}$.

The counterpart result in discrete time is given in the following theorem and it also defines two different classes of multipliers:

Theorem II.8 (Discrete-time Zames-Falb multipliers, [14], [43]) Consider the discrete-time feedback system in Fig. 2 with G a stable LTI system and ϕ memoryless and slope-restricted in S[0,k]. Suppose that there exists an LTI multiplier $M: \ell_2 \to \ell_2$ whose transfer function has the form

$$M(z) = 1 - H(z) \tag{14}$$

such that the impulse response h of H satisfies that $h_0 = 0$ and

$$\sum_{i=-\infty}^{\infty} |h_i| < 1. \tag{15}$$

Moreover, let us assume that either the nonlinearity is odd or $h_i \ge 0$. Suppose further

$$Re\{M(z)(1+kG(z))\}>0, \quad \forall |z|=1.$$
 (16)

Then the feedback interconnection (2) is ℓ_2 -stable.

Remark II.9 Inequality (8) is evaluated over $\omega \in \mathbb{R}$ whereas inequality (16) is evaluated over the frequency interval $\omega \in [0,2\pi]$. Hence, by the Extreme Value Theorem [58], it is unnecessary to define any $\delta > 0$ for the discrete case corresponding to that used in the continuous case.

Similarly to the previous definitions, we can define the classes of multipliers \mathcal{M}^d and \mathcal{M}^d_{odd} .

D. Off-axis circle criterion

The continuous-time off-axis circle criterion is given.

Lemma II.10 (Off-axis circle criterion for continuous-time systems, [3]) Consider the feedback system in Fig. 2 with G LTI stable and ϕ is slope-restricted in S[0,k]. Suppose that the Nyquist plot of the linear part of the system $G(j\omega)$ lies entirely to the right of a straight line passing through the point $(-\frac{1}{k} + \delta, 0)$ where $\delta > 0$ and ϕ is monotonically increasing. Then the feedback interconnection (2) is \mathcal{L}_2 -stable.

For discrete time, only a highly restrictive version is proposed.

Lemma II.11 (Reduced off-axis circle criterion for discrete-time systems, [42]) Let the Nyquist plot of $G(e^{j\omega})$ for all $0 \le \omega \le \pi$ lie entirely to the right of a straight line, whose slope k is nonnegative passing through $(-\frac{1}{K_2},0)$. Let ω_0 be such that $Re\ G(e^{j\omega_0}) = -\frac{1}{K_2}$ and $Re\ G(e^{j\omega}) \ge -\frac{1}{K_2}$ for $\omega \ge \omega_0$ and $Im\ G(e^{j\omega}) \le 0$ for $\omega_0 \ge \omega \ge 0$. Then the system is asymptotically stable for all monotone ϕ with slope restriction K_2 in the feedback path if

$$\theta \le -\frac{1}{2}\omega_0 + \frac{\pi}{2},\tag{17}$$

where θ is the angle made by the straight line and the imaginary axis, i.e., $\theta = \cot^{-1} k$. If Im $G(e^{j\omega}) \ge 0$ for $\omega_0 \ge \omega \ge 0$, the same argument can be used to prove the asymptotic stability of the system with nonpositive k and

$$\theta \ge \frac{1}{2}\omega_0 - \frac{\pi}{2}.\tag{18}$$

E. Further mathematical notation

For the convenience of solving potential numerical issues, the notation of $O(\cdot)$ is given.

Definition II.12 The condition

$$f(t) = g(t) + O(t^n), \text{ as } t \to 0.$$
 (19)

means that there exist M and to such that

$$|f(t) - g(t)| < Mt^n \text{ on } [0, t_0].$$
 (20)

The floor function, denoted by |v|, is defined by

$$\lfloor v \rfloor = \max\{m \in \mathbb{Z} \mid m \le v\}. \tag{21}$$

III. CONTINUOUS PHASE LIMITATIONS AND THE KALMAN CONJECTURE

Megretski presents in [46] a phase limitation for continuoustime Zames-Falb multipliers. In this section we generalise the result to a wider set of frequency intervals, and derive separate results for both \mathcal{M}^c and \mathcal{M}^c_{odd} . Although it is stated in [46] that the result there is valid for \mathcal{M}^c_{odd} (in the terminology of this paper) we show by counterexample that it is in fact valid for \mathcal{M}^c only. Finally, we bridge the limitation of [46] with the Kalman conjecture; this is the key motivation to develop a different set of phase limitations for the discrete-time Zames-Falb multipliers.

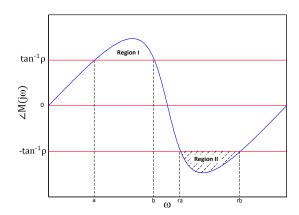


Fig. 3. Illustration of Theorem III.4 with the choice of interval from [46]: $\kappa = 1$, c = ra and d = rb. The result is given in terms of ρ while the phase of the multiplier M is $\tan^{-1} \rho$.

A. Phase limitations

Definition III.1 Let 0 < a < b < c < d, $\kappa > 0$, $\lambda > 0$ and $\mu > 0$. Define

$$\rho^{c} = \sup_{t > 0} \frac{|\psi(t)|}{\phi(t)},\tag{22}$$

and

$$\rho_{odd}^{c} = \sup_{t>0} \frac{|\psi(t)|}{\tilde{\phi}(t)},\tag{23}$$

where

$$\psi(t) = \frac{\lambda \cos(at)}{t} - \frac{\lambda \cos(bt)}{t} - \frac{\mu \cos(ct)}{t} + \frac{\mu \cos(dt)}{t}, \quad (24)$$

$$\phi(t) = \lambda(b-a) + \kappa\mu(d-c) + \phi_1(t), \tag{25}$$

ana

$$\tilde{\phi}(t) = \lambda (b-a) + \kappa \mu (d-c) - |\phi_1(t)|, \tag{26}$$

with

$$\phi_1(t) = \frac{\lambda \sin(at)}{t} - \frac{\lambda \sin(bt)}{t} + \frac{\kappa \mu \sin(ct)}{t} - \frac{\kappa \mu \sin(dt)}{t}.$$
(27)

Lemma III.2 If λ and μ are chosen such that

$$\frac{\lambda}{\mu} = \frac{d^2 - c^2}{b^2 - a^2},\tag{28}$$

then ρ^c and ρ^c_{odd} in Definition III.1 are well-defined; that is to say $\rho^c < \infty$ and $\rho^c_{odd} < \infty$.

Remark III.3 The direct calculation of the ratios $\psi(t)/\phi(t)$ and $\psi(t)/\tilde{\phi}(t)$ is numerically ill-conditioned for small t since, with the choice (28), we have $\psi(t) = 0 + O(t^3)$, $\phi(t) = 0 + O(t^2)$ and $\tilde{\phi}(t) = 0 + O(t^2)$, all as $t \to 0$. Nevertheless, the same construction ensures we can write

$$\frac{\psi(t)}{\phi(t)} = \gamma t + O(t^3) \text{ and } \frac{\psi(t)}{\tilde{\phi}(t)} = \gamma t + O(t^3) \text{ as } t \to 0, \quad (29)$$

with

$$\gamma = -\frac{1}{4} \frac{\lambda (b^4 - a^4) - \mu (d^4 - c^4)}{\lambda (b^3 - a^3) + \kappa \mu (d^3 - c^3)}.$$
 (30)

We use this relation for small t in the numerical examples below.

Theorem III.4 (Continuous-time phase limitations) *Let M be a continuous-time Zames-Falb multiplier. Suppose*

$$Im(M(j\omega)) > \rho Re(M(j\omega)) \text{ for all } \omega \in [a,b],$$
 (31)

and

$$Im(M(j\omega)) < -\kappa \rho Re(M(j\omega)) \text{ for all } \omega \in [c,d],$$
 (32)

for some $\rho > 0$. Then under the conditions of Lemma III.2.

(i)
$$\rho < \rho^c$$
 if $M \in \mathcal{M}^c$,

(ii)
$$\rho < \rho_{odd}^c$$
 if $M \in \mathcal{M}_{odd}^c$.

Proof: See Appendix.

Lemma III.2 and Theorem III.4, with the choice $\kappa=1$, c=ra, d=rb and hence $\lambda/\mu=r^2$, are in [46]. An interpretation of Theorem III.4 with these values is illustrated in Fig. 3 (see also [45]). According to the constraints on the coefficients of continuous Zames-Falb multipliers, if the phase is simultaneously greater than $\tan^{-1}\rho$ on $\omega\in[a,b]$ (in Region I) and smaller than $-\tan^{-1}\rho$ on $\omega\in[ra,rb]$ (in Region II), then $\rho<\rho^c$ if $M\in \mathscr{M}^c$ and $\rho<\rho^c_{odd}$ if $M\in \mathscr{M}^c_{odd}$.

Remark III.5 It is straightforward to produce the counterpart of Theorem III.4 with (31) and (32) replaced by $Im(M(j\omega)) < -\rho Re(M(j\omega))$ for all $\omega \in [a,b]$ and $Im(M(j\omega)) > \kappa \rho Re(M(j\omega))$ for all $\omega \in [c,d]$ respectively.

Remark III.6 In [46], Megretski uses a positive sign in the exponential of the Laplace transform:

$$M(j\omega) = 1 - \int_{-\infty}^{\infty} e^{j\omega t} h(t) dt.$$
 (33)

This is the standard convention in the Physics literature (see for example [59]) but opposite to that used in [12]. The apparent discrepancy has no significant consequence for the analysis of phase limitations since if M(s), with impulse response m(t), is a Zames-Falb multiplier then M(-s), with impulse response m(-t), is also a Zames-Falb multiplier.

It is natural to ask whether a phase limitation over a single frequency range can be constructed in a similar manner. This is not possible in continuous time, as any corresponding definition of ρ^c or ρ^c_{odd} would be unbounded as t approaches 0. Loosely speaking, we can generate a multiplier in \mathcal{M}^c with phase arbitrarially close to $\pm 90^\circ$ over an arbitrararily large frequency inteterval by selecting $h(t) = (1-\varepsilon)\delta(t-t^*)$ with $\varepsilon > 0$ arbitrarially close to 0 and t^* arbitrarially close to 0. But we construct such phase limitations for discrete-time multipliers below, in Section IV.

B. Numerical example

Here we illustrate Theorem III.4 with a numerical example. Let a = 1.6 and b = 2.25. Let $\kappa = 1$, c = ra and d = rb with r = 2.1. Then a sweep over time intervals followed by local numerical search gives

$$\rho^c \approx 0.6069, \tan^{-1} \rho^c \approx 31.25^\circ,$$
 (34)

and

$$\rho_{odd}^c \approx 1.4928, \ \tan^{-1} \rho_{odd}^c \approx 56.18^{\circ}.$$
 (35)

Now consider the multiplier

$$M(j\omega) = 1 - \int_{-\infty}^{\infty} e^{-j\omega t} h(t) dt$$
 (36)

with $h(t) = -0.9\delta(t+1)$. Figure 4 shows that the relations (31) and (32), or equivalently

$$\angle M(j\omega) > \tan^{-1} \rho$$
 over the interval $[a,b]$, (37)

and

$$\angle M(j\omega) < -\tan^{-1}\rho$$
 over the interval $[ra, rb]$, (38)

are satisfied simultaneously for $\rho = \rho^c$ but not for $\rho = \rho^c_{odd}$. This is consistent with Theorem III.4 as $M \in \mathcal{M}^c_{odd}$ but $M \notin \mathcal{M}^c$. It is a counterexample to the false claim in [46] that the phase limitation of Theorem III.4 part (i) is applicable to the wider class $M \in \mathcal{M}^c_{odd}$.

Remark III.7 Both in this numerical example and at the end of Section III-A we consider multipliers where h takes the form $h(t) = (1 - \varepsilon)\delta(t - \tau)$ for some τ . There is a close link with Theorem III.4. Specifically if $|\psi(\tau)|/\phi(\tau) = \rho^c$ and $\varepsilon \to 0$ then

$$\int_{-\infty}^{\infty} \psi(t)h(t)dt = \rho^{c}[\lambda(b-a) + \kappa\mu(d-c)] + \rho^{c}\int_{-\infty}^{\infty} \phi_{1}h(t)dt.$$
(39)

Compare (80) in the proof of Theorem III.4. Similarly if $|\psi(\tau)|/\tilde{\phi}(\tau)=\rho_{odd}^c$ then

$$\int_{-\infty}^{\infty} \psi(t)h(t)dt =$$

$$= \rho_{odd}^{c} [\lambda(b-a) + \kappa\mu(d-c)] + \rho_{odd}^{c} \int_{-\infty}^{\infty} \phi_{1}h(t) dt. \quad (40)$$

We discuss the corresponding relations at greater length in Section IV-B for the discrete-time case.

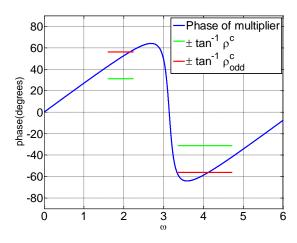


Fig. 4. Phase of the multiplier (36) and the continuous phase limitations $\pm \tan^{-1} \rho^c \approx \pm 31.25^\circ$ and $\pm \tan^{-1} \rho^c_{odd} \approx \pm 56.18$ evaluated on [1.6, 2.25] and [3.36, 4.725].

C. Counterexamples to Kalman conjecture via phase limitations

It is instructive to interpret the phase limitations of Theorem III.4 in a manner consistent with known results about the Kalman conjecture.

On the one hand, it is well-known that first, second and third order plants hold the Kalman conjecture [49]. The phase of such plants cannot reach both Regions I and II in Fig. 3. So the phase limitations cannot apply to these plants. First-order plants do not require a dynamic multiplier, second-order plants require a dynamics multiplier with a tunable zero and a pole at infinity, i.e. a Popov multiplier, and-third order plant requires both a tunable pole and zero, i.e. first order RL/RC multipliers. In all these cases, only a first-order multiplier is required, and we know that there is no phase limitation in the selection of such multipliers [39].

Remark III.8 Although the off-axis circle criterion is also based on RL/RC multipliers, it is not sufficient to show all third-order plants hold the Kalman conjecture. For example, the off-axis circle criterion with the plant

$$G = \frac{s^2}{s^3 + 1.002s^2 + s + 0.998},\tag{41}$$

guarantees stability with k < 3.928, whereas the multiplier $M = (s+1)/(s+\varepsilon)$ guarantees stability for any positive k with a sufficiently small value $\varepsilon > 0$.

On the other hand the phase limitations may be applied to fourth-order plants, and these in turn may be counterexamples to the Kalman conjecture by: a) showing numerically that a phase limitation can be applied to a well-known plant, and b) showing that the Lur'e system with this plant and a slope-restricted nonlinearity may be unstable.

Specifically we will consider the phase limitation of Theorem III.4 part (i) with $\kappa = 1$, c = ra, and d = rb; that is to say the original result of [46] applied to \mathcal{M}^c . A particularly suitable example to show this limitation is O'Shea example [11], [45]:

$$G(s) = \frac{s^2}{(s^2 + 2\xi s + 1)^2},\tag{42}$$

since the symmetry of the problem simplifies the selection of the parameters. In this example, O'Shea showed that there is a Zames-Falb multiplier for any k if $\xi > 0.5$. The following result shows that it is not possible to reach an arbitrary large k for any $\xi \leq 0.25$. For the case $\xi = 0.25$, the phase of G(s) is above 177.98° over the interval [a,b] where a=0.02249 and b=0.03511; hence it is below -177.98° over the interval [1/b,1/a] by using the symmetry of the plant. Then a suitable Zames-Falb multiplier for this plant would require a phase below -87.98° over the interval [a,b] and above 87.98° over the interval [1/b,1/a]. The phase limitation ensures that there is no Zames-Falb multiplier with such a phase characteristic, since $\tan^{-1}\rho^c\approx 87.79^\circ$. Strictly speaking, we have used the counterpart of Theorem III.4 mentioned in Remark III.5.

Although numerical reliability can be problematic in the discussion of the Kalman conjecture [57], simulations of the plant with asymmetrical saturation show a time evolution that

does not appear to settle to zero, supporting the validity of Conjecture I.2. The simulation shown in Figure 5 has been run in MATLAB R2013, using the solver ode45, with maximum step size of 0.0001 s, and relative tolerance of 10^{-3} . The nonlinearity ϕ is described by the nonlinear function

$$N(x) = \begin{cases} -1000 & x < -1; \\ 1000x & -1 \le x \le 0; \\ 0 & x > 0; \end{cases}$$
 (43)

the input g is given by

$$g(t) = \begin{cases} 100 & t \le 20s; \\ 0 & t > 20s; \end{cases}$$
 (44)

and f(t) = 0. The relevance of this counterexample to the Kalman conjecture is that we can show that there is no Zames-Falb multiplier with $h(t) \ge 0$ for the system. The asymmetry of the nonlinearity seems to be a key factor as simulations with symmetric saturations show stable behaviour. The importance of asymmetry in the stability of Lur'e systems with saturation has been discussed recently [36], [60].

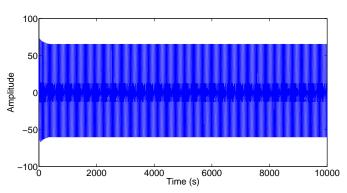


Fig. 5. Amplitude of the signal v in a simulation of the feedback interconnection depicted in Fig. 2 where G is given by (42) with $\xi = 0.25$, ϕ is described by (43), g(t) is given by (44), and f(t) = 0.

Remark III.9 The magnitude of the response in Fig. 5 is bounded. Since the plant G is stable and ϕ is sector-bounded it follows that if $g \in \mathcal{L}_{\infty}$ then all signals must be in \mathcal{L}_{∞} .

IV. DISCRETE-TIME PHASE LIMITATION

In this section we develop phase limitations for discretetime Zames-Falb multipliers. Their derivation is in the spirit of Megretski's limitation [46] and Theorem III.4 for continuoustime multipliers. However their properties are simpler and consistent with the existence of second-order discrete-time counterexamples to the Kalman conjecture [50], [51]. In particular, and by contrast with their continuous-time counterparts, they are concerned with properties over a single interval $\omega \in [a,b]$.

It is worth highlighting that if a discrete-time multiplier preserves the positivity of all monotone and bounded non-linearities then either it is a Zames-Falb multiplier or there exists a Zames-Falb multiplier with the same phase [14], [43]. Hence any phase limitation on the discrete-time Zames-Falb multipliers is also a limitation for any discrete-time multiplier.

A. Phase limitations

Definition IV.1 Let $0 \le a \le b \le \pi$. Define

$$\rho^d = \max_{n \in \mathbb{Z}^+} \frac{|\psi_d(n)|}{\phi_d(n)},\tag{45}$$

and

$$\rho_{odd}^d = \max_{n \in \mathbb{Z}^+} \frac{|\psi_d(n)|}{\tilde{\phi}_d(n)},\tag{46}$$

where

$$\psi_d(n) = \frac{\cos(an)}{n} - \frac{\cos(bn)}{n},$$

$$\phi_d(n) = (b-a) + \phi_{d,1}(n),$$
(47)

$$\phi_d(n) = (b-a) + \phi_{d,1}(n),$$
 (48)

and

$$\tilde{\phi}_d(n) = (b-a) - |\phi_{d,1}(n)|,$$
 (49)

with

$$\phi_{d,1}(n) = \frac{\sin(an)}{n} - \frac{\sin(bn)}{n}.$$
 (50)

Lemma IV.2 Both ρ^d and ρ^d_{odd} in Definition IV.1 are well-defined; that is to say $\rho^d < \infty$ and $\rho^d_{odd} < \infty$.

Theorem IV.3 (Discrete-time phase limitations) Let M be a discrete-time Zames-Falb multiplier. Suppose

$$Im(M(e^{j\omega})) > \rho Re(M(e^{j\omega})) \text{ for all } \omega \in [a,b],$$
 (51)

for some $\rho > 0$. Then

(i)
$$\rho < \rho^d$$
 if $M \in \mathcal{M}^d$,
(ii) $\rho < \rho^d_{odd}$ if $M \in \mathcal{M}^d_{odd}$.

(ii)
$$\rho < \rho_{odd}^d$$
 if $M \in \mathcal{M}_{odd}^d$.

An interpretation of Theorem IV.3 is illustrated in Fig. 6. According to the constraints on the coefficients of discretetime Zames-Falb multipliers, if the phase is greater than $\tan^{-1} \rho$ on $\omega \in [a,b]$ (in Region A), then $\rho < \rho^d$ if $M \in \mathcal{M}^d$ and $\rho < \rho^d_{odd}$ if $M \in \mathcal{M}^d_{odd}$.

Remark IV.4 It is straightforward to produce the counterpart of Theorem IV.3 with (51) replaced by

$$Im(M(e^{j\omega})) < -\rho Re(M(e^{j\omega})) \text{ for all } \omega \in [a,b].$$
 (52)

An algorithm for finding the phase limitation in Theorem IV.3 part (i) for a second order plant is given in [54]. For a given stable plant G and a value of k such that $0 < k < k_N$ the phase of an ideal multiplier is obtained as

$$\angle M_d = \begin{cases} \angle (G+1/k) - 90 & \text{if } \angle (G+1/k) > 90 \\ \angle (G+1/k) + 90 & \text{if } \angle (G+1/k) < -90 \\ 0 & \text{otherwise.} \end{cases}$$
 (53)

Then the algorithm increases k until the existence of such a multiplier can be discarded by using the limitation presented in Theorem IV.3.

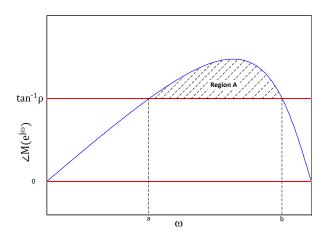


Fig. 6. Illustration of Theorem IV.3. The result is given in terms of ρ while the phase of the multiplier M is $\tan^{-1} \rho$.

B. Integral bound and sparsely parametrized multipliers

Theorem IV.3 gives relative bounds on the real and imaginary parts of a Zames-Falb multiplier's frequency response over an interval [a,b]. It is straightforward to derive a closely related result in terms of the integrals over the same interval.

Theorem IV.5 Let M be a discrete-time Zames-Falb multiplier. Suppose

$$\int_{a}^{b} Im(M(e^{j\omega})) d\omega > \rho \int_{a}^{b} Re(M(e^{j\omega})) d\omega \qquad (54)$$

for some $\rho > 0$. Then

(i)
$$\rho < \rho^d$$
 if $M \in \mathcal{M}^d$

(i)
$$\rho < \rho^d$$
 if $M \in \mathcal{M}^d$,
(ii) $\rho < \rho^d_{odd}$ if $M \in \mathcal{M}^d_{odd}$

Remark IV.6 Theorem IV.5 is stronger than Theorem IV.3 in the sense that condition (51) is sufficient for condition (54) but not necessary. Theorem IV.3 may be derived as a Corollary of Theorem IV.5 by applying the Mean Value Theorem [58].

Theorem IV.5 gives a tight phase limitation in the sense that we can associate a set of sparsely parameterized multipliers with Theorem IV.5 as follows.

Proposition IV.7

(i) For a given a and b, define the set $\mathcal{N}^d \subset \mathbb{Z}$ as the set of integers n such that $\psi_d(n)/\phi_d(n) = \rho^d$. Then multipliers of the form

$$M(z) = 1 - \sum_{n \in \mathcal{N}^d} h_n z^{-n}$$
(55)

with

$$h_0 = 0, \ h_n \ge 0 \ and \ \sum_{n \in \mathcal{N}([a,b])} h_n = 1 - \varepsilon$$
 (56)

satisfy (54) with ρ arbitrarily close to ρ^d in the limit as $\varepsilon \to 0$.

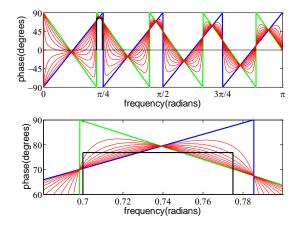


Fig. 7. Phases of the limiting cases $M(z)=1-z^8$, $M(z)=1-z^{-9}$ and linear combinations of the form $M(z)=1-\lambda z^8-(1-\lambda)z^{-9}$ with $0<\lambda<1$. The phase limitation $\tan^{-1}\rho^d\approx 76.8^\circ$ over the interval [a,b]=[0.7,0.77501] is also shown. The top figure shows the phase over the frequency range from 0 to π radians, while the bottom figure shows the same data in the frequency range from 0.68 to 0.8 radians.

(ii) For a given a and b, define the set $\mathcal{N}_{odd}^d \subset \mathbb{Z}$ as the set of integers n such that $\psi_d(n)/\tilde{\phi}_d(n) = \rho_{odd}^d$. Then multipliers of the form

$$M(z) = 1 - \sum_{n \in \mathcal{N}^d} h_n z^{-n} \tag{57}$$

with

$$h_0 = 0$$
 and $\sum_{n \in \mathcal{N}([a,b])} h_n = 1 - \varepsilon$ (58)

satisfy (54) with ρ arbitrarily close to ρ_{odd}^d in the limit as $\varepsilon \to 0$.

Remark IV.8 It is, once again, straightforward to produce the counterpart of Theorem IV.5 with (54) replaced by

$$\int_{a}^{b} Im(M(e^{j\omega})) d\omega < -\rho \int_{a}^{b} Re(M(e^{j\omega})) d\omega.$$
 (59)

Similarly for Theorem IV.7 with (i) $\psi_d(n)/\phi_d(n) = -\rho^d$ and (ii) $\psi_d(n)/\tilde{\phi}_d(n) = -\rho^d_{odd}$.

As an illustrative example, suppose a = 0.7 and b = 0.77501 (approx.). Then

$$\tan^{-1} \rho^d \approx 76.8^\circ, \tag{60}$$

and

$$\mathcal{N}^d = \{-8, 9\}. \tag{61}$$

Fig 7 shows the phases of the limiting cases $M(z) = 1 - z^8$, $M(z) = 1 - z^{-9}$ and linear combinations of the form $M(z) = 1 - \lambda z^8 - (1 - \lambda)z^{-9}$ with $0 < \lambda < 1$. It can be seen that the phases are near to $\tan^{-1} \rho^d$ over the interval [a, b]. However they always have values both above and below, indicating that Theorem IV.3 is not tight in the same sense as Theorem IV.5.

Remark IV.9 A similar analysis is possible for continuoustime multipliers. Compare Remark III.7. C. Discrete-time counterparts of the off-axis circle criterion

The off-axis circle criterion [3] (Theorem II.10) is a useful frequency-based graphical stability test for continuous-time systems. It is sometimes assumed (e.g. in [52]) that its discrete-time counterpart is true. We state this as a conjecture:

Conjecture IV.10 Consider the feedback system in Fig. 2 with $G \in \mathbf{RH}_{\infty}$, and ϕ is slope-restricted in S[0,k]. Suppose that the Nyquist plot of the linear part of the system $G(e^{j\omega})$ lies entirely to the right of a straight line passing through the point $(-\frac{1}{k} + \delta, 0)$ where $\delta > 0$ and ϕ is monotonically increasing. Then the feedback interconnection (2) is ℓ_2 -stable.

A geometrical interpretation of both Theorem II.10 for continuous-time systems and Conjecture IV.10 for discrete-time systems is given in Fig 8.

The phase-limitation on discrete-time Zames-Falb multipliers carries the implication that there can be no multiplier construction corresponding to that for RL/RC multipliers of [42] used to prove Theorem II.10. We summarise the argument as follows:

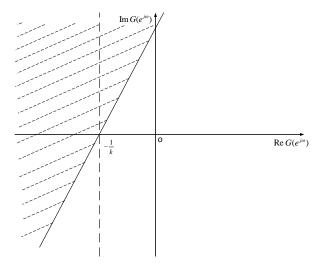


Fig. 8. Geometrical interpretation of the off-axis circle criterion considering the plant G (Theorem II.10 for continuous-time systems and Conjecture IV.10 for discrete-time systems). The Theorem for continuous-time systems is true but the Conjecture for discrete-time systems is false in general.

- 1) Under the conditions of Conjecture IV.10 there is some θ in (-90,90) degrees such that the phase of 1+kG always lies in the interval $(-90-\theta,90-\theta)$ degrees. Hence an ideal LTI multiplier with constant phase θ would render the real part of M(1+kG) positive over all frequencies.
- 2) In their proof of the continuous off-axis circle criterion Cho and Narendra [3] show that it is possible to construct RL/RC multipliers whose phase is arbitrarily close to some constant θ degrees over an arbitrarily large interval. We show that for some values of θ this may not be possible for any discrete-time LTI multiplier.
- 3) If a discrete-time LTI multiplier preserves the positivity of a slope-restricted nonlinearity then there is a Zames-Falb multiplier with the same phase [14], [43], so we can

limit our set of multipliers to the class of LTI Zames-Falb multipliers.

4) If $\theta > \tan^{-1}(2/\pi) \approx 32.48^{\circ}$ then Theorem IV.3 precludes any such construction of a Zames-Falb multiplier since if $a \to 0^+$ and $b \to \pi^-$ then $\rho^d \to \tan^{-1}(-2/\pi)$.

Hence the phase limitation can be used to invalidate Conjecture IV.10 when $\theta > \tan^{-1}(2/\pi)$. Smaller values can be obtained by using different values of a and b. It follows that any counterpart of the off-axis circle criterion in discrete-time must take into account specific information about frequency intervals. This is true of the more limited result originally derived by Narendra and Cho [42] (Theorem II.11). In fact it can be shown that the counterexamples to the Kalman conjecture of [50] and [51] are also counterexamples to Conjecture IV.10.

D. Finite search in discrete-time domain

Here we provide a result which simplifies the numerical implementation. Although the definitions of ρ^d and ρ^d_{odd} are given with an infinite number of terms, they can be calculated using a finite number $n = n_N$ given in Lemma IV.11.

Lemma IV.11 Let $0 \le a < b \le \pi$, then

$$\rho^d = \max_{1 \le n \le n_N} \frac{|\psi_d(n)|}{\phi_d(n)},\tag{62}$$

and

$$\rho_{odd}^{d} = \max_{1 \le n \le n_N} \frac{|\psi_d(n)|}{\tilde{\phi}_d(n)},\tag{63}$$

with

$$n_N = |\mathbf{v}|, \tag{64}$$

where

$$v = \frac{2(b-a) - 2\sin b + 2\sin a - 2\cos b + 2\cos a}{(a-b)(\cos b - \cos a)}.$$
 (65)

Proof: See Appendix.

Suppose we wish to find a phase limitation over the interval $\omega \in [0.7, 0.75]$. Applying Lemma IV.11 we find $\nu = 55.2$ and hence $|\psi_d(n)|/\phi_d(n) < |\psi_d(1)|/\phi_d(1)$ for all $n > n_N$, with

$$n_N = 55.$$
 (66)

Hence it is sufficient to search over the integers $1 \le n \le 55$ for ρ^d . The numerical results shown in Fig. 9 demonstrate that $|\psi_d(n)|/\phi_d(n) < |\psi_d(1)|/\phi_d(1)$ for all n > 18. In fact the maximum occurs at n = -9.

E. Numerical example

Let us consider the negative feedback interconnection between the plant

$$G(z) = \frac{z}{z^2 - 1.8z + 0.81},\tag{67}$$

and a slope-restricted nonlinearity. This second-order plant is a counterexample to the discrete-time Kalman conjecture as there is a periodic solution when k = 2.1 [50], and the Nyquist value is $k_N = 3.61$. Using the algorithm of [24] we find there

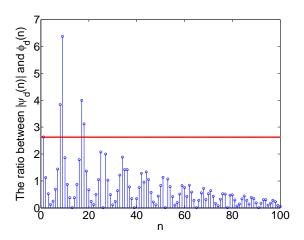


Fig. 9. The value of $f_d(n)$ with different value of n

exists a Zames-Falb multiplier for non-odd nonlinearities when $\hat{k}_{ZF} = 1.3028$.

Using the phase limitation result given in Theorem IV.3 part (i), it is possible to show that there is no Zames-Falb multiplier for any $k > k_{PL} = 1.4603$. Fig. 10 illustrates that the phase limitation results indicate there can be no appropriate Zames-Falb multiplier when k = 1.5. The phase limitation is given by $\tan^{-1} \rho^{\hat{d}} = 66.7137^{\circ}$, where ρ^{d} is obtained using Definition IV.1 with a = 0.7198 and b = 0.8996. By contrast, Fig. 11 shows that this limitation is not active when $k = \hat{k}_{ZF}$; this is expected since we have been able to find a suitable Zames-Falb multiplier for this value of the gain. A complete list of slope restriction results of G(z) in (67) is given in Table II. The result of the reduced off-axis circle criterion k_R shows conservativeness compared to all the other results in the Table. The (false) result from the direct discrete-time counterpart of the off-axis circle criterion is greater than the slope obtained by phase limitation, i.e. $k_O > k_{PL}$; this demonstrates that Conjecture IV.10 is false.

Finally, using combination of deadzone and saturation as nonlinearity, we are able to find periodic solution with $\hat{k}_C = 1.3666$. These results are consistent with Conjecture I.2, i.e. $\hat{k}_{ZF} < \hat{k}_C < k_{PL}$.

TABLE II
RESULTS OF DIFFERENT SLOPE RESTRICTIONS (NON-ODD NONLINEARITY)

	k	k_{RO}	\hat{k}_{ZF}	\hat{k}_C	k_{PL}	k_O	k_N
ſ	Result	0.8962	1.3028	1.3666	1.4603	3.61	3.61

V. CONCLUSIONS

In this paper we have demonstrated the connection between phase limitations of Zames-Falb multipliers and the Kalman conjecture.

In continuous time, we have generalised a limitation proposed by Megretski, clarified its remit and illustrated its effect with a numerical example. In particular we show it can be applied to a fourth-order plant where the resulting numerical implementation shows instability. It remains open which

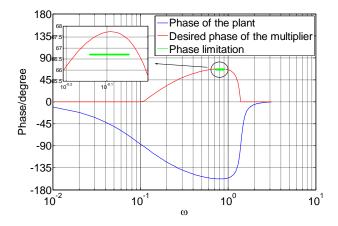


Fig. 10. Phase of (1+1.5G), desired phase of the multiplier and the phase limitation

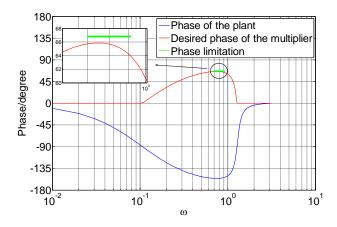


Fig. 11. Phase of (1+1.3028G), desired phase of the multiplier and the phase limitation

choice of intervals [a,b] and [c,d] and scaling parameter κ in Theorem III.4 provides most insight.

Motivated by this connection and recent results on the Kalman conjecture in discrete time, we have derived a more simple phase limitation for discrete-time Zames-Falb multipliers. Numerical results in discrete time are easier to obtain and we show that the slope restriction obtained by using phase limitation theorems can be about 40% of the Nyquist value even for some second-order examples. Thus the phase limitation can be directly useful when forming benchmarks for searches over Zames-Falb multipliers. Further, the phase limitation can be used to show there can be no direct counterpart in discrete time (Conjecture IV.10) to the off-axis circle criterion for continuous-time systems (Theorem II.10).

Based on the results of this paper, we propose Conjecture I.2, which seems to be compatible with current state-of-the-art knowledge and results for both continuous and discrete-time domains.

There is plenty of scope for future work. It seems possible that the phase limitations might be used to provide a more computationally efficient search fo appropriate multipliers. Phase limitations for the class of Zames-Falb multipliers available when the nonlinearity is quasi-odd [36] require further research.

VI. ACKNOWLEDGEMENTS

We would like to thank the anonymous reviewers for their helpful suggestions.

VII. APPENDIX

A. Proof of Lemma III.2

Both the functions

$$f_1(\omega) = \omega - \frac{\sin \omega t}{t} \tag{68}$$

and

$$f_2(\omega) = \omega + \frac{\sin \omega t}{t} \tag{69}$$

are monotone non-decreasing in ω when t > 0. It follows that $\phi(t) > 0$ and $\tilde{\phi}(t) > 0$ when t > 0. In addition

$$\lim_{t \to \infty} \phi(t) = \lambda(b-a) + \kappa \mu(d-c) > 0, \tag{70}$$

and

$$\lim_{t \to \infty} \tilde{\phi}(t) = \lambda(b-a) + \kappa \mu(d-c) > 0.$$
 (71)

Finally

$$\phi_1(t) = -[\lambda(b-a) + \kappa\mu(d-c)] + \lambda \frac{(b^3 - a^3)t^2}{6} + \kappa\mu \frac{(d^3 - c^3)t^2}{6} + O(t^4) \text{ as } t \to 0, \quad (72)$$

and

$$\psi(t) = \lambda \frac{(b^2 - a^2)t}{2} - \mu \frac{(d^2 - c^2)t}{2} - \lambda \frac{(b^4 - a^4)t^3}{24} + \mu \frac{(d^4 - c^4)t^3}{24} + O(t^5) \text{ as } t \to 0, \quad (73)$$

so the choice (28) ensures

$$\lim_{t \to 0} \frac{|\psi(t)|}{\phi(t)} = 0,\tag{74}$$

and

$$\lim_{t \to 0} \frac{|\psi(t)|}{\tilde{\phi}(t)} = 0. \tag{75}$$

B. Proof of Theorem III.4

Suppose (31) and (32) hold for some multiplier $M(j\omega) = 1 - H(j\omega)$. Then

$$\operatorname{Im}(M(j\omega)) = \int_{-\infty}^{\infty} \sin(\omega t) h(t) dt, \tag{76}$$

and

$$\operatorname{Re}(M(j\omega)) = 1 - \int_{-\infty}^{\infty} \cos(\omega t) h(t) dt, \tag{77}$$

where h in the impulse response of H. Hence integrating (31) and (32) over their respective intervals gives

$$\int_{-\infty}^{\infty} \frac{\cos(at) - \cos(bt)}{t} h(t) dt >$$

$$\rho(b-a) + \rho \int_{-\infty}^{\infty} \frac{\sin(at) - \sin(bt)}{t} h(t) dt, \quad (78)$$

and

$$\int_{-\infty}^{\infty} \frac{\cos(ct) - \cos(dt)}{t} h(t) dt < -\kappa \rho (d-c) - \kappa \rho \int_{-\infty}^{\infty} \frac{\sin(ct) - \sin(dt)}{t} h(t) dt. \quad (79)$$

Summing the two inequalities, multiplied by λ and $-\mu$ respectively, gives

$$\int_{-\infty}^{\infty} \psi(t)h(t)dt > \rho[\lambda(b-a) + \kappa\mu(d-c)] + \rho\int_{-\infty}^{\infty} \phi_1h(t)dt.$$
(80)

(i) If $M \in \mathcal{M}^c$ then $||h||_1 < 1$ and $h(t) \ge 0$ for all t. So

$$\rho[\lambda(b-a) + \kappa\mu(d-c)] > \int_{-\infty}^{\infty} \rho[\lambda(b-a) + \kappa\mu(d-c)]h(t) dt. \quad (81)$$

and hence we can write (80) as

$$\int_{-\infty}^{\infty} (\psi(t) - \rho \phi(t)) h(t) dt > 0.$$
 (82)

But, since ϕ is an even function and ψ is an odd function,

$$\psi(t) - \rho^c \phi(t) \le 0 \text{ for all } t. \tag{83}$$

Further, since ϕ is non-negative,

$$\psi(t) - \rho \phi(t) \le 0 \text{ for all } t \text{ when } \rho \ge \rho^c.$$
 (84)

Hence $\rho < \rho^c$.

(ii) If $M \in \mathcal{M}_{odd}^c$ then we can only say $||h||_1 < 1$. Nevertheless

$$\rho[\lambda(b-a) + \kappa\mu(d-c)] > \int_{-\infty}^{\infty} \rho[\lambda(b-a) + \kappa\mu(d-c)]|h(t)|dt. \quad (85)$$

and hence (80) leads to

$$\int_{-\infty}^{\infty} (|\psi(t)| - \rho \tilde{\phi}(t)) |h(t)| dt > 0.$$
 (86)

But, since $\tilde{\phi}$ is also an even function and (as before) ψ is an odd function,

$$|\psi(t)| - \rho_{odd}^c \tilde{\phi}(t) < 0 \text{ for all } t. \tag{87}$$

Further, since $\tilde{\phi}$ is non-negative,

$$|\psi(t)| - \rho \tilde{\phi}(t) \le 0 \text{ for all } t \text{ when } \rho \ge \rho_{odd}^c.$$
 (88)

Hence $\rho < \rho_{odd}^c$.

C. Proof of Lemma IV.2

The result is immediate following a similar argument to the proof of Lemma III.2. In particular, as ϕ_d and $\tilde{\phi}_d$ are evaluated for discrete values of $n \ge 1$, their limiting behaviour as $n \to 0$ need not be considered.

D. Proof of Theorem IV.3

Suppose (51) holds for some multiplier $M(e^{j\omega}) = 1 - H(e^{j\omega})$. Then

$$\operatorname{Im}(M(e^{j\omega})) = \sum_{n=-\infty}^{\infty} \sin(\omega n) h_n$$
 (89)

and

$$\operatorname{Re}(M(e^{j\omega})) = 1 - \sum_{n=-\infty}^{\infty} \cos(\omega n) h_n, \tag{90}$$

where h is the impulse response of H. Hence integrating (51) over the interval [a,b] gives

$$\sum_{n=-\infty}^{\infty} \frac{\cos(an) - \cos(bn)}{n} h_n > \rho(b-a) + \rho \sum_{n=-\infty}^{\infty} \frac{\sin(an) - \sin(bn)}{n} h_n. \quad (91)$$

(i) If $M \in \mathcal{M}^d$ then $h_0 = 0$, $||h||_1 < 1$ and $h_n \ge 0$ for all n.

$$\rho(b-a) > \sum_{n=-\infty}^{\infty} \rho(b-a)h_n, \tag{92}$$

and hence we can write (91) as

$$\sum_{n=-\infty}^{\infty} (\psi_d(n) - \rho \phi_d(n)) h_n > 0.$$
 (93)

But, since ϕ_d is an even function and ψ_d is an odd function,

$$\psi_d(n) - \rho^d \phi_d(n) \le 0 \text{ for all } n \ge 1.$$
 (94)

Further, since ϕ_d is non-negative,

$$\psi_d(n) - \rho \phi_d(n) \le 0$$
 for all $n \ge 1$ when $\rho \ge \rho^d$. (95)

Hence $\rho < \rho^d$.

(ii) If $M \in \mathcal{M}_{odd}^d$ then we can only say $h_0 = 0$ and $||h||_1 < 1$. Nevertheless.

$$\rho(b-a) > \sum_{n=-\infty}^{\infty} \rho(b-a)|h_n|, \tag{96}$$

and hence (91) leads to

$$\sum_{n=-\infty}^{\infty} (|\psi_d(n)| - \rho \tilde{\phi}_d(n)) |h_n| > 0.$$
 (97)

But, since $\tilde{\phi}_d$ is also an even function and (as before) ψ_d is an odd function,

$$|\psi_d(n)| - \rho_{odd}^d \tilde{\phi}_d(n) \le 0 \text{ for all } n \ge 1.$$
 (98)

Further, since $\tilde{\phi}_d$ is non-negative,

$$|\psi_d(n)| - \rho \tilde{\phi}_d(n) \le 0$$
 for all $n \ge 1$ when $\rho \ge \rho_{odd}^d$. (99)
Hence $\rho < \rho_{odd}^d$.

E. Proof of Theorem IV.5

Substituting

$$\operatorname{Im}(M(e^{j\omega})) = \sum_{n=-\infty}^{\infty} \sin(\omega n) h_n$$
 (100)

and

$$\operatorname{Re}(M(e^{j\omega})) = 1 - \sum_{n=-\infty}^{\infty} \cos(\omega n) h_n, \tag{101}$$

into (54) leads to (91). The proof is then identical to that of Theorem IV.3.

F. Proof of Proposition IV.7

(i) Let
$$M_n(z) = 1 - z^{-n}$$
 with $n \in \mathcal{N}([a,b])$. Then

$$Im(M_n(e^{j\omega})) = \sin(\omega n), \tag{102}$$

$$Re(M_n(e^{j\omega})) = 1 - \cos(\omega n). \tag{103}$$

Integrating over the interval yields

$$\int_{a}^{b} \operatorname{Im}(M(e^{j\omega})) d\omega = \rho^{c} \int_{a}^{b} \operatorname{Re}(M(e^{j\omega})) d\omega. \quad (104)$$

Furthermore, if

$$M(z) = 1 - \sum_{n \in \mathcal{N}([a,b])} \lambda_n z^{-n}, \qquad (105)$$

with

$$\lambda_n \ge 0 \text{ and } \sum_{n \in \mathcal{N}([a,b])} \lambda_n = 1,$$
 (106)

then we may write

$$M(z) = \sum_{n \in \mathcal{N}([a,b])} M_n(z). \tag{107}$$

The proof follows straightforwardly.

(ii) Similar.

G. Proof of Lemma IV.11

Let

$$\varepsilon = \frac{|\psi_d(1)|}{\phi_d(1)} = \frac{|\cos a - \cos b|}{b - a - (\sin b - \sin a)}$$

$$= -\frac{\cos b - \cos a}{b - a - (\sin b - \sin a)} = -\frac{\psi_d(1)}{\phi_d(1)},$$
(108)

where we have used that $(x - \sin x)$ is a monotonically increasing function; and

$$v = \frac{2}{(b-a)} \frac{1+\varepsilon}{\varepsilon} = \frac{2-2\psi_d(1)/\phi_d(1)}{-(b-a)\psi_d(1)/\phi_d(1)}$$
$$= \frac{2(b-a)-2\sin b + 2\sin a - 2\cos b + 2\cos a}{(a-b)(\cos b - \cos a)}.$$
 (109)

For n > v, we know (b-a)n-2 > 0. In addition,

$$\varepsilon(b-a)n > 2 + 2\varepsilon,\tag{110}$$

so

$$\frac{|\psi_d(n)|}{\phi_d(n)} = \frac{|\cos(bn) - \cos(an)|}{(b-a)n - [\sin(bn) - \sin(an)]} < \frac{2}{(b-a)n - 2} < \varepsilon.$$

$$\tag{111}$$

As a result,

$$\frac{|\psi_d(n)|}{\phi_d(n)} < \frac{|\psi_d(1)|}{\phi_d(1)} \qquad \forall n > \nu. \tag{112}$$

Finally, it is easy to check that

$$\frac{|\psi_d(1)|}{\tilde{\phi}_d(1)} = \frac{|\psi_d(1)|}{\phi_d(1)} \tag{113}$$

and hence the same relation holds for $|\psi_d(n)|/\tilde{\phi}_d(n)$.

REFERENCES

- [1] M. Vidyasagar, Nonlinear systems analysis. Prentice Hall, 1978.
- [2] H. K. Khalil, "Nonlinear systems, 3rd ed." Prentice Hall Upper Saddle River, 2002.
- [3] Y.-S. Cho and K. Narendra, "An off-axis circle criterion for stability of feedback systems with a monotonic nonlinearity," *IEEE Transactions on Automatic Control*, vol. 13, no. 4, pp. 413–416, 1968.
- [4] C. J. Harris and J. Valenca, The stability of input-output dynamical systems. Elsevier, 1983.
- [5] Y. Z. Tsypkin, "On the stability in the large of nonlinear sampled-data systems," *Doklady Akademii Nauk SSSR*, vol. 145, pp. 52–55, 1962.
- [6] —, "A criterion for absolute stability of automatic pulse systems with monotonic characteristics of the nonlinear element," in *Soviet Physics Doklady*, vol. 9, 1964, p. 263.
- [7] E. Jury and B. Lee, "On the stability of a certain class of nonlinear sampled-data systems," *IEEE Transactions on Automatic Control*, vol. 9, no. 1, pp. 51–61, 1964.
- [8] —, "On the absolute stability of nonlinear sampled-data systems," IEEE Transactions on Automatic Control, vol. 9, no. 1, pp. 551–554, 1964.
- [9] N. S. Ahmad, W. P. Heath, and G. Li, "LMI-based stability criteria for discrete-time Lur'e systems with monotonic, sector-and slope-restricted nonlinearities," *IEEE Transactions on Automatic Control*, vol. 58, no. 2, pp. 459–465, 2013.
- [10] N. S. Ahmad, J. Carrasco, and W. P. Heath, "A less conservative LMI condition for stability of discrete-time systems with slope-restricted nonlinearities," *IEEE Transactions on Automatic Control*, vol. 60, no. 6, pp. 1692–1697, 2015.
- [11] R. O'Shea, "An improved frequency time domain stability criterion for autonomous continuous systems," *IEEE Transactions on Automatic Control*, vol. 12, no. 6, pp. 725–731, 1967.
- [12] G. Zames and P. L. Falb, "Stability conditions for systems with monotone and slope-restricted nonlinearities," SIAM Journal on Control, vol. 6, no. 1, pp. 89–108, 1968.
- [13] R. O'Shea and M. Younis, "A frequency-time domain stability criterion for sampled-data systems," *IEEE Transactions on Automatic Control*, vol. 12, no. 6, pp. 719–724, 1967.
- [14] J. Willems and R. Brockett, "Some new rearrangement inequalities having application in stability analysis," *IEEE Transactions on Automatic Control*, vol. 13, no. 5, pp. 539–549, 1968.
- [15] C. Desoer and M. Vidyasagar, Feedback Systems: InputOutput Properties. Academic Press, Orlando, FL, USA, 1975.
- [16] M. Safonov and G. Wyetzner, "Computer-aided stability analysis renders Popov criterion obsolete," *IEEE Transactions on Automatic Control*, vol. 32, no. 12, pp. 1128–1131, 1987.
- [17] X. Chen and J. T. Wen, "Robustness analysis of LTI systems with structured incrementally sector bounded nonlinearities," in *Proceedings* of the American Control Conference, vol. 5, 1995, pp. 3883–3887.
- [18] ——, "Robustness analysis for linear time-invariant systems with structured incrementally sector bounded feedback nonlinearities," *Applied Mathematics and Computer Science*, vol. 6, pp. 623–648, 1996.
- [19] M. C. Turner, M. Kerr, and I. Postlethwaite, "On the existence of stable, causal multipliers for systems with slope-restricted nonlinearities," *IEEE Transactions on Automatic Control*, vol. 54, no. 11, pp. 2697–2702, 2009
- [20] M. Chang, R. Mancera, and M. Safonov, "Computation of Zames-Falb multipliers revisited," *IEEE Transactions on Automatic Control*, vol. 57, no. 4, p. 1024, 2012.
- [21] J. Carrasco, W. P. Heath, G. Li, and A. Lanzon, "Comments on "On the existence of stable, causal multipliers for systems with slope-restricted nonlinearities"," *IEEE Transactions on Automatic Control*, vol. 57, no. 9, pp. 2422–2428, 2012.

- [22] J. Carrasco, M. Maya-Gonzalez, A. Lanzon, and W. P. Heath, "LMI searches for anticausal and noncausal rational Zames–Falb multipliers," *Systems & Control Letters*, vol. 70, pp. 17–22, 2014.
- [23] N. S. Ahmad, J. Carrasco, and W. P. Heath, "LMI searches for discretetime Zames-Falb multipliers," in 52nd IEEE Conference on Decision and Control, 2013, pp. 5258–5263.
- [24] S. Wang, W. P. Heath, and J. Carrasco, "A complete and convex search for discrete-time noncausal FIR Zames-Falb multipliers," in *Proceedings* of the 53rd IEEE Conference on Decision and Control, 2014, pp. 3918– 3923.
- [25] A. Megretski and A. Rantzer, "System analysis via Integral Quadratic Constraints," *IEEE Transactions on Automatic Control*, vol. 42, no. 6, pp. 819–830, 1997.
- [26] C.-Y. Kao, A. Megretski, U. Jönsson, and A. Rantzer, "A MATLAB toolbox for robustness analysis," in *IEEE International Symposium on Computer Aided Control Systems Design*, 2004, pp. 297–302.
- [27] J. Veenman, C. W. Scherer, and H. Köroğlu, "Robust stability and performance analysis based on integral quadratic constraints," *European Journal of Control*, vol. 31, pp. 1–32, 2016.
- [28] D. Altshuller, Frequency Domain Criteria for Absolute Stability: A Delay-integral-quadratic Constraints Approach. Springer, 2013.
- [29] M. G. Safonov and V. V. Kulkarni, "Zames-falb multipliers for MIMO nonlinearities," in *Proceedings of the American Control Conference*, vol. 6. IEEE, 2000, pp. 4144–4148.
- [30] F. J. D'Amato, M. A. Rotea, A. V. Megretski, and U. T. Jönsson, "New results for analysis of systems with repeated nonlinearities," *Automatica*, vol. 37, no. 5, pp. 739–747, 2001.
- [31] R. Mancera and M. G. Safonov, "All stability multipliers for repeated MIMO nonlinearities," Systems & Control Letters, vol. 54, no. 4, pp. 389–397, 2005.
- [32] M. C. Turner, M. L. Kerr, and I. Postlethwaite, "On the existence of multipliers for MIMO systems with repeated slope-restricted nonlinearities," in *ICCAS-SICE*, 2009, pp. 1052–1057.
- [33] M. Fetzer and C. W. Scherer, "Full-block multipliers for repeated, sloperestricted scalar nonlinearities," *International Journal of Robust and Nonlinear Control*, 2017, DOI: 10.1002/rnc.3751.
- [34] A. Rantzer, "Friction analysis based on integral quadratic constraints," Int. J. Robust Nonlinear Control, vol. 11, no. 7, pp. 645–652, 2001.
- [35] D. Materassi and M. V. Salapaka, "A generalized Zames-Falb multiplier," *IEEE Transactions on Automatic Control*, vol. 56, no. 6, pp. 1432–1436, 2011.
- [36] W. P. Heath, J. Carrasco, and D. A. Altshuller, "Stability analysis of asymmetric saturation via generalised Zames-Falb multipliers," in Proceedings of the 54th IEEE Conference on Decision and Control, 2015, pp. 3748–3753.
- [37] M. C. Turner and M. L. Kerr, "L₂ gain bounds for systems with sector bounded and slope-restricted nonlinearities," *International Journal of Robust and Nonlinear Control*, vol. 22, no. 13, pp. 1505–1521, 2012.
- [38] B. Hu, M. J. Lacerda, and P. Seiler, "Robustness analysis of uncertain discrete-time systems with dissipation inequalities and integral quadratic constraints," *International Journal of Robust and Nonlinear Control*, 2016, DOI: 10.1002/rnc.3646.
- [39] J. Carrasco, W. P. Heath, and A. Lanzon, "Equivalence between classes of multipliers for slope-restricted nonlinearities," *Automatica*, vol. 49, no. 6, pp. 1732–1740, 2013.
- [40] W. P. Heath and A. G. Wills, "Zames-Falb multipliers for quadratic programming," in *Proceedings of the 44th IEEE Annual Conference on Decision and Control (CDC)*. IEEE, 2005, pp. 963–968.
- [41] L. Lessard, B. Recht, and A. Packard, "Analysis and design of optimization algorithms via integral quadratic constraints," SIAM Journal on Optimization, vol. 26, no. 1, pp. 57–95, 2016.
- [42] K. S. Narendra and Y.-S. Cho, "Stability analysis of nonlinear and timevarying discrete systems," SIAM Journal on Control, vol. 6, no. 4, pp. 625–646, 1968
- [43] J. C. Willems, The analysis of feedback systems. The MIT Press, 1971.
- [44] J. Carrasco, W. P. Heath, and A. Lanzon, "On multipliers for bounded and monotone nonlinearities," *Systems & Control Letters*, vol. 66, pp. 65–71, 2014.
- [45] J. Carrasco, M. C. Turner, and W. P. Heath, "Zames–Falb multipliers for absolute stability: From O'Shea's contribution to convex searches," *European Journal of Control*, vol. 28, pp. 1–19, 2016.
- [46] A. Megretski, "Combining L1 and L2 methods in the robust stability and performance analysis of nonlinear systems," in *Proceedings of the 34th IEEE Conference on Decision and Control*, vol. 3, 1995, pp. 3176–3181.
- [47] U. Jönsson and M.-C. Laiou, "Stability analysis of systems with nonlinearities," in *Proceedings of the 35th IEEE Conference on Decision and Control*, vol. 2, 1996, pp. 2145–2150.

- [48] U. Jönsson, "Robustness analysis of uncertain and nonlinear systems," Department of Automatic Control, Lund Institute of Technology, vol. 1047, 1996.
- [49] N. Barabanov, "On the Kalman problem," Siberian Mathematical Journal, vol. 29, no. 3, pp. 333–341, 1988.
- [50] J. Carrasco, W. P. Heath, and M. de la Sen, "Second order counterexample to the Kalman conjecture in discrete-time," in *Proceeding of the European Control Conference*, 2015.
- [51] W. P. Heath, J. Carrasco, and M. de la Sen, "Second-order counterexamples to the discrete-time Kalman conjecture," *Automatica*, vol. 60, pp. 140–144, 2015.
- [52] A. R. Plummer and C. Ling, "Stability and robustness for discrete-time systems with control signal saturation," *Proceedings of the Institution* of Mechanical Engineers, Part 1: Journal of Systems and Control Engineering, vol. 214, no. 1, pp. 65–76, 2000.
- [53] Y. Okuyama, T. Kosaka, and F. Takemori, "Robust stability analysis for nonlinear sampled-data control systems in a frequency domain," in *Proceedings of the European Control Conference (ECC)*,, 1999, pp. 2668–2673.
- [54] S. Wang, J. Carrasco, and W. P. Heath, "Phase limitations of discretetime Zames-Falb multipliers," in *Proceedings of the 54th IEEE Confer*ence on Decision and Control, 2015, pp. 5707–5712.
- [55] R. E. Kalman, "Physical and mathematical mechanisms of instability in nonlinear automatic control systems," *Trans. ASME*, vol. 79, no. 3, pp. 553–566, 1957.
- [56] R. Fitts, "Two counterexamples to Aizerman's conjecture," *IEEE Transactions on Automatic Control*, vol. 11, no. 3, pp. 553–556, 1966.
- [57] G. A. Leonov and N. V. Kuznetsov, "Hidden attractors in dynamical systems. From hidden oscillations in Hilbert–Kolmogorov, Aizerman, and Kalman problems to hidden chaotic attractor in Chua circuits," *International Journal of Bifurcation and Chaos*, vol. 23, no. 01, p. 1330002, 2013.
- [58] T. M. Apostol, *Mathematical analysis*, 2nd ed. Addison Wesley, 1974.
- [59] J. Bechhoefer, "Kramers-Kronig, Bode, and the meaning of zero," American Journal of Physics, vol. 79, no. 10, pp. 1053–1059, 2011.
- [60] Y. Li and Z. Lin, "On the estimation of the domain of attraction for linear systems with asymmetric actuator saturation via asymmetric Lyapunov functions," in *Proceedings of the American Control Conference*, 2016, pp. 1136–1141.



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