# A Regularized and Smoothed Fischer-Burmeister Method for Quadratic Programming with Applications to Model Predictive Control 

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#### Abstract

This paper considers solving convex quadratic programs (QPs) in a real-time setting using a regularized and smoothed FischerBurmeister method (FBRS). The Fischer-Burmeister function is used to map the optimality conditions of the quadratic program to a nonlinear system of equations which is solved using Newton's method. Regularization and smoothing are applied to improve the practical performance of the algorithm and a merit function is used to globalize convergence. FBRS is simple to code, easy to warmstart, robust to early termination, and has attractive theoretical properties, making it appealing for realtime and embedded applications. Numerical experiments using several predictive control examples show that the proposed method is competitive with other state of the art solvers.


Index Terms—Quadratic Programming, Semismooth, Model Predictive Control, Newton's Method, Non-smooth Analysis, Embedded Optimization, Convex Optimization

## I. INTRODUCTION

Real-time optimization has the potential to dramatically improve the capabilities of many systems. A key class of optimization problems in real-time applications involves convex quadratic programs (QPs); many practical problems in control, signal processing, machine learning and other domains can be posed as convex QPs [1]. In recent years significant progress has been made developing fast, reliable algorithms for solving both QPs and more general optimization problems online. However, many applications, especially fast systems with limited computing power, remain challenging.

An important instance of a real-time, embedded optimization problem is the one in model predictive control (MPC) [2] [3] [4], where an optimal control problem over a receding horizon is solved during each sampling period. The optimal control problem for a discrete time linear-quadratic MPC formulation can be expressed as a convex QP. Furthermore, convex QPs form the basis for many algorithms used in nonlinear model predictive control (NMPC) such as sequential quadratic programming (SQP) [5], and the real-time iteration scheme [6] which solves just one QP per timestep.

Interest in embedded optimization has motivated extensive research into fast, reliable solvers tailored for embedded systems. Algorithms and solvers specialized for MPC include the algorithms of Wang et al. [7] and FORCES [8], which are based on interior point methods, and qpOASES [9] which is based on the active set method. Other algorithms, which are often used for MPC but which can solve more general convex QPs, include GPAD [10], CVXGEN [11], NNLS [12], piecewise smooth Newton's methods [13], [14], and PQP [15]. In addition, solvers and algorithms for embedded second order cone programs, have begun to appear [16] [17].

This paper considers the application of the Fischer-Burmeister (FB) function and a smoothing Newton's method to solving convex QPs. The necessary conditions for optimality are mapped to a system of non-smooth equations using the Fisher-Burmeister function. The

[^0]equations are then smoothed and Newton's method is then applied to solve the resulting root finding problem. Regularization and a line search are added to control the numerical conditioning of the linear subproblems and enforce global convergence.

The regularized and smoothed FB algorithm, which we will refer to as FBRS (Fischer Burmeister Regularized Smoothed), has nice properties which make it attractive for embedded optimization. Firstly, FBRS displays global convergence and quadratic asymptotic convergence, properties it inherits from its nature as a damped generalized Newton's method. Secondly, it is simple to implement, a complete embeddable implementation is possible in under 100 lines of MATLAB code. Finally, it can be effectively warmstarted when solving sequences of related QPs, which is beneficial in many realtime optimization problems, including MPC.

Fischer-Burmeister (FB) functions in conjunction with both smoothing and semismooth Newton's method have been investigated in the past; a version of this algorithm, without smoothing or regularization, applied to general nonlinear programs was studied in [18]. In addition, some smoothing methods for linear complementarity problems, which subsume box constrained convex QPs, have been proposed, see e.g., [19] and the references therein. However, this paper investigates its use for quadratic programming at a level of depth and detail not present in the previous literature. Furthermore, we consider its suitability for embedded use and perform numerical experiments demonstrating its applicability to predictive control. In addition, FBRS includes practical improvements such as regularization to handle ill-conditioned Jacobians.

FBRS also has some advantages when compared with other methods for solving convex QPs. Firstly, FBRS smooths the complementarity conditions in a manner similar to an interior point algorithm; however, unlike an interior point method, FBRS is locally quadratically convergent with no smoothing. This is the key property of FBRS which makes it attractive for solving sequences of related QPs. Secondly, FBRS has a faster convergence rate than first-order methods such as dual accelerated gradient projection (GPAD), the alternating direction method of multipliers (ADMM), or multiplicative update methods such as PQP, and does not require that the QP be strictly convex. Finally, in contrast with active set or primal barrier interior point methods, the initial guess need not be feasible.

Fisher [20] used the eponymous function to map the Karush-KuhnTucker (KKT) conditions for a nonlinear program to a nonlinear system of equations which is then solved with a non-smooth Newton's method based on Clarke's generalized Jacobian. Local convergence results are obtained but globalization is not considered. In [21] a semismooth Newton's method which uses a penalized FB function to solve nonlinear complementarity problems is presented which uses a version of the C-differential and globalizes the algorithm using a line-search. The application of the FB function to diesel engine MPC was considered in [22].

Some key papers concerning generalized Newton's methods and their convergence are [23] and [24]. Reference [25] concerns the convergence of inexact generalized Newton's methods. Reference
[26] is a useful survey on the topic.
Notation: Our notations are standard. $\mathbb{R}^{n}$ denotes the set of $n$ dimensional real vectors. For a vector $x \in \mathbb{R}^{n} x_{i}$ denotes its $i$-th entry and the relations $\leq, \geq,<,>$ are understood component wise. For a matrix $A \in \mathbb{R}^{m \times n} A_{i}$ denotes the $i$-th row of the matrix; if $I$ is an index set $A_{I}$ denotes the concatenation of all $A_{i}, i \in I$. Let $S_{+(+)}^{n}$ denote the set of $n \times n$ (strictly) symmetric positive definite matrices, $A \succ 0, A \succeq 0$ denote positive definiteness and semi-definiteness, respectively. The kernel of a map $T$ is denoted by $\operatorname{ker} T$ and $I$ is used to denote the identify matrix, the dimensions of which should be clear from context. Set unions, intersections, and subtractions are denoted by $\cup, \cap$, and $\backslash$ respectively. The cardinality of a set $S$ is denoted by $|S|$. A matrix or vector norm $\|\cdot\|$ will be taken to indicate the two norm unless otherwise indicated. Let $h: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ and $g(x): \mathbb{R}^{n} \rightarrow \mathbb{R}_{\geq 0}$. We write $h(x)=O(g(x))$ as $x \rightarrow \bar{x}$ if $\exists M>0$ such that $\|h(x)\| \leq M g(x)$ for all $x$ sufficiently close to $\bar{x}$. If $\|h(x)\| \leq \varepsilon g(x), \forall \varepsilon>0$ holds for all $x$ sufficiently close to $\bar{x}$ then we say $h(x)=o(g(x))$.

## II. Problem formulation

This paper considers solving convex QPs in $n$ variables with $q$ constraints of the form

$$
\begin{array}{cl}
\min _{z} & g(z)=\frac{1}{2} z^{T} H z+f^{T} z, \\
\text { s.t } A z \leq b, \tag{1b}
\end{array}
$$

where $H \in S_{+}^{n}$ is the Hessian matrix, $f \in \mathbb{R}^{n}, z \in \mathbb{R}^{n}, A \in \mathbb{R}^{q \times n}$, and $b \in \mathbb{R}^{q}$. For simplicity we consider the case where there are no equality constraints. The extension to equality and inequality constrained problems is straightforward, alternatively polyhedral inequality and equality constrained problems can always be converted into a purely inequality constrained problem provided the equality constraints are feasible, see e.g., [1] Section 4.1.3]. The Lagrangian for this problem is

$$
\begin{equation*}
L(z, v)=\frac{1}{2} z^{T} H z+f^{T} z+v^{T}(A z-b), \tag{2}
\end{equation*}
$$

and the KKT conditions for (1) are

$$
\begin{gather*}
\nabla_{z} L=H z+f+A^{T} v=0  \tag{3a}\\
v_{i} \cdot y_{i}=0, \quad i=1 \ldots q  \tag{3b}\\
v \geq 0, \quad y \geq 0 \tag{3c}
\end{gather*}
$$

where we let,

$$
\begin{equation*}
y=b-A z, \tag{4}
\end{equation*}
$$

denote the constraint residual. Since $H$ is positive semidefinite these conditions are necessary and sufficient for optimality under an appropriate constraint qualification. We are interested in the case where (1) has a unique primal-dual solution, so we impose some additional assumptions on the problem.

Let $x^{*}=\left(z^{*}, v^{*}\right)$ denote a point satisfying (3), referred to as a critical or KKT point, and let $I_{a}(z)=\left\{i \in 1 \ldots q \mid A_{i} z=b_{i}\right\}$ denote the set of active constraints at $z$. Recall that for a system of linear inequalities of the form $A z \leq b$, the linear independence constraint qualification (LICQ) is said to hold at a point $\bar{z}$ if

$$
\begin{equation*}
\operatorname{rank} A_{I_{a}(\bar{z})}=\left|I_{a}(\bar{z})\right|, \tag{5}
\end{equation*}
$$

and that if

$$
\begin{equation*}
u^{T} H u>0, \forall u \neq 0 \text { such that } A_{i} u=0, \forall i \in I_{a}^{+}\left(z^{*}, v^{*}\right), \tag{6}
\end{equation*}
$$

where $I_{a}^{+}(z, v)=\left\{i \in 1 \ldots q \mid A_{i} z=b_{i}, v_{i}>0\right\}$, then $x^{*}$ is said to satisfy the strong second order sufficient conditions (SSOSC). The following two assumptions are then sufficient for local primal-dual uniqueness.

Assumption 1. (A1) There exists a point $x^{*}=\left(z^{*}, v^{*}\right)$ that satisfies the strong second order sufficient conditions.
Assumption 2. (A2) The linear independence constraint qualification (LICQ) holds at $x^{*}$.

Since (1) is convex, the SSOSC implies that $z^{*}$ is the unique global minimizer of (11. The LICQ is used in place of Slater's condition, because the LICQ implies that the dual variable $v^{*}$ associated with $z^{*}$ is unique and thus the primal-dual solution is isolated; this simplifies the convergence analysis. In the degenerate case where either the primal or dual solution is not isolated an algorithm in the same vein as the stabilized Josephy-Newton method may be applicable, see e.g., [27) Chapter 7].

Since $H$ is assumed to be positive semidefinite rather than strictly positive definite we add an additional assumption to ensure that intermediate iterations are well defined.

## Assumption 3. (A3) $\operatorname{ker} H \cap \operatorname{ker} A=\{0\}$

Intuitively, this condition requires all directions along which the cost function has no curvature to be constrained. For a QP this condition can be checked numerically. It is often desirable for (A3) to be satisfied by construction in an embedded context; typically using regularization. Note that $H$ is only assumed to be positive semidefinite, rather than strictly positive definite; it allows (1) to capture a wider range of QP problems. If $H$ is strictly positive definite then the SSOSC and (A3) are satisfied automatically.

## III. SOME CONCEPTS FROM NON-SMOOTH ANALYSIS

In this section we review some key concepts from non-smooth analysis which are required to motivate and analyze FBRS. We begin with generalized differentiation. Suppose a function $G: \mathbb{R}^{N} \mapsto \mathbb{R}^{M}$ is locally Lipschitz on a set $U \subseteq \mathbb{R}^{N}$, i.e. $\exists L>0$ s.t $\| G(x+\xi)-$ $G(x)\|\leq L\| \xi \|, \forall \xi \in U$. Then Rademacher's theorem [28] states that $G$ is differentiable almost everywhere. Letting $D_{G}$ denote the set of points where $G$ is differentiable, the B-differential is defined as

$$
\begin{equation*}
\partial_{B} G(x)=\left\{J \in \mathbb{R}^{M \times N} \mid \exists\left\{x^{k}\right\} \subset D_{G}:\left\{x^{k}\right\} \rightarrow x,\left\{\nabla G\left(x_{k}\right)\right\} \rightarrow J\right\}, \tag{7}
\end{equation*}
$$

and Clarke's generalized Jacobian, $\partial G(x)=\operatorname{convh} \partial_{B} G(x)$, can be defined as the convex hull of the B-differential [29]. The generalized Jacobian is a set of matrices, wherever $G$ is differentiable $\nabla G(x) \in \partial G(x)$ and $\partial G(x)=\{\nabla G(x)\}$ wherever $G$ is continuously differentiable [27].
A mapping $G: \mathbb{R}^{N} \mapsto \mathbb{R}^{M}$ is said to be semismooth at $x \in \mathbb{R}^{N}$ if $G$ is locally Lipschitz at $x$, directionally differentiable in every direction and the estimate

$$
\begin{equation*}
\sup _{J \in \partial G(x+\xi)}\|G(x+\xi)-G(x)-J \xi\|=o(\|\xi\|) \tag{8}
\end{equation*}
$$

holds. If the right hand side is replaced by the stronger bound $O\left(\|\xi\|^{2}\right)$ then $G$ is said to be strongly semismooth at $x$ [27]. The concept of semismoothness plays a key role in the analysis of nonsmooth Newton's methods [23].

In this paper the C -differential of a mapping, denoted $\partial_{C} G$, is defined as

$$
\begin{equation*}
\partial_{C} G=\partial G_{1} \times \partial G_{2} \times \ldots \partial G_{N}, \tag{9}
\end{equation*}
$$

where $\partial G_{i}$ is the transpose of the so-called generalized gradient of $G_{i}$, which is simply the generalized Jacobian [29] of the component mapping $G_{i}: \mathbb{R}^{N} \mapsto \mathbb{R}$. Note that $\partial G_{i}$ is a row vector, thus $\partial_{C} G$ is a set of matrices whose rows are the transposed generalized gradients of the component functions $G_{i}$. This form of the C-differential was introduced in [30] and [31], it possesses many of the properties of the generalized Jacobian but can be easier to compute and characterize.

## IV. The Algorithm and its Properties

FBRS (Fischer-Burmeister Regularized and Smoothed) is a complimentary mapping algorithm based on Newton's method which uses smoothing to ensure robust globalization. FBRS, and complementarity mapping methods in general, function by bijectively mapping the complementarity conditions 3b and (3c) to a system of equations using what is known as a nonlinear complementarity (NCP) function [32]. We will use the generalized (smoothed) Fischer-Burmeister function

$$
\begin{equation*}
\phi_{\varepsilon}(a, b)=a+b-\sqrt{a^{2}+b^{2}+\varepsilon^{2}} \tag{10}
\end{equation*}
$$

which has the property that

$$
\begin{equation*}
\phi_{\varepsilon}(a, b)=0 \Leftrightarrow a \geq 0, b \geq 0, \sqrt{2} a b=\varepsilon \tag{11}
\end{equation*}
$$

and that

$$
\begin{equation*}
\left\|\phi_{\varepsilon}(a, b)-\phi(a, b)\right\| \leq \varepsilon, \quad \phi(a, b)=\phi_{0}(a, b) \tag{12}
\end{equation*}
$$

Applying the FB function to the complementarity conditions in (3) yields the following nonlinear mapping,

$$
F_{\varepsilon}(x)=\left[\begin{array}{c}
\nabla_{z} L(z, v)  \tag{13}\\
\phi_{\varepsilon}(v, y)
\end{array}\right], \quad F(x)=F_{0}(x)
$$

where $x=(z, v)$ denotes the primal-dual pair. The following two results are direct consequences of the properties of the FB function and relate the smoothed and original problems.

Corollary 1. Under the assumptions in Section $I \square$ the mapping $F_{0}(x)=F(x)=0$ if and only if $x=x^{*}$, is the solution to (1). Further, this root is unique.

Proof. Since the problem is convex, the SSOSC implies uniqueness of $z^{*}$ and the LICQ implies uniqueness of $v^{*}$. Roots of $F(x)$ coincide with points which satisfy the KKT conditions [20]. Thus $x^{*}$ uniquely satisfies $F\left(x^{*}\right)=0$.
Corollary 2. For all $x \in \mathbb{R}^{n+q}$ and $\varepsilon \geq 0$ we have that $\left\|F_{\varepsilon}-F\right\| \leq$ $\sqrt{q} \varepsilon$ and

$$
\begin{equation*}
\|F(x)\| \leq\left\|F_{\varepsilon}(x)\right\|+\sqrt{q} \varepsilon \tag{14}
\end{equation*}
$$

Proof. Direct computation yields,

$$
\left\|F(x)_{\varepsilon}-F(x)\right\|=\left\|\left[\begin{array}{c}
0  \tag{15}\\
\phi_{\varepsilon}(y, v)-\phi(y, v)
\end{array}\right]\right\|
$$

applying 12 , the properties of the two norm, and the reverse triangle inequality then yields the result.

The general FBRS algorithm approximately solves a sequence of sub-problems $F_{\varepsilon_{k}}\left(x_{k}\right)=0$ for a decreasing sequence, $\varepsilon_{k} \rightarrow 0$. Each sub-problem is solved using Newton's method

$$
\begin{equation*}
x_{k+1}=x_{k}-t_{k} V_{k}^{-1} F_{\varepsilon_{k}}\left(x_{k}\right) \tag{16}
\end{equation*}
$$

where $V_{k}=\nabla_{x} F_{\varepsilon_{k}}\left(x_{k}\right)$ and $t_{k} \in(0,1]$ is a steplength chosen by a linesearch to enforce convergence far from a solution.

The iteration matrix or Jacobian $V_{k}$ is always non-singular for any $\varepsilon \geq 0$ (see Section V Theorem 1) but can become ill-conditioned, so a regularization term is added. Defining $R(x, \delta)=\left[\begin{array}{ll}0^{T} & \delta\left(v^{T}+y^{T}\right)\end{array}\right]^{T}$, where $\delta \geq 0$ is the regularization strength, we can replace $V_{k}$ with $K_{k}=V_{k}+\nabla_{x} R\left(x_{k}, \delta_{k}\right)$ in , leading to the smoothed and regularized Newton iteration

$$
\begin{equation*}
x_{k+1}=x_{k}-t_{k} K_{k}^{-1} F_{\varepsilon_{k}}\left(x_{k}\right) \tag{17}
\end{equation*}
$$

[^1]which forms the core of the FBRS method. Expressions for computing $K$ and an analysis of its properties are presented in Section $\nabla$
Remark 1. A semismooth version of FBRS results if in 16 $V_{k}$ is redefined as $V_{k} \in \partial_{C} F_{\varepsilon}\left(x_{k}\right)$. This version allows $\varepsilon=0$ and reduces to the smoothed version for $\varepsilon>0$ (since then $\partial_{C} F_{\varepsilon}=\left\{\nabla_{x} F_{\varepsilon}\right\}$ ). A semismooth $F B$ method without regularization has been previously proposed in the literature, see e.g., $\boxed{18]}$ or [27] Section 5.1.2] and is globally convergent for convex QPs. However, we have observed that both smoothing and regularization improve the numerical performance of the algorithm and avoid the need to compute generalized derivatives. The semismooth variant of FBRS can be shown (see Section VII] Theorem 2] to be locally quadratically convergent when $\varepsilon \geq 0$. This property distinguishes NCP function based smoothing methods from interior points methods, where the barrier strength can only approach zero in the limit. This property means that the FBRS subproblems do not become ill-conditioned even when $\varepsilon$ is very small, facilitating warmstarting.

FBRS is summarized in Algorithm 1 and is simply Newton's method globalized using a linesearch and homotopy. The merit function used to globalize each subproblem in the algorithm is defined as

$$
\begin{equation*}
\theta_{\varepsilon}(x)=\frac{1}{2}\left\|F_{\varepsilon}(x)\right\|_{2}^{2} \tag{18}
\end{equation*}
$$

and the parameter $\sigma \in(0,0.5)$ encodes how much reduction we require in the merit function. The desired solution tolerance is denoted $\tau$, and $\beta \in(0,1)$ controls reduction in the backtracking linesearch. More sophisticated algorithms for computing $t_{k}$, e.g., polynomial interpolation, can be used in place of the backtracking linesearch; however we found this backtracking to be effective in practice. Typical values for the fixed parameters are $\sigma \approx 10^{-4}$ and $\beta \approx 0.7$.

```
Algorithm 1 FBRS
Input: \(H, A, f, b, x_{0}, \sigma, \beta, \tau, \delta_{0}\), max_iters
Output: \(x\)
    \(x \leftarrow x_{0}, \varepsilon \leftarrow \frac{\tau}{2 \sqrt{q}}, \delta \leftarrow \delta_{0}\)
    for \(k=0\) to max_iters-1 do
        \(\delta \leftarrow \min \left(\delta,\left\|F_{\varepsilon}(x)\right\|\right)\)
        if \(\left\|F_{0}(x)\right\| \leq \tau\) then
            break;
        end if
        Solve \(K(x, \varepsilon, \delta) \Delta x=-F_{\mathcal{E}}(x)\) for \(\Delta x\)
        \(t \leftarrow 1\)
        while \(\theta_{\varepsilon}(x+t \Delta x) \geq(1-2 t \sigma) \theta_{\varepsilon}(x)\) do
            \(t \leftarrow \beta t\)
        end while
        \(x \leftarrow x+t \Delta x\)
    end for
    return \(x\)
```

For any fixed $\varepsilon>0$ FBRS, under the assumptions in Section $\Pi$ exhibits global linear convergence (see Section VII Theorem 3) and local quadratic convergence (see Section VII Theorems 2 and 4) to the unique point satisfying $F_{\varepsilon}(x)=0$. For simplicity, the assumption that $\delta_{k}$ is always "small enough" that global convergence is not impeded is implicit in Algorithm If $\delta_{k}$ is too big at any iteration the linesearch may fail. In this case $\delta_{k}$ can be reduced and a new step computed or a gradient descent step on the merit function can be taken.

Remark 2. For embedded applications we often only require moderate precision solutions, e.g., $\|F\| \approx 10^{-6}$ to $10^{-8}$ in double precision. In these situations we found fixing $\varepsilon$ at a small value is sufficient
and simplifies warmstarting by removing the need to reinitialize $\varepsilon$. In this paper we simply take $\varepsilon_{k}=\frac{\tau}{2 \sqrt{q}}$. More sophisticated strategies for updating $\varepsilon$ may be helpful for improving numerical performance. Inspiration could potentially be drawn from the varied barrier update rules used in interior point methods [33]. If high precision solutions are required the semismooth $(\varepsilon=0)$ variant of FBRS can be used and could be warmstarted using the smoothed algorithm.

Remark 3. In practice since $K_{k}$ is guaranteed to be nonsingular even for $\delta=0$ the regularization parameter exists solely to handle numerical ill-conditioning and a small fixed value can be used throughout. For example $\delta_{k}=\delta_{0}=10^{-8}$ was used throughout this paper.

## V. The Iteration Matrix and Computation of Step Directions

This section presents a more detailed analysis of the properties of the iteration matrix $K$, whose factorization is the main computational burden of FBRS. The Newton step system is

$$
\left[\begin{array}{cc}
H & A^{T}  \tag{19}\\
-C A & D
\end{array}\right]\left[\begin{array}{c}
\Delta z \\
\Delta v
\end{array}\right]=\left[\begin{array}{c}
-\nabla_{z} L(z, v) \\
-\phi_{\varepsilon}(v, y)
\end{array}\right]=\left[\begin{array}{l}
r_{s} \\
r_{c}
\end{array}\right],
$$

the matrices $C=\operatorname{diag}(\gamma)$ and $D=\operatorname{diag}(\mu)$ are diagonal matrices constructed by regularizing the gradient of the smoothed FB function. If $\varepsilon>0$ the diagonal elements of $C$ and $D$ are,

$$
\begin{equation*}
\gamma_{i}=1-\frac{y_{i}}{\sqrt{y_{i}^{2}+v_{i}^{2}+\varepsilon^{2}}}+\delta, \mu_{i}=1-\frac{v_{i}}{\sqrt{y_{i}^{2}+v_{i}^{2}+\varepsilon^{2}}}+\delta, \tag{20}
\end{equation*}
$$

further, it is evident that $C \succ 0$ and $D \succ 0$ if $\varepsilon>0$.
Remark 4. The generalized gradient of the FB function is well known, see e.g., [34]; when using the semismooth version of FBRS $K$ can be computed by noting that $\gamma_{i}$ and $\mu_{i}$ are multivalued only if $\varepsilon=0$ and $\left(v_{i}, y_{i}\right)=0$; then any choice of $\gamma_{i}$ and $\mu_{i}$ that statisfies

$$
\begin{equation*}
\gamma_{i}=1-\alpha+\delta, \mu_{i}=1-\beta+\delta \quad \text { s.t } \quad \alpha^{2}+\beta^{2} \leq 1, \tag{21}
\end{equation*}
$$

implies that $K \in \partial_{C} F_{\mathcal{\varepsilon}}(x, \varepsilon)+\nabla_{x} R(x, \delta)$.
The Jacobian matrix $V=\nabla_{x} F_{\varepsilon}$ can be shown to be nonsingular.
Theorem 1. Let Assumption 3 hold and pick $\varepsilon>0$. Then $V=\nabla_{x} F_{\varepsilon}(x)$ is non-singular $\forall x \in \mathbb{R}^{n+q}$.

Proof. Recall that since $V$ is smoothed but not regularized ( $\delta=0$ ) if $\varepsilon>0$ then $D \succ 0$, and thus $V$ can be factored blockwise as

$$
V=\left[\begin{array}{cc}
I & A^{T} D^{-1}  \tag{22}\\
0 & I
\end{array}\right]\left[\begin{array}{cc}
H+A^{T} D^{-1} C A & 0 \\
0 & D
\end{array}\right]\left[\begin{array}{cc}
I & 0 \\
-D^{-1} C A & I
\end{array}\right] .
$$

The upper and lower triangular factors are necessarily invertible, as is $D$, thus only $T=H+A^{T} D^{-1} C A$ must to be analyzed; we will show that $T \succ 0$. Letting $L=\sqrt{D^{-1} C} A$, it's clear that $T=H+L^{T} L \succeq 0$. To show positive definiteness assume there exists

$$
\begin{equation*}
z \neq 0 \text { such that } z^{T} T z=z^{T} H z+(L z)^{T} L z=0 \tag{23}
\end{equation*}
$$

since $H \succeq 0$ and $L^{T} L \succeq 0, z^{T} T z=0$ implies that $H z=0$ and $L z=\sqrt{D^{-1} C} A z=0$. Since $D^{-1} C$ is diagonal and positive definite this contradicts Assumption 3 ( $\operatorname{ker} H \cap \operatorname{ker} A=\{0\}$ ). As a result $H+A^{T} D^{-1} C A$ must be positive definite and each factor in is invertible.
Remark 5. It turns out that for the special case of convex QPs all elements of $\partial_{C} F_{\varepsilon}$ are non-singular when $\varepsilon \geq 0$ [18] Theorem 4.4]. The proof is involved and we have observed that the smoothed version works well in practice; as a result we have elected to present a selfcontained proof of Theorem $\square$ for the smoothed version.

Corollary 3. Let assumption $A$ hold, $\delta>0$, and $\varepsilon>0$, then all $K=V+\nabla R(x, \delta)$ are non-singular.
Proof. Recall that if $\delta>0$ then $C$ and $D$ are both positive definite for all $\varepsilon>0$. Then repeat the proof of Theorem 1

The main computational burden of FBRS is solving the linear system, $K \Delta x=-F_{\varepsilon}$, see Algorithm 1 If there is no exploitable structure then an LU decomposition is a practical choice. If $K$ is sparse or matrix-vector products can be computed quickly then an iterative method may be appropriate. Alternatively, using the same block-LU decomposition as in the proof of Theorem 1 a condensed decomposition can be derived

$$
\begin{gather*}
\left(H+A^{T} C D^{-1} A\right) \Delta z=r_{s}-A^{T} D^{-1} r_{c},  \tag{24a}\\
D \Delta v=r_{c}+C A \Delta z . \tag{24b}
\end{gather*}
$$

Under ( $\mathrm{A} \sqrt{3}$ ) we have that $H+A^{T} C D^{-1} A \succ 0$ and thus we are able to reduce a general $q+n \times q+n$ system to a $n \times n$ dense symmetric positive definite system, which can be solved using a Cholesky factorization or the conjugate gradient method, and a diagonal $q \times q$ system.

## VI. Numerical experiments

In this section we compare FBRS against other methods in terms of execution time on both real-time hardware and on a regular computer. Execution times on a computer are of relevance during initial controller development and allow us to compare FBRS against a wider array of solvers. FBRS was compared against several state of the art solvers: (i) quadprog (MATLAB 2015a SP1) Interior Point (IP), (ii) QPKWIK (dual active set) [35], (iii) ECOS (self-dual interior point) [16], (iv) GPAD (dual accelerated gradient projection) [10], and (v) PDIP (primal-dual interior point) [33, Algorithm 14.3]. We use the norm of the natural residual,

$$
F_{N R}(x)=\left[\begin{array}{ll}
\nabla_{z} L(z, v)^{T} & \min (y, v)^{T} \tag{25}
\end{array}\right]^{T}
$$

in this section to quantify the quality of a solution.

## A. Comparisons against other methods

Four example problems of varying sizes were considered: A convex MPC controller for asteroid circumnavigation [36], a diesel engine Economic MPC problem [37], an extended command governor for an F16 control problem, similar to the reference governor presented in [38], and an MPC controller for a spacecraft attitude control problem, similar to the example presented in [39]. Each control problem generates a sequence of related, feasible, QPs. The solution of the previous QP is used to initialize the next for all algorithms that accept a warmstart. Table $\square$ summarizes the size of each problem and the number of QPs in the sequence 2 .

The result: $\sqrt[3]{ }$ are shown in Tables $\square$ and $\square I I$ which summarize the average and maximum execution times for each QP sequence,

[^2]TABLE I
SUMMARY OF PROBLEM SIZES.

|  | Asteroid | Diesel | F16 | S/C |
| :---: | :---: | :---: | :---: | :---: |
| Number of variables | 280 | 30 | 12 | 31 |
| Number of constraints | 490 | 70 | 1010 | 181 |
| Number of QPs in sequence | 300 | 3000 | 599 | 50 |

TABLE II
Average execution time for each sequence of QPs. Entries in EACH COLUMN HAVE BEEN NORMALIZED BY THE FIRST ELEMENT.

|  | Asteroid | Diesel | F16 | S/C |
| :---: | :---: | :---: | :---: | :---: |
| Normalization [msec] | 17.75 | 0.11 | 2.14 | 0.18 |
| FBRS | 1 | 1 | 1 | 1 |
| PDIP | 4.06 | 9.62 | 0.45 | 7.40 |
| QPKWIK | 4.28 | 1.12 | 0.08 | 0.54 |
| GPAD | $N / A$ | 17.94 | 10.97 | 0.81 |
| Quadprog IP | 3.60 | 75.04 | 16.23 | 121.77 |
| ECOS | 9.01 | 23.99 | 8.54 | 54.87 |

respectively. FBRS performed well on the smaller problems, with performance similar to the dual active set method on the diesel and spacecraft examples. FBRS was soundly beaten on the F16 problem which has many constraints and few variables. Two interior point methods (PDIP and QUADPROG IP) had the best worst case execution times on the asteroid example (the largest considered) by a small margin. FBRS had the best average execution time and was competitive in terms of worst case execution time. Overall, FBRS, despite its simplicity, was found to be competitive with state of the art solvers for both large and small scale problems.

## B. Comparisons on embedded hardware

The performance of the FBRS solver was compared against other methods on embedded hardware. Specifically the economic MPC controller from [37] was placed in closed loop with a high fidelity model of a diesel engine. The model and controller, including the QP solvers which were implemented using embedded MATLAB, were implemented in Simulink (2010b SP2) and loaded onto a DS1006 rapid prototyping unit using real-time workshop ( 2.8 GHz CPU , 1 GB RAM). We used $\left\|F_{N R}\right\| \leq 10^{-4}$ as a stopping criterion and measured the turnaround time of the QP solvers using the profiling tools supplied with the processor board. ECOS and quadprog could not be loaded onto the DS1006 board since they are not compatible with the real-time workshop build process. FBRSacc and PDIPacc disable all safeguards and use structured linear algebra to speed up computation of matrix operations, e.g., (24).

Remark 6. Note that the DS1006 runs a real-time OS, and thus the turnaround time is the precise metric used to judge if an application is executable in real-time.

TABLE III
MAXIMUM EXECUTION TIME FOR EACH SEQUENCE OF QPs. ENTRIES IN EACH COLUMN HAVE BEEN NORMALIZED BY THE FIRST ELEMENT.

|  | Asteroid | Diesel | F16 | S/C |
| :---: | :---: | :---: | :---: | :---: |
| Normalization [msec] | 164.67 | 1.16 | 11.40 | 0.26 |
| FBRS | 1 | 1 | 1 | 1 |
| PDIP | 0.94 | 2.26 | 0.18 | 5.56 |
| QPKWIK | 3.24 | 0.73 | 0.06 | 0.91 |
| GPAD | $N / A$ | 63.13 | 8.80 | 1.06 |
| Quadprog IP | 0.96 | 8.22 | 5.77 | 112.42 |
| ECOS | 1.31 | 3.95 | 2.40 | 44.82 |



Fig. 1. Measured turnaround time, residual, and the excitation sequence for the embedded comparisons. All methods were cold started.


Fig. 2. Measured turnaround time, residual, and the excitation sequence for the embedded comparisons. All methods were warm started.

Figures 11 and 2 illustrate the results when the solver are coldstarted and warm-started respectively. GPAD was unable to converge when cold-started and QPKWIK performed very poorly, this is surprising given that it performed well during the tests in Section VI-A A summary of the results are given in Table IV FBRSacc performed best on average and was able to efficiently exploit warmstarting; PDIPacc was the most efficient cold-start algorithm. FBRSacc and PDIPacc had effectively (within 5 microseconds) identical worst case turnaround times. Note that the sampling period of the diesel airpath control application is 8 msec ; both FBRSacc and PDIPacc are thus real-time executable.

## VII. Convergence of the algorithm

In this section we analyze the convergence properties of FBRS. We consider the semismooth variant of FBRS (see Remark 11) which allows $\varepsilon \geq 0$. Several properties of the mapping $F_{\varepsilon}$ and the merit function are established and then local and global convergence results are obtained.
${ }^{4}$ We attribute this to subtleties in the automatic codegeneration process which we are still investigating. The code used to implement the QPKWIK algorithm on the DS1006 board is identical to the code used during the numerical trials detailed in Section VI-A

TABLE IV
Summary of the embedded testing. Times are in msec. ERR INDICATES THE MAXIMUM VALUE OF $\left\|F_{N R}\right\|$.

|  | WARM |  |  | COLD |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | AVE | MAX | ERR | AVE | MAX | ERR |
| FBRS | 1.02 | 17.68 | $10^{-4}$ | 14.8 | 23.8 | $10^{-4}$ |
| PDIP | 7.04 | 11.65 | $10^{-4}$ | 9.84 | 12.7 | $10^{-4}$ |
| GPAD | 28.6 | 162.1 | 0.35 | 146 | 163 | 1.8 |
| FBRSacc | 0.31 | 2.04 | $10^{-4}$ | 2.25 | 3.02 | $10^{-4}$ |
| PDIPacc | 1.23 | 1.99 | $10^{-4}$ | 1.67 | 2.27 | $10^{-4}$ |
| QPKWIK | 465 | 712 | 5.4 | 649 | 841 | $1.2 \cdot 10^{-4}$ |

## A. Key properties of the mapping

Here we establish some properties which will be needed to prove the convergence of FBRS. These properties hold for all $\varepsilon \geq 0$; in this section we focus on the case where $\varepsilon=0$, if $\varepsilon>0$ then strong semismoothness is implied by continuous differentiability and CD regularity is implied by Jacobian non-singularity.

Proposition 1. The mapping $F_{\mathcal{E}}: \mathbb{R}^{n+q} \times \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^{n+q}$ has the following properties.

1) $F_{\varepsilon}$ is locally Lipschitz continuous i.e., for every $x \in \mathbb{R}^{n+q}$ there exists $L_{F}(x)>0$ and a neighbourhood $O$ of $x$ such that

$$
\begin{equation*}
\left\|F_{\varepsilon}(x+\xi)-F_{\varepsilon}(x)\right\| \leq L_{F}\|\xi\| \quad \forall \xi \in O \tag{26}
\end{equation*}
$$

2) The mapping $F_{\varepsilon}$ is strongly semismooth.
3) The mapping $F_{\varepsilon}$ is $C D$ (Clarke Differential) regular $\boxed{24}$ in the vicinity of a root $\bar{x}$ which satisfies $F_{\mathcal{E}}(\bar{x})=0$. This implies that there exists $L_{I}$ and a neighbourhood $S$ of $\bar{x}$ such that

$$
\begin{equation*}
\|x-\bar{x}\| \leq L_{I}\|F(x)\| \quad \forall x \in S \tag{27}
\end{equation*}
$$

4) Define the error matrix $E=V-K$ as the difference between the regularized and unregularized iteration matrices and let $\delta$ denote the regularization parameter. Then

$$
\begin{equation*}
\exists \gamma>0 \text { such that }\|E\| \leq \gamma \delta, \forall x \in \mathbb{R}^{n+q} \tag{28}
\end{equation*}
$$

Proof. Result 1: This follows from the Lipschitz continuity of affine functions and of the FB function [20].

Result 2: The mapping $F_{\mathcal{E}}$ is the concatenation of an affine function and the composition of an affine function and the Fischer-Burmeister transform. Affine functions are strongly semismooth as is the FB function [40, Lemma 16]. Further, the concatenation and composition of (strong) semismooth mappings are (strongly) semismooth, see, e.g., [27, Propositions 1.73 and 1.74].

Result 3: CD regularity can be established by noting that the CD regularity of the Fisher-Burmeister mapping applied to the KKT conditions of a general nonlinear program, assuming the SSOSC and the LICQ, was proven in [20, Lemma 4.2]. The CD regularity of the convex QPs considered in this paper then follows as a special case.

Result 4: This bound can be directly computed by inspecting 19 and applying the properties of norms with $\gamma=1+\|A\|$.

## B. Key properties of the merit function

Proposition 2. The merit function $\theta_{\varepsilon}=\frac{1}{2}\left\|F_{\mathcal{E}}\right\|_{2}^{2}: \mathbb{R}^{n+q} \times \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ has the following properties.

1) For any $\varepsilon \geq 0$ the merit function $\theta_{\varepsilon}(x)$ is continuously differentiable.
2) The gradient of the merit function can be computed as

$$
\begin{equation*}
\nabla_{x} \theta_{\varepsilon}(x)=V^{T} F_{\varepsilon}(x) \tag{29}
\end{equation*}
$$

for any $V \in \partial_{C} F_{\varepsilon}(x)$.
3) For any $\varepsilon \geq 0$ the merit function has a unique minimizer $\bar{x}$ which corresponds to $F_{\varepsilon}(\bar{x})=0$.
Proof. Result 1: If $\varepsilon>0$ then $F_{\varepsilon}$ is continuously differentiable so we consider the case when $\varepsilon=0$. Following [41], consider $\partial \theta^{T}=$ $\partial F^{T} F \subseteq \partial_{C} F^{T} F$, which holds by the calculus of the Generalized Jacobian ( |29], Theorems 2.6.6, 2.2.4). Expanding the product we obtain

$$
\partial \theta^{T}=\left[\begin{array}{c}
H \nabla_{z} L+A^{T} C \phi  \tag{30}\\
A \nabla_{z} L+D \phi
\end{array}\right]
$$

the product $w=C \phi$, when written elementwise, is of the form $w_{i}=$ $\gamma_{i} \phi\left(y_{i}, v_{i}\right)$. Since $\gamma_{i}$ is multivalued only if $\left(v_{i}, y_{i}\right)=0$, (see Remark 5) which in turn implies that $\phi\left(v_{i}, y_{i}\right)=0$, the multivalued elements of $w_{i}=\gamma_{i} \phi\left(y_{i}, v_{i}\right)$ are "zeroed out". The same argument holds for $D \phi$ and thus the products $C \phi$ and $D \phi$ must be single valued implying that $\partial \theta^{T}=\nabla_{x} \theta$ is continuously differentiable 29] Corollary to Theorem 2.2.4].

Result 2: Since $\partial \theta=\partial_{C} \theta \subseteq F^{T} \partial_{C} F$ and $F^{T} V$ is a singleton for any $V \in \partial_{C} F$, we must have that $\nabla_{x} \theta=\partial \theta^{T}=V^{T} F$ for any $V \in \partial_{C} F$.

Result 3: The necessary conditions for minimizing the merit function are $\nabla_{x} \theta_{\varepsilon}=V^{T} F_{\varepsilon}(x)=0$ for any $V \in \partial_{C} F_{\varepsilon}(x)$. If $\varepsilon>0$ then $V$ is always nonsingular (Theorem 1) and $V^{T} F_{\mathcal{E}}=0$ if and only if $F_{\mathcal{E}}(x)=0$. If $\varepsilon=0$ then [18] Theorem 4.4] can be invoked in place of Theorem 1 to obtain the same result.

## C. Asymptotic convergence

This section focuses on local convergence of FBRS for some fixed $\varepsilon \geq 0$. We will use $k \in \mathbb{Z}_{+}$as a superscript as the iteration counter for the algorithm. Let $x^{*}$ denote the root of $F_{\varepsilon}$ which exists and is unique under our assumptions so that $F_{\varepsilon}^{*}=F_{\varepsilon}\left(x^{*}\right)=0$. We define the error $e^{k}=x^{k}-x^{*}$, and the matrices $V \in \partial_{C} F_{\varepsilon}\left(x^{k}, \varepsilon\right)$ and $K=$ $V+\nabla_{x} R\left(x^{k}, \delta\right)$
Theorem 2. Let $\left\{x^{k}\right\},\left\{\Delta x^{k}\right\}$ be generated by FBRS and pick any fixed $\varepsilon \geq 0$. Then there exists $\eta>0$ and a neighbourhood $U$ of the root $x^{*}$ of $F_{\mathcal{E}}$, such that if $x^{0} \in U$ then the bound $\left\|e^{k+1}\right\| \leq \eta\left\|e^{k}\right\|^{2}$ holds, and $\left\{x^{k}\right\} \in U$ converges quadratically to $x^{*}$.

Proof. Consider the update equation

$$
\begin{equation*}
\left\|e^{k+1}\right\|=\left\|x^{k+1}-x^{*}\right\|=\left\|x^{k}-x^{*}+\Delta x^{k}\right\| \tag{31a}
\end{equation*}
$$

and recall that $\Delta x^{k}=-K^{-1} F_{\varepsilon}$. Combining these two and performing some algebraic manipulations we obtain

$$
\begin{align*}
\left\|e^{k+1}\right\| & \leq\left\|e^{k}-K^{-1} F_{\varepsilon}\right\|  \tag{31b}\\
& \leq\left\|K^{-1}\right\|\left\|K e^{k}-F_{\varepsilon}+F_{\varepsilon}^{*}\right\|  \tag{31c}\\
& \leq M\left\|K e^{k}-V e^{k}+V e^{k}-F_{\varepsilon}+F_{\varepsilon}^{*}\right\|  \tag{31d}\\
& \leq M\left\|K e^{k}-V e^{k}\right\|+M\left\|V e^{k}-F_{\varepsilon}+F_{\varepsilon}^{*}\right\|  \tag{31e}\\
& \leq M\|K-V\|\left\|e^{k}\right\|+M\left\|V e^{k}-F_{\varepsilon}+F_{\varepsilon}^{*}\right\| \tag{31f}
\end{align*}
$$

where $M \geq\left\|K^{-1}(x)\right\|, \forall x \in U_{3}$, where $U_{3}$ is any compact neighbourhood of $x^{*}$. The existence of $M$ is guaranteed by the non-singularity of $K$ (Corollary 3). The first term in (31f) represents the error induced by regularization, using (28) we have the following bound

$$
\begin{equation*}
M\|K-V\|\left\|e^{k}\right\| \leq \gamma M \delta^{k}\left\|e^{k}\right\| \tag{31~g}
\end{equation*}
$$

For the second term, following [23] and [30], the strong semismoothness of $F_{\varepsilon}$ implies that there exists a neighbourhood $U_{1}$ of $x^{*}$ and a constant $T>0$ such that

$$
\begin{align*}
\left\|V e^{k}-F_{\varepsilon}+F_{\varepsilon}^{*}\right\| & \leq \sqrt{\sum_{i=1}^{n+q}\left\|V_{i} e^{k}-F_{\varepsilon, i}+F_{\varepsilon, i}^{*}\right\|^{2}}  \tag{31h}\\
& \leq T\left\|e^{k}\right\|^{2}, \quad \forall x \in U_{1} \tag{31i}
\end{align*}
$$

Combining 31f), 31h), 31g, that $\delta^{k} \leq\left\|F_{\varepsilon}(x)\right\|$ by the construction of FBRS (Step 3), and the Lipshitz continuity of $F$ (26) yields

$$
\begin{equation*}
\left\|e^{k+1}\right\| \leq M\left(\gamma L_{F}^{2}+T\right)\left\|e^{k}\right\|^{2} \quad \forall x \in U \tag{31j}
\end{equation*}
$$

where $U=U_{1} \cap U_{2} \cap U_{3}$, and $U_{2}, L_{F}=L_{F}\left(x^{*}\right)$ are the neighbourhood and Lipschitz constant in 26). Letting $\eta=M\left(\gamma L_{F}^{2}+T\right)$ completes the proof.

## D. Global convergence

This section provides a short proof of the global convergence properties of FBRS.
Lemma 1. Let $\left\{x^{k}\right\}$ and $\left\{\Delta x^{k}\right\}$ be generated by FBRS. Assume that $\theta_{\varepsilon}\left(x^{k}\right) \neq 0$ and $1-\gamma\left\|K^{-1}\right\| \delta^{k}>\sigma$, where $\gamma$ is the constant in 28. Then there exists a step length $t^{k} \in(0,1]$ such that the Armijo condition,

$$
\begin{equation*}
\theta_{\varepsilon}\left(x^{k}+t^{k} \Delta x^{k}\right) \leq\left(1-2 t^{k} \sigma\right) \theta_{\varepsilon}\left(x^{k}\right) \tag{32}
\end{equation*}
$$

is satisfied.
Proof. Consider a fixed but arbitrary iteration $k$; from this point forward we drop the iteration superscript to steamline the presentation of the proof. As $\theta_{\varepsilon}$ is continuously differentiable (see section VII-B) we can invoke the fundamental theorem of calculus to write that

$$
\begin{align*}
\theta_{\varepsilon}(x+t \Delta x) & =\theta_{\varepsilon}(x)+t \nabla_{x} \theta_{\varepsilon}(x)^{T} \Delta x \\
& +t \int_{0}^{1}\left[\nabla_{x} \theta(x+t \Delta x \lambda)-\nabla \theta_{\varepsilon}(x)\right]^{T} \Delta x d \lambda \tag{33a}
\end{align*}
$$

defining $\Delta \theta(t)=\theta_{\varepsilon}(x+t \Delta x)-\theta_{\varepsilon}(x)$ and using that $\nabla_{x} \theta_{\varepsilon}(x)=$ $V^{T} F(x)$, for any $V \in \partial_{C} F(x)$ (see section VII-B) yields that

$$
\begin{align*}
\Delta \theta(t) & =t \nabla_{x} \theta_{\varepsilon}^{T} \Delta x+t \int_{0}^{1}\left[\nabla_{x} \theta_{\varepsilon}(x+t \Delta x \lambda)-\nabla \theta_{\varepsilon}\right]^{T} \Delta x d \lambda  \tag{33b}\\
= & -t F^{T} V K^{-1} F+t \int_{0}^{1}\left[\nabla_{x} \theta_{\varepsilon}(x+t \Delta x \lambda)-\nabla \theta_{\varepsilon}\right]^{T} \Delta x d \lambda \tag{33c}
\end{align*}
$$

Substituting in $V K^{-1}=I+E K^{-1}$ yields

$$
\begin{align*}
\Delta \theta(t) & \leq-t\|F\|^{2}\left(1-\left\|E K^{-1}\right\|\right) \\
& +t \int_{0}^{1}\left[\nabla_{x} \theta_{\varepsilon}(x+t \Delta x \lambda)-\nabla \theta_{\varepsilon}\right]^{T} \Delta x d \lambda, \tag{33d}
\end{align*}
$$

rearranging the bound on $\|E\|$ 28), letting $M(\delta)=\left\|K^{-1}\right\|$, and taking norms of the remaining positive terms yields the following estimate

$$
\begin{align*}
\Delta \theta(t) & \leq-2 t \theta_{\varepsilon}(1-\gamma M \delta) \\
& +t \int_{0}^{1}\left\|\nabla_{x} \theta_{\varepsilon}(x+t \Delta x \lambda)-\nabla \theta_{\varepsilon}\right\|\|\Delta x\| d \lambda \tag{33e}
\end{align*}
$$

Since $\nabla_{\varepsilon} \theta$ is Lipschitz, see 30, letting $L_{\theta}$ be its Lipschitz constant and integrating we obtain that

$$
\begin{align*}
\Delta \theta(t) & \leq-2 t \theta_{\varepsilon}(1-\gamma M \delta)+t \int_{0}^{1} t L_{\theta}\|\Delta x\|^{2} d \lambda  \tag{33f}\\
& \leq-2 t \theta_{\varepsilon}(1-\gamma M \delta)+\frac{1}{2} t^{2} L_{\theta}\left\|K^{-1} F\right\|^{2}  \tag{33g}\\
& \leq-2 t \theta_{\varepsilon}(1-\gamma M \delta)+t^{2} L_{\theta} M^{2} \theta_{\varepsilon} \tag{33h}
\end{align*}
$$

From the last inequality we can conclude that there exists a sufficiently small $t$ such that the Armijo condition is satisfied, in particular any $t<\hat{t}$ where

$$
\begin{equation*}
\hat{t}=\frac{2(1-\gamma M \delta-\sigma)}{L_{\theta} M^{2}} \tag{34}
\end{equation*}
$$

will be accepted by the algorithm. Further, FBRS uses a backtracking line search with backtracking factor $\beta \in(0,1)$ so we can conclude that $t \geq \beta \hat{t}$ bounding $t$ away from zero.

Corollary 4. For all $x \in\left\{x^{k}\right\}$ there exists $\bar{\delta}$ such that $\Delta x$ will be a direction of sufficient decrease for $\theta_{\varepsilon}$ if $\delta<\bar{\delta}$.

Proof. A sufficient condition for $\Delta x$ to be a direction of sufficient descent for $\theta_{\varepsilon}$ is $1-\gamma M \delta=1-\gamma \delta\left\|(V-E(\delta))^{-1}\right\|>\sigma$. Since $V$ is always invertible (Theorem 11) and $K=V-E$ is invertible for any $\delta \geq 0$ (Corollary 3) then $\left\|(V-E)^{-1}\right\| \rightarrow\left\|V^{-1}\right\|$ as $\delta \rightarrow 0$ and $\Gamma(\delta)=$ $\gamma \delta\left\|(V-E)^{-1}\right\| \rightarrow 0$ as $\delta \rightarrow 0$. The existence of $\bar{\delta}$ then follows from the continuity of $\Gamma(\delta)$.

Theorem 3. Let the assumptions in section $\mathbb{1 \square}$ and Lemma $\square$ hold and let the sequence $\left\{x^{k}\right\}$ be generated by FBRS. Then for all initial points, $x^{0} \in \mathbb{R}^{n+q}$, the sequence $\left\{x^{k}\right\}$ is well defined and $\left\{x^{k}\right\} \rightarrow x^{*}$ as $k \rightarrow \infty$.

Proof. We begin by noting that, by Corollary 3 the iteration matrix $K(x, \varepsilon)$ is always non-singular; as a result the sequence $\left\{x^{k}\right\}$ generated by FBRS is unique and well defined for any initial condition.

Let $\Delta x\left(x^{k}, \delta^{k}\right)$ be generated by FBRS. Consider the merit function $\theta_{\varepsilon}$; if $\theta_{\varepsilon}(x)>0$ then if $\delta$ is chosen sufficiently small, which is always possible by Corollary 4 then $\Delta x$ will be a direction of sufficient descent for $\theta_{\varepsilon}$. Thus invoking Lemma we have that

$$
\begin{equation*}
\theta_{\varepsilon}\left(x^{k+1}\right)<\left(1-2 t_{k} \sigma\right) \theta_{\varepsilon}\left(x^{k}\right) \tag{35}
\end{equation*}
$$

as $t_{k} \in(0,1]$ and $\sigma \in(0,0.5)\left\{\theta_{\varepsilon}\left(x^{k}\right)\right\}$ is a strictly decreasing sequence. Since $\theta_{\varepsilon}$ is bounded from below by zero $\left\{\theta_{\varepsilon}\left(x^{k}\right)\right\}$ must converge to some $\theta^{*} \geq 0$ as $k \rightarrow \infty$ and, as $1-2 t_{k} \sigma<1$, we must have that $\theta^{*}=0$. Noting that $\theta_{\varepsilon}(x)=0$ if and only if $F_{\varepsilon}(x)=0$ and that $F_{\mathcal{E}}(x)=0$ if and only if $x=x^{*}$ completes the proof.

## E. Acceptance of unit steps

In this section we prove that once the iterates are sufficiently close to the solution then the linesearch will accept unit steps, allowing FBRS to recover the fast asymptotic convergence rates of Theorem 2
Theorem 4. Let the assumptions in section IIT hold and let $\left\{x^{k}\right\}$ and $\left\{\Delta x^{k}\right\}$ be generated by FBRS. Then there exists a neighbourhood $X$ of the solution $x^{*}$ such that $\theta_{\varepsilon}\left(x^{k}+\Delta x^{k}\right) \leq(1-2 \sigma) \theta_{\varepsilon}\left(x^{k}\right), \forall x \in X$, implying that the linesearch will accept unit steps.

Proof. Choose a fixed but arbitrary iteration $k$; from this point forward we drop the iteration superscript to steamline the presentation of the proof. Consider,

$$
\begin{equation*}
\theta_{\varepsilon}(x+\Delta x)=\frac{1}{2}\left\|F_{\mathcal{\varepsilon}}(x+\Delta x)-F_{\varepsilon}\left(x^{*}\right)\right\|^{2} \tag{36a}
\end{equation*}
$$

using the Lipshitz continuity of $F_{\varepsilon}$ and that, by Theorem 2 there exists $\eta>0$ such that $\left\|x+\Delta x-x^{*}\right\| \leq \eta\left\|x-x^{*}\right\|^{2}$ in some neighbourhood $U$ of $x^{*}$ we can conclude that

$$
\begin{align*}
\theta_{\varepsilon}(x+\Delta x) & \leq \frac{1}{2} L_{F}^{2}\left\|x+\Delta x-x^{*}\right\|^{2}  \tag{36b}\\
& \leq \frac{1}{2} L_{F}^{2} \eta\left\|x-x^{*}\right\|^{4} \tag{36c}
\end{align*}
$$

for all $x \in U$. The CD regularity of $F_{\mathcal{\varepsilon}}$, see section VII-A implies that there exists a neighbourhood $S$ of $x^{*}$ and $L_{I}>0$ such that $\left\|x-x^{*}\right\| \leq$ $L_{I}\left\|F_{\varepsilon}(x)\right\| \quad \forall x \in S$, thus

$$
\begin{align*}
\theta_{\varepsilon}(x+\Delta x) & \leq \frac{1}{2} L_{F}^{2} \eta L_{I}^{2}\left\|x-x^{*}\right\|^{2}\left\|F_{\varepsilon}(x)\right\|^{2}  \tag{36d}\\
& \leq L_{F}^{2} \eta L_{I}^{2}\left\|x-x^{*}\right\|^{2} \theta_{\varepsilon}(x), \tag{36e}
\end{align*}
$$

for all $x \in U \cap S$. By continuity of $\left\|x-x^{*}\right\|$ there then must exist $\hat{x}$ such that

$$
\begin{equation*}
L_{F}^{2} \eta L_{I}^{2}\left\|\hat{x}-x^{*}\right\|^{2}=1-2 \sigma \tag{36f}
\end{equation*}
$$

and thus we have that

$$
\begin{equation*}
\theta_{\varepsilon}(x+\Delta x) \leq(1-2 \sigma) \theta_{\varepsilon}(x) \tag{36~g}
\end{equation*}
$$

for all $x$ such that $\left\|x-x^{*}\right\| \leq\left\|\hat{x}-x^{*}\right\|$. Setting $X=\{x \in U \cap S \mid \| x-$ $\left.x^{*}\|\leq\| \hat{x}-x^{*} \|\right\}$ completes the proof.

## VIII. Conclusion

This paper presented a regularized and smoothed FischerBurmeister method for solving convex QPs. The method is attractive for real-time and embedded applications since its simple to code, easy to warmstart, and its performance is competitive with other state of the art solvers. Future work includes extending the method to more general convex problems e.g., SOCPs, and considering problems with non-unique dual solutions.

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[^1]:    ${ }^{1}$ Throughout this paper the application of the FB function to vectors i.e., $\phi_{\varepsilon}(x, y)$ for $x, y \in \mathbb{R}^{n}$, will be understood to be elementwise.

[^2]:    ${ }^{2}$ For all experiments performed on a laptop each QP in each sequence is solved 25 times and the measured execution time is averaged to attenuate variability in execution time caused by the operating system (OS).
    ${ }^{3}$ All experiments were performed on a 2015 i7 Macbook Pro with 16GB of memory running MATLAB 2015a SP1. FBRS, QPKWIK, PDIP, and GPAD were implemented in the MATLAB language and compiled into mex functions using MATLAB Coder. We implemented matrix factorization updating for QPKWIK. All second order methods were limited to 30 major iterations except for QPKWIK which used up to 500; GPAD was allowed up to 3000 iterations. FBRS, and PDIP were terminated when $\left\|F_{N R}\right\| \leq 10^{-4}$. The solution tolerances for QUADPROG, GPAD and ECOS were tuned until the solution errors, using the same metric, were of the same order of magnitude as the other methods. QPKWIK does not have an adjustable error tolerance, $10^{-4}$ was used as the constraint tolerance.

