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***Abstract*—In this paper, we introduce an event-based boundary control for a 2×2 coupled linear hyperbolic system. We use a well established backstepping controller which stabilizes the system along with a dynamic triggering condition which determines when the controller must be updated. The main contributions rely on the definition of an event-based controller under which global exponential stability of the system is achieved and furthermore, the existence of a minimal dwell-time between two triggering times is guaranteed. The well-posedness of the system under the event-based controller is stated. A simulation example is presented to illustrate the results.**

***Index Terms*—Linear hyperbolic systems, Backstepping control design, dynamic triggering condition, event-based control.**

I. Introduction

Event-based control is a computer control strategy which has become an active research area. One reason is because this new paradigm on sampled control aims to use communications, computational and actuating resources efficiently by updating control inputs aperiodically (only when needed). Another reason is due to its rigorous way to implement digitally continuous time controllers. Several contributions in this field have been developed for networked control systems, ranging from seminal works in [2], [1] until more recent ones in [15], [21], [23] and the references therein. In general, event-based control includes two main components to be designed: a feedback control law which stabilizes the system and a triggering strategy which determines the time instants when the control needs to be updated while evaluating continuously the behavior of the system. Among triggering strategies, a static rule obtained by an Input-to-State Stability (ISS)-based property is introduced in [27], a dynamic rule is introduced in [13] whereas strategies relying on the time derivative of the Lyapunov function are developed for instance in [22], [25].

On the other hand, for control of infinite dimensional systems, namely those governed by partial differential equations (PDEs), digital control synthesis commonly relies on reducing the model by discretizing the space in order to get ordinary differential equations, thus finite dimensional approaches can be applied. Although the design of event-based control, by tackling directly the PDE, has not been widely studied, few approaches on sampled data and event-triggered control of parabolic PDEs, are considered in [12] and [24], [31]. For hyperbolic PDEs, close frameworks to event-based control are the work on switched hyperbolic systems as in [14], [20] and the work on sampled-data systems as in [18]. However, a recent work has introduced two event-based boundary controllers for linear hyperbolic systems of conservation laws. Indeed, inspired by two of the main strategies developed for finite dimensional systems, an extension by means of Lyapunov techniques for stability of linear hyperbolic systems has been done in [10]. An extension of it, using a dynamic triggering condition has been discussed in [11]. It is worth recalling that stability analysis and continuous stabilization of hyperbolic PDEs have been considered for a long time in the literature. Lyapunov techniques ([6], [5]) and backstepping boundary control design ([8], [7], [28]) are the most commonly used. It is worth also mentioning that backstepping method was initially developed for parabolic equations [26] and it was firstly introduced to first-order hyperbolic PDEs in [19]. In practical scenarios, hydraulic networks using balance laws are studied e.g. in [3, Chapter 8]. However, actuation on those systems may be expensive due to actuators inertia when regulating the water level and the water flow rate by using gates opening as the control actions. Then, event-based control would suggest to modulate efficiently the gates opening, only when needed. As periodic sampling schemes may produce unnecessary updates of the sampled controllers, which will cause high utilization of computational and communication resources, as well as actuator sollicitation, event-based control may show benefits with respect to periodic schemes.

In both [10] and [11], event-based controllers using output feedback are studied by following Lyapunov techniques and taking into account the *dissipativity condition* on the boundary for stability. In this paper, we use rather a full state-feedback control which is designed following the backstepping approach for stabilizing a system of balance laws.

The main contribution of this work relies on the study of a event-based controller using a dynamic triggering condition. We introduce such a triggering policy using the Lyapunov function candidate for the so-called *target system* along with the deviation between continuous time controller and the event-based one when sampling. We prove then that a minimal dwell-time between triggering times exists. Consequently, we prove the well-posedness of the system and finally the global exponential stability of the closed-loop system.

This paper is organized as follows. In Section II, we introduce the class of linear hyperbolic system, the backstepping transformation and some preliminaries on stability and backstepping boundary control. Section III provides the event-based stabilization results. Section IV provides a numerical example to illustrate the main results. Finally, conclusions and perspectives are given in Section V.

Notation : $\mathbb{R}^+$ will denote the set of nonnegative real numbers. The usual Euclidean norm in $\mathbb{R}^n$ is denoted by $|\cdot|$ and the associated matrix norm is denoted $\|\cdot\|$. The set of all functions $g:[0,1]\to\mathbb{R}^n$ such that $\int_0^1 |g(x)|^2dx<\infty$ is denoted by $L^2([0,1],\mathbb{R}^n)$ and for g and $\tilde{g}\in L^2([0,1],\mathbb{R}^n)$, the inner product is $<g,\tilde{g}>_{L^2}=\int_0^1 g^T\tilde{g}dx$ and is equipped with the norm $\|\cdot\|_{L^2([0,1],\mathbb{R}^n)}$. Given a topological set S, and an interval $I\subseteq\mathbb{R}$, the set $\mathscr{C}^0(I;S)$ (resp. $\mathscr{C}_{pw}(I;S)$) is the set of continuous functions (resp. piecewise contiuous functions) $g:I\to S$.

II. Preliminaries on backstepping boundary control of 2×2 linear hyperbolic PDEs

Let us consider the linear hyperbolic system

$$u_t(t,x)+\lambda_1 u_x(t,x) = c_1 v(t,x) \quad (1)$$

$$v_t(t,x)-\lambda_2 v_x(t,x) = c_2 u(t,x) \quad (2)$$

*This work has been partially supported by the LabEx PERSYVAL-Lab (ANR-11-LABX-0025-01). N. Espitia, N. Marchand and C. Prieur are with Univ. Grenoble Alpes, CNRS, Gipsa-lab, F-38000 Grenoble, France (e-mail: nicolas.espitia-hoyos@gipsa-lab.fr, nicolas.marchand@gipsa-lab.fr, christophe.prieur@gipsa-lab.fr)

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along with the following boundary conditions:

$$u(t,0) = qv(t,0) \tag{3}$$
$$v(t,1) = U(t) \tag{4}$$

where $u,v : \mathbb{R}^+ \times [0,1] \to \mathbb{R}$ are the system states with $x \in [0,1]$, $t \geq 0$, $U(t)$ is the control input and $\lambda_1 > 0$, $\lambda_2 > 0$. In addition, for technical issues related to the existence of solutions, we assume that $c_1, c_2 \neq 0$, $q \neq 0$, $\cos(w) - q\frac{\lambda_1}{c_1}w\sin(w) \neq 0$ if $c_1c_2 > 0$ and $\cosh(w) + q\frac{\lambda_1}{c_1}w\sinh(w) \neq 0$ if $c_1c_2 < 0$, where $w = \sqrt{\frac{|c_1c_2|}{\lambda_1\lambda_2}}$.

In order to stabilize this system, the backstepping method has been considered for instance in [30] and [7]. Roughly, the idea of the backstepping method is to use an invertible Volterra integral transformation to convert the unstable linear hyperbolic PDE (1)-(4) into a stable linear hyperbolic of conservation laws, which is usually called *target system* and is given as follows:

$$\begin{aligned} \alpha_t(t,x) + \lambda_1\alpha_x(t,x) &= 0 & (5)\\ \beta_t(t,x) - \lambda_2\beta_x(t,x) &= 0 & (6) \end{aligned}$$

with the following boundary conditions:

$$\begin{aligned} \alpha(t,0) &= q\beta(t,0) & (7)\\ \beta(t,1) &= 0 & (8) \end{aligned}$$

where $\alpha,\beta : \mathbb{R}^+ \times [0,1] \to \mathbb{R}$. Hence, $U(t)$ can be chosen to realize the transformation.

A. Backstepping transformation and kernel equations

The following backstepping Volterra transformation has been used to map the system (1)-(4) into the system (5)-(8):

$$\begin{aligned} \alpha(t,x) &= u(t,x) - \int_0^x K^{uu}(x,\xi)u(t,\xi)d\xi \\ &\quad - \int_0^x K^{uv}(x,\xi)v(t,\xi)d\xi & (9) \end{aligned}$$

$$\begin{aligned} \beta(t,x) &= v(t,x) - \int_0^x K^{vu}(x,\xi)u(t,\xi)d\xi \\ &\quad - \int_0^x K^{vv}(x,\xi)v(t,\xi)d\xi & (10) \end{aligned}$$

It has been shown that by introducing (9)-(10) into (5)-(6), integrating by parts and using the boundary conditions, the original system is transformed to the target system with the kernel $K = \begin{pmatrix} K^{uu}(x,\xi) & K^{uv}(x,\xi) \\ K^{vu}(x,\xi) & K^{vv}(x,\xi) \end{pmatrix}$, of the Volterra transformation, satisfying the following linear hyperbolic equations:

$$\begin{aligned} \lambda_1 K_x^{uu}(x,\xi) + \lambda_1 K_\xi^{uu}(x,\xi) &= -c_2K^{uv}(x,\xi) & (11)\\ \lambda_1 K_x^{uv}(x,\xi) - \lambda_2 K_\xi^{uv}(x,\xi) &= -c_1K^{uu}(x,\xi) & (12)\\ \lambda_2 K_x^{vu}(x,\xi) - \lambda_1 K_\xi^{vu}(x,\xi) &= c_2K^{vv}(x,\xi) & (13)\\ \lambda_2 K_x^{vv}(x,\xi) + \lambda_2 K_\xi^{vv}(x,\xi) &= c_1K^{vu}(x,\xi) & (14) \end{aligned}$$

with boundary conditions

$$\begin{aligned} K^{uu}(x,0) &= \tfrac{\lambda_2}{q\lambda_1}K^{uv}(x,0) & (15)\\ K^{uv}(x,x) &= \tfrac{c_1}{\lambda_1+\lambda_2} & (16)\\ K^{vu}(x,x) &= -\tfrac{c_2}{\lambda_1+\lambda_2} & (17)\\ K^{vv}(x,0) &= \tfrac{q\lambda_1}{\lambda_2}K^{vu}(x,0) & (18) \end{aligned}$$

The kernel equations evolve in a triangular domain given by $\mathscr{T} = \{(x,\xi) : 0 \leq \xi \leq x \leq 1\}$. It is known that there exists a unique solution to (11)-(18), that the transformation is invertible, and that the inverse transformation, which maps the target system into the original system (1)-(4), is given by [7]:

$$\begin{aligned} u(t,x) &= \alpha(t,x) + \int_0^x L^{\alpha\alpha}(x,\xi)\alpha(t,\xi)d\xi \\ &\quad + \int_0^x L^{\alpha\beta}(x,\xi)\beta(t,\xi)d\xi & (19) \end{aligned}$$

$$\begin{aligned} v(t,x) &= \beta(t,x) + \int_0^x L^{\beta\alpha}(x,\xi)\alpha(t,\xi)d\xi \\ &\quad + \int_0^x L^{\beta\beta}(x,\xi)\beta(t,\xi)d\xi & (20) \end{aligned}$$

Moreover, the kernel $L = \begin{pmatrix} L^{\alpha\alpha}(x,\xi) & L^{\alpha\beta}(x,\xi) \\ L^{\beta\alpha}(x,\xi) & L^{\beta\beta}(x,\xi) \end{pmatrix}$ of this transformation satisfies the following linear hyperbolic equations whose solution exists and is unique:

$$\begin{aligned} \lambda_1 L_x^{\alpha\alpha}(x,\xi) + \lambda_1 L_\xi^{\alpha\alpha}(x,\xi) &= c_1L^{\beta\alpha}(x,\xi) & (21)\\ \lambda_1 L_x^{\alpha\beta}(x,\xi) - \lambda_2 L_\xi^{\alpha\beta}(x,\xi) &= c_1L^{\beta\beta}(x,\xi) & (22)\\ \lambda_2 L_x^{\beta\alpha}(x,\xi) - \lambda_1 L_\xi^{\beta\alpha}(x,\xi) &= -c_2L^{\alpha\alpha}(x,\xi) & (23)\\ \lambda_2 L_x^{\beta\beta}(x,\xi) + \lambda_2 L_\xi^{\beta\beta}(x,\xi) &= -c_2L^{\alpha\beta}(x,\xi) & (24) \end{aligned}$$

with boundary conditions

$$\begin{aligned} L^{\alpha\alpha}(x,0) &= \tfrac{\lambda_2}{q\lambda_1}L^{\alpha\beta}(x,0) & (25)\\ L^{\alpha\beta}(x,x) &= \tfrac{c_1}{\lambda_1+\lambda_2} & (26)\\ L^{\beta\alpha}(x,x) &= -\tfrac{c_2}{\lambda_1+\lambda_2} & (27)\\ L^{\beta\beta}(x,0) &= \tfrac{q\lambda_1}{\lambda_2}L^{\beta\alpha}(x,0) & (28) \end{aligned}$$

Definition 1: *[L^2-norm stability] The linear hyperbolic system* (1)-(4) *with controller U is globally exponentially stable (GES) if there exist $\tilde{\upsilon} > 0$ and $C > 0$ such that, for every $(u^0, v^0)^T \in L^2([0,1];\mathbb{R}^2)$, the solution satisfies, for all t in $\mathbb{R}^+$,*

$$\|(u(t,\cdot), v(t,\cdot))^T\|_{L^2([0,1];\mathbb{R}^2)} \leq Ce^{-\tilde{\upsilon}t}\|(u^0,v^0)^T\|_{L^2([0,1];\mathbb{R}^2)} \tag{29}$$

As it can be seen in [7], $U(t)$ is a continuous full-state feedback control which is designed to ensure that the closed-loop system is GES in L^2 norm. The aforementioned backstepping transformation is used to get $U(t)$ under the form

$$U(t) = \int_0^1 K^{vu}(1,\xi)u(t,\xi)d\xi + \int_0^1 K^{vv}(1,\xi)v(t,\xi)d\xi \tag{30}$$

Equivalently, (30) can be expressed as follows:

$$U(t) = \int_0^1 L^{\beta\alpha}(1,\xi)\alpha(t,\xi)d\xi + \int_0^1 L^{\beta\beta}(1,\xi)\beta(t,\xi)d\xi \tag{31}$$

Note that the gains of the controller are the kernels satisfying (21)-(28).

Furthermore, in [7], the following Lyapunov function candidate is considered to show that the system (5)-(8) is GES:

$$V(\alpha,\beta) = \int_0^1 (A\alpha^2(x)\frac{e^{-\mu x}}{\lambda_1} + B\beta^2(x)\frac{e^{\mu x}}{\lambda_2})dx \tag{32}$$

with $A = e^{\mu}$, $B = q^2e^{\mu} + 1$ and $\mu > 0$. Since the system (5)-(8) is GES, so is the system (1)-(4). Indeed, since the transformation (9)-(10) is invertible, when applying either the continuous control (30) or (31), the original system has the same stability properties as the target system.

III. Event-based stabilization

In this section, we introduce an event-based control scheme for stabilization of the hyperbolic system (1)-(2). It relies on both the backstepping continuous-time control (31) that will be sampled on events and a triggering condition which determines when the event should occur. For that, we slightly modify the boundary conditions in both systems (1)-(4) and (5)-(7) by considering a perturbation on one of the boundaries. More precisely, let us consider the following linear hyperbolic system,

$$u_t(t,x)+\lambda_1 u_x(t,x) = c_1 v(t,x) \tag{33}$$
$$v_t(t,x)-\lambda_2 v_x(t,x) = c_2 u(t,x) \tag{34}$$
$$u(t,0) = qv(t,0) \tag{35}$$
$$v(t,1) = U_d(t) \tag{36}$$

where $U_d(t)=U(t)+d(t)$ with $U(t)$ given by (31) and $d(t)$ can be seen as a disturbance that will be rigorously characterized later on. It is worth remarking that here, d will not be an external disturbance (as considered for instance in [28] where the equations considered there are similar to (33)-(40) but the problem statement is quite different to the one in this paper) and is not intended to be rejected. Here, d can be viewed as a deviation between a continuous controller and an event-based one.

Then, applying the backstepping transformation (9)-(10), one has the equivalent system (*Target perturbed system*):

$$\alpha_t(t,x)+\lambda_1\alpha_x(t,x) = 0 \tag{37}$$
$$\beta_t(t,x)-\lambda_2\beta_x(t,x) = 0 \tag{38}$$
$$\alpha(t,0) = q\beta(t,0) \tag{39}$$
$$\beta(t,1) = d(t) \tag{40}$$

In addition, the function (32) will be used in the sequel in order to introduce the triggering condition. In fact, the event triggering law can be achieved using a strict Lyapunov condition along with an ISS property with respect to a deviation between the continuous controller and the event-based one, as introduced in [10]. Actually, developing ideas from that work, we can end up with a triggering condition which depends only on the current state and the deviation between controllers. For that reason, it can be called static triggering condition. However, in the present framework, it turned out that it is very difficult to find a minimal dwell-time between two event times when considering a static triggering condition. To overcome this problem, we will propose a dynamic triggering condition for which we are able to prove the existence of a minimal dwell-time and in turn, the well-posedness of the system under investigation.

It is worth mentioning that guaranteeing the existence of a minimal dwell-time avoids the so-called Zeno phenomenon that means infinite triggering times in a finite-time interval [1]. In practice, Zeno phenomenon would represent infeasible implementation into digital platforms since one would require to sample infinitely fast.

Therefore, inspired by [13] and [11], let us define the event-based controller considered in this paper. In the sequel we will call it φ and it encloses both the triggering condition and the backstepping feedback controller. Lyapunov analysis will be carried out for the target perturbed system.

Definition 2: *[Definition of φ] Let $\sigma\in(0,1)$, $\theta\geq 0$ $\eta>0$, $\mu>0$, $\upsilon=\mu\min\{\lambda_1,\lambda_2\}$, $\kappa_1,\kappa_2>0$, $m^0\in\mathbb{R}^-$, $B=q^2e^{\mu}+1$. Let L the kernel of the inverse backstepping transformation* (19)-(20) *which is solution to the system* (21)-(28). *Let $t\mapsto V(\alpha(t,\cdot),\beta(t,\cdot))$ be given by* (32).

We define φ the functional from $\mathscr{C}^0(\mathbb{R}^+;L^2([0,1];\mathbb{R}^2))$ to $\mathscr{C}_{pw}(\mathbb{R}^+,\mathbb{R})$ that maps $(\alpha,\beta)^T$ to U_d as follows:

- *Let the increasing sequence of time instants (t_k) be defined iteratively by $t_0=0$, and for all $k\geq 1$,*

$$\begin{aligned}t_{k+1}=\inf\{t\in\mathbb{R}^+|t>t_k\wedge\\ \theta Be^{\mu}\Big(\textstyle\int_0^1 L^{\beta\alpha}(1,\xi)(\alpha(t_k,\xi)-\alpha(t,\xi))d\xi\\ +\textstyle\int_0^1 L^{\beta\beta}(1,\xi)(\beta(t_k,\xi)-\beta(t,\xi))d\xi\Big)^2\\ \geq\theta\sigma\upsilon V(t)-m(t)\}\end{aligned} \tag{41}$$

  where m satisfies the ordinary differential equation,

$$\begin{aligned}\dot m(t) &= -\eta m(t)\\ &+\Bigg(Be^{\mu}\Big(\int_0^1 L^{\beta\alpha}(1,\xi)(\alpha(t_k,\xi)-\alpha(t,\xi))d\xi\\ &+\int_0^1 L^{\beta\beta}(1,\xi)(\beta(t_k,\xi)-\beta(t,\xi))d\xi\Big)^2\\ &-\sigma\upsilon V(t)-\kappa_1\alpha^2(t,1)-\kappa_2\beta^2(t,0)\Bigg)\end{aligned} \tag{42}$$

  for a given $\eta\geq\upsilon(1-\sigma)$ and $m(0)=m^0$.
- *Let the control function be defined by:*

$$U_d(t)=\int_0^1 L^{\beta\alpha}(1,\xi)\alpha(t_k,\xi)d\xi+\int_0^1 L^{\beta\beta}(1,\xi)\beta(t_k,\xi)d\xi \tag{43}$$

  for all $t\in[t_k,t_{k+1})$.

Remark 1: Let us remark that d in (40), given by

$$\begin{aligned}d(t) &= \textstyle\int_0^1 L^{\beta\alpha}(1,\xi)\alpha(t_k,\xi)d\xi+\int_0^1 L^{\beta\beta}(1,\xi)\beta(t_k,\xi)d\xi\\ &-\textstyle\int_0^1 L^{\beta\alpha}(1,\xi)\alpha(t,\xi)d\xi-\int_0^1 L^{\beta\beta}(1,\xi)\beta(t,\xi)d\xi\end{aligned} \tag{44}$$

for all $t\in[t_k,t_{k+1})$, can be seen as a deviation between the continuous controller (31) and the event-based controller (43). As in [10], we follow the perturbed approach inspired by [27], [21] and [15] from finite-dimensional systems. In this setting, the event triggering condition ensures that, for all $t\geq 0$, $\theta Be^{\mu}d^2(t)\leq\theta\sigma\kappa V(t)-m(t)$ which in turn guarantees $m(t)\leq 0$ as stated in the following lemma. In addition, $m(t)$ can be seen as a weighted averaged value of $Be^{\mu}d^2-\sigma\upsilon V-\kappa_1\alpha^2(\cdot,1)-\kappa_2\beta^2(\cdot,0)$.

Lemma 1: *Under the definition of φ, it holds that $\theta Be^{\mu}d^2(t)-\theta\sigma\upsilon V(t)+m(t)\leq 0$ and $m(t)\leq 0$.*

Proof: By construction, from Definition 2, with (44), events are triggered to guarantee, for all $t\geq 0$,

$$\theta Be^{\mu}d^2(t)-\theta\sigma\upsilon V(t)\leq -m(t) \tag{45}$$

If $\theta=0$, we obtain $m(t)\leq 0$. In the case $\theta>0$, it follows from (45) that

$$Be^{\mu}d^2(t)-\sigma\upsilon V(t)\leq -\frac{1}{\theta}m(t) \tag{46}$$

Then, using (42), we have that for all $t\geq 0$,

$$\dot m\leq -\eta m-\frac{1}{\theta}m-\kappa_1\alpha^2(\cdot,1)-\kappa_2\beta^2(\cdot,0)$$

Hence, by the Comparison principle, we conclude that $m(t)\leq 0$, for all $t\geq 0$. ■

Proposition 1: *There exists a unique solution $(u,v)^T\in\mathscr{C}^0([t_k,t_{k+1}];L^2([0,1];\mathbb{R}^2))$ to the system* (33)-(36) *between two time instants t_k and t_{k+1}.*

Proof: For a constant input $U_d(t)=U(t_k)$ for all $t\in[t_k,t_{k+1})$, the system admits a unique equilibrium point $\{u^*,v^*\}$ satisfying:

$$u_x^*=\tfrac{c_1}{\lambda_1}v^*$$

[1] We refer the reader to [17], [21] for further details and examples.

$$v_x^* = \frac{-c_2}{\lambda_2} u^*$$

$$u^*(0) = q v^*(0) \tag{47}$$

$$v^*(1) = U_d = U(t_k) \tag{48}$$

Let us consider $u_{xx}^*(x) = -w^2 u^*(x)$, with $w = \sqrt{\frac{|c_1 c_2|}{\lambda_1 \lambda_2}}$, whose solution is given, in the case when $c_1 c_2 > 0$, by $u^*(x) = a\cos(wx) + b\sin(wx)$. Similarly, we can obtain that $v^*(x) = \frac{\lambda_1}{c_1}(-aw\sin(wx) + bw\cos(wx))$. Using (47) and (48) one can uniquely obtain a and b, that is, $a = q\frac{U(t_k)}{\cos w - q\frac{\lambda_1}{c_1} w \sin w}$ and $b = \frac{c_1}{\lambda_1 w}\frac{U(t_k)}{\cos w - q\frac{\lambda_1}{c_1} w \sin w}$. In the case when $c_1 c_2 < 0$, we would obtain $u^*(x) = a\cosh(wx) + b\sinh(wx)$ and $v^*(x) = \frac{\lambda_1}{c_1}(aw\sinh(wx) + bw\cosh(wx))$ with $a = q\frac{U(t_k)}{\cosh w + q\frac{\lambda_1}{c_1} w \sinh w}$ and $b = \frac{c_1}{\lambda_1 w}\frac{U(t_k)}{\cosh w + q\frac{\lambda_1}{c_1} w \sinh w}$.

By performing the change of variable $\tilde{u} = u - u^*$ and $\tilde{v} = v - v^*$, we obtain the following hyperbolic system of balance laws, for all $t \in [t_k, t_{k+1})$:

$$\tilde{u}_t(t,x) + \lambda_1 \tilde{u}_x(t,x) = c_1 \tilde{v}(t,x) \tag{49}$$

$$\tilde{v}_t(t,x) - \lambda_2 \tilde{v}_x(t,x) = c_2 \tilde{u}(t,x) \tag{50}$$

$$\tilde{u}(t,0) = q\tilde{v}(t,0) \tag{51}$$

$$\tilde{v}(t,1) = 0 \tag{52}$$

This system is a particular case of the system considered in [9]. Therefore, the classical definition of solution in L^2 can be applied, thus $(\tilde{u}, \tilde{v})^T \in \mathscr{C}^0([t_k, t_{k+1}]; L^2([0,1];\mathbb{R}^2))$ (see [9, Definition 1]). Hence, for the original variables, it holds that $(u,v)^T \in \mathscr{C}^0([t_k, t_{k+1}]; L^2([0,1];\mathbb{R}^2))$. It concludes the proof. ■

Using (9)-(10), it follows straightforwardly that there exists a unique solution $(\alpha, \beta)^T \in \mathscr{C}^0([t_k, t_{k+1}]; L^2([0,1];\mathbb{R}^2))$ to the system (37)-(40) between two time instants t_k and t_{k+1}. This allows to state the following result which will be useful for the sequel.

Proposition 2: *The function d given by* (44) *and the function V given by* (32)*, are continuous on* $[t_k, t_{k+1}]$.

Proof: On one hand, by the definition of the inner product, it can be noticed that d in (44) is as follows:

$$\begin{aligned} d(t) \;=\; & \left\langle \begin{pmatrix} L^{\beta\alpha}(1,\cdot) \\ L^{\beta\beta}(1,\cdot) \end{pmatrix}, \begin{pmatrix} \alpha(t_k,\cdot) \\ \beta(t_k,\cdot) \end{pmatrix} \right\rangle_{L^2([0,1];\mathbb{R}^2)} \\ & - \left\langle \begin{pmatrix} L^{\beta\alpha}(1,\cdot) \\ L^{\beta\beta}(1,\cdot) \end{pmatrix}, \begin{pmatrix} \alpha(t,\cdot) \\ \beta(t,\cdot) \end{pmatrix} \right\rangle_{L^2([0,1];\mathbb{R}^2)} \end{aligned}$$

for all $t \in [t_k, t_{k+1})$. Since $\alpha(t,\cdot)$ and $\beta(t,\cdot)$ are continuous with respect to time due to Proposition 1, and the inner product preserves the continuity, it follows that d is in $\mathscr{C}^0([t_k, t_{k+1}], \mathbb{R})$. On the other hand, V given by (32), can be viewed as

$$V(\alpha(t,\cdot), \beta(t,\cdot)) = \left\| \begin{pmatrix} \sqrt{\frac{Ae^{-\mu\cdot}}{\lambda_1}}\alpha(t,\cdot) \\ \sqrt{\frac{Be^{\mu\cdot}}{\lambda_2}}\beta(t,\cdot) \end{pmatrix} \right\|^2_{L^2([0,1];\mathbb{R}^2)}$$

Again, due to continuity arguments for $\alpha(t,\cdot)$ and $\beta(t,\cdot)$, and the L^2-norm preserving the continuity, we conclude that $V(\alpha(t,\cdot), \beta(t,\cdot))$ is a continuous function with respect to t. ■

Lemma 2: *For d given by* (44) *and V given by* (32)*, it holds that*

$$(\dot{d}(t))^2 \le \varepsilon_1 \alpha^2(t,1) + \varepsilon_2 d^2(t) + \varepsilon_3 V(t) \tag{53}$$

for $\varepsilon_1, \varepsilon_2$ and $\varepsilon_3 > 0$ and for all $t \in (t_k, t_{k+1})$.

Proof: From (44), let us take its time derivative as follows:

$$\dot{d}(t) = -\int_0^1 L^{\beta\alpha}(1,\xi)\alpha_t(t,\xi)d\xi - \int_0^1 L^{\beta\beta}(1,\xi)\beta_t(t,\xi)d\xi$$

Using the dynamics (37) and (38), it clearly follows that

$$\dot{d}(t) = \lambda_1 \int_0^1 L^{\beta\alpha}(1,\xi)\alpha_x(t,\xi)d\xi - \lambda_2 \int_0^1 L^{\beta\beta}(1,\xi)\beta_x(t,\xi)d\xi$$

Integrating by parts, one gets

$$\begin{aligned} \dot{d}(t) \;=\; & \lambda_1 \alpha(t,1) L^{\beta\alpha}(1,1) - \lambda_1 \alpha(t,0) L^{\beta\alpha}(1,0) \\ & -\lambda_1 \int_0^1 L_x^{\beta\alpha}(1,\xi)\alpha(t,\xi)d\xi - \lambda_2 \beta(t,1) L^{\beta\beta}(1,1) \\ & +\lambda_2 \beta(t,0) L^{\beta\beta}(1,0) + \lambda_2 \int_0^1 L_x^{\beta\beta}(1,\xi)\beta(t,\xi)d\xi \end{aligned} \tag{54}$$

Due to (39), we have

$$\begin{aligned} \dot{d}(t) \;=\; & \lambda_1 \alpha(t,1) L^{\beta\alpha}(1,1) - \lambda_2 \beta(t,1) L^{\beta\beta}(1,1) \\ & +\beta(t,0)(-\lambda_1 q L^{\beta\alpha}(1,0) + \lambda_2 L^{\beta\beta}(1,0)) \\ & -\lambda_1 \int_0^1 L_x^{\beta\alpha}(1,\xi)\alpha(t,\xi)d\xi \\ & +\lambda_2 \int_0^1 L_x^{\beta\beta}(1,\xi)\beta(t,\xi)d\xi \end{aligned} \tag{55}$$

Recalling from (27)-(28) that $L^{\beta\alpha}(1,1) = -\frac{c_2}{\lambda_1+\lambda_2}$ and $L^{\beta\beta}(1,0) = q\frac{\lambda_1}{\lambda_2} L^{\beta\alpha}(1,0)$, we replace them into (55), thus

$$\begin{aligned} \dot{d}(t) \;=\; & \lambda_1 \alpha(t,1)\frac{-c_2}{\lambda_1+\lambda_2} - \lambda_2 \beta(t,1) L^{\beta\beta}(1,1) \\ & -\lambda_1 \int_0^1 L_x^{\beta\alpha}(1,\xi)\alpha(t,\xi)d\xi \\ & +\lambda_2 \int_0^1 L_x^{\beta\beta}(1,\xi)\beta(t,\xi)d\xi \end{aligned} \tag{56}$$

Now, taking the square of $\dot{d}$ and using the *Young's inequality*, we can bound it as follows:

$$\begin{aligned} (\dot{d}(t))^2 \;\le\; & 2\left(\frac{\lambda_1 c_2}{\lambda_1+\lambda_2}\alpha(t,1) + \lambda_2 L^{\beta\beta}(1,1)\beta(t,1)\right)^2 \\ & +2\Big(-\lambda_1 \int_0^1 L_x^{\beta\alpha}(1,\xi)\alpha(t,\xi)d\xi \\ & \qquad +\lambda_2 \int_0^1 L_x^{\beta\beta}(1,\xi)\beta(t,\xi)\Big)^2 \\ \le\; & 4(\tfrac{\lambda_1 c_2}{\lambda_1+\lambda_2}\alpha(t,1))^2 + 4(\lambda_2 L^{\beta\beta}(1,1)\beta(t,1))^2 \\ & +4\lambda_1^2\left(\int_0^1 L_x^{\beta\alpha}(1,\xi)\alpha(t,\xi)d\xi\right)^2 \\ & +4\lambda_2^2\left(\int_0^1 L_x^{\beta\beta}(1,\xi)\beta(t,\xi)d\xi\right)^2 \end{aligned}$$

By the *Cauchy Schwarz inequality*, one gets

$$\begin{aligned} (\dot{d}(t))^2 \;\le\; & 4(\tfrac{\lambda_1 c_2}{\lambda_1+\lambda_2})^2\alpha^2(t,1) + 4\lambda_2^2(L^{\beta\beta}(1,1))^2\beta^2(t,1) \\ & +4\lambda_1^2\int_0^1\left(L_x^{\beta\alpha}(1,\xi)\right)^2 d\xi \int_0^1 \alpha^2(t,\xi)d\xi \\ & +4\lambda_2^2\int_0^1\left(L_x^{\beta\beta}(1,\xi)\right)^2 d\xi \int_0^1 \beta^2(t,\xi)d\xi \end{aligned}$$

Let us remark that, $\int_0^1 (L_x^{\beta\alpha}(1,\xi))^2 d\xi$ and $\int_0^1 (L_x^{\beta\beta}(1,\xi))^2 d\xi$ exist and let us call them $L_x^{\tilde{\beta}\alpha}$ and $L_x^{\tilde{\beta}\beta}$ respectively. In fact, this is due to the regularity of the Kernels on the domain $\mathscr{T}$ as proved in [30, Theorem 5]. Therefore,

$$\begin{aligned} (\dot{d}(t))^2 \;\le\; & (\tfrac{2\lambda_1 c_2}{\lambda_1+\lambda_2})^2\alpha^2(t,1) + (2\lambda_2 L^{\beta\beta}(1,1))^2\beta^2(t,1) \\ & +4\max\{\lambda_1^2 L_x^{\tilde{\beta}\alpha}, \lambda_2^2 L_x^{\tilde{\beta}\beta}\} \\ & \qquad \times\left(\int_0^1 \alpha^2(t,\xi) + \beta^2(t,\xi)d\xi\right) \end{aligned}$$

In addition, let us remark that for (32), there exists $r_1>0$ (depending on μ) such that $\frac{1}{r_1}\int_0^1(\alpha^2(t,x)+\beta^2(t,x))dx \le V(\alpha(t,\cdot),\beta(t,\cdot)) \le r_1\int_0^1(\alpha^2(t,x)+\beta^2(t,x))dx$ (see e.g. [29] for a more general quadratic Lyapunov function candidate). Hence $(\dot d)^2$ is finally bounded as follows:

$$\begin{aligned}(\dot d(t))^2 \quad &\le \quad (\tfrac{2\lambda_1 c_2}{\lambda_1+\lambda_2})^2\alpha^2(t,1)+(2\lambda_2 L^{\beta\beta}(1,1))^2 d^2(t)\\ &\quad +4\max\{\lambda_1^2 L_x^{\tilde\beta\alpha},\lambda_2^2 L_x^{\tilde\beta\beta}\}r_1 V \end{aligned} \tag{57}$$

with $d^2=\beta^2(t,1)$ due to (40). Setting $\varepsilon_1=(\frac{2\lambda_1 c_2}{\lambda_1+\lambda_2})^2$, $\varepsilon_2=(2\lambda_2 L^{\beta\beta}(1,1))^2$ and $\varepsilon_3=4\max\{\lambda_1^2 L_x^{\tilde\beta\alpha},\lambda_2^2 L_x^{\tilde\beta\beta}\}r_1$, we finish the proof. ■

Theorem 1: *Under the event-based controller φ in Defintion 2, with positive scalars θ, σ, μ, υ, B, κ_1, κ_2 and ε_1 (from Lemma 2) satisfying the following conditions,*

$$\kappa_1 \ge \max\{2\theta Be^{\mu}\varepsilon_1, 2\theta\sigma\upsilon\} \tag{58}$$

$$\kappa_2 \ge 2\theta\sigma\upsilon \tag{59}$$

There exists a minimal dwell-time $\tau>0$ between two triggering times, i.e. $t_{k+1}-t_k\ge\tau$, for all $k\ge 0$.

Proof: From the definition of φ, events are triggered to guarantee, for all $t\ge 0$,

$$\theta Be^{\mu}d^2(t) \le \theta\sigma\upsilon V(t) - m(t) \tag{60}$$

Let us consider the following function involving the functions in (60).

$$\psi = \frac{\theta Be^{\mu}d^2+\frac12 m}{\theta\sigma\upsilon V-\frac12 m}$$

A lower bound for the inter-execution times according to (41) is given by the time it takes for the function ψ to go from $\psi(t_k)$ to $\psi(t_{k+1})=1$, where $\psi(t_k)\le 0$ (virtue of $m(t_k)\le 0$ due to Lemma 1 and $d(t_k)=0$). Note that ψ is a continuous function on $[t_k,t_{k+1}]$ thanks to Proposition 2 and the fact that $m\in\mathscr{C}^0(\mathbb{R}^+,\mathbb{R}^-)$. Then, by the intermediate value theorem, there exists $t_k^{'}>t_k$ such that for all $t\in[t_k^{'},t_{k+1}]$, $\psi(t)\in[0,1]$. We have then that for all $t\in[t_k^{'},t_{k+1}]$, the time derivative of ψ is given as follows:

$$\dot\psi = \frac{2\theta Be^{\mu}d\dot d+\frac12\dot m}{\theta\sigma\upsilon V-\frac12 m} - \frac{(\theta\sigma\upsilon\dot V-\frac12\dot m)}{\theta\sigma\upsilon V-\frac12 m}\psi$$

Using the *Young's inequality* as $2d\dot d\le d^2+(\dot d)^2$, and from (42) we have that

$$\begin{aligned}\dot\psi \quad \le \quad & \frac{\theta Be^{\mu}d^2}{\theta\sigma\upsilon V-\frac12 m}+\frac{\theta Be^{\mu}(\dot d)^2}{\theta\sigma\upsilon V-\frac12 m}\\ &+\frac{\frac12\left(-\eta m+Be^{\mu}d^2-\sigma\upsilon V-\kappa_1\alpha^2(\cdot,1)-\kappa_2\beta^2(\cdot,0)\right)}{\theta\sigma\upsilon V-\frac12 m}\\ &-\frac{\theta\sigma\upsilon\dot V}{\theta\sigma\upsilon V-\frac12 m}\psi+\frac{\frac12\left(-\eta m+Be^{\mu}d^2-\sigma\upsilon V\right)}{\theta\sigma\upsilon V-\frac12 m}\psi\\ &+\frac{\frac12\left(-\kappa_1\alpha^2(\cdot,1)-\kappa_2\beta^2(\cdot,0)\right)}{\theta\sigma\upsilon V-\frac12 m}\psi \end{aligned} \tag{61}$$

where $\dot V$ in (61) is the time derivative of (32) along the solutions (37)-(38). Indeed, by integrating by parts and using the boundary conditions (39)-(40), $\dot V$ is given as follows:

$$\begin{aligned}\dot V \quad = \quad & -\alpha^2(t,1)Ae^{-\mu}+\beta^2(t,0)(q^2A-B)\\ &+Be^{\mu}d^2(t)-\mu\int_0^1(\alpha^2(x)Ae^{-\mu x}+\beta^2(x)Be^{\mu x})dx\end{aligned}$$

with $A=e^{\mu}$ and $B=q^2e^{\mu}+1$. Replacing $\dot V$ in (61) and using (53) we obtain

$$\begin{aligned}\dot\psi \quad \le \quad & \frac{\theta Be^{\mu}d^2}{\theta\sigma\upsilon V-\frac12 m}+\frac{\theta Be^{\mu}\varepsilon_1\alpha^2(\cdot,1)}{\theta\sigma\upsilon V-\frac12 m}\\ &+\frac{\theta Be^{\mu}\varepsilon_2 d^2}{\theta\sigma\upsilon V-\frac12 m}+\frac{\theta Be^{\mu}\varepsilon_3 V}{\theta\sigma\upsilon V-\frac12 m}\\ &+\frac{\frac12\left(-\eta m+Be^{\mu}d^2-\sigma\upsilon V-\kappa_1\alpha^2(\cdot,1)-\kappa_2\beta^2(\cdot,0)\right)}{\theta\sigma\upsilon V-\frac12 m}\\ &-\frac{\theta\sigma\upsilon\Big(-(\alpha^2(\cdot,1)+\beta^2(\cdot,0))\Big)}{\theta\sigma\upsilon V-\frac12 m}\psi\\ &-\frac{\theta\sigma\upsilon\Big(Be^{\mu}d^2-\mu\int_0^1(\alpha^2Ae^{-\mu x}+\beta^2Be^{\mu x})dx\Big)}{\theta\sigma\upsilon V-\frac12 m}\psi\\ &+\frac{\frac12\left(-\eta m+Be^{\mu}d^2-\sigma\upsilon V\right)}{\theta\sigma\upsilon V-\frac12 m}\psi\\ &+\frac{\frac12\left(-\kappa_1\alpha^2(\cdot,1)-\kappa_2\beta^2(\cdot,0)\right)}{\theta\sigma\upsilon V-\frac12 m}\psi \end{aligned} \tag{62}$$

Re-organizing terms and knowing that $\mu\int_0^1(\alpha^2Ae^{-\mu x}+\beta^2Be^{\mu x})\le \mu\max\{\lambda_1,\lambda_2\}V$, (62) is rewritten as follows

$$\begin{aligned}\dot\psi \quad \le \quad & \frac{\theta Be^{\mu}(1+\varepsilon_2+\frac{1}{2\theta})d^2}{\theta\sigma\upsilon V-\frac12 m}+\frac{(\theta Be^{\mu}\varepsilon_1-\frac12\kappa_1)\alpha^2(\cdot,1)}{\theta\sigma\upsilon V-\frac12 m}\\ &+\frac{(\theta Be^{\mu}\varepsilon_3-\frac12\sigma\upsilon)V}{\theta\sigma\upsilon V-\frac12 m}-\frac{\frac12\eta m}{\theta\sigma\upsilon V-\frac12 m}-\frac{\frac12\kappa_2\beta^2(\cdot,0)}{\theta\sigma\upsilon V-\frac12 m}\\ &+\frac{\left(\theta\sigma\upsilon\alpha^2(\cdot,1)-\frac12\kappa_1\alpha^2(\cdot,1)\right)}{\theta\sigma\upsilon V-\frac12 m}\psi\\ &+\frac{\left(\theta\sigma\upsilon\beta^2(\cdot,0)-\frac12\kappa_2\beta^2(\cdot,0)\right)}{\theta\sigma\upsilon V-\frac12 m}\psi\\ &+\frac{(-\theta\sigma\upsilon+\frac12)Be^{\mu}d^2}{\theta\sigma\upsilon V-\frac12 m}\psi\\ &+\frac{(\theta\sigma\upsilon\mu\max\{\lambda_1,\lambda_2\}-\frac12\sigma\upsilon)V}{\theta\sigma\upsilon V-\frac12 m}\psi-\frac{\frac12\eta m}{\theta\sigma\upsilon V-\frac12 m}\psi\end{aligned}$$

Setting $\kappa_1\ge\max\{2\theta Be^{\mu}\varepsilon_1,2\theta\sigma\upsilon\}$ and $\kappa_2\ge 2\theta\sigma\upsilon$ in light of (58)-(59), we have

$$\begin{aligned}\dot\psi \quad \le \quad & \frac{\theta Be^{\mu}(1+\varepsilon_2+\frac{1}{2\theta})d^2}{\theta\sigma\upsilon V-\frac12 m}+\frac{(\theta Be^{\mu}\varepsilon_3-\frac12\sigma\upsilon)V}{\theta\sigma\upsilon V-\frac12 m}\\ &-\frac{\frac12\eta m}{\theta\sigma\upsilon V-\frac12 m}+\frac{(-\theta\sigma\upsilon+\frac12)Be^{\mu}d^2}{\theta\sigma\upsilon V-\frac12 m}\psi\\ &+\frac{(\theta\sigma\upsilon\mu\max\{\lambda_1,\lambda_2\}-\frac12\sigma\upsilon)V}{\theta\sigma\upsilon V-\frac12 m}\psi\\ &-\frac{\frac12\eta m}{\theta\sigma\upsilon V-\frac12 m}\psi\end{aligned} \tag{63}$$

By remarking that $\left(-\frac{\frac12\eta m}{\theta\sigma\upsilon V-\frac12 m}\right)\le\eta$, $\left(\frac{(\theta Be^{\mu}\varepsilon_3-\frac12\sigma\upsilon)V}{\theta\sigma\upsilon V-\frac12 m}\right)\le \frac{\theta Be^{\mu}\varepsilon_3-\frac12\sigma\upsilon}{\theta\sigma\upsilon}$ and $\frac{(\theta\sigma\upsilon\mu\max\{\lambda_1,\lambda_2\}-\frac12\sigma\upsilon)V}{\theta\sigma\upsilon V-\frac12 m}\le\frac{(\theta\sigma\upsilon\mu\max\{\lambda_1,\lambda_2\}-\frac12\sigma\upsilon)}{\theta\sigma\upsilon}$,

(63) yields

$$\begin{aligned}\dot{\psi} &\leq \frac{\theta B e^{\mu}(1+\varepsilon_2+\frac{1}{2\theta})d^2}{\theta\sigma\upsilon V-\frac{1}{2}m}+\frac{\theta B e^{\mu}\varepsilon_3-\frac{1}{2}\sigma\upsilon}{\theta\sigma\upsilon}\\ &+\eta+\frac{(-\theta\sigma\upsilon+\frac{1}{2})Be^{\mu}d^2}{\theta\sigma\upsilon V-\frac{1}{2}m}\psi\\ &+\frac{(\theta\sigma\upsilon\mu\max\{\lambda_1,\lambda_2\}-\frac{1}{2}\sigma\upsilon)}{\theta\sigma\upsilon}\psi+\eta\psi\end{aligned}$$

which is rewritten as follows,

$$\begin{aligned}\dot{\psi} &\leq \frac{(1+\varepsilon_2+\frac{1}{2\theta})(\theta B e^{\mu}d^2+\frac{1}{2}m-\frac{1}{2}m)}{\theta\sigma\upsilon V-\frac{1}{2}m}\\ &+\left(\frac{\theta B e^{\mu}\varepsilon_3-\frac{1}{2}\sigma\upsilon}{\theta\sigma\upsilon}+\eta\right)\\ &+\frac{(-\theta\sigma\upsilon+\frac{1}{2})(\theta Be^{\mu}d^2+\frac{1}{2}m-\frac{1}{2}m)}{\theta(\theta\sigma\upsilon V-\frac{1}{2}m)}\psi\\ &+\left(\frac{\theta\sigma\upsilon\mu\max\{\lambda_1,\lambda_2\}-\frac{1}{2}\sigma\upsilon}{\theta\sigma\upsilon}+\eta\right)\psi\end{aligned}$$

By remarking that $\frac{-\frac{1}{2}m(1+\varepsilon_2+\frac{1}{2\theta})}{\theta\sigma\upsilon V-\frac{1}{2}m} \leq (1+\varepsilon_2+\frac{1}{2\theta})$, $\frac{-\frac{1}{2}m}{\theta\sigma\upsilon V-\frac{1}{2}m}\frac{(-\theta\sigma\upsilon+\frac{1}{2})}{\theta} \leq \frac{(-\theta\sigma\upsilon+\frac{1}{2})}{\theta}$ and that ψ is given by $\frac{\theta Be^{\mu}d^2+\frac{1}{2}m}{\theta\sigma\upsilon V-\frac{1}{2}m}$, it can be finally deduced that

$$\begin{aligned}\dot{\psi} \leq &\left(\frac{\theta B e^{\mu}\varepsilon_3-\frac{1}{2}\sigma\upsilon}{\theta\sigma\upsilon}+\eta+(1+\varepsilon_2+\tfrac{1}{2\theta})\right)\\ &+\Bigg(\frac{(-\theta\sigma\upsilon+\frac{1}{2})}{\theta}+\frac{\theta\sigma\upsilon\mu\max\{\lambda_1,\lambda_2\}-\frac{1}{2}\sigma\upsilon}{\theta\sigma\upsilon}\\ &+\eta+(1+\varepsilon_2+\tfrac{1}{2\theta})\Bigg)\psi+\frac{(-\theta\sigma\upsilon+\frac{1}{2})}{\theta}\psi^2\end{aligned}$$

This differential inequality has the form

$$\dot{\psi} \leq a_0+a_1\psi+a_2\psi^2$$

where, after some simplifications,

$$\begin{aligned}a_0 &= \tfrac{Be^{\mu}\varepsilon_3}{\sigma\upsilon}+\eta+\varepsilon_2+1\\ a_1 &= -\sigma\upsilon+\mu\max\{\lambda_1,\lambda_2\}+\eta+\varepsilon_2+1+\tfrac{1}{2\theta}\\ a_2 &= -\sigma\upsilon+\tfrac{1}{2\theta}\end{aligned}$$

where a_0, a_1 are a_2 turn out to be positive scalars (as soon as $\theta < \frac{1}{2\sigma\upsilon}$).

Then, by the Comparison principle, it follows that the time needed by ψ to go from $\psi(t_k^{'})=0$ to $\psi(t_{k+1})=1$ is at least

$$\tau=\int_0^1\frac{1}{a_0+a_1s+a_2s^2}ds$$

Thus, $t_{k+1}-t_k^{'}\geq\tau$. Consequently, as $t_{k+1}-t_k\geq t_{k+1}-t_k^{'}$, we achieve that $t_{k+1}-t_k\geq\tau$, being then τ a lower bound of the inter-execution times or minimal dwell-time. It concludes the proof. ■

Now that we have proved that there is a minimal dwell-time, no Zeno solution can appear. Therefore we are able to state the following result on the existence of solutions of the system (33)-(36) for all $t\in\mathbb{R}^+$.

Corollary 1: *There exists a unique solution* $(u,v)^T \in \mathscr{C}^0(\mathbb{R}^+;L^2([0,1];\mathbb{R}^2))$ *to the system* (33)-(36).

Proof: This is an immediate consequence of Proposition 1 and Theorem 1. The solution is iteratively built between successive triggering times. ■

Remark 2: Due to the backstepping transformation (9)-(10), the well-posedness of the target perturbed system (37)-(40) immediately follows as well.

Let us state the main result of the paper.

Theorem 2: *Let* $\sigma\in(0,1)$, $\mu>0$, $\upsilon=\mu\min\{\lambda_1,\lambda_2\}$, $A=e^{\mu}$, $B=e^{\mu}q^2+1$, ε_1 *(from Lemma 2). Let* $\eta\geq\upsilon(1-\sigma)$ *and* $0<\theta\leq\min\{\frac{1}{2\sigma\upsilon},\frac{1}{2Be^{\mu}\varepsilon_1}\}$, κ_1 *and* κ_2 *such that*

$$\max\{2\theta Be^{\mu}\varepsilon_1,2\theta\sigma\upsilon\}\leq\kappa_1\leq1 \tag{64}$$

$$2\theta\sigma\upsilon\leq\kappa_2\leq1 \tag{65}$$

holds. Let V *be given by* (32) *and* d *given by* (44). *Then the system* (33)-(36) *with event-based controller* $U_d=\varphi$ *has a unique solution and is globally exponentially stable.*

Proof: The existence and uniqueness of a solution to the system (33)-(36) with controller φ is given by Corollary 1. Let us show that the system is globally exponential stable.

Consider the following Lyapunov function candidate for the augmented system (37)-(40) with (42), defined for all $(\alpha(t,\cdot),\beta(t,\cdot))\in L^2([0,1];\mathbb{R}^2)$ and $m\in\mathbb{R}^-$ by

$$W(\alpha,\beta,m)=V(\alpha,\beta)-m \tag{66}$$

Taking the time derivative of (66) along the solutions, it yields,

$$\begin{aligned}\dot{W} &= -\alpha^2(\cdot,1)Ae^{-\mu}+\beta^2(\cdot,0)(q^2A-B)+Be^{\mu}d^2\\ &\quad -\mu\int_0^1(\alpha^2(x)Ae^{-\mu x}+\beta^2(x)Be^{\mu x})dx-\dot{m}\end{aligned} \tag{67}$$

Setting $\upsilon=\mu\min\{\lambda_1,\lambda_2\}$, note that $-\mu\int_0^1(\alpha^2(x)Ae^{-\mu x}+\beta^2(x)Be^{\mu x})dx < -\upsilon\int_0^1(\alpha^2(x)A\frac{e^{-\mu x}}{\lambda_1}+\beta^2(x)B\frac{e^{\mu x}}{\lambda_2})dx$. Moreover, setting $A=e^{\mu}$, $B=q^2e^{\mu}+1$, and using (42), from (67) one gets,

$$\begin{aligned}\dot{W} \leq &-\upsilon V-\alpha^2(\cdot,1)-\beta^2(\cdot,0)+Be^{\mu}d^2\\ &+\eta m-Be^{\mu}d^2+\sigma\upsilon V\\ &+\kappa_1\alpha^2(\cdot,1)+\kappa_2\beta^2(\cdot,0)\end{aligned} \tag{68}$$

which can be rewritten as follows:

$$\begin{aligned}\dot{W} \leq &-\upsilon(1-\sigma)W+(-\upsilon(1-\sigma)+\eta)m\\ &+(\kappa_1-1)\alpha^2(\cdot,1)+(\kappa_2-1)\beta^2(\cdot,0)\end{aligned}$$

Setting κ_1 and κ_2 in light of (64)-(65) we have that $\kappa_1\leq1$ and $\kappa_2\leq1$ and that meet the constraints (58)-(59) i.e. $\kappa_1\geq\max\{2\theta Be^{\mu}\varepsilon_1,2\theta\sigma\upsilon\}$ and $\kappa_2\geq2\theta\sigma\upsilon$ (conditions to be satisfied to guarantee the existence of a minimal dwell-time).

Therefore, it follows that

$$\dot{W}\leq-\upsilon(1-\sigma)W+(-\upsilon(1-\sigma)+\eta)m$$

From the definition of φ, events are triggered to guarantee, for all $t>0$, $\theta Be^{\mu}d^2(t)\leq\theta\sigma\upsilon V(t)-m(t)$. Then, by Lemma 1, we guarantee also that $m\leq0$. Recalling that $\eta\geq\upsilon(1-\sigma)$, we obtain

$$\dot{W}\leq-\upsilon(1-\sigma)W$$

By the Comparison principle, and remarking that $V(\alpha,\beta)\leq W(\alpha,\beta,m)$ we have, for all $t\geq0$,

$$V(\alpha(t,\cdot),\beta(t,\cdot))\leq e^{-\upsilon(1-\sigma)t}W(\alpha^0,\beta^0,m^0)$$

With $m^0=0$, we just obtain

$$V(\alpha(t,\cdot),\beta(t,\cdot))\leq e^{-\upsilon(1-\sigma)t}V(\alpha^0,\beta^0) \tag{69}$$

which in fact proves that the system (37)-(40) is GES in L^2 norm. Therefore, as it has been well established in backstepping approach for hyperbolic PDEs, using the inverse transformation of (9)-(10) (i.e. (19)-(20)), the system (33)-(36) is also GES in L^2 norm. More precisely, an estimate of the the L^2 norm of system (33)-(36) in terms of the L^2 norm of system (37)-(40) can be done as follows (see e.g. [8] for further details):

$$\|(u(t,\cdot),v(t,\cdot))^T\|^2_{L^2([0,1];\mathbb{R}^2)} \leq$$
$$(1+2\|L\|_\infty)^2(1+2\|K\|_\infty)^2$$
$$\times r_1^2 e^{-\upsilon(1-\sigma)t}\|(u^0(\cdot),v^0(\cdot))^T\|^2_{L^2([0,1];\mathbb{R}^2)}$$

where $\|K\|_\infty = \max_{(x,\xi)\in\mathscr{T}}|K(x,\xi)|$, $\|L\|_\infty = \max_{(x,\xi)\in\mathscr{T}}|L(x,\xi)|$. Hence, this concludes the proof. ■

Comments on the choice of parameters.
Note that while υ and B are given by stability issues, and σ is related to the decay rate, θ is a free parameter to be properly chosen as given in hypothesis of Theorem 2, then one can set κ_1 and κ_2 meeting (64)-(65). Let us remark however that in this work, an optimal choice of parameters regarding conservatism or sampling speed, is not tackled. We leave the study of the influence of parameters to the performance of the system for future investigations. In this paper we were namely focus on the stability result and well-posedness.

Remark 3: Let us remark that if a periodic sampling scheme is intended to be applied to the system (33)-(36) instead of an event-based scheme as presented throughout the paper, one suitable period could be the minimal dwell-time τ obtained from Theorem 1.

Remark 4: Results in this paper may be extended to systems with space-varying coefficients (based on e.g. [30] for the computation of Kernels L to be used in Definition 2) or even to $m+n$ hyperbolic equations (inspired by e.g. [16]). However, the result on the existence of a minimal dwell-time provided in Theorem 1 must be carefully addressed due to complexity of technical details and some assumptions that may be given in terms of matrix inequalities.

IV. Numerical simulations

Consider the system (33)-(36) with $\lambda_1 = 1$, $\lambda_2 = \sqrt{2}$, $c_1 = 1.5$, $c_2 = 2$ and $q = 1/4$. The initial conditions are $u^0(x) = qv^0(x)$ with $v^0(x) = 10(1-x)$ for all $x \in [0,1]$.

A. Event-based stabilization

The boundary conditions are $u(t,0) = qv(t,0)$ and $v(t,1) = U_d(t)$ where $U_d(t) = U(t) + d(t)$. In addition, $\upsilon = 0.1$, $\mu = 0.0707$ and $B = 0.533$, $\varepsilon_1 = 2.745$. Concerning the triggering algorithm, we choose the following parameters: $\sigma = 0.9$, $\theta = 8\times\ 10^{-3}$, $\eta = 0.1$, $\kappa_1 = 2.75\times\ 10^{-2}$ and $\kappa_2 = 7.8723\times\ 10^{-4}$. They satisfy the constraints (64)-(65).

The number of events under this approach is 9 on a frame of 4s meaning that the control value needed to be updated only 9 times.

Figure 1 shows the second component of solution $v(t,x)$ when stabilizing with continuous time controller U (left) and the event-based controller U_d (right). Note that attractivity to the origin is achieved and the overall behavior for both solutions is similar. Nevertheless, for the continuous case, it is well known that the system converges to the origin in finite time. In the event-based case, no conclusion in this issue can be provided yet. Note also the discontinuities introduced on the right boundary according to U_d and the propagation from the right to the left across the spacial domain. Figure 2 shows the time evolution of the functions appearing in the triggering condition (41). Once the trajectory $\theta Be^{\mu}d^2$ reaches the trajectory $\theta\sigma nuV - m$, an event is generated, the control value is updated and d is reset to zero. Figure 3 shows the continuous-time backstepping controller U and the discontinuous backstepping controller (event-based one) U_d.

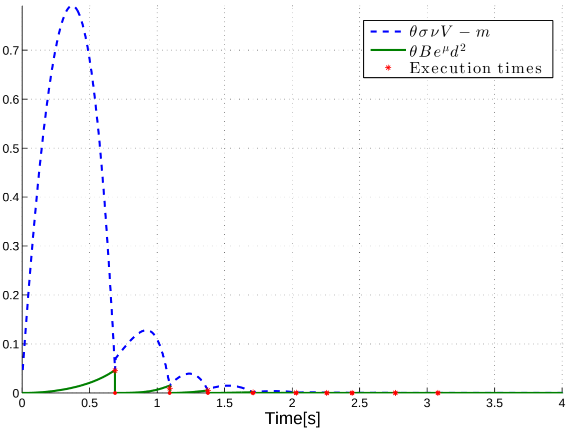

Fig. 2. Trajectories involved in triggering condition (41) for controller $U_d = \varphi(\alpha,\beta)$.

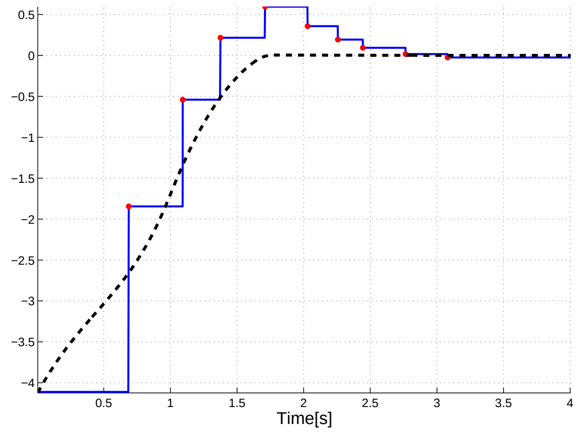

Fig. 3. Time-evolution of the continuous-time control U (black dashed line) and the event-based control U_d (blue line with red circle marker)

V. Conclusion

While in literature it is not sufficiently clear how fast boundary continuous time controllers of hyperbolic PDEs must be sampled in a periodic fashion so as to implement them into a digital platform, event-based control might propose a rigorous way of sampling aperiodically, by updating control inputs (when needed) while guaranteeing stability. In this paper, an event-based boundary controller to stabilize a 2×2 coupled linear hyperbolic system is introduced. It is proved that no Zeno phenomenon is present and then the well-posedness and global exponential stability of the hyperbolic system are obtained. The event-based controller is based on Lyapunov analysis and backstepping design method. To the authors knowledge, this is the first event based control for coupled hyperbolic system under the backstepping design, proposed in literature.

This work leaves some open questions. Since in more realistic scenarios, backstepping controllers are designed using observed states, for event-based control under backstepping, triggering laws should also include an estimate of the state. Based on [30], the output feedback control can be used as a continuous control to be sampled on events. It is important however to guarantee that under the triggering condition depending only on the observed states, there is no Zeno phenomenon.

It could be fruitful to study the impact of parameters to the performance as well as robustness with respect to exogenous disturbances while studying carefully the triggering condition along with the minimal inter-execution time. Indeed, exogenous disturbances

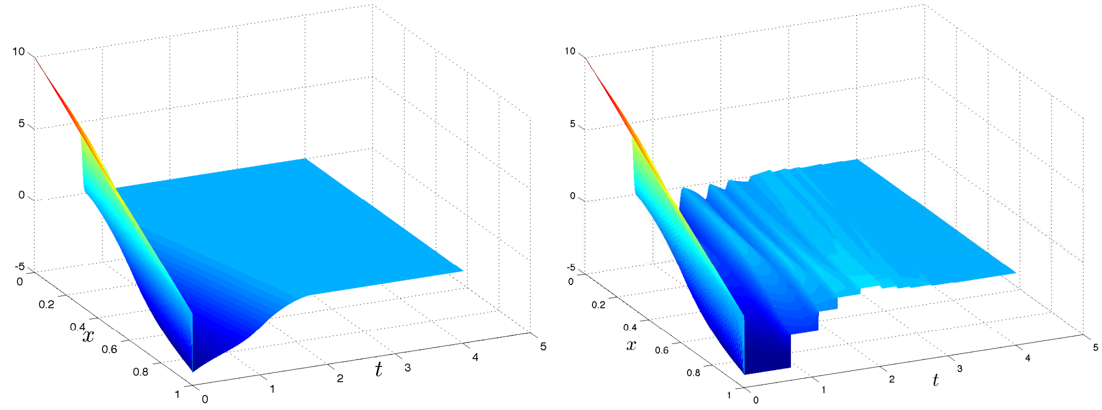

Fig. 1. Numerical solution of the second component v of the closed-loop system with continuous time controller U (left) and with event-based controller U_d (right).

may introduce Zeno phenomenon. Event-separation properties of the event-based scheme might be useful to tackle that issue by following for instance [4].

Acknowledgement

Authors would like to thank Florent Di Meglio for valuable comments on a preliminary version of this paper.

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