Average Consensus by Graph Filtering: New Approach, Explicit Convergence Rate, and Optimal Design

Jing-Wen Yi , Li Chai , *Member, IEEE*, and Jingxin Zhang , *Member, IEEE*

Abstract—**This paper revisits the problem of multiagent consensus from a graph signal processing perspective. Describing a consensus protocol as a graph spectrum filter, we present an effective new approach to the analysis and design of consensus protocols in the graph spectrum domain for the uncertain networks, which are difficult to handle by the existing time-domain methods. This novel approach has led to the following new results: 1) explicit connection between the time-varying consensus protocol and the graph filter; 2) new necessary and sufficient conditions for both finite-time and asymptotic average consensus of multiagent systems (MASs); 3) direct link between the consensus convergence rate and periodic consensus protocols, and conversion of fast consensus problem to the polynomial design of the graph filter; 4) two explicit design methods of the periodic consensus protocols with a predictable convergence rate for MASs on uncertain graphs; and 5) explicit formulas for the convergence rate of designed protocols. Several numerical examples are given to demonstrate the validity, effectiveness, and advantages of these results.**

Index Terms—**Average consensus, graph filter, graph signal processing (GSP), multiagent systems (MASs).**

I. INTRODUCTION

CONSENSUS of multiagent systems (MASs) is a fundamental problem in collective behaviors of autonomous individuals, which has been extensively studied in the last decades [1]–[5]. The key problem is to design appropriate distributed protocols (control sequences) such that each agent only get information from its local neighbors, and the whole network of agents may coordinate to reach an agreement on certain quantities of interest eventually. Many results have been

published for MASs, considering different physical phenomena such as network communication delay [6], [7], switching topology [8], [9], communication channel noise [10], [11], quantized data [12], [13], and nonlinear dynamics [14], [15].

The convergence rate of consensus is crucial to practical applications. Many scholars have studied the fast consensus problem of MASs [16]–[22]. Xiao and Boyd [16], [17] cast the fastest distributed linear averaging and least-mean-square consensus problems into the optimal weight design problems to minimize the asymptotic convergence rate or the total mean-square deviation, and proposed computational methods to solve the corresponding convex optimization problems. Olfati-Saber [18] adopted the "random rewiring" procedure to increase the algebraic connectivity of small-world networks and solved the ultrafast consensus. Aysal *et al.* [19] accelerated the convergence rate of the distributed average consensus by changing the state update to a convex combination of the standard consensus iteration and a linear prediction. Erseghe *et al.* [20] utilized the alternating direction multipliers method to provide an effective indication on how to choose a network matrix for optimized consensus performance. Kokiopoulou and Frossard [21] applied a polynomial filter on the network matrix to shape its spectra and, hence, increase the convergence rate and used a semidefinite program to optimize the polynomial coefficients. Montijano *et al.* [22] proposed a fast and stable distributed algorithm based on the Chebyshev polynomials to solve the consensus problem and accelerate the convergence rate.

For fixed and known graph topologies, the MASs can reach consensus in finite time under delicately designed control strategies. Sundaram and Hadjicostis [23] presented a simple linear iteration to calculate the consensus value and proposed a finite-time consensus algorithm using the notion of the minimal polynomial of each node. Hendrickx *et al.* [24] utilized the matrix factorization approach to investigate the finite-time consensus and obtained the algebraic conditions for the minimum polynomial and eigenvalues of the weight matrix to satisfy the "definitive consensus conjecture." Kibangou [25] proposed the minimum polynomial approach to design finite-time average consensus protocols and showed that the smallest possible number of steps to reach consensus is equal to the diameter of the graph. Using the finite-time Lyapunov stability theory, the authors of [26]–[28] studied the finite-time consensus problem and provided some discontinuous or nonlinear consensus protocols.

Graph signal processing (GSP) has recently been emerged as a powerful new paradigm for high-dimensional data analysis and processing [29]–[34]. Fundamental operations, such as

graph Fourier transform, translation, modulation, filtering, and convolution, have been defined for graph signals based on the graph topology described by the adjacency or Laplacian matrix, and some effective design methods have been developed [29]–[35]. Some of these new techniques have been applied to the analysis and design of MAS consensus in the last few years.

Sandryhaila *et al.* [36] designed a matrix polynomial as a graph filter to solve the average consensus in finite time. By solving a semidefinite program, some approximation algorithms were obtained to guarantee the finite-time consensus. Segarra *et al.* [37] proposed an optimal design of graph filters to implement arbitrary linear transformations between graph signals and showed its application to the finite-time consensus and analog network coding. Izumi *et al.* [38] showed that the multiagent consensus corresponds to the low-pass filtering of the graph signal and designed a low-pass filter by polynomial approximation of an exponential function. These results have provided new insights into MASs and opened a new avenue to solving the consensus problem of MASs.

Despite the excellent works discussed above, there are still some fundamental problems unsolved in MAS consensus.

1) The explicit connection between the graph filter and the time-varying control protocol has not been revealed.
2) The designs of graph filters for MAS consensus are only for the MASs with known graph spectra and are mostly numerical and approximate, lacking analytical and explicit solutions.
3) The analysis and design of control protocols are still limited. There are few necessary and sufficient conditions on the average consensus of MASs.
4) The convergence rate of consensus protocols is yet to be fully understood.
5) The Lyapunov-function-based analysis of finite-time consensus only provides sufficient conditions that can be conservative, and the resulting discontinuous or nonlinear consensus protocols are difficult to realize.

To address the above problems, in this paper, we study systematically the consensus problem using the recent results of GSP theory and present the following new results:

1) the explicit connection between the general time-varying control protocol and the graph filter;
2) the new necessary and sufficient conditions for both finite-time and asymptotic average consensus, in terms of the corresponding graph filter of control sequence. These conditions are simple and encompass, as special cases, the MAS consensus under the constant control sequence [1], [16], [17], the MAS consensus under the time-varying control sequence [8], [39], and the finite-time consensus [24], [25], [36], [38];
3) the direct link between the exact convergence rate of consensus and the maximum magnitude of the graph filter of an M-periodic control sequence. This result converts the fast consensus problem to a polynomial design problem of graph filter and gives the exact convergence rate instead of the upper bound of the convergence rate given in the existing literature;
4) for any uncertain graph $\mathcal{G}$, with the second smallest eigenvalue λ_2 and the largest eigenvalue λ_N of the graph Laplacian $[\lambda_2, \lambda_N] \subseteq [\alpha, \beta]$, the consensus can be achieved

by simply using the control sequence $\varepsilon(jM + k) = \frac{1}{r_k}$, $k = 0, \ldots, M - 1, j \in \mathbb{N}$, with r_k distributed uniformly on $[\alpha, \beta]$. The upper bound of the convergence rate of this protocol is smaller than that of the existing result in [16];
5) the explicit formulas for designing the unique M-periodic control sequence $\{\varepsilon^*(k + jM)\}$ to attain the optimal worst-case convergence rate for any uncertain graph $\mathcal{G}$ with $[\lambda_2, \lambda_N] \subseteq [\alpha, \beta]$, the explicit formula for calculating the optimal worst-case convergence rate resulting from $\{\varepsilon^*(k + jM)\}$, and the explicit formula for calculating the worst-case convergence rate under the general time-varying control sequences. All the formulas are analytical and the design and calculations are precise, without the iterative approximation used in the existing literature.

Part of these results was presented in [45] as a conference paper without full proof and interpretation. This paper is a substantial extension of [45] with extended new results, full proof and interpretation, and extensive new examples.

The rest of this paper is organized as follows. Section II introduces some background about spectral graph theory and GSP. Section III presents necessary and sufficient conditions for average consensus from a graph signal filtering perspective. Upon casting the problem of average consensus to a graph filter design, Section IV investigates the convergence rate of asymptotic consensus on the known networks under periodic control sequences. For the uncertain networks, Section V presents a Lagrange polynomial (LP) interpolation method to design the periodic control sequences and compares the convergence performance with the existing result. To pursue the optimal consensus on uncertain networks, Section VI proposes a worst-case optimal (WO) interpolation method to obtain the explicit formulas for the design and convergence rate evaluation of periodic control sequences. The validity and performance of the proposed methods are demonstrated in Section VII by extensive simulation and numerical experiment results. Section VIII concludes this paper with remarks on the presented results and future work.

II. PRELIMINARIES

A. Spectral Graph Theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted undirected graph with the set of vertices $\mathcal{V} = \{\nu_1, \nu_2, \ldots, \nu_N\}$, the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$. The vertex indices belong to a finite index set $\mathcal{I} = \{1, 2, \ldots, N\}$. An edge of $\mathcal{G}$ is denoted by $e_{ij} = (\nu_i, \nu_j) \in \mathcal{E}$ if and only if there exist information exchanges between vertex ν_i and vertex ν_j. The adjacency elements corresponding to the edges of the graph are positive, i.e., $e_{ij} \in \mathcal{E} \Leftrightarrow a_{ij} = a_{ji} > 0$. Assume $a_{ii} = 0$ for all $i \in \mathcal{I}$. The set of neighbors of vertex ν_i is denoted by $\mathcal{N}_i = \{\nu_j \in \mathcal{V} : (\nu_i, \nu_j) \in \mathcal{E}\}$. For any $\nu_i, \nu_j \in \mathcal{V}, a_{ij} > 0 \Leftrightarrow j \in \mathcal{N}_i$. The degree of vertex ν_i is represented by $d_i = \sum_{j=1}^{N} a_{ij}$. The Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} := \text{diag}\{d_1, \ldots, d_N\}$.

Lemma 1 (see[40]): For an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, the Laplacian matrix $\mathcal{L}$ satisfies the following properties.

1) $\mathcal{L}$ is symmetric matrix and positive semidefinite.
2) All the eigenvalues of $\mathcal{L}$ are real in an ascending order as

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \leq 2\bar{d} \tag{1}$$

where $\bar{d} = \max_i\{d_i\}$ is the maximum degree of the graph.

3) $\mathcal{L}$ has the following singular value decomposition:

$$\mathcal{L} = V\Lambda V^T \qquad (2)$$

where $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, and $V = [v_1, v_2, \ldots, v_N] \in \mathbb{R}^{N \times N}$ is an unitary matrix.

4) Zero is a simple eigenvalue of $\mathcal{L}$ if and only if $\mathcal{G}$ is connected, and the associated eigenvector is $v_1 = \frac{1}{\sqrt{N}}\vec{1}$, where $\vec{1} \in \mathbb{R}^N$ is the vector of all ones.

B. Graph Signal Processing

Consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with Laplacian matrix $\mathcal{L}$; the graph signal x is the collection of the signal values on the vertices of the graph, i.e., $x = [x_1, x_2, \ldots, x_N]^T \in \mathbb{R}^N$. The graph Fourier transform $\hat{x}$ of $x \in \mathbb{R}^N$ on $\mathcal{G}$ is defined as the expansion of x in terms of eigenfunctions of the graph Laplacian [29], [30], [32]

$$\hat{x}_{\lambda_i} := \langle x, v_i \rangle = \sum_{\ell=1}^{N} x_\ell v_i^{(\ell)} \qquad (3)$$

where $\{v_i\}_{1,2,\ldots,N}$ are orthonormal eigenvectors of $\mathcal{L}$. The inverse graph Fourier transform is then given by

$$x_\ell = \sum_{i=1}^{N} \hat{x}_{\lambda_i} v_i^{(\ell)}. \qquad (4)$$

Note that $\hat{x} = [\hat{x}_{\lambda_1}, \hat{x}_{\lambda_2}, \ldots, \hat{x}_{\lambda_N}]^T = V^T x$, where V is the unitary matrix defined in (2). The inverse graph Fourier transform is then given by $x = V\hat{x}$. In graph Fourier analysis, the graph Laplacian eigenvalues correspond to the frequency in the spatial frequency domain. The eigenvectors associated with smaller eigenvalues (low frequency) vary slowly across the graph, while those with larger eigenvalues oscillate more rapidly.

Denote a graph spectral filter $h(\cdot)$ as a real-valued function on the spectrum of graph Laplacian; the graph spectral filtering is defined as [32]

$$\hat{y}_{\lambda_i} = h(\lambda_i)\hat{x}_{\lambda_i} \qquad (5)$$

and $\hat{y} = [\hat{y}_{\lambda_1}, \hat{y}_{\lambda_2}, \ldots, \hat{y}_{\lambda_N}]^T$ is the filtered graph signal represented in the frequency domain.

Taking the inverse graph Fourier transform, we have

$$y = V \begin{bmatrix} h(\lambda_1) & & \\ & \ddots & \\ & & h(\lambda_N) \end{bmatrix} V^T x \qquad (6)$$

where $y = [y_1, y_2, \ldots, y_N]^T$ is the filtered graph signal in the spatial domain. When the graph spectral filter (5) is an Mth-order polynomial $h(\lambda_i) = \sum_{k=0}^{M} b_k \lambda_i^k$, where $\{b_k\}_{k=0,1,\ldots,M}$ are real coefficients, the filtered signal y_ℓ at vertex ℓ is a linear combination of the components of the input signal at agents within an M-hop local neighborhood of agent ℓ [32]

$$y_\ell = \sum_{i=1}^{N} h(\lambda_i)\hat{x}_{\lambda_i} v_i^{(\ell)} = \sum_{j=1}^{N} x_j \sum_{k=0}^{M} b_k (\mathcal{L}^k)_{\ell,j}. \qquad (7)$$

III. GRAPH-FILTER-BASED NECESSARY AND SUFFICIENT CONDITIONS FOR AVERAGE CONSENSUS

This section presents the explicit connection between general time-varying consensus protocol and graph filtering and uses this connection to derive the necessary and sufficient conditions of average consensus in terms of a graph filter.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be an undirected graph of N nodes with the adjacency matrix $\mathcal{A} = [a_{ij}]_{N \times N}$. Suppose each vertex of the graph is an agent described by

$$x_i(k+1) = x_i(k) + u_i(k), \quad i = 1, \ldots, N \qquad (8)$$

where $x_i(k) \in \mathbb{R}$ is the state and $u_i(k) \in \mathbb{R}$ is the control input. We consider a general time-varying control protocol

$$u_i(k) = \varepsilon(k) \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(k) - x_i(k)) \qquad (9)$$

where $\varepsilon(k) > 0$ is the control gain at time k, and denote $x(k) = [x_1(k), \ldots, x_N(k)]^T \in \mathbb{R}^N$.

Lemma 2: The control protocol (9) for the MAS (8) functions as a graph filter of the system initial state $x(0)$, and the corresponding graph filter can be written as

$$h(\lambda, T) := \prod_{k=0}^{T-1}(1 - \varepsilon(k)\lambda) \qquad (10)$$

where λ is the eigenvalue of the graph Laplacian matrix $\mathcal{L}$.

Proof: It follows from (8) and (9) that

$$x(T) = (I - \varepsilon(T-1)\mathcal{L})x(T-1)$$
$$= \prod_{k=0}^{T-1}(I - \varepsilon(k)\mathcal{L})x(0) \qquad (11)$$

where I is the identity matrix and $\mathcal{L} = \mathcal{D} - \mathcal{A}$ is the graph Laplacian matrix. From (2), (11) can be written as

$$x(T) = V\left(\prod_{k=0}^{T-1}(I - \varepsilon(k)\Lambda)\right)V^T x(0)$$

with $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. Using (10), we get

$$x(T) = V\text{diag}\{h(\lambda_1, T), \ldots, h(\lambda_N, T)\}V^T x(0). \qquad (12)$$

Comparing (12) with (6), it is obvious that $x(T)$ can be viewed as the output of the graph filter $h(\lambda, T)$ with the input $x(0)$, and the $h(\lambda, T)$ given in (10) is the corresponding graph filter of the control protocol (9) that does the filtering. ∎

Lemma 2 is an extension of [38, Th. 1] for the constant control protocol to the general time-varying control protocol (9). It has revealed in (10) the explicit connection between the general time-varying control protocol and the graph filter, which plays a key role in the analysis of MAS dynamics. Theorem 1 shows that it characterizes the necessary and sufficient conditions of consensus. To present the theorem, the following definition is introduced.

Definition 1: The average consensus of MAS (8) under the control protocol (9) is said to be reached asymptotically if

$$\lim_{k \to \infty} x_i(k) = \frac{1}{N}\sum_{j=1}^{N} x_j(0) := \bar{x}(0), \quad i = 1, 2, \ldots, N$$

for any initial state $x(0) \in \mathbb{R}^N$. The average consensus is said to be reached at time T if

$$x_i(k) = \bar{x}(0), \quad i = 1, 2, \ldots, N$$

for any $k \geq T$.

Theorem 1: For the MAS (8) on a connected graph $\mathcal{G}$ under the control sequence $\varepsilon(k)$ in (9), let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$ be the eigenvalues of the graph Laplacian matrix and $h(\lambda, T)$ be the corresponding graph filter defined in (10). Then, we have the following.

i) The MAS reaches average consensus at time $T < \infty$ if and only if $h(\lambda_i, T) = 0$ for $i = 2, \ldots, N$.

ii) The MAS reaches average consensus asymptotically if and only if

$$h(\lambda_i, \infty) := \lim_{T \to \infty} h(\lambda_i, T)$$
$$= \lim_{T \to \infty} \left\{ \prod_{k=0}^{T-1} (1 - \varepsilon(k)\lambda_i) \right\} = 0 \quad (13)$$

for $i = 2, \ldots, N$.

iii) Assume that the control sequence satisfies $\lim_{k \to \infty} \varepsilon(k) = 0$, and the MAS cannot reach consensus in finite time. Then, the consensus is reached asymptotically if and only if $\sum_{k=0}^{\infty} \varepsilon(k) = \infty$.

Proof: i) Using (12) with $\lambda_1 = 0$ and the fact that V is unitary and $\mathrm{v}_1 = \frac{1}{\sqrt{N}} \vec{1}$, we have

$$x(T) = \mathrm{V diag}\{1, h(\lambda_2, T), \ldots, h(\lambda_N, T)\} V^T x(0)$$
$$= \bar{x}(0)\vec{1} + h(\lambda_2, T)\mathrm{v}_2 \mathrm{v}_2^T x(0)$$
$$+ \cdots + h(\lambda_N, T)\mathrm{v}_N \mathrm{v}_N^T x(0). \quad (14)$$

The "if" part is obvious from (14). Now, suppose $h(\lambda_i, T) \neq 0$ for some i. Setting $x(0) = \mathrm{v}_i$, we know from the orthogonality $\vec{1}^T \mathrm{v}_i = 0$ that the average of v_i is zero. However, it follows from (14) that $x(T) = h(\lambda_i, T)\mathrm{v}_i$ is not zero. This contradiction proves the "only if" part.

ii) It follows from (14) that

$$\lim_{T \to \infty} x(T) = \bar{x}(0)\vec{1}$$
$$+ \lim_{T \to \infty} \left(h(\lambda_2, T)\mathrm{v}_2 \mathrm{v}_2^T + \cdots + h(\lambda_N, T)\mathrm{v}_N \mathrm{v}_N^T \right) x(0).$$
$$(15)$$

Then, the "if" part follows from (15) and the "only if" part can be proved by contradiction similar to that in (i).

iii) It follows from $\lim_{k \to \infty} \varepsilon(k) = 0$ that there exists $K > 0$ such that $\varepsilon(k)\lambda_N < 1$ for any $k \geq K$. Then, we have $0 < \varepsilon(k)\lambda_i < 1$ for $2 \leq i \leq N$ and $k \geq K$. Let $z(k) = \varepsilon(k)\lambda$. Then, using the fact that $1 - z(k) < e^{-z(k)}$ for any $z(k) > 0$ and the assumption $\sum_{k=0}^{\infty} \varepsilon(k) = \infty$, we have

$$\prod_{k=K}^{\infty} (1 - \varepsilon(k)\lambda) \leq e^{-(\lambda \sum_{k=K}^{\infty} \varepsilon(k))} = 0. \quad (16)$$

Therefore, $h(\lambda_i, \infty) = 0$ for $i = 2, \ldots, N$, and according to (ii), the MAS reaches average consensus asymptotically.

We use contradiction to prove the "only if" part. Suppose that the MAS system reaches average consensus asymptotically but not at any finite time under a control protocol satisfying $\sum_{k=0}^{\infty} \varepsilon(k) < \infty$. Note that $\varepsilon(k) > 0$ and $\sum_{k=0}^{\infty} \varepsilon(k) < \infty$ imply that $\lim_{K \to \infty} \sum_{k=K}^{\infty} \varepsilon(k) = 0$. Hence, there exists an integer $K_2 > 0$ such that $\sum_{k=K_2}^{\infty} \varepsilon(k) < \frac{1}{2\lambda_N}$. Then, for $0 < \lambda \leq \lambda_N$, we have

$$\prod_{k=K_2}^{\infty} (1 - \varepsilon(k)\lambda) > 1 - \sum_{k=K_2}^{\infty} \varepsilon(k)\lambda > \frac{1}{2}.$$

Since the system cannot reach finite-time average consensus, we know from (i) that there exists an $i, 2 \leq i \leq N$, such that $h(\lambda_i, K_2) \neq 0$. Then, $h(\lambda_i, \infty) = h(\lambda_i, K_2) \prod_{k=K_2}^{\infty} (1 - \varepsilon(k)\lambda_i) \neq 0$. This contradicts the result of (ii) that $h(\lambda_i, \infty) = 0$ for $i = 2, \ldots, N$, since it is supposed that the system reaches consensus asymptotically. ∎

Remark 1: Theorem 1 shows that the MAS reaches consensus if and only if the graph filter $h(\lambda, T)$ (or $h(\lambda, \infty)$) is a "low-pass" filter, which annihilates "high-frequency" components at $\lambda_2, \ldots, \lambda_N$. In particular, we can set $\varepsilon(k) = \frac{1}{\lambda_{k+2}}$, $k = 0, \ldots, N - 2$, to achieve consensus at time $N - 1$, since the corresponding graph filter $h(\lambda, N-1) = \prod_{k=0}^{N-2} \left(1 - \frac{\lambda}{\lambda_{k+2}}\right) = 0$ at $\lambda_i, i = 2, \ldots, N$.

Remark 2: The infinite product $h(\lambda_i, \infty) = 0$ is called "diverge to zero," which isolates the case there is an exact zero term. The infinite product $h(\lambda_i, \infty) = 0$ not only provides novel insights into the asymptotic consensus, but also turns out to be instrumental in the analysis of convergence rate, as shown in the next section. The Lyapunov-based methods for asymptotic consensus usually provide sufficient conditions only [8]–[10], which are somewhat conservative. To our knowledge, (13) is the simplest necessary and sufficient condition for asymptotic consensus that has established the direct link between the control sequence and the graph topology.

Remark 3: Time-varying control protocols with $\lim_{k \to \infty} \varepsilon(k) = 0$ are used to address the time-varying topologies and stochastic communication noises in [8], where $\sum_{k=0}^{\infty} \varepsilon(k) = \infty$ is an assumption. Here, we show that it is a necessary condition in the fixed topology and noise-free scenarios.

When the Laplacian matrix $\mathcal{L}$ has multiple eigenvalues, the consensus time can be shorter than $N - 1$. This is stated in the following corollary, which follows directly from Theorem 1(i).

Corollary 1: For a connected graph $\mathcal{G}$ with N vertices, assume that its Laplacian matrix has K distinct nonzero eigenvalues $\lambda_{p_k}, k = 0, \ldots, K - 1$. Then, K is the minimum time for the MAS to reach consensus. Moreover, by applying

$$\varepsilon(k) = \frac{1}{\lambda_{p_k}}, \quad 0 \leq k \leq K - 1 \quad (17)$$

the MAS (8) can reach average consensus at time K to obtain $x(K) = \bar{x}(0)\vec{1}$.

Based on Corollary 1, we summarize below some results of finite-time consensus for the MAS on special graphs.

1) For a complete graph $\mathcal{G}$ with N vertices, the eigenvalues of its Laplacian matrix are

$$\lambda_i = \begin{cases} 0, & i = 1 \\ N, & 2 \leq i \leq N \end{cases}.$$

Hence, the MAS can reach consensus at the finite time $T = 1$ by choosing $\varepsilon(0) = \frac{1}{N}$, that is,

$$u_i(0) = \frac{1}{N} \sum_{j \in N_i} a_{ij}(x_j(0) - x_i(0)).$$

2) For a complete bipartite graph $\mathcal{G}$ with $M + N$ vertices, the eigenvalues of its Laplacian matrix are

$$\lambda_i = \begin{cases} 0, \ i = 1 \\ M, \ 2 \le i \le N \\ N, \ N + 1 \le i \le M + N - 1 \\ M + N, \ i = M + N \end{cases}.$$

Hence, the MAS can reach consensus at the finite time $T = 3$ by choosing the consensus protocol as follows:

$$u_i(0) = \frac{1}{M} \sum_{j \in N_i} a_{ij}(x_j(0) - x_i(0))$$

$$u_i(1) = \frac{1}{N} \sum_{j \in N_i} a_{ij}(x_j(1) - x_i(1))$$

$$u_i(2) = \frac{1}{M + N} \sum_{j \in N_i} a_{ij}(x_j(2) - x_i(3)).$$

3) For a star graph $\mathcal{G}$ with N vertices, the eigenvalues of its Laplacian matrix are

$$\lambda_i = \begin{cases} 0, \ i = 1 \\ 1, \ 2 \le i \le N - 1 \\ N, \ i = N \end{cases}.$$

Hence, the MAS can reach consensus at the finite time $T = 2$ by choosing the consensus protocol as

$$u_i(0) = \sum_{j \in N_i} a_{ij}(x_j(0) - x_i(0))$$

$$u_i(1) = \frac{1}{N} \sum_{j \in N_i} a_{ij}(x_j(1) - x_i(1)).$$

4) For a cycle graph $\mathcal{G}$ with N vertices, the eigenvalues of its Laplacian matrix are $\lambda_1 = 0$, $\lambda_i = 2 - 2 \cos \frac{2\pi(i-1)}{N}$, $i = 2, \ldots, \text{floor}(\frac{N+1}{2})$ with multiplicity 2, and $\lambda_N = 4$ with multiplicity 1 if N is even. Hence, the MAS can reach consensus at the finite time $T = \text{ceil}(\frac{N-1}{2})$ by choosing $\varepsilon(k) = \frac{1}{\lambda_{k+2}}, k = 0, \ldots, \text{floor}(\frac{N+1}{2}) - 2$, and $\varepsilon(\frac{N}{2} - 1) = \frac{1}{4}$ if N is even.

5) For a path graph $\mathcal{G}$ with N vertices, the eigenvalues of its Laplacian matrix are $\lambda_1 = 0$, $\lambda_i = 2 - 2 \cos \frac{\pi(i-1)}{N}$, $i = 2, \ldots, N$. Hence, the MAS can reach consensus at the finite time $T = N - 1$ by choosing $\varepsilon(k) = \frac{1}{\lambda_{k+2}}, k = 0, \ldots, N - 2$.

Remark 4: The fact that finite-time consensus can be achieved by choosing control gains equal to the reciprocal of eigenvalues of the Laplacian matrix is not new. It has been obtained by using different methods, for example, matrix factorization method [24], minimal polynomial method [25], and

graph filter method [36]. However, by using the notion of the graph filter $h(\lambda, T)$, Theorem 1(i) and Corollary 1 have established the explicit connection between the average consensus and the graph filter. That is, average consensus is reached if and only if $h(\lambda, T) = 0$ at $\lambda_2, \ldots, \lambda_N$.

Corollary 1 has shown that finite-time consensus can be achieved by the control strategy (17) using the eigenvalue information of the graph Laplacian matrix. This result not only gives a new interpretation of average consensus, but also provides a systematic method to characterize the convergence rate explicitly. In particular, we can show that the exact convergence rate is equal to $\rho = \max_{\{\lambda_i\}} |h(\lambda_i, \cdot)|$ for periodic control sequences. Following this, we can further convert the fast consensus design problem to a polynomial interpolation problem, which can be readily solved by various methods. The polynomial interpolation problem will be discussed in Section V in detail.

IV. EXACT CONVERGENCE RATE OF PERIODIC CONTROL SEQUENCES

This section analyzes the convergence rate of asymptotic consensus under the M-periodic control sequence $\varepsilon(k + M) = \varepsilon(k), \forall k \in \mathbb{Z}$, with the aid of the following definitions.

For an M-periodic control sequence $\varepsilon(k)$, define

$$h(\lambda, M) = \prod_{k=0}^{M-1} (1 - \varepsilon(k)\lambda) \tag{18}$$

as its corresponding graph filter, $e(k) := x(k) - \bar{x}(0)\vec{1}$ as its error of consensus at time k, and

$$\rho_M := \sup_{\|x(0)\|_2 = 1} \frac{\|e(M)\|_2}{\|e(0)\|_2} \tag{19}$$

as its per-period convergence rate, which is an extension of per-step convergence rate in [16].

Theorem 2: For the MAS (8) on a connected graph $\mathcal{G}$ under the control of an M-periodic control sequence $\varepsilon(k)$, let $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N$ be the eigenvalues of the graph Laplacian matrix and $h(\lambda, M)$ be as given by (18). Then, the MAS reaches consensus asymptotically if and only if the exact convergence rate $\rho^* < 1$, where

$$\rho^* := \max_{\{\lambda_i\}} |h(\lambda_i, M)|. \tag{20}$$

Moreover, the per-period convergence rate ρ_M equals the exact convergence rate ρ^*, that is,

$$\rho_M = \sup_{\|x(0)\|_2 = 1} \frac{\|e(M)\|_2}{\|e(0)\|_2} = \max_{\{\lambda_i\}} |h(\lambda_i, M)| = \rho^*. \tag{21}$$

Proof: Since $\varepsilon(k)$ is M-periodic

$$h(\lambda, jM) = \prod_{k=0}^{jM-1} (1 - \varepsilon(k)\lambda) = \left(\prod_{k=0}^{M-1} (1 - \varepsilon(k)\lambda) \right)^j$$

$$= (h(\lambda, M))^j. \tag{22}$$

From $|h(\lambda_i, jM)| \le (\rho^*)^j$ and $\rho^* < 1$, $\lim_{j \to \infty} h(\lambda_i, jM + l) = 0$ for $l = 0, \ldots, M - 1, i = 2, \ldots, N$. Then, the average consensus is guaranteed by the "if" part of Theorem 1(ii). Contrarily, if the MAS reaches average consensus, then by Theorem 1(ii), $\lim_{j \to \infty} h(\lambda_i, jM) = \lim_{j \to \infty} (h(\lambda_i, M))^j = 0$

for $i = 2, \ldots, N$. This holds only if $\rho^* = \max_{\{\lambda_i\}} |h(\lambda_i, M)| < 1$.

From (14) and the unitariness of V, we have

$$
\begin{aligned}
\rho_M &= \sup_{\|x(0)\|_2 = 1} \frac{\left\| x(M) - \bar{x}(0)\vec{1} \right\|_2}{\left\| x(0) - \bar{x}(0)\vec{1} \right\|_2} \\
&= \sup_{\|x(0)\|_2 = 1} \frac{\left\| \sum_{i=2}^{N} h(\lambda_i, M)\mathrm{v}_i \mathrm{v}_i^T x(0) \right\|_2}{\left\| \sum_{i=2}^{N} \mathrm{v}_i \mathrm{v}_i^T x(0) \right\|_2} \\
&\leq \sup_{\|x(0)\|_2 = 1} \frac{\max_{\{\lambda_i\}} |h(\lambda_i, M)| \left\| \sum_{i=2}^{N} \mathrm{v}_i \mathrm{v}_i^T x(0) \right\|_2}{\left\| \sum_{i=2}^{N} \mathrm{v}_i \mathrm{v}_i^T x(0) \right\|_2} \\
&= \max_{\{\lambda_i\}} |h(\lambda_i, M)| = \rho^*.
\end{aligned}
\tag{23}
$$

Assume that λ_j is the eigenvalue such that $|h(\lambda_j, M)| = \max_{\{\lambda_i\}} |h(\lambda_i, M)|$. Setting $x(0) = \mathrm{v}_j$, we have $\|\mathrm{v}_j\|_2 = 1$, $\mathrm{v}_j \vec{1} = 0$, $x(M) = h(\lambda_j, M)\mathrm{v}_j$, and

$$
\frac{\|e(M)\|_2}{\|e(0)\|_2} = \frac{\|h(\lambda_j, M)\mathrm{v}_j\|_2}{\|\mathrm{v}_j\|_2} = |h(\lambda_j, M)|.
\tag{24}
$$

It then follows from the definition (19) that

$$
\rho_M \geq |h(\lambda_j, M)|.
\tag{25}
$$

Combining (23) and (25) proves (21). ∎

Remark 5: The ρ_M in (21) represents the exact convergence rate in the sense that $\rho_M \leq \gamma$ for any γ satisfying $\|e(jM)\|_2 \leq \gamma^j \|e(0)\|_2$, whereas many results based on the Lyapunov function [8], [9], [11], [12] can only give the upper bound of the convergence rate, which is usually much larger than ρ_M.

Remark 6: Theorem 2 turns the problem of finding a faster consensus algorithm to the problem of designing a polynomial $h(\lambda, M)$ such that ρ_M (or ρ^*) in (21) is small. Various polynomial interpolation techniques can be employed to solve this problem, as detailed in Sections V and VI.

V. CONSENSUS ON UNCERTAIN NETWORKS BY LP INTERPOLATION

In practical applications, it is usually difficult, if not impossible, to obtain the exact eigenvalues of the Laplacian matrix, especially when the network is large and complex. However, there are efficient methods to estimate λ_2 and λ_N [41]. It is, therefore, sensible to consider the MAS on an uncertain graph in $\{\mathcal{G}\}_{[\alpha,\beta]}$, where $\{\mathcal{G}\}_{[\alpha,\beta]}$ is the set of all connected graphs with $[\lambda_2, \lambda_N] \subseteq [\alpha, \beta]$, α the lower bound of the algebraic connectivity, and β the upper bound of the Laplacian spectral radius.

Define the worst-case convergence rate as

$$
\gamma_M := \sup_{\{\mathcal{G}\}_{[\alpha,\beta]}} \rho_M
\tag{26}
$$

where ρ_M is the per-period convergence rate defined in (19). Then, the following lemma is immediate.

Lemma 3: The worst-case convergence rate satisfies

$$
\gamma_M = \sup_{\{\mathcal{G}\}_{[\alpha,\beta]}} \rho^* = \max_{\lambda \in [\alpha,\beta]} |h(\lambda, M)|
\tag{27}
$$

where ρ^* is the exact convergence rate defined in (20).

Since the eigenvalues of graph Laplacian are unknown, we may naively assume them distributed uniformly on $[\alpha, \beta]$ and hence choose an M-periodic control sequence by distributing $\frac{1}{\varepsilon(k)}$, $k = 0, 1, \ldots, M - 1$, uniformly on $[\alpha, \beta]$. The graph filter can then be obtained by LP interpolation as described in the following.

Choose sequentially M points in $[\alpha, \beta]$ with equal distance

$$
r_k = \alpha + \frac{\beta - \alpha}{M + 1}(k + 1), \quad k = 0, \ldots, M - 1.
\tag{28}
$$

Set the M-periodic control sequence as

$$
\varepsilon(k + jM) = \frac{1}{r_k} = \frac{1}{\alpha + \frac{\beta - \alpha}{M + 1}(k + 1)}
\tag{29}
$$

for $k = 0, 1, \ldots, M - 1, j = 0, 1, \ldots$, which gives the corresponding graph filter in the form of LP

$$
h(\lambda, M) = \prod_{k=0}^{M-1} \left(1 - \frac{\lambda}{r_k} \right).
\tag{30}
$$

Theorem 3 shows that the M-periodic control sequence (29) does provide consensus. It also presents an explicit expression of the worst-case convergence rate ρ_M, which asserts the better performance of (29) as compared to the existing result [16].

Theorem 3: For any MAS (8) on a connected graph $\mathcal{G} \in \{\mathcal{G}\}_{[\alpha,\beta]}$, let the control sequence be given by (29) with the corresponding graph filter (30) and r_k given by (28). Then, the MAS reaches average consensus, and the per-period convergence rate satisfies $\rho_M \leq \gamma_M < 1$, where

$$
\gamma_M = \frac{M!}{\prod_{k=1}^{M}(k + \frac{M+1}{\beta-\alpha}\alpha)}.
\tag{31}
$$

Proof: For any $\lambda \in [\alpha, \beta]$, there exists an integer $i, 0 \leq i \leq M$, such that $\lambda \in [\alpha + \frac{\beta-\alpha}{M+1}i, \alpha + \frac{\beta-\alpha}{M+1}(i+1)]$. From (30), we have

$$
\begin{aligned}
&|h(\lambda, M)| \\
&= \prod_{k=0}^{i-1} \left(\frac{\lambda}{r_k} - 1 \right) \cdot \prod_{k=i}^{M-1} \left(1 - \frac{\lambda}{r_k} \right) \\
&\leq \prod_{k=0}^{i-1} \frac{\alpha + \frac{\beta-\alpha}{M+1}(i+1) - \left(\alpha + \frac{\beta-\alpha}{M+1}(k+1) \right)}{\alpha + \frac{\beta-\alpha}{M+1}(k+1)} \\
&\quad \cdot \prod_{k=i}^{M-1} \frac{\alpha + \frac{\beta-\alpha}{M+1}(k+1) - \left(\alpha + \frac{\beta-\alpha}{M+1}i \right)}{\alpha + \frac{\beta-\alpha}{M+1}(k+1)} \\
&= \prod_{k=0}^{i-1} \frac{\frac{\beta-\alpha}{M+1}(i-k)}{\alpha + \frac{\beta-\alpha}{M+1}(k+1)} \prod_{k=i}^{M-1} \frac{\frac{\beta-\alpha}{M+1}(k+1-i)}{\alpha + \frac{\beta-\alpha}{M+1}(k+1)k} \\
&= \prod_{k=0}^{i-1} \frac{(i-k)}{(k+1+\frac{M+1}{\beta-\alpha}\alpha)} \prod_{k=i}^{M-1} \frac{(k+1-i)}{(k+1+\frac{M+1}{\beta-\alpha}\alpha)} \\
&= \frac{i!(M-i)!}{\prod_{k=1}^{M}(k + \frac{M+1}{\beta-\alpha}\alpha)} \leq \frac{M!}{\prod_{k=1}^{M}(k + \frac{M+1}{\beta-\alpha}\alpha)} = \gamma_M
\end{aligned}
$$

and the equality holds if and only if $\lambda = \alpha$ or $\lambda = \beta$. It then follows from (21) that

$$\rho_M = \max_{\{\lambda_i\}} |h(\lambda_i, M)| \le \max_{\lambda \in [\alpha, \beta]} |h(\lambda, M)|$$

$$\le \frac{M!}{\prod_{k=1}^{M} (k + \frac{M+1}{\beta - \alpha}\alpha)} = \gamma_M$$

and $\gamma_M < 1$ since $\frac{M+1}{\beta - \alpha}\alpha > 0$. By Theorem 2, the MAS reaches average consensus asymptotically. ∎

Remark 7: In [16], it is proved that the fastest convergence rate by constant control sequence is $\rho = \frac{\beta - \alpha}{\beta + \alpha}$, with $\varepsilon(k) \equiv \varepsilon_c = \frac{2}{\alpha + \beta}$. This is a special case of $M = 1$ in (29). Because

$$\gamma_M = \frac{M!}{\prod_{k=1}^{M} (k + \frac{M+1}{\beta - \alpha}\alpha)} = \prod_{k=1}^{M} \frac{k(\beta - \alpha)}{k(\beta - \alpha) + (M+1)\alpha}$$

$$= \prod_{k=1}^{M} \frac{\beta - \alpha}{\beta + \frac{M+1-k}{k}\alpha}$$

and $\prod_{k=1}^{M} (\beta + \frac{M+1-k}{k}\alpha) > (\beta + \alpha)^M$, we have

$$\gamma_M \le \left(\frac{\beta - \alpha}{\beta + \alpha}\right)^M = \left(1 - \frac{2\alpha}{\beta + \alpha}\right)^M$$

and the equality holds if and only if $M = 1$. Therefore, for $M > 1$, the convergence performance of M-periodic control sequence (29) is better than that of the fastest constant control gain in [16].

When the algebraic connectivity α and the Laplacian spectral radius β are uncertain, we have the following result.

Corollary 2: For the MAS (8) on an uncertain connected graph $\mathcal{G}$ with $\lambda_N < \bar{\beta}$, set the M-periodic control sequence as

$$\varepsilon(k + jM) = \frac{M+1}{\bar{\beta}(k+1)} \tag{32}$$

for $k = 0, 1, \dots, M-1, j = 0, 1, \dots$. Then, the MAS reaches average consensus asymptotically. Furthermore, if M is large enough such that all the nonzero eigenvalues of the graph Laplacian are located in the interval $[\frac{\bar{\beta}}{M+1}, \frac{M\bar{\beta}}{M+1}]$, then the convergence rate satisfies $\rho_M \le \gamma_M = \frac{1}{M}$.

Proof: For the M-periodic control gain in (32), the corresponding graph filter can be obtained as

$$h(\lambda, M) = \prod_{k=0}^{M-1} \left(1 - \frac{(M+1)\lambda}{\bar{\beta}(k+1)}\right).$$

For any $\lambda \in [\frac{\bar{\beta}}{M+1}, \frac{\bar{\beta}M}{M+1}]$, there exists an integer $i, 1 \le i \le M-1$, such that $\lambda \in [\frac{\bar{\beta}}{M+1}i, \frac{\bar{\beta}}{M+1}(i+1)]$. Thus, we have

$$|h(\lambda, M)|$$

$$= \prod_{k=0}^{i-1} \left(\frac{(M+1)\lambda}{\bar{\beta}(k+1)} - 1\right) \cdot \prod_{k=i}^{M-1} \left(1 - \frac{(M+1)\lambda}{\bar{\beta}(k+1)}\right)$$

$$\le \prod_{k=0}^{i-1} \frac{\bar{\beta}(i+1) - \bar{\beta}(k+1)}{\bar{\beta}(k+1)} \cdot \prod_{k=i}^{M-1} \frac{\bar{\beta}(k+1) - \bar{\beta}i}{\bar{\beta}(k+1)}$$

$$= \prod_{k=0}^{i-1} \frac{i-k}{k+1} \prod_{k=i}^{M-1} \frac{k+1-i}{k+1}$$

$$= \frac{i!(M-i)!}{M!} < 1.$$

For $\lambda \in (0, \frac{\bar{\beta}}{M+1}), |h(\lambda, M)| = \prod_{k=0}^{M-1} (1 - \frac{(M+1)\lambda}{\bar{\beta}(k+1)}) < 1$. For $\lambda \in (\frac{\bar{\beta}M}{M+1}, \bar{\beta})$, we have

$$|h(\lambda, M)| = \prod_{k=0}^{M-1} \left(\frac{(M+1)\lambda}{\bar{\beta}(k+1)} - 1\right)$$

$$< \prod_{k=0}^{M-1} \left(\frac{(M+1)}{(k+1)} - 1\right) = 1.$$

Therefore, we have

$$\rho_M \le \max_{\lambda \in (0, \bar{\beta})} |h(\lambda, M)| < 1.$$

From Theorem 2, we know that the MAS reaches average consensus asymptotically.

For $\lambda \in [\frac{\bar{\beta}}{M+1}, \frac{M\bar{\beta}}{M+1}]$, it is easy to verify that

$$\gamma_M = \max_{i \in \{1, \cdots, M-1\}} \frac{i!(M-i)!}{M!} = \frac{1}{M}.$$

It follows from Theorem 2 that $\rho_M = \max_{\{\lambda_i\}} |h(\lambda_i, M)| \le \gamma_M = \frac{1}{M}$. ∎

VI. EXPLICIT SOLUTIONS OF THE OPTIMAL CONSENSUS ON UNCERTAIN NETWORKS

This section studies the design of optimal M-periodic control sequence $\varepsilon(k)$ with the optimal worst-case convergence rate γ_M for any graph $\mathcal{G} \in \{\mathcal{G}\}_{[\alpha, \beta]}$, where γ_M is defined in (26). Specifically, we will solve the following polynomial interpolation problem:

$$\min_{\varepsilon(k)} \gamma_M = \min_{\varepsilon(k), 0 \le k \le M-1} \max_{\lambda \in [\alpha, \beta]} \left| \prod_{k=1}^{M} (1 - \varepsilon(k)\lambda) \right| \tag{33}$$

to obtain the optimal M-periodic control sequence $\{\varepsilon^*(k)\}$ and derive the analytic formula for the optimal value γ_M^*.

The authors of [42] have used Chebyshev polynomial as the orthogonal bases to construct the least-squares approximation $h(\lambda) = \Sigma_{i=0}^{M} h_i \lambda^i$ for a desired filter. However, different from [42], (33) is a uniform interpolation problem that can-

TABLE I
CHEBYSHEV POLYNOMIALS

M	$\mathcal{T}_M(\chi)$, $\chi \in [-1,1]$	$g_M(\lambda)$, $\lambda \in [\alpha, \beta]$
$M=1$	χ	$\frac{2}{\beta-\alpha}\lambda - \frac{\beta+\alpha}{\beta-\alpha}$
$M=2$	$2\chi^2 - 1$	$2\left(\frac{2}{\beta-\alpha}\lambda - \frac{\beta+\alpha}{\beta-\alpha}\right)^2 - 1$
$M=3$	$4\chi^3 - 3\chi$	$4\left(\frac{2}{\beta-\alpha}\lambda - \frac{\beta+\alpha}{\beta-\alpha}\right)^3 - 3\left(\frac{2}{\beta-\alpha}\lambda - \frac{\beta+\alpha}{2\beta-\alpha}\right)$
$M=4$	$8\chi^4 - 8\chi^2 + 1$	$8\left(\frac{2}{\beta-\alpha}\lambda - \frac{\beta+\alpha}{\beta-\alpha}\right)^4 - 8\left(\frac{2}{\beta-\alpha}\lambda - \frac{\beta+\alpha}{\beta-\alpha}\right)^2 + 1$
$M=5$	$16\chi^5 - 20\chi^3 + 5\chi$	$16\left(\frac{2}{\beta-\alpha}\lambda - \frac{\beta+\alpha}{\beta-\alpha}\right)^5 - 20\left(\frac{2}{\beta-\alpha}\lambda - \frac{\beta+\alpha}{\beta-\alpha}\right)^3 + 5\left(\frac{2}{\beta-\alpha}\lambda - \frac{\beta+\alpha}{\beta-\alpha}\right)$
$M=6$	$32\chi^6 - 48\chi^4 + 18\chi^2 - 1$	$32\left(\frac{2}{\beta-\alpha}\lambda - \frac{\beta+\alpha}{\beta-\alpha}\right)^6 - 48\left(\frac{2}{\beta-\alpha}\lambda - \frac{\beta+\alpha}{\beta-\alpha}\right)^4 + 18\left(\frac{2}{\beta-\alpha}\lambda - \frac{\beta+\alpha}{\beta-\alpha}\right)^2 - 1$

not be solved by orthogonal basis approximation. A possible fix to this problem is to solve (33) numerically by linear programming. Besides computational complexity, this numerical approach has at least two drawbacks: 1) it does not allow direct computation of γ_M^* from α, β, M, and cannot reveal the essential relationship between γ_M^* and α, β, M; and 2) the optimal polynomial thus computed might have complex roots, giving complex valued $\varepsilon(k)$. Such a solution cannot be implemented by the control protocol (9), which requires $\varepsilon(k) > 0$. In fact, all the polynomial-approximation-based methods in the literature [21], [22], [36]–[38] cannot guarantee the designed polynomial to have only positive real roots.

To overcome the above difficulties, we use Chebyshev polynomial interpolation to derive the explicit and analytical solution to the problem (33). We first construct the polynomial $h(\lambda, M)$ and then prove that it is the unique and optimal solution of (33).

For $\chi \in [-1, 1]$, the Chebyshev polynomials $\mathcal{T}_M(\chi)$, $M = 0, 1, 2, \ldots$, are defined as [43], [44]

$$\mathcal{T}_M(\chi) := \begin{cases} 1, & M = 0 \\ \chi, & M = 1 \\ 2\chi\mathcal{T}_{M-1}(\chi) - \mathcal{T}_{M-2}(\chi), & M \geq 2 \end{cases} \quad (34)$$

which have the following properties.

i) $\mathcal{T}_M(\chi) = \cos(M \arccos \chi)$ for $\chi \in [-1, 1]$.
ii) $\mathcal{T}_M\left(\cos \frac{2i+1}{2M}\pi\right) = 0$ for $i = 0, 1, \ldots, M-1$.
iii) $\max_{x \in [-1,1]} |\mathcal{T}_M(\chi)| = 1$, and $\mathcal{T}_M(\chi) = \pm 1$ alternately at $\chi_i = \cos \frac{i}{M}\pi$, $i = 0, 1, \ldots, M$.

Setting $\chi = \frac{2}{\beta-\alpha}\lambda - \frac{\beta+\alpha}{\beta-\alpha}$ in $\mathcal{T}_M(\chi)$ yields a new polynomial $g_M(\lambda)$ on $\lambda \in [\alpha, \beta]$

$$g_M(\lambda) := \mathcal{T}_M(\chi) = \cos\left[M \arccos\left(\frac{2}{\beta-\alpha}\lambda - \frac{\beta+\alpha}{\beta-\alpha}\right)\right]. \quad (35)$$

Six example pairs of $\mathcal{T}_M(\chi)$ and $g_M(\lambda)$, derived using $\chi = \frac{2}{\beta-\alpha}\lambda - \frac{\beta+\alpha}{\beta-\alpha}$ and (34) and (35), are shown in Table I.

Lemma 4: The polynomial $g_M(\lambda)$ has the following properties.

i) $g_M\left(\frac{\beta-\alpha}{2}\cos\frac{2i+1}{2M}\pi + \frac{\beta+\alpha}{2}\right) = 0$ for $i = 0, 1, \ldots, M-1$.

ii) $\max_{\lambda \in [\alpha, \beta]} |g_M(\lambda)| = 1$ and $g_M(\lambda_i) = \pm 1$ alternatively at $\lambda_i = \frac{\beta-\alpha}{2}\chi_i + \frac{\beta+\alpha}{2}$, $i = 0, 1, \ldots M$.

iii) $g_M(0)$ can be written as

$$g_M(0) = \frac{1}{2}(-1)^M \left(\frac{\sqrt{\beta/\alpha} - 1}{\sqrt{\beta/\alpha} + 1}\right)^M$$
$$+ \frac{1}{2}(-1)^M \left(\frac{\sqrt{\beta/\alpha} + 1}{\sqrt{\beta/\alpha} - 1}\right)^M \quad (36)$$

and $\lim_{M \to \infty} |g_M(0)| = \infty$ monotonically.

Proof: (i) and (ii) follow immediately from the relation $\lambda = \frac{\beta-\alpha}{2}\chi + \frac{\beta+\alpha}{2}$. We now prove (iii). Note that $g_M(\lambda)$ can be written as

$$g_M(\lambda) = \begin{cases} 1, & M = 0 \\ \frac{2\lambda}{\beta-\alpha} - \frac{\beta+\alpha}{\beta-\alpha}, & M = 1 \\ 2\frac{2\lambda-(\beta+\alpha)}{\beta-\alpha}g_{M-1}(\lambda) & M \geq 2 \\ \quad -g_{M-2}(\lambda), & \end{cases} \quad (37)$$

Hence, we have

$$g_M(0) = \begin{cases} 1, & M = 0 \\ -\frac{\beta+\alpha}{\beta-\alpha}, & M = 1 \\ -\frac{2(\beta+\alpha)}{\beta-\alpha}g_{M-1}(0) - g_{M-2}(0), & M \geq 2 \end{cases} \quad (38)$$

By direct verification, $g_M(0)$ in (36) is the explicit formula of iteration (38). It then follows directly from (36) that

$$|g_M(0)| = \frac{1}{2}\left(1 - \frac{2}{\sqrt{\beta/\alpha} + 1}\right)^M + \frac{1}{2}\left(1 + \frac{2}{\sqrt{\beta/\alpha} - 1}\right)^M \quad (39)$$

and

$$|g_{M+1}(0)| - |g_M(0)|$$

$$= \frac{1}{2}\left(1 - \frac{2}{\sqrt{\beta/\alpha} + 1}\right)^{M+1} + \frac{1}{2}\left(1 + \frac{2}{\sqrt{\beta/\alpha} - 1}\right)^{M+1}$$

$$- \frac{1}{2}\left(1 - \frac{2}{\sqrt{\beta/\alpha} + 1}\right)^{M} - \frac{1}{2}\left(1 + \frac{2}{\sqrt{\beta/\alpha} - 1}\right)^{M}$$

$$= \frac{1}{2}\left(1 + \frac{2}{\sqrt{\beta/\alpha} - 1}\right)^{M} \frac{2}{\sqrt{\beta/\alpha} - 1} -$$

$$\frac{1}{2}\left(1 - \frac{2}{\sqrt{\beta/\alpha} + 1}\right)^{M} \frac{2}{\sqrt{\beta/\alpha} + 1}$$

$$> \frac{1}{\sqrt{\beta/\alpha} - 1} - \frac{1}{\sqrt{\beta/\alpha} + 1}$$

$$= \frac{2}{\beta/\alpha - 1} > 0.$$

Therefore, the sequence $|g_M(0)|$ approaches infinity monotonically as M tends to infinity. ∎

Theorem 4: The optimal solution to the problem (33) is given by

$$\varepsilon^*(k) = \frac{1}{r_k}$$

where

$$r_k = \frac{\beta - \alpha}{2}\cos\frac{2k+1}{2M}\pi + \frac{\beta + \alpha}{2}, \quad k = 0, 1, \ldots, M - 1 \tag{40}$$

are the roots of the unique Mth-order polynomial

$$\frac{1}{g_M(0)}g_M(\lambda) = \prod_{k=0}^{M-1}\left(1 - \frac{1}{r_k}\lambda\right) =: h(\lambda, M) \tag{41}$$

with $g_M(\lambda)$ and $g_M(0)$ defined in (35) and (36), respectively. This optimal solution yields the optimal worst-case convergence rate

$$\gamma_M^* = \frac{1}{|g_M(0)|}.$$

Proof: From Lemma 4, $h(\lambda, M)$ is an Mth-order polynomial with roots as given in (40). Hence, it can be written as (41). Since $\max_{\lambda \in [\alpha, \beta]} |g_M(\lambda)| = 1$ and $g_M(\lambda_i) = \pm 1$ alternatively at $\lambda_i = \frac{\beta - \alpha}{2}\chi + \frac{\beta + \alpha}{2}$, $i = 0, 1, \ldots M$, the polynomial $h(\lambda, M)$ satisfies $\max_{\lambda \in [\alpha, \beta]} |h(\lambda, M)| = \frac{1}{|g_M(0)|}$ and $h(\lambda, M) = \pm\frac{1}{|g_M(0)|}$ alternatively at $M + 1$ points in $[\alpha, \beta]$. By the Chebyshev alternation theorem (see [44, p. 30]), we known that this $h(\lambda, M)$ is the unique Mth-order polynomial that solves the optimal interpolation problem (33). Finally, the optimality of $\gamma_M^* = \frac{1}{|g_M(0)|}$ is obvious. ∎

Combining Theorems 2 and 4, we immediately obtain the following result.

Theorem 5: For any MAS (8) on a connected graph $\mathcal{G} \in \{\mathcal{G}\}_{[\alpha, \beta]}$, set the M-periodic control sequence as

$$\varepsilon^*(k + jM) = \frac{1}{r_k}, \quad k = 0, 1, \ldots, M - 1, j = 0, 1, \ldots \tag{42}$$

where r_k is defined in (40). Then, the MAS reaches average consensus asymptotically, and the exact convergence rate ρ_M satisfies

$$\rho_M \leq \gamma_M^* = \frac{1}{|g_M(0)|}. \tag{43}$$

Moreover, for any other M-periodic control sequence $\varepsilon(k)$, there always exists a connected graph $\tilde{\mathcal{G}} \in \{\mathcal{G}\}_{[\alpha, \beta]}$ such that its convergence rate under the $\varepsilon(k)$ is $\rho_{\varepsilon(k)} > \gamma_M^*$.

Remark 8: Although the proof is simple, the implication of Theorem 5 is important. It means that γ_M^* is the fastest convergence rate for the worst-case scenario if α and β are the only information we know about the network. $\gamma_M^* = \frac{1}{|g_M(0)|}$ also provides the direct relation between the bound of convergence rate and the graph topology. It is obvious that $\gamma_M^* \approx 2(\frac{\sqrt{\beta/\alpha} - 1}{\sqrt{\beta/\alpha} + 1})^M$ for large M.

Next, we present the performance limitation of the optimal worst-case convergence rate under the general time-varying control sequences.

For a given graph $\mathcal{G}$, the asymptotic convergence rate is defined as

$$\rho_{\text{asym}} := \sup_{\|x(0)\|_2 = 1} \lim_{M \to \infty}\left(\frac{\|e(M)\|_2}{\|e(0)\|_2}\right)^{1/M}. \tag{44}$$

For a set of graphs $\{\mathcal{G}\}_{[\alpha, \beta]}$, the worst-case asymptotic convergence rate is defined as

$$\gamma_{\text{asym}} := \sup_{\{\mathcal{G}\}_{[\alpha, \beta]}} \rho_{\text{asym}}. \tag{45}$$

It is easy to verify that $\rho_{\text{asym}} = \lim_{M \to \infty}\rho_M^{1/M}$ and $\gamma_{\text{asym}} = \lim_{M \to \infty}\gamma_M^{1/M}$, where ρ_M is defined in (19) and γ_M is defined in (26).

Theorem 6: For the MAS (8) on the set of connected graphs $\{\mathcal{G}\}_{[\alpha, \beta]}$ and under the control of a general time-varying protocol (9), we have

$$\min_{\{\varepsilon(k)\}} \gamma_{\text{asym}} = \frac{\sqrt{\beta/\alpha} - 1}{\sqrt{\beta/\alpha} + 1}. \tag{46}$$

Proof: It is obvious that

$$\left(\frac{\sqrt{\beta/\alpha} + 1}{\sqrt{\beta/\alpha} - 1}\right)^M \geq |g_M(0)| \geq \frac{1}{2}\left(\frac{\sqrt{\beta/\alpha} + 1}{\sqrt{\beta/\alpha} - 1}\right)^M. \tag{47}$$

Therefore, we have

$$\frac{\sqrt{\beta/\alpha} + 1}{\sqrt{\beta/\alpha} - 1} \geq \lim_{M \to \infty}|g_M(0)|^{\frac{1}{M}}$$

$$\geq \lim_{M \to \infty}\left(\frac{1}{2}\right)^{\frac{1}{M}}\frac{\sqrt{\beta/\alpha} + 1}{\sqrt{\beta/\alpha} - 1} = \frac{\sqrt{\beta/\alpha} + 1}{\sqrt{\beta/\alpha} - 1}$$
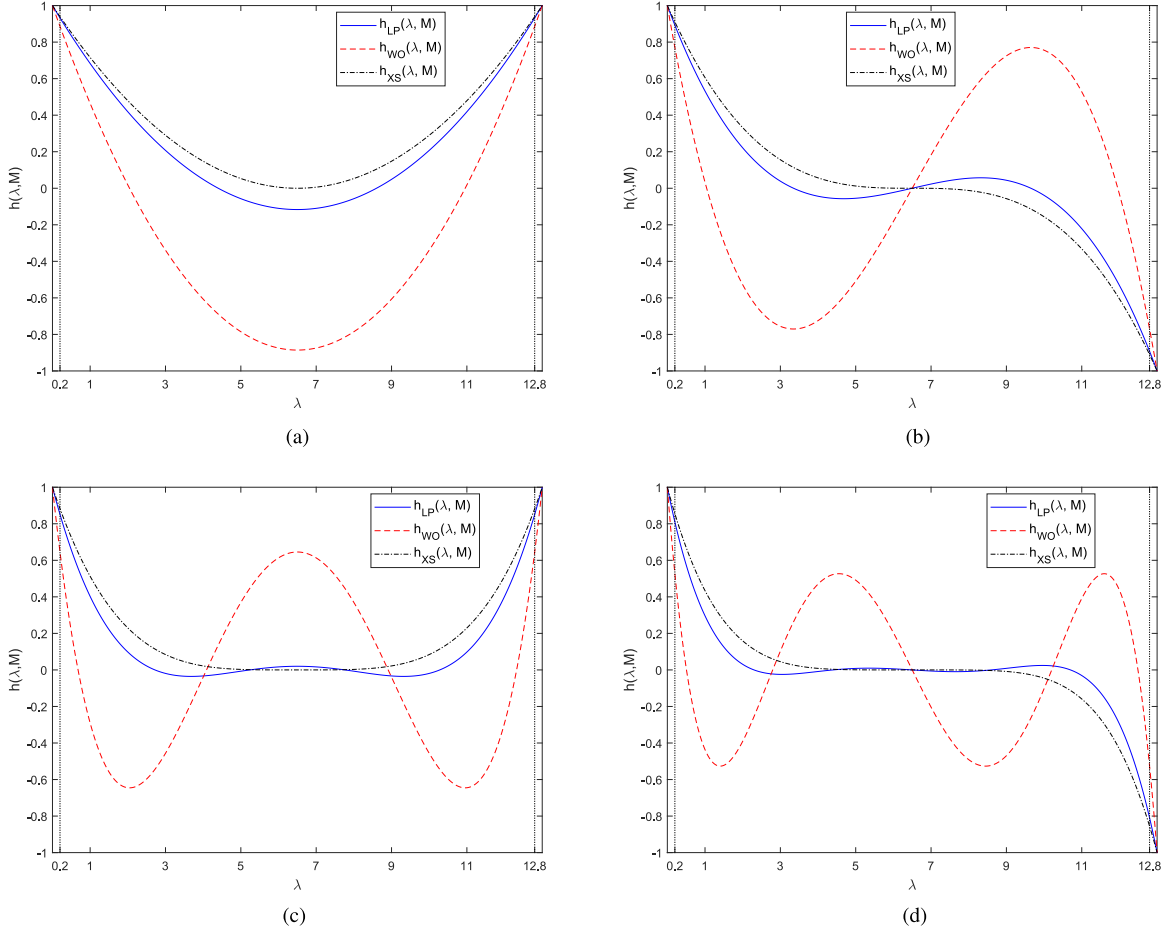
Fig. 1.   Frequency responses of graph filters designed by the listed methods. (a) Designed filters with $M = 2$. (b) Designed filters with $M = 3$. (c) Designed filters with $M = 4$. (d) Designed filters with $M = 5$.

which implies

$$\lim_{M \to \infty} |g_M(0)|^{\frac{1}{M}} = \frac{\sqrt{\beta/\alpha} + 1}{\sqrt{\beta/\alpha} - 1}. \tag{48}$$

Let γ^*_{asym} be the optimal value of $\min_{\{\varepsilon(k)\}} \gamma_{\text{asym}}$. Then, we have

$$\gamma^*_{\text{asym}} = \lim_{M \to \infty} (\gamma^*_M)^{\frac{1}{M}} = \lim_{M \to \infty} \left| \frac{1}{g_M(0)} \right|^{\frac{1}{M}} = \frac{\sqrt{\beta/\alpha} - 1}{\sqrt{\beta/\alpha} + 1}.$$

■

VII. ILLUSTRATIVE EXAMPLES

In this section, we evaluate our methods by simulation and numerical experiments. Experiments have been designed to study the behavior of the graph filters (30) in Theorem 3 and (41) in Theorem 4, respectively. We also evaluate the convergence rates on different networks and compare the two proposed methods with the best constant control gain proposed in [16]. We use the evaluation results to draw the guidelines for the proper choices of the design method and design parameters of graph filters, irrespective of the network structures.

A. Worst-Case Convergence Rate

We start with MASs on connected graphs in the set $\{\mathcal{G}\}_{[0.2, 12.8]}$. According to Theorems 3 and 4, the graph fil-

ters designed by LP interpolation method (30) and the WO interpolation method (41) are, respectively,

$$h_{\text{LP}}(\lambda, M) = \prod_{k=1}^{M} \left(1 - \frac{1}{0.2 + \frac{12.6}{M+1}k} \lambda \right) \tag{49}$$

and

$$h_{\text{WO}}(\lambda, M) = \prod_{k=1}^{M} \left(1 - \frac{1}{6.3 \cos \frac{2k-1}{2M}\pi + 6.5} \lambda \right). \tag{50}$$

It is proved in [16] that the best constant control gain is $\varepsilon(k) \equiv \varepsilon_c = \frac{2}{0.2+12.8} = \frac{1}{6.5}$, which can be regarded as a first-order graph filter $h_{XS}(\lambda, 1) = (1 - \frac{\lambda}{6.5})$ giving

$$h_{XS}(\lambda, M) = \left(1 - \frac{\lambda}{6.5} \right)^M. \tag{51}$$

For $M = 2, 3, 4, 5$, Fig. 1 shows the frequency responses of the graph filters (49)–(51), and Table II gives their worst-case convergence rates computed by (27). For the LP interpolation method and the fast linear iteration approach, it can be seen that the exact values of $\gamma_M = \max_{\lambda \in [0.2, 12.8]} |h(\lambda, M)|$ are taken from $\lambda = 0.2$. For the WO interpolation method, the frequency responses are equal amplitude oscillations in interval $[0.2, 12.8]$, and the first peaks are still taken from $\lambda = 0.2$, i.e., $\gamma_M = h_{\text{WO}}(0.2, M)$. Comparing the amplitudes at $\lambda = 0.2$,

TABLE II
WORST-CASE CONVERGENCE RATE γ_M OF THREE METHODS FOR DIFFERENT M ($\{\mathcal{G}\}_{[0.2,12.8]}$)

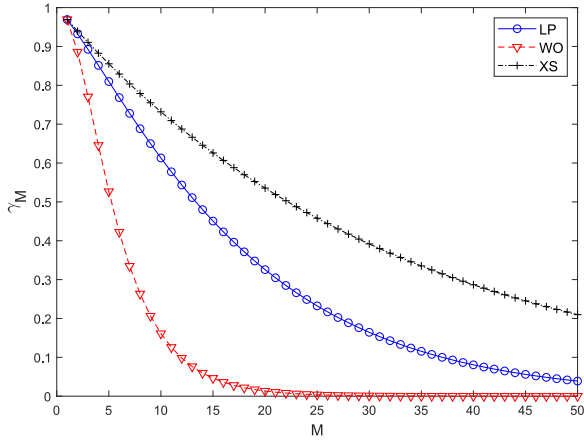| M	$M=2$	$M=3$	$M=4$	$M=5$
Lagrange polynomial interpolation method (49) $h_{LP}(\lambda,M) = \prod\limits_{k=1}^{M} \left(1 - \frac{1}{0.2 + \frac{12.6}{M+1}k}\lambda\right)$	0.9324	0.8925	0.8513	0.8097
Worst-case optimal interpolation method (50) $h_{WO}(\lambda,M) = \prod\limits_{k=1}^{M} \left(1 - \frac{1}{6.3\cos\frac{2k-1}{2M}\pi + 6.5}\lambda\right)$	**0.8858**	**0.7706**	**0.6456**	**0.5268**
Fast linear iterations approach (51) [16] $h_{XS}(\lambda,M) = (1-\lambda/6.5)^M$	0.9394	0.9105	0.8824	0.8554

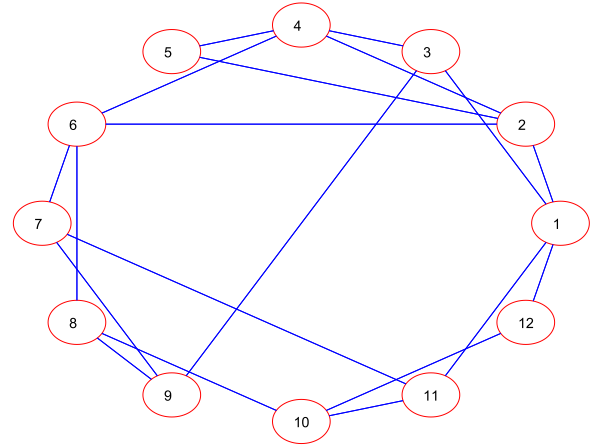Fig. 2.   Worst-case convergence rate γ_M by different methods.

Fig. 3.   Small-world graph $\mathcal{G}_3$.

it is clear that the WO interpolation method attains the fastest worst-case convergence rate γ_M.

The worst-case convergence rates for $M \in [1,50]$ are plotted in Fig. 2. As seen from Fig. 2, for all these three filters, γ_M decreases monotonically as M increases, and the decreasing rates become slower when M exceeds some values. Since a smaller γ_M means a faster worst-case convergence rate, the convergence rates resulting from our methods are faster than that of the best constant control gain proposed in [16]. And the WO interpolation method gives the optimal control strategy in the worst case, with the fastest convergence rate.

B. Per-Period Convergence Rate on Some Sample Graphs

Consider the MASs on four sample graphs: (a) the star graph $\mathcal{G}_1$ with 12 agents, (b) the cycle graph $\mathcal{G}_2$ with 12 agents, (c) the graph $\mathcal{G}_3$ generated by a small-world network model shown in Fig. 3, and (d) the path graph $\mathcal{G}_4$ with six agents. It is easy to verify that the four sample graphs are in the set $\mathcal{G}_{[0.2,12.8]}$. We now investigate the per-period convergence rate ρ_M of the three filters (49)–(51) on these four graphs.

1) Star Graph $\mathcal{G}_1$: The nonzero eigenvalues of the graph Laplacian $\mathcal{L}_{\mathcal{G}_1}$ are $\{\lambda_i\}_{\mathcal{G}_1} = \{1,12\}$. The per-period convergence rates of MAS on $\mathcal{G}_1$, calculated by (21), are shown in Fig. 4(a). It can be seen that as M increases, the per-period convergence rates resulting from the LP interpolation method (30) and the best constant control gain in [16] both monotonically decrease to zero, but that of the WO interpolation method (41) oscillating to zero. It appears that the LP interpolation method

is always better than the fast linear iterations approach, but the WO interpolation method may be worse than the latter, or even the worst of all the three methods when $M=5$.

The per-period convergence rate ρ_M resulting from the WO interpolation method drops very fast when $M=3$ because the corresponding roots of the graph filter $h_{WO}(\lambda,3)$ are $\{r_k\} = \{1.044, 6.5, 11.956\}$, which are close to the nonzero eigenvalues $\{\lambda_i\}_{\mathcal{G}_1}$. Thus, for the star graph $\mathcal{G}_1$, there is no need to increase M to accelerate the convergence rate. One only needs to select the 3-periodic control sequence $\varepsilon(k) = \{\frac{1}{11.956}, \frac{1}{6.5}, \frac{1}{1.044}\}$ to achieve fast consensus.

2) Cycle Graph $\mathcal{G}_2$: The nonzero eigenvalues of the graph Laplacian matrix $\mathcal{L}_{\mathcal{G}_2}$ are $\{\lambda_i\}_{\mathcal{G}_2} = \{0.2679, 1, 2, 3, 3.7321, 4\}$. The per-period convergence rates of MASs on $\mathcal{G}_2$ calculated by (21) are shown in Fig. 4(b). As the eigenvalues are distributed uniformly on $[0,4]$ and $\lambda_2 = 0.2679$, which is close to $\alpha = 0.2$, the curves of ρ_M by the three methods are similar to those of the worst-case convergence rate γ_M shown in Fig. 2, and the WO interpolation method gives the fastest per-period convergence rate.

3) Graph $\mathcal{G}_3$ Generated by a Small-World Network Model: We randomly generated 50 small-world networks with 12 agents and choose a rare case $\mathcal{G}_3$ to demonstrate that the WO interpolation method may not show its advantages for specific graphs. The nonzero eigenvalues of the graph Laplacian matrix $\mathcal{L}_{\mathcal{G}_3}$ are $\{\lambda_i\}_{\mathcal{G}_3} = \{0.655, 1.2694, 1.9964, 2.8578, 3.6319, 3.8860, 5.0364, 5.2884, 5.7759, 6.4118, 7.1909\}$. The per-period convergence rates of the MAS on $\mathcal{G}_3$, calculated by (21), are shown in Fig. 4(c). It can be seen that the control sequence designed by the LP
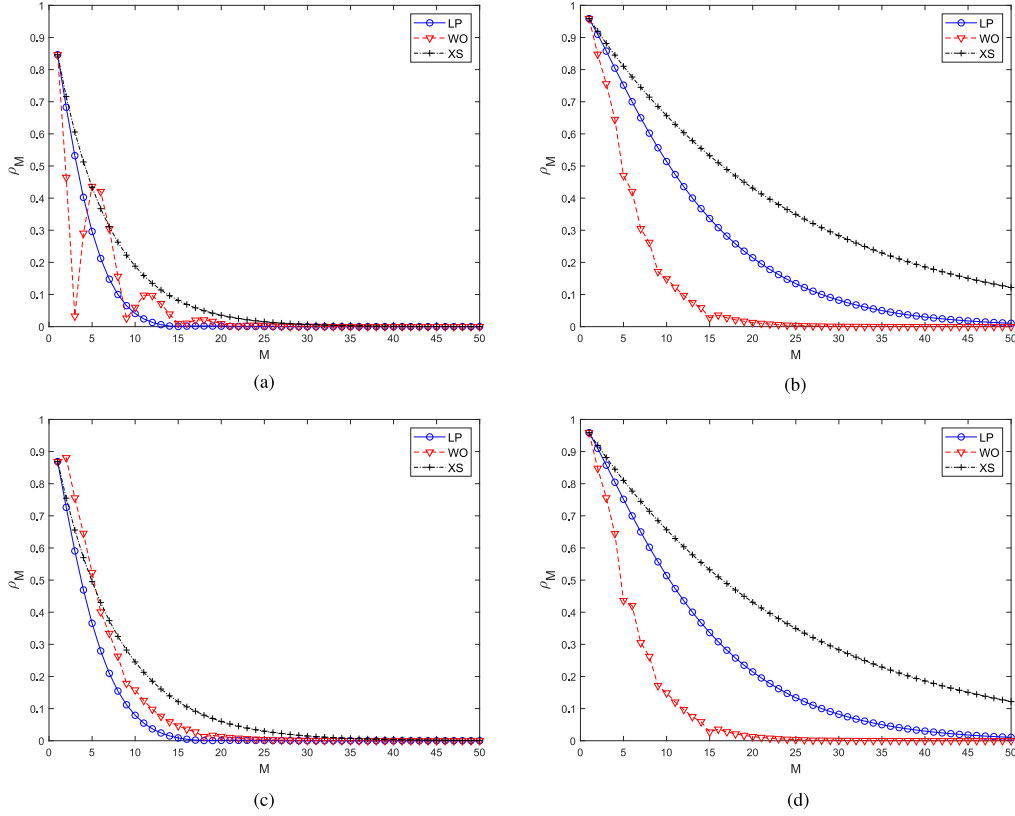
Fig. 4.   Per-period convergence rate ρ_M by different methods. (a) Per-period convergence rate for $\mathcal{G}_1$. (b) Per-period convergence rate for $\mathcal{G}_2$. (c) Per-period convergence rate for $\mathcal{G}_3$. (d) Per-period convergence rate for $\mathcal{G}_4$.

TABLE III
PER-PERIOD CONVERGENCE RATE ρ_M OF THREE METHODS ON FOUR SAMPLE GRAPHS

Method	Star graph $\mathcal{G}_1$			Cycle graph $\mathcal{G}_2$			Small-World graph $\mathcal{G}_3$			Path graph $\mathcal{G}_4$		
	LP	WO	XS	LP	WO	XS	LP	WO	XS	LP	WO	XS
$M=2$	0.6829	**0.4645**	0.7160	0.9099	**0.8478**	0.9193	**0.7264**	0.8807	0.7549	0.9099	**0.8478**	0.9193
$M=3$	0.5321	**0.0328**	0.6059	0.8577	**0.7556**	0.8814	**0.5906**	0.7556	0.6559	0.8577	**0.7556**	0.8814
$M=4$	0.4024	**0.2907**	0.5127	0.8044	**0.6449**	0.8451	**0.4693**	0.6454	0.5699	0.8044	**0.6449**	0.8451
$M=5$	**0.2961**	0.4363	0.4338	0.7515	**0.4696**	0.8103	**0.3656**	0.5231	0.4952	0.7515	**0.4362**	0.8103

interpolation method has the fastest convergence rate, and for $M \leq 5$, the control sequence designed by the WO interpolation method has the slowest convergence rate. This shows that for a given graph, the convergence rates resulting from interpolation methods are closely related to the spectral distribution of the graph Laplacian matrix.

4) Path Graph $\mathcal{G}_4$: The nonzero eigenvalues of the graph Laplacian matrix $\mathcal{L}_{\mathcal{G}_4}$ are $\{\lambda_i\}_{\mathcal{G}_4} = \{0.2679, 1, 2, 3, 3.7321\}$. It can be seen that $\{\lambda_i\}_{\mathcal{G}_4} = \{\lambda_i\}_{\mathcal{G}_2} \setminus \{4\}$. This means that although the network structure and the number of agents are different, the Laplacian spectral distributions of the graph $\mathcal{G}_4$ and the graph $\mathcal{G}_2$ are nearly the same. Hence, the per-period convergence rates of the MAS on $\mathcal{G}_4$ calculated by (21) are similar to those on $\mathcal{G}_2$, as shown in Fig. 4(d). This shows that for different graphs with similar Laplacian spectral distributions, the convergence performance of the three methods is almost the same.

Table III gives the exact values of the per-period convergence rate ρ_M on the four sample graphs for the control period $M = 2, 3, 4, 5$. Compared to the best constant control gain pro-

posed in [16], the LP interpolation method always has a faster convergence rate, which corroborates the analysis in Remark 7. For most cases, the WO interpolation method can attain the fastest convergence rate.

C. Consensus Performance

To compare the evolution of the MAS states of the three methods on the four sample graphs, we take the control period $M = 3$ as an example. Then, the graph filters (49)–(51) become

$$h_{\text{LP}}(\lambda, 3) = \left(1 - \frac{\lambda}{3.35}\right)\left(1 - \frac{\lambda}{6.5}\right)\left(1 - \frac{\lambda}{9.65}\right)$$

$$h_{\text{WO}}(\lambda, 3) = \left(1 - \frac{\lambda}{1.044}\right)\left(1 - \frac{\lambda}{6.5}\right)\left(1 - \frac{\lambda}{11.956}\right)$$

$$h_{XS}(\lambda, 3) = \left(1 - \frac{\lambda}{6.5}\right)^3.$$
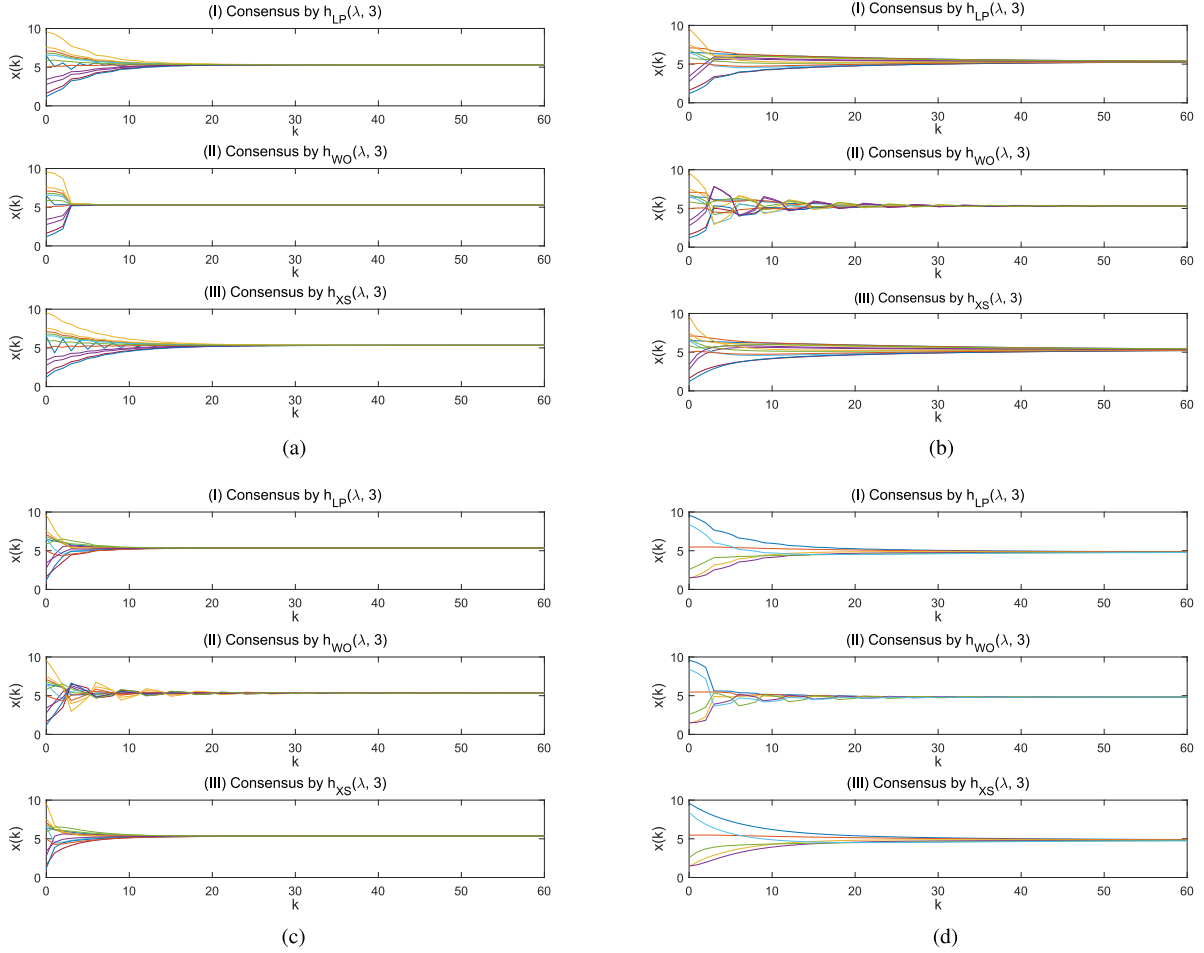
Fig. 5. Agent state trajectories for sample graphs by the designed graph filters with $M = 3$. (a) Consensus for the star graph $\mathcal{G}_1$. (b) Consensus for the cycle graph $\mathcal{G}_2$. (c) Consensus for the small-world graph $\mathcal{G}_3$. (d) Consensus for the path graph $\mathcal{G}_4$.

For the initial states of the agents randomly taken from the interval $[0, 10]$, Fig. 5 shows that the MASs on the four sample graphs reach average consensus asymptotically under the control protocols designed by the above graph filters.

For the star graph $\mathcal{G}_1$, the MAS under the control protocol designed by the WO interpolation method converges very fast, whereas for the cycle graph $\mathcal{G}_2$ and the path graph $\mathcal{G}_4$, the convergence rates are much slower due to the distributions of Laplacian spectra. For the graph $\mathcal{G}_3$, the consensus performance of the WO interpolation method is the worst of all the three methods. The curves shown in Fig. 5 are consistent with the exact convergence rates calculated in Table III.

D. Consensus on Large-Scale Networks

This subsection illustrates the effectiveness and performance of the graph filters given in Theorems 3 and 4 on large-scale networks.

First, we consider the MAS (8) with 100 agents on a connected graph $\mathcal{G}_L \in \{\mathcal{G}\}_{[0.2,12.8]}$ shown in Fig. 6(a). The graph $\mathcal{G}_L$ is generated by a small-world network model, which is constructed by a 4-regular network under a random rewiring probability $p = 0.6$. The second smallest eigenvalue and the largest eigenvalue are $\lambda_2 = 0.4761$ and $\lambda_{100} = 9.1865$, respectively. Take the control period $M = 5$ as an example; the graph filters

(49)–(51) [with the frequency responses shown in Fig. 1(d)] can be derived as

$$h_{\text{LP}}(\lambda, 5) = \left(1 - \frac{\lambda}{10.7}\right)\left(1 - \frac{\lambda}{8.6}\right)\left(1 - \frac{\lambda}{6.5}\right)$$
$$\left(1 - \frac{\lambda}{4.4}\right)\left(1 - \frac{\lambda}{2.3}\right)$$

$$h_{\text{WO}}(\lambda, 5) = \left(1 - \frac{\lambda}{12.492}\right)\left(1 - \frac{\lambda}{10.203}\right)\left(1 - \frac{\lambda}{6.5}\right)$$
$$\left(1 - \frac{\lambda}{2.797}\right)\left(1 - \frac{\lambda}{0.508}\right)$$

$$h_{XS}(\lambda, 5) = \left(1 - \frac{\lambda}{6.5}\right)^5.$$

For the initial states of the agents randomly taken from the interval $[0, 10]$, the states of the closed-loop systems under the control of these three protocols are plotted in Fig. 6(b). It can be seen that all the three protocols asymptotically achieve the average consensus on the large-scale networks. The per-period convergence rates of the three methods are $\rho_{5\{\text{LP}\}} = 0.5916$, $\rho_{5\{\text{WO}\}} = 0.5266$, and $\rho_{5\{XS\}} = 0.6837$, respectively. As expected, the per-period convergence rates of our methods are faster than that of the
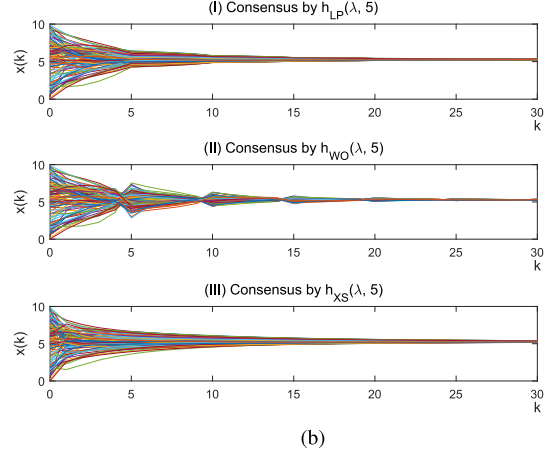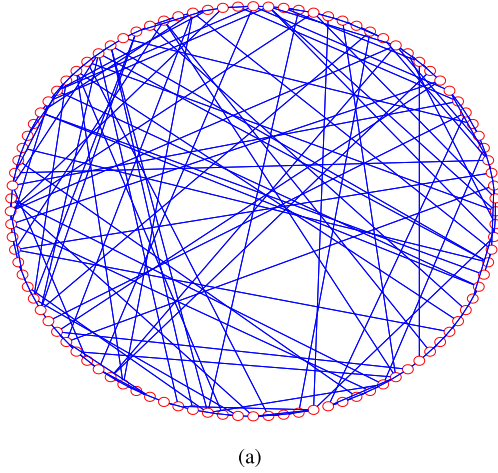
Fig. 6.   Graph topology for the MAS with 100 agents and the agent state trajectories under control protocols designed by different methods. (a) Small-world graph $\mathcal{G}_L$ with 100 agents. (b) Agent state trajectories of the MAS on $\mathcal{G}_L$.
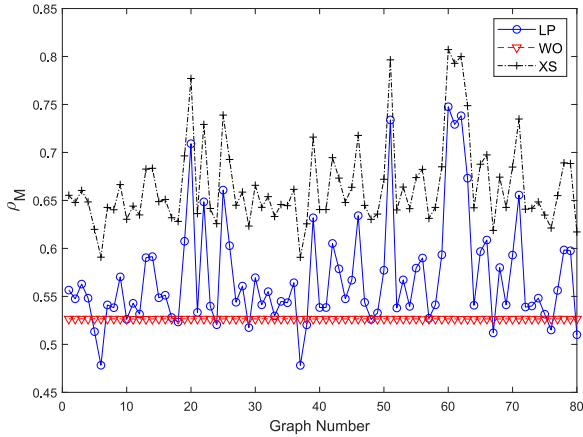

Fig. 7.   Per-period convergence rate ρ_M for 80 random graphs by different methods with $M = 5$.

fast linear iteration approach in [16], and the WO interpolation method has the fastest per-period convergence rate.

Next, we generate again 80 random connected graphs of 100 nodes in the set $\{\mathcal{G}\}_{[0.2, 12.8]}$. To verify the effectiveness of the graph filters shown above, we plot in Fig. 7 the per-period convergence rates ρ_M resulting from these filters. It can be seen that the per-period convergence rate ρ_M of the LP interpolation method is always smaller than that of the fast linear iteration approach. This indicates that the periodic control sequence from the LP interpolation method has better consensus performance compared with the best constant control gain proposed in [16]. For the WO interpolation method, the per-period convergence rate $\rho_{5_{\{WO\}}}$ is almost a straight line around the worst-case convergence rate $\gamma_{5_{\{WO\}}} = 0.5268$, with very little fluctuation. It is noted that in about 15% of the 80 graphs, the exact values of ρ_M for the WO interpolation method are larger than those of the LP interpolation method. This shows that the WO approximation method is optimal in the worst case, but not always good for some specific examples.

E. Relation Between the Worst-Case Convergence Rate and α, β, M

We analyze how to use the graph filter to find better design parameters or the performance limitation regardless of the network structure.

In Fig. 2, we have evaluated the worst-case convergence rate γ_M for the given $\alpha = 0.2$ and $\beta = 12.8$, and have shown that γ_M decreases monotonically as the control period M increases. Setting the ratios of the spectral radius and the algebraic connectivity, β/α, at different values, we compare in Fig. 8(a) and (b) the worst-case convergence rate γ_M of the LP interpolation method and that of the WO interpolation method.

First, we inspect the effect of the control period M. Although γ_M of both methods always decreases as M increases, the deceasing rate reduces when M exceeds a certain threshold. This means that beyond a certain threshold, there is little gain in improving the consensus performance by increasing M. Thus, we need to choose the control period M properly to achieve a fast convergence rate.

Next, we inspect the effect of β/α. As seen from Fig. 8, with the increase of β/α, the worst-case convergence rate γ_M of both methods becomes larger for a given control period M. This indicates that the more dispersed the distribution of the graph Laplacian spectra are, the harder it is to achieve the average consensus. Therefore, for graphs with large spectral radius and small algebraic connectivity, we should choose a larger control period M to get a faster convergence rate and use the WO interpolation method to get a much faster decreasing rate of γ_M than that of LP interpolation method.

With $M = 10$ and $\beta = 12.8$, Fig. 9(a) and (b) depicts the frequency responses of the graph filters from the two interpolation methods at different values of β/α. It can be seen that when β/α is small, such as $\beta/\alpha = 2, 4$, the frequency responses of both graph filters are similar. As β/α increases, the graph filter designed by the WO interpolation method exhibits obvious equal amplitude oscillations in the interval $[\alpha, \beta]$, while the graph filter designed by the LP interpolation method has large amplitude at both ends of the interval $[\alpha, \beta]$.
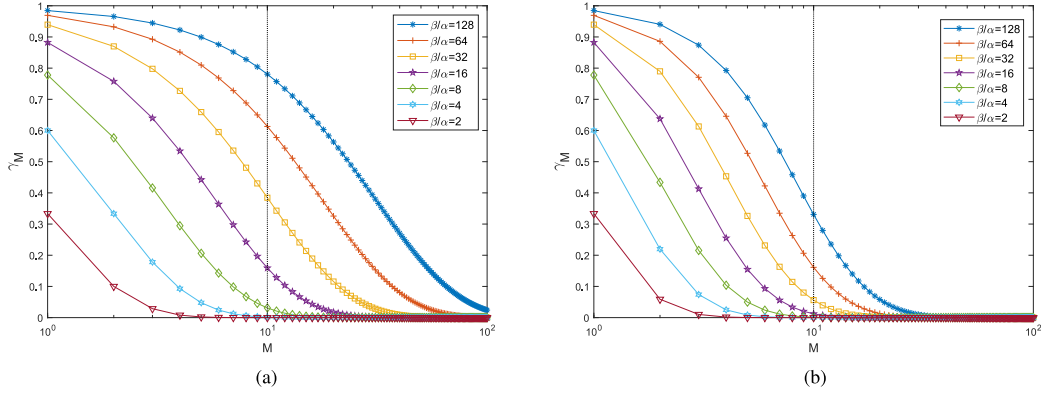
Fig. 8.   Worst-case convergence rate γ_M for different M and β/α based on (a) the graph filter $h_{\mathsf{LP}}(\lambda, M)$ by the LP interpolation method and (b) the graph filter $h_{\mathsf{WO}}(\lambda, M)$ by the WO interpolation method.
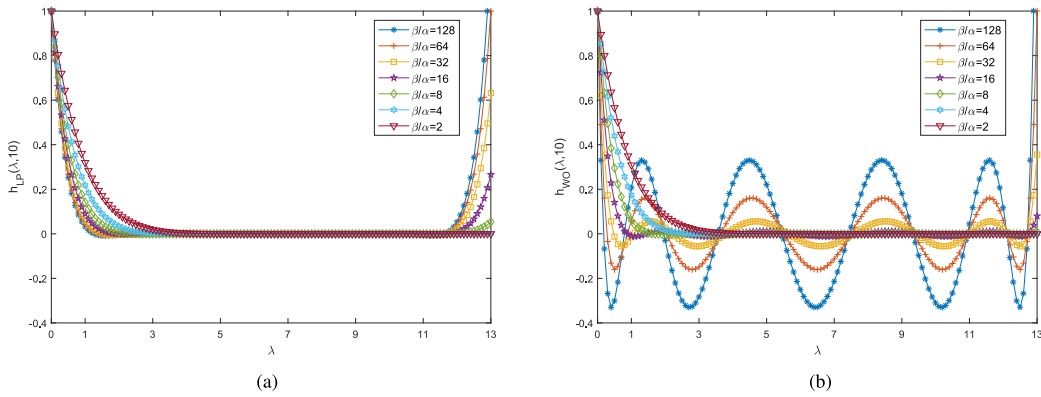

Fig. 9.   Frequency responses of the designed graph filters with $M = 10$. (a) LP interpolation method. (b) WO interpolation method.

VIII. CONCLUSION

The direct and explicit connection between MAS consensus and spectral filtering of graph signals has been established. Based on this connection, a novel approach has been presented to analyze MAS consensus and design effective control protocols in the graph spectrum domain. This novel approach has led to the new analysis results and design methods listed in the Abstract for the consensus of MASs on the uncertain graphs. Numerical examples have demonstrated the validity, effectiveness, and advantages of these results and methods.

The presented approach can overcome the difficulties of existing time-domain methods in handling uncertain MASs and, hence, can offer more precise analysis, deeper insights, more effective design methods, and better performance for MAS consensus.

For simplicity and also due to space limit, the presented results are only for the first-order deterministic MAS on undirected graphs. All the results presented in this paper can be extended to the high-order and stochastic MASs on the directed graphs, which will be reported elsewhere.

REFERENCES

[1] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[2] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, no. 1, pp. 215–233, Jan. 2007.

[3] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proc. IEEE*, vol. 95, no. 1, pp. 138–162, Jan. 2007.

[4] Y. Cao, W. Yu, W. Ren, and G. Chen, "An overview of recent progress in the study of distributed multi-agent coordination," *IEEE Trans. Ind. Inform.*, vol. 9, no. 1, pp. 427–438, Feb. 2013.

[5] S. Knorn, Z. Chen, and R. H. Middleton, "Overview: Collective control of multiagent systems," *IEEE Trans. Control Netw. Syst.*, vol. 3, no. 4, pp. 334–347, Dec. 2016.

[6] Y. P. Tian and C. L. Liu, "Consensus of multi-agent systems with diverse input and communication delays," *IEEE Trans. Autom. Control*, vol. 53, no. 9, pp. 2122–2128, Oct. 2008.

[7] L. Li, M. Fu, H. Zhang, and R. Lu, "Consensus control for a network of high order continuous-time agents with communication delays," *Automatica*, vol. 89, pp. 144–150, 2018.

[8] T. Li and J. F. Zhang, "Consensus conditions of multi-agent systems with time-varying topologies and stochastic communication noises," *IEEE Trans. Autom. Control*, vol. 55, no. 9, pp. 2043–2057, Sep. 2010.

[9] K. You, Z. Li, and L. Xie, "Consensus condition for linear multi-agent systems over randomly switching topologies," *Automatica*, vol. 49, no. 10, pp. 3125–3132, 2013.

[10] T. Li, F. Wu, and J. F. Zhang, "Multi-agent consensus with relative-state-dependent measurement noises," *IEEE Trans. Autom. Control*, vol. 59, no. 9, pp. 2463–2468, Sep. 2014.

[11] S. Liu, L. Xie, and H. Zhang, "Distributed consensus for multi-agent systems with delays and noises in transmission channels," *Automatica*, vol. 47, no. 5, pp. 920–934, 2011.

[12] T. Li, M. Fu, L. Xie, and J. F. Zhang, "Distributed consensus with limited communication data rate," *IEEE Trans. Autom. Control*, vol. 56, no. 2, pp. 279–292, Feb. 2011.

[13] Z. Qiu, L. Xie, and Y. Hong, "Quantized leaderless and leader-following consensus of high-order multi-agent systems with limited data rate," *IEEE Trans. Autom. Control*, vol. 61, no. 9, pp. 2432–2447, Sep. 2016.

[14] Z. Li, W. Ren, X. Liu, and M. Fu, "Consensus of multi-agent systems with general linear and Lipschitz nonlinear dynamics using distributed adaptive protocols," *IEEE Trans. Autom. Control*, vol. 58, no. 7, pp. 1786–1791, Jul. 2013.

[15] X. Chen and Z. Chen, "Robust sampled-data output synchronization of nonlinear heterogeneous multi-agents," *IEEE Trans. Autom. Control*, vol. 62, no. 3, pp. 1458–1464, Mar. 2017.

[16] L. Xiao and S. Boyd, "Fast linear iterations for distributed averaging," *Syst. Control Lett.*, vol. 53, no. 1, pp. 65–78, 2004.

[17] L. Xiao, S. Boyd, and S. J. Kim, "Distributed average consensus with least-mean-square deviation," *J. Parallel Distrib. Comput.*, vol. 67, no. 1, pp. 33–46, 2007.

[18] R. Olfati-Saber, "Ultrafast consensus in small-world networks," in *Proc. Amer. Control Conf.*, Jun. 2005, pp. 2371–2378.

[19] T. C. Aysal, B. N. Oreshkin, and M. J. Coates, "Accelerated distributed average consensus via localized node state prediction," *IEEE Trans. Signal Process.*, vol. 57, no. 4, pp. 1563–1576, Apr. 2009.

[20] T. Erseghe, D. Zennaro, E. Dall'Anese, and L. Vangelista, "Fast consensus by the alternating direction multipliers method," *IEEE Trans. Signal Process.*, vol. 59, no. 11, pp. 5523–5537, Nov. 2011.

[21] E. Kokiopoulou and P. Frossard, "Polynomial filtering for fast convergence in distributed consensus," *IEEE Trans. Signal Process.*, vol. 57, no. 1, pp. 342–354, Jan. 2009.

[22] E. Montijano, J. I. Montijano, and C. Sagues, "Chebyshev polynomials in distributed consensus applications," *IEEE Trans. Signal Process.*, vol. 61, no. 3, pp. 693–706, Feb. 2013.

[23] S. Sundaram and C. N. Hadjicostis, "Finite-time distributed consensus in graphs with time-invariant topologies," in *Proc. Amer. Control Conf.*, Jul. 2007, pp. 711–716.

[24] J. M. Hendrickx, R. M. Jungers, A. Olshevsky, and G. Vankeerberghen, "Graph diameter, eigenvalues, and minimum-time consensus," *Automatica*, vol. 50, no. 2, pp. 635–640, 2014.

[25] A. Y. Kibangou, "Step-size sequence design for finite-time average consensus in secure wireless sensor networks," *Syst. Control Lett.*, vol. 67, pp. 19–23, 2014.

[26] J. Corts, "Finite-time convergent gradient flows with applications to network consensus," *Automatica*, vol. 42, no. 11, pp. 1993–2000, 2006.

[27] L. Wang and F. Xiao, "Finite-time consensus problems for networks of dynamic agents," *IEEE Trans. Autom. Control*, vol. 55, no. 4, pp. 950–955, Apr. 2010.

[28] S. Li, H. Du, and X. Lin, "Finite-time consensus algorithm for multi-agent systems with double-integrator dynamics," *Automatica*, vol. 47, no. 8, pp. 1706–1712, 2011.

[29] A. Sandryhaila and J. M. F. Moura, "Discrete signal processing on graphs," *IEEE Trans. Signal Process.*, vol. 61, no. 7, pp. 1644–1656, Apr. 2013.

[30] A. Sandryhaila and J. M. F. Moura, "Discrete signal processing on graphs: Frequency analysis," *IEEE Trans. Signal Process.*, vol. 62, no. 12, pp. 3042–3054, Jun. 2014.

[31] S. Chen, R. Varma, A. Sandryhaila, and J. Kovačević, "Discrete signal processing on graphs: Sampling theory," *IEEE Trans. Signal Process.*, vol. 63, no. 24, pp. 6510–6523, Dec. 2015.

[32] D. I. Shuman, S. K. Narang, P. Frossard, A. Ortega, and P. Vandergheynst, "The emerging field of signal processing on graphs: Extending high-dimensional data analysis to networks and other irregular domains," *IEEE Signal Process. Mag.*, vol. 30, no. 3, pp. 83–98, May 2013.

[33] D. I. Shuman, B. Ricaud, and P. Vandergheynst, "Vertex-frequency analysis on graphs," *Appl. Comput. Harmonic Anal.*, vol. 40, no. 2, pp. 260–291, 2016.

[34] D. I. Shuman, M. J. Faraji, and P. Vandergheynst, "A multiscale pyramid transform for graph signals," *IEEE Trans. Signal Process.*, vol. 64, no. 8, pp. 2119–2134, Apr. 2016.

[35] A. Gadde, S. K. Narang, and A. Ortega, "Bilateral filter: Graph spectral interpretation and extensions," in *Proc. IEEE Int. Conf. Image Process.*, Sep. 2013, pp. 1222–1226.

[36] A. Sandryhaila, S. Kar, and J. M. F. Moura, "Finite-time distributed consensus through graph filters," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, May 2014, pp. 1080–1084.

[37] S. Segarra, A. G. Marques, and A. Ribeiro, "Optimal graph-filter design and applications to distributed linear network operators," *IEEE Trans. Signal Process.*, vol. 65, no. 15, pp. 4117–4131, Aug. 2017.

[38] S. Izumi, S. I. Azuma, and T. Sugie, "On a relation between graph signal processing and multi-agent consensus," in *Proc. IEEE 55th Conf. Decis. Control*, Dec. 2016, pp. 957–961.

[39] T. Li and J. F. Zhang, "Mean square average-consensus under measurement noises and fixed topologies: Necessary and sufficient conditions," *Automatica*, vol. 45, no. 8, pp. 1929–1936, 2009.

[40] C. Godsil and G. Royle, *Algebraic Graph Theory*. Cambridge, U.K.: Cambridge Univ. Press, 1974.

[41] F. R. K. Chung, *Spectral Graph Theory*. Providence, RI, USA: Amer. Math. Soc., 1997.

[42] D. I. Shuman, P. Vandergheynst, and P. Frossard, "Distributed signal processing via Chebyshev polynomial approximation," *IEEE Trans. Signal Inf. Process. Netw.*, 2017.

[43] J. C. Mason and D. C. Handscomb, *Chebyshev Polynomials*. Berlin, Germany: Springer, 2015.

[44] G. G. Lorentz, *Approximation of Functions*. New York, NY, USA: Holt, Rinehart and Winston, 1966.

[45] J. W. Yi and L. Chai, "Graph filter design for multi-agent system consensus," in *Proc. IEEE 56th Conf. Decis. Control*, Dec. 2017, pp. 1082–1087.

Jing-Wen Yi received the B.E. degree from Yantai University, Yantai, China, in 2010, and the Ph.D. degree from the Huazhong University of Science and Technology, Wuhan, China, in 2016.

She is currently an Associate Professor with the School of Information Science and Engineering, Wuhan University of Science and Technology, where she was a Postdoctoral Research Fellow from April 2016 to January 2018. Her research interests include multiagent systems, networked control systems, and graph signal processing.

Li Chai (S'00–M'03) received the B.S. degree in applied mathematics and the M.S. degree in control science and engineering from Zhejiang University, Hangzhou, China, in 1994 and 1997, respectively, and the Ph.D. degree in electrical engineering from the Hong Kong University of Science and Technology, Hong Kong, in 2002.

In September 2002, he joined Hangzhou Dianzi University, Hangzhou. He was a Postdoctoral Research Fellow with Monash University, Melbourne, Australia, from May 2004 to June 2006. In 2008, he joined the Wuhan University of Science and Technology, Wuhan, China, where he is currently a Chutian Chair Professor. He has been a Visiting Scholar with Newcastle University, Australia, and Harvard University. His research interests include distributed optimization, filter bank frames, graph signal processing, and networked control systems.

Prof. Chai is the recipient of the Distinguished Young Scholar of the National Science Foundation of China. He is currently an Associate Editor for the *Decision and Control*.

Jingxin Zhang (M'02) received the M.E. and Ph.D. degrees in electrical engineering from Northeastern University, Shenyang, China.

Since 1989, he has held research and academic positions with Northeastern University, the University of Florence, Italy, the University of Melbourne, the University of South Australia, Deakin University, and Monash University, Australia. He is currently an Associate Professor of Electrical Engineering with the Swinburne University of Technology, Melbourne, Australia, and an Adjunct Associate Professor of Electrical and Computer Systems Engineering with Monash University, Melbourne. His research interests include signals and systems and their applications to biomedical and industrial systems.

Dr. Zhang is the recipient of the 1989 Fok Ying Tong Educational Foundation (Hong Kong) for the Outstanding Young Faculty Members in China and the 1992 China National Education Committee Award for the Advancement of Science and Technology.