Stabilization of Two-port Networked Systems with Simultaneous Uncertainties in Plant, Controller, and Communication Channels

Di Zhao, Li Qiu, and Guoxiang Gu

***Abstract*—In this paper, we study robust stabilization for the networked control system (NCS) over the communication channels described by cascaded two-port networks. Such an NCS involves simultaneous uncertainties in the plant, controller, and two-port communication channels. The cascaded two-port connections arise when signals in the NCS are transmitted through bidirectional communication channels separated by a sequence of relays. Distortions and interferences occurring in communications are taken into account. We consider $\mathcal{H}_\infty$-norm bounded uncertainties in the transmission matrices of the two-port channels, and the gap-type uncertainties in the plant and controller models. A necessary and sufficient condition for the robust stability of the NCS is presented in the form of an "arcsine" inequality, which states that the NCS is stable whenever the uncertainties quantified by the aforementioned metrics are well bounded and satisfy the "arcsine" inequality. A stability margin related to the Gang of Four transfer matrix is obtained, based on which, the controller synthesis problem can be solved through a special $\mathcal{H}_\infty$ optimization with favourable properties. Furthermore, a generalized stability condition is studied in terms of frequency-wise bounded uncertainties, and the corresponding controller synthesis problem is proposed and solved.**

Fig. 1: A standard closed-loop system.

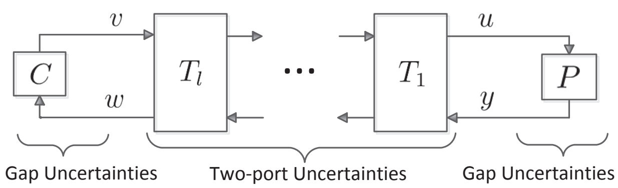

Fig. 2: Cascaded two-port networked feedback system with simultaneous uncertainties.

I. Introduction

UNCERTAINTIES permeate every physical system to be controlled due to the inadequacies of the mathematical models. Indeed it is difficult to model many physical phenomena mathematically; Even if a physical system can be modeled accurately, the mathematical model can be too complex to be used in analysis and synthesis for control system design. To combat against uncertainties, the control robustness naturally comes into the picture. The robust stabilization problem has been studied over decades and shown to be critical in many applications [1]–[4]. It becomes more important and challenging for networked control systems (NCSs), because of the distortions and interferences induced by communication channels. We study uncertain NCSs in this paper, focusing on those with inaccurate or partially known models and those involving distortions and interferences in communication channels.

In order to characterize uncertain systems, it is crucial to define an appropriate metric or distance between a pair of systems with one for the true physical system, and the other for the plant model. A geometric method with the input-output point of view to model the uncertain dynamics has been developed through the gap metric and its variations, among which the gap [5]–[8], the pointwise gap [9], [10] and the ν-gap [11], [12] have been intensively studied. It is noticed in [11] that among these metrics, the ν-gap characterizes the largest set of uncertain systems tolerable by a nominal closed-loop system. In this study, we adopt the gap metric for its simplicity. Since the variations of the gap metric share many common properties, most of the results in this paper hold true for the ν-gap and the pointwise gap with similar arguments.

Under the aforementioned metrics, the corresponding robust stabilization problems can be formulated. First, given a nominal plant and a controller, the "largest" amount of uncertainties can be characterized and analyzed for which the feedback system maintains the stability. In addition, given the nominal plant only, an optimal robust controller can be designed, which maximizes the robust stability margin. Most of the robust stabilization problem in terms of the standard feedback system as in Fig. 1 has been well studied and neatly solved in the last three decades [7], [8], [10]–[14]. Concerning the simultaneous uncertainties both in the plant and controller, [8], [10] and [12] establish tight stability conditions by introducing the angular gap metric, which is the "arcsine" of the gap, the pointwise gap and the ν-gap, respectively.

Perfect communication is assumed in the literature of conventional control, including robust control. In the network era, signals are transmitted through imperfect communication channels for most practical systems. As such, we have NCSs, differing from conventional control systems, as the information exchanged between the plant and controller is

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through communication networks [15]. As the quality of control heavily relies on the conditions of communication channels, the channel uncertainties will have to be taken into account in modeling and analyzing the feedback system. The communication channels in an NCS can be modeled in various ways so as to reveal actual situations. In this paper, we present a two-port NCS model by extending the standard closed-loop system (Fig. 1) to the feedback system with cascaded two-port connections (Fig. 2). Based on the architecture of the two-port NCS, we measure the dynamic uncertainties in the plant and controller with the gap metric, and measure those in the transmission matrices of the two-port networks with $\mathcal{H}_\infty$-norm bounds. Our problem formulation for robust stabilization is mainly motivated by the application scenario in stabilizing a class of feedback systems under communication constraints. Within the feedback loop, the plant and controller cannot communicate directly, and the signals can only pass through the communication network consisting of a sequence of relays. The direct motivating examples are satellite networks [16], wireless sensor networks [17] and other large-scale networks with multiple routings. Moreover, each communication channel between a pair of neighboring relays can be regarded as a subsystem that involves not only multiplicative distortions on the transmitted signal itself, but also additive interferences induced by the signal in the reverse direction. This phenomenon is usually encountered in a bidirectional wireless network subject to channel fading or under malicious attacks [18].

The two-port networks are not new, and have been studied over decades for different purposes. They are first introduced in the electrical circuit theory [19], [20]. Later, they are borrowed to characterize linear time-invariant (LTI) systems with a chain-scattering representation in [21]. In the application of teleoperation in robotics, the two-port networks are used to model communication blocks between the human operator and the environment [22]. The two-port representations have also been used for studying feedback robustness from the perspective of the ν-gap metric [23]. Recently, the two-port networks have been employed in [24]–[26] to model communication channels, and study feedback stabilization for a class of NCSs.

In this paper, we continue the investigation on robust networked control. Partial results of this study have been reported in conference papers [25], [26]. Our main contribution is a clean result on analyzing the stability of the feedback system with uncertainties from multiple sources, including model uncertainties and cascaded two-port uncertainties. As it is known, a general approach to deal with the robust stability problem with multiple sources of uncertainties is through μ analysis [4], [27]. In general, the μ analysis is computationally difficult and even NP hard in the case of multiple (more than two) block-diagonal uncertainties. However, for this specific two-port problem, we can mitigate these difficulties by exploring the special structures of the two-port networks and taking advantage of the geometric insights into the angles and rotations of the subspaces. By generalizing the "arcsine" theorem [8] for a standard feedback system, we are able to obtain a concise necessary and sufficient robust stability condition for the two-port NCS. Furthermore, as the stability margin coincides with that of the standard closed-loop system, the optimal robust controller synthesis problem can be solved by the same one-block $\mathcal{H}_\infty$ optimization [4], [12].

It is worth noting there exist previous works on robust stabilization of NCSs with special communication architectures and various uncertainty descriptions. For example, [22] considers teleoperation of robots through two-port communication networks with time delays, [28] considers a plant with parametric uncertainties over networks subject to data packet loss, [29] considers a plant with polytopic uncertainties in its coefficients over a communication channel subject to fading, and etc. Our work differs from the previous ones mainly in that we model not only the dynamic uncertainties in the plant and controller, but also the distortions and interferences induced by bidirectional communication channels.

The rest of the paper is organized as follows. In Section II, we introduce the notation system and preliminaries of the robust control problem related to the standard closed-loop system. In Section III, the physical meaning of a two-port network is discussed, and a two-port NCS is modeled. In Section IV, we present a necessary and sufficient stability condition for the two-port NCS, and prove its sufficiency. In Section V, we introduce the techniques in the analysis of subspaces and prove the necessity of the robust stability condition. In Section VI, we extend the main stability condition to the case where the uncertainties are frequency-wise bounded, and study the corresponding synthesis problem. In Section VII, we give an analytic example to illustrate the tightness of the stability condition, a simulated example involving nonlinearity and time delay, and a controller synthesis example involving frequency-wise bounded uncertainties. Last, we summarize our contribution and discuss future work in Section VIII.

II. Preliminary Results

A. Notation

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ be the real or complex field, respectively, and $\mathbb{F}^n$ be the linear space of n-dimensional vectors over the field $\mathbb{F}$. For $x \in \mathbb{F}^n$, its Euclidean norm is denoted by $\|x\|$. For matrix $A \in \mathbb{F}^{m\times n}$, its transpose is denoted by A^T, its conjugate (Hermitian) transpose is by A^*, and its ith singular value is by $\sigma_i(A)$, $i = 1, 2, \ldots, \min\{m, n\}$, in a nonincreasing order. The largest singular value is specially denoted by $\bar{\sigma}(A) := \sigma_1(A)$. The operator norm (spectral norm) of A is denoted by $\|A\| := \bar{\sigma}(A)$, and the column range of A is by $\mathcal{R}(A)$. Given matrices

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{F}^{(m+p)\times(m+p)} \text{ and } A \in \mathbb{F}^{p\times m},$$

we define the linear fractional transformation (LFT) as

$$\mathrm{LFT}(M, A) := (M_{22}A + M_{21})(M_{11} + M_{12}A)^{-1},$$

assuming the existence of the above inverse.

The system under consideration is continuous-time and LTI, represented by its transfer matrix. The Laplace variable s may be omitted for simplicity. Denote $\mathcal{L}_2/\mathcal{L}_\infty$ and $\mathcal{H}_2/\mathcal{H}_\infty$ as the standard Lebesgue and Hardy 2-spaces/∞-spaces,

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respectively, and $\mathcal{RL}_\infty/\mathcal{RH}_\infty$ as the set consisting of all real rational members of $\mathcal{L}_\infty/\mathcal{H}_\infty$. Let $\mathcal{P}$ be the field of real rational transfer functions. For transfer matrix $P \in \mathcal{P}^{p\times m}$, its para-Hermitian is denoted by $P^\sim(s) = P^T(-s)$. We say P is stable, if its $\mathcal{H}_\infty$-norm

$$\|P\|_\infty := \sup_{\mathrm{Re}(s)>0} \bar{\sigma}[P(s)]$$

is bounded, in which case, $P \in \mathcal{RH}_\infty^{p\times m}$. The superscript may be omitted for simplicity if it can be inferred from the context. A vector signal $v(t) \in \mathbb{F}^n$ is said to be causal, if $v(t) = 0$ $\forall t < 0$, and to have bounded energy, if its 2-norm

$$\|v\|_2 := \sqrt{\int_0^\infty \|v(t)\|^2 dt}$$

is bounded. The set of Laplace transforms of all causal and energy bounded signals $v(t) \in \mathbb{F}^n$ is precisely $\mathcal{H}_2^n$. The superscript can again be omitted, if the context is clear.

For a possibly unstable system $P \in \mathcal{P}^{p\times m}$, denote its input as u and its output as y, then its graph is defined by the following set

$$\mathcal{G}_P := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} :\ u \in \mathcal{H}_2^m,\ y = Pu \in \mathcal{H}_2^p \right\}.$$

Two transfer matrices M and N in $\mathcal{RH}_\infty$ are (right) coprime if there exist transfer matrices X and Y in $\mathcal{RH}_\infty$ such that

$$XM + YN = I.$$

It is known [14] that every $P \in \mathcal{P}^{p\times m}$ admits a right coprime factorization (RCF):

$$P = NM^{-1},$$

where $M, N \in \mathcal{RH}_\infty$ are right coprime. It has been shown [4] that if $P = NM^{-1}$ is a RCF, then

$$\mathcal{G}_P = \begin{bmatrix} M \\ N \end{bmatrix} \mathcal{H}_2, \tag{1}$$

where $\begin{bmatrix} M \\ N \end{bmatrix}$ is called the graph symbol of P.

B. Robust Stability in Gap Metric

After George Zames introduced the gap metric into the control field [5], there has been a major development in the last few decades as evidenced by [7]–[11]. Next we briefly introduce some key concepts and an application of the gap metric on the robust stability of a standard closed-loop system. Let $\mathcal{X}$ and $\mathcal{Y}$ be two subspaces of a Hilbert space $\mathcal{H}$ and let $\Pi_\mathcal{X}$ and $\Pi_\mathcal{Y}$ be the orthogonal projections on $\mathcal{X}$ and $\mathcal{Y}$, respectively. The gap metric between the two subspaces is defined as

$$\gamma(\mathcal{X}, \mathcal{Y}) := \|\Pi_\mathcal{X} - \Pi_\mathcal{Y}\|, \tag{2}$$

where $\|\cdot\|$ stands for the induced operator norm.

The gap between LTI systems P_1 and $P_2 \in \mathcal{P}$ is defined to be the gap between their system graphs, i.e.,

$$\delta(P_1, P_2) := \gamma(\mathcal{G}_{P_1}, \mathcal{G}_{P_2}).$$

Given system $P \in \mathcal{P}$, denote the gap ball with center P and radius $r \in [0, 1)$ as

$$\mathcal{B}(P, r) := \left\{ \tilde{P} \in \mathcal{P} :\ \delta(P, \tilde{P}) \le r \right\}.$$

The standard closed-loop system in Fig. 1 is denoted as $[P, C]$, where $P \in \mathcal{P}^{p\times m}$ represents the plant and $C \in \mathcal{P}^{m\times p}$ the controller.

The standard closed-loop system $[P, C]$ is well-posed when $I - CP$ has full normal rank, i.e., it only loses ranks for some but not all $s \in \mathbb{C}$. Under this mild condition, the well-known Gang of Four transfer matrix [30] can be represented as

$$\mathrm{GoF}(P, C) := \begin{bmatrix} I \\ P \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} I & -C \end{bmatrix},$$

which will be sometimes abbreviated as GoF for simplicity. The closed-loop system $[P, C]$ is said to be stable, or C is said to stabilize P, if GoF(P, C) is stable, i.e., GoF$(P, C) \in \mathcal{RH}_\infty^{(m+p)\times(m+p)}$.

In robust stability analysis, a fundamental problem is to characterize the largest amount of uncertainty in both the plant and controller for which the closed-loop system maintains its stability. One important robust stability result, with the stability condition given in terms of an "arcsine" inequality, was obtained in [8]. We quote it in the following lemma.

Lemma 1. *Assume that the nominal system $[P, C]$ is stable. For $r_p, r_c \in [0, 1)$, the perturbed system $[\tilde{P}, \tilde{C}]$ is stable for all $\tilde{P} \in \mathcal{B}(P, r_p)$ and $\tilde{C} \in \mathcal{B}(C, r_c)$ if and only if*

$$\arcsin r_p + \arcsin r_c < \arcsin \|\mathrm{GoF}(P, C)\|_\infty^{-1}. \tag{3}$$

Lemma 1 precisely quantifies the largest simultaneous uncertainties in the plant and the controller which the feedback system in Fig. 1 can tolerate while the stability is maintained. The introduction of "arcsine" function in the stability condition (3) is closely related to the fact that $\arcsin \delta(P_1, P_2)$ is a metric for $P_1, P_2 \in \mathcal{P}$, called the angular metric [8]. It turns out that the angular metric is a length [8], [31]. Hence its triangle inequality is tight, which is the essential reason why condition (3) is both necessary and sufficient. Naturally, the value $\|\mathrm{GoF}(P, C)\|_\infty^{-1}$ is regarded as the stability margin of the closed-loop system $[P, C]$. The design of the optimal robust controller can be attributed to solving an $\mathcal{H}_\infty$ problem with respect to the Gang of Four transfer matrix. Specifically, the optimization problem is aimed at computing the following maximum stability margin:

$$\max_C \left\{ \|\mathrm{GoF}(P, C)\|_\infty^{-1} :\ C \text{ stabilizes } P \right\}. \tag{4}$$

This is a special $\mathcal{H}_\infty$ problem, and can be reduced to the Nehari problem, namely, the one-block $\mathcal{H}_\infty$ model matching problem, which has been well studied and neatly solved [2], [7], [13].

III. A Cascaded Two-port Model for NCSs

A. Two-port Networks as Communication Channels

Modeling electrical devices and their connections by two-port networks has been widely accepted and commonly used

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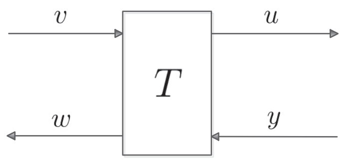
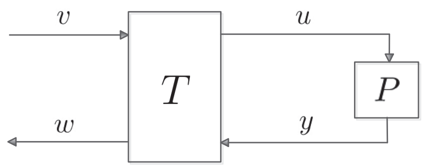
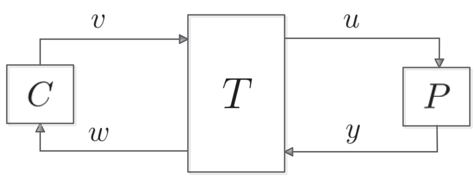


(a) A single two-port network.



(b) A one-stage two-port connection.



(c) Two-port NCS with one-stage of two-port network.

Fig. 3: Two-port network T.

[19], [20]. In our study, we use two-port networks to model uncertain bidirectional communication channels. The network T in Fig. 3(a) has two external ports, with one port composed of signals v, w and the other of signals u, y. A two-port network T has various representations with respect to different parameters, such as impedance parameters, admittance parameters, hybrid parameters, transmission parameters and so on. In this study, we will focus on the representation with the transmission parameters, which will be simply called the transmission representation. Then, we let u, v be the downlink signals both with dimension m and let w, y be the uplink signals both with dimension p. With a little abuse of notation, we denote the transmission (parameter) matrix of the two-port network as T, which fully characterizes the communication channel as follows:

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \mathcal{P}^{(m+p)\times(m+p)} \text{ and } \begin{bmatrix} v \\ w \end{bmatrix} = T \begin{bmatrix} u \\ y \end{bmatrix}. \quad (5)$$

Here, the symbol T stands for both the two-port network itself and its transmission matrix for notational simplicity. When we cascade two channels, their transmission matrices will simply multiply together, which is one major advantage to utilize the transmission representation for communication channels. It is worth noting that the transmission representation of a two-port network is also called the chain-scattering representation and has been used to solve the $\mathcal{H}_\infty$ control problems in [21].

When the communication channel is perfect, i.e., the channel involves no distortions or interferences, the transmission matrix is simply

$$T = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix}.$$

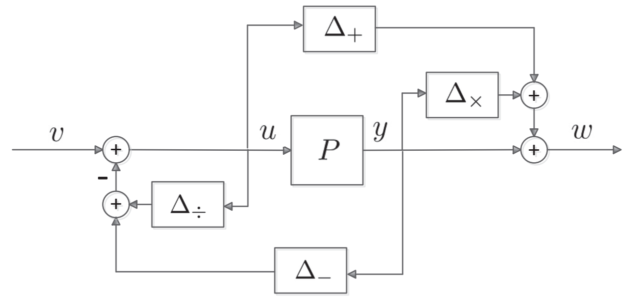


Fig. 4: Plant with the uncertainty quartet.

When the channel admits both distortions and interferences, we model the perturbed transmission matrix as

$$T = I + \Delta = \begin{bmatrix} I_m + \Delta_\div & \Delta_- \\ \Delta_+ & I_p + \Delta_\times \end{bmatrix},$$

and throughout the paper, it is assumed that

$$\Delta = \begin{bmatrix} \Delta_\div & \Delta_- \\ \Delta_+ & \Delta_\times \end{bmatrix} \in \mathcal{RH}_\infty \text{ and } \|\Delta\|_\infty < 1.$$

The subscripts $\div, -, +, \times$ for Δ representing different uncertainties were firstly used in [32] for the purpose to describe various connections vividly, as is shown momentarily. The four-block matrix Δ will be called the uncertainty quartet in this study.

As shown in Fig. 3(b), we connect a two-port network T to the plant P and denote its transmission matrix as $T = I + \Delta$. It is shown in [21], [32] that combining equation (5) with $y = Pu$ yields a perturbed plant $\tilde{P}$ from v to w via the LFT,

$$\begin{aligned} \tilde{P} &= \text{LFT}\left(\begin{bmatrix} I_m + \Delta_\div & \Delta_- \\ \Delta_+ & I_p + \Delta_\times \end{bmatrix}, \ P \right) \\ &= [(I_p + \Delta_\times)P + \Delta_+][I_m + \Delta_\div + \Delta_- P]^{-1}. \end{aligned} \quad (6)$$

The diagram in Fig. 4 shows how the plant P is affected by uncertainties in a quartet. Specifically, we can assign each of the four blocks in an uncertainty quartet a detailed explanation [32], namely, the uncertainty of inverse multiplication ($\div$), inverse addition ($-$), addition ($+$) and multiplication ($\times$). The diagonal blocks $\Delta_\div, \Delta_\times$ and the off-diagonal blocks Δ_-, Δ_+ model two types of perturbations in a two-port network. To be precise, the diagonal blocks represent multiplicative linear distortions of the transmitted signals, mostly due to signal attenuations in the fading channel. The off-diagonal blocks represent additive interference from the reverse signals, which occurs mostly in bidirectional communication channels.

An uncertainty quartet can model some types of malicious or intelligent attacks as well. For instance, $\Delta_\div$ and $\Delta_\times$ can model packet loss processes or denial of service (DoS) attacks within bidirectional communication channels. All four blocks in an uncertainty quartet can be used to model false data injection attacks to the controller or plant.

In order to keep the perturbed system well-defined, we add a mild condition on the channel uncertainty Δ, so that the transfer matrix $I_m + \Delta_\div + \Delta_- P$ has full normal rank. Actually, this can be guaranteed as long as Δ is small enough.

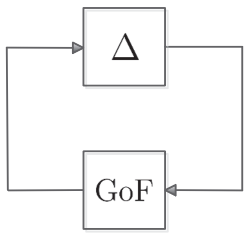


Fig. 5: Standard closed-loop system equivalent to one-stage two-port NCS.

It is worth noting that describing an uncertain system using a linear fractional transformation is not new in robust control [4], [21]. Traditionally, an uncertain system takes the form of LFT(G, Δ), a fixed LFT of an uncertain component Δ. Nevertheless, in our study, an uncertain system takes the form of LFT$(I + \Delta, P)$ that is an uncertain LFT of a possibly known plant P. The main difference between the fixed LFT and the uncertain one relies on how the uncertainty appears and functions within the NCS.

Consider the situation when we close the loop in Fig. 3(b) with controller C that stabilizes P, as in Fig. 3(c). Following the derivation in [24], we separate the uncertainty quartet Δ and the nominal closed-loop system $[P, C]$. Then we obtain a feedback connection as in Fig. 5. The robust stability of this system can be simply analyzed through the well-known small-gain theorem [4, Theorem 8.1], resulting in the following stability condition.

Lemma 2. *Assume that the nominal system $[P, C]$ is stable. For $r \in [0, 1)$, the two-port NCS in Fig. 3(c) is stable for all $\Delta \in \mathcal{RH}_\infty$ with $\|\Delta\|_\infty \le r$ if and only if*

$$r < \|\mathrm{GoF}(P, C)\|_\infty^{-1}. \tag{7}$$

Here, the stability margin $\|\mathrm{GoF}(P, C)\|_\infty^{-1}$ comes into the picture again, as it appears in Lemma 1 for the gap-type uncertainties. This motivates us to explore a unified robust stability condition with combined gap-type model uncertainties and two-port uncertainty quartets, which will be shown in the next section. At this moment, one may wonder if the small-gain theorem can be further utilized in a general cascaded two-port NCS. Actually, neither the small-gain theorem nor the more general method via μ-analysis, is applicable to the robust stabilization of a general two-port NCS due to the existence of simultaneous uncertainties from multiple sources. We will revisit this point soon in the next subsection.

B. Graph Analysis on Cascaded Two-port NCSs

We already know that a system perturbed by an uncertainty quartet can be expressed by LFT$(I+\Delta, P)$. In this subsection, we aim to express the system perturbed by cascaded two-port uncertainties with its graph. As illustrated in Fig. 2, the plant $P = NM^{-1}$ and controller $C = VU^{-1}$ expressed by RCFs communicate with each other through cascaded two-port networks. Considering the input and output of P, we can

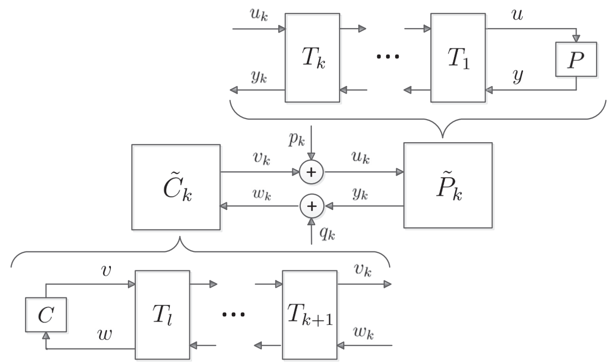


Fig. 6: Equivalent closed-loop system.

represent every element in the graph of P through its graph symbol as

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix} x, \tag{8}$$

for every $x \in \mathcal{H}_2$.

Consider the transmission representations of the two-port networks $\{T_k\}_{k=1}^{l}$. If the kth stage of the two-port network admits an uncertainty $\Delta_k \in \mathcal{RH}_\infty$, then the corresponding transmission matrix is denoted as $T_k = I + \Delta_k$. For each integer $k \in [1, l-1]$, we can associate the first k stages of the cascaded two-port networks with the plant P, and the remaining $l-k$ stages with the controller C. Then the diagram in Fig. 2 is equivalently transformed into that in Fig. 6 to form a standard closed-loop system $[\tilde{P}_k, \tilde{C}_k]$. Signals in Fig. 6 are described by

$$\begin{bmatrix} u_k \\ y_k \end{bmatrix} = T_k \cdots T_1 \begin{bmatrix} u \\ y \end{bmatrix} = (I + \Delta_k) \cdots (I + \Delta_1) \begin{bmatrix} u \\ y \end{bmatrix},$$
$$\begin{bmatrix} v_k \\ w_k \end{bmatrix} = T_{k+1}^{-1} \cdots T_l^{-1} \begin{bmatrix} v \\ w \end{bmatrix} = (I + \Delta_{k+1})^{-1} \cdots (I + \Delta_l)^{-1} \begin{bmatrix} v \\ w \end{bmatrix}.$$

Take equations (1) and (8) into account. If we view P together with $\{T_j\}_{j=1}^{k}$ as a perturbed plant $\tilde{P}_k$ with uncertainties $\{\Delta_j\}_{j=1}^{k}$, $\tilde{P}_k = N_k M_k^{-1}$ can be determined by its graph

$$\mathcal{G}_{\tilde{P}_k} = \begin{bmatrix} M_k \\ N_k \end{bmatrix} \mathcal{H}_2 = (I + \Delta_k) \cdots (I + \Delta_1) \mathcal{G}_P. \tag{9}$$

Similarly, if we view C together with $\{T_j\}_{j=k+1}^{l}$ as a perturbed controller $\tilde{C}_k$ with uncertainties $\{\Delta_j\}_{j=k+1}^{l}$, $\tilde{C}_k = V_k U_k^{-1}$ can be determined by its inverse graph

$$\mathcal{G}'_{\tilde{C}_k} = \begin{bmatrix} V_k \\ U_k \end{bmatrix} \mathcal{H}_2 = (I + \Delta_{k+1})^{-1} \cdots (I + \Delta_l)^{-1} \mathcal{G}'_C, \tag{10}$$

where the inverse graph $\mathcal{G}'_C$ of $C = VU^{-1}$ is defined as

$$\mathcal{G}'_C = \begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H}_2. \tag{11}$$

Note that the system in Fig. 6 is an equivalent standard closed-loop system $[\tilde{P}_k, \tilde{C}_k]$ with $\tilde{P}_k$ as plant and $\tilde{C}_k$ as controller. For convenience, we include $k = 0$ and $k = l$ with the interpretation that the entire two-port network is grouped with controller C for $k = 0$, and with plant P for $k = l$. Since

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we assume $\Delta_k \in \mathcal{RH}_\infty$ and $\|\Delta_k\|_\infty < 1$, both $I + \Delta_k$ and $(I + \Delta_k)^{-1}$ are in $\mathcal{RH}_\infty$. Hence (M_k, N_k) is right coprime, and so is (U_k, V_k). As the systems are well-posed here, both M_k and U_k have full normal rank. Therefore, the perturbed plants and controllers $\tilde{P}_k$ and $\tilde{C}_k$ are well-defined by their respective graphs for each $k = 0, 1, \ldots, l$.

C. Stability of Two-port NCSs

The stability of the two-port NCS is defined in a usual way.

Definition 1. *The NCS in Fig. 6 is said to be stable if for all simultaneously injected signals p_k and $q_k \in \mathcal{H}_2$, $k = 0, 1, \ldots, l$, it holds that the signals on all ports, namely, u_k, y_k, v_k and w_k, $k = 0, 1, \ldots, l$, are in $\mathcal{H}_2$.*

In brief, to verify stability of an NCS, we inject energy-bounded signals from all possible inputs, then check if all the output signals are energy-bounded. Note that the NCS is composed of LTI systems so that it possesses the superposition principle. Hence the procedures to determine the stability can be simplified as follows.

Lemma 3. *The NCS is stable if and only if the equivalent closed-loop system $[\tilde{P}_k, \tilde{C}_k]$ is stable for $k = 0, 1, \ldots, l$.*

Proof. The necessity is obvious as $[\tilde{P}_k, \tilde{C}_k]$ will be stable if the NCS is stable. For sufficiency, assume that $[\tilde{P}_k, \tilde{C}_k]$ is stable. Let $p_k, q_k \in \mathcal{H}_2$, then it follows that $u_k, w_k \in \mathcal{H}_2$. It is also clear from the well-posedness assumption that $I + \Delta_k$ and $(I + \Delta_k)^{-1} \in \mathcal{RH}_\infty$. Hence every signal in the NCS can be regarded as an output of a stable system with u_k and $w_k \in \mathcal{H}_2$ as its input signals. Then the signals on all ports are in $\mathcal{H}_2$, which completes the proof. $\square$

Naturally, one would ask what the exact condition is for the two-port NCS in Fig. 2 to be stable, if the plant, controller and communication channels involve simultaneous uncertainties introduced previously. It is worth noting that the uncertainties in the plant and controller are modeled differently from those in the two-port communication channels, which introduces a possible technical difficulty. If only the uncertainty quartets are assumed to exist in the two-port NCS, then one may attempt to solve this particular robust stability problem through μ-analysis [4], [27]. However, in general, the exact computation of μ-value is NP hard and only a numerical upper bound is obtainable via the convex optimization [33].

IV. Robust Stability of Two-port NCSs

The main result of this paper concerns the robust stability condition of the two-port NCS, which is stated as follows.

Theorem 1. *Assume that the nominal system $[P, C]$ is stable. For $r_p,\ r_c,\ r_k \in [0, 1)$, the NCS in Fig. 2 is stable for all $\tilde{P} \in \mathcal{B}(P, r_p)$, $\tilde{C} \in \mathcal{B}(C, r_c)$ and $\Delta_k \in \mathcal{RH}_\infty$ with $\|\Delta_k\|_\infty \le r_k,\ k = 1, 2, \ldots, l$, if and only if*

$$\arcsin r_p + \arcsin r_c + \sum_{k=1}^{l} \arcsin r_k < \arcsin \|\mathrm{GoF}(P, C)\|_\infty^{-1}. \quad (12)$$

This theorem characterizes the trade-off between two types of uncertainties from the system modeling and the communication channels, as revealed in equation (12). Theorem 1 reduces to Lemma 1 by letting $r_k = 0$ for each integer $k \in [1, l]$. It reduces to Lemma 2 by letting $r_k = 0$ for each integer $k \in [2, l]$, $r_p = 0$ and $r_c = 0$. The stability margin $\|\mathrm{GoF}(P, C)\|_\infty^{-1}$ given here is the same as that in Lemmas 1 and 2, resulting in the same controller synthesis problem.

Theorem 1 was first announced in conference paper [25] without proof. In the rest of this section, we prove its sufficiency part, and in the next section, we prove the necessity part after introducing the needed mathematical background.

An uncertainty quartet in a two-port network introduces two special uncertainty neighborhoods, which are related to the graphs of systems, shown in equations (9) and (10). The formal definition of the neighborhoods are as follows.

Definition 2. *Let $P = NM^{-1}$ be an RCF. Two uncertainty neighborhoods, centered at P and with radius $r \in [0, 1)$, are defined as*

$$\mathcal{N}_1(P, r) = \left\{ \tilde{P} = \tilde{N}\tilde{M}^{-1} : \ \begin{bmatrix} \tilde{M} \\ \tilde{N} \end{bmatrix} = (I + \Delta) \begin{bmatrix} M \\ N \end{bmatrix}, \right. \left. \Delta \in \mathcal{RH}_\infty, \|\Delta\|_\infty \le r \right\},$$

$$\mathcal{N}_2(P, r) = \left\{ \tilde{P} = \tilde{N}\tilde{M}^{-1} : \ \begin{bmatrix} \tilde{M} \\ \tilde{N} \end{bmatrix} = (I + \Delta)^{-1} \begin{bmatrix} M \\ N \end{bmatrix}, \right. \left. \Delta \in \mathcal{RH}_\infty, \|\Delta\|_\infty \le r \right\}.$$

It can be seen that $\mathcal{N}_1(P, r)$ corresponds to the uncertainty neighborhood at the plant side as in (9), and $\mathcal{N}_2(P, r)$ corresponds to the neighborhood at the controller side as in (10). These two types of uncertainty neighborhoods are also closely related to the gap balls. The following result is true in light of [34, Theorem 1].

Lemma 4. *For every $r \in [0, 1)$, it holds*

$$\mathcal{N}_1(P, r) \cup \mathcal{N}_2(P, r) \subset \mathcal{B}(P, r).$$

The above lemma shows that the two-port neighborhoods are contained in the gap ball with the same nominal system P and radius r. Using this result, we can show the sufficiency part of Theorem 1 as follows.

Let r_p, r_c and $\{r_k\}_{k=1}^{l}$ satisfy condition (12). For each integer $k \in [0, l]$, excite the network at only the kth stage, as in Fig. 6. A slight difference in the following is that the plant and controller are replaced with the perturbed ones, $\tilde{P}$ and $\tilde{C}$, respectively. Denote their RCFs as $\tilde{P} = \tilde{N}\tilde{M}^{-1}$ and $\tilde{C} = \tilde{V}\tilde{U}^{-1}$, respectively, and let

$$\begin{bmatrix} M_k \\ N_k \end{bmatrix} = (I + \Delta_k) \cdots (I + \Delta_1) \begin{bmatrix} \tilde{M} \\ \tilde{N} \end{bmatrix},$$

$$\begin{bmatrix} V_k \\ U_k \end{bmatrix} = (I + \Delta_{k+1})^{-1} \cdots (I + \Delta_l)^{-1} \begin{bmatrix} \tilde{V} \\ \tilde{U} \end{bmatrix}.$$

From the well-posedness assumption, we know that the kth perturbed plant $\tilde{P}_k = N_k M_k^{-1}$ is well-defined. So is the perturbed controller $\tilde{C}_k = V_k U_k^{-1}$. Denote $\tilde{P}_0 = \tilde{P}$ and $\tilde{C}_l = \tilde{C}$ for convenience. It follows from Lemma 4 that

$$\tilde{P}_k \in \mathcal{N}_1(\tilde{P}_{k-1}, r_k) \subset \mathcal{B}(\tilde{P}_{k-1}, r_k),$$
$$\tilde{C}_k \in \mathcal{N}_2(\tilde{C}_{k+1}, r_{k+1}) \subset \mathcal{B}(\tilde{C}_{k+1}, r_{k+1}).$$

By iteratively utilizing the triangular inequality of the angular metric [8, Proposition 1], we obtain inequalities

$$\arcsin \delta(\tilde{P}_k, \tilde{P}) \leq \sum_{j=1}^{k} \arcsin \delta(\tilde{P}_j, \tilde{P}_{j-1}) \leq \sum_{j=1}^{k} \arcsin r_j,$$

$$\arcsin \delta(\tilde{C}_k, \tilde{C}) \leq \sum_{j=k+1}^{l} \arcsin \delta(\tilde{C}_{j-1}, \tilde{C}_j) \leq \sum_{j=k+1}^{l} \arcsin r_j.$$

Recall $\tilde{P} \in \mathcal{B}(P, r_p)$ and $\tilde{C} \in \mathcal{B}(C, r_c)$. Applying the triangular inequality again yields

$$\arcsin \delta(\tilde{P}_k, P) \leq \arcsin r_p + \sum_{j=1}^{k} \arcsin r_j,$$
$$\arcsin \delta(\tilde{C}_k, C) \leq \arcsin r_c + \sum_{j=k+1}^{l} \arcsin r_j.$$

It follows from Lemma 1 and condition (12) that the equivalent closed-loop system $[\tilde{P}_k, \tilde{C}_k]$ in Fig. 6 is stable for each integer $k \in [0, l]$, and thus the NCS in Fig. 2 is robustly stable, in light of Lemma 3. This completes the proof for the sufficiency part of Theorem 1.

V. Necessity of the Robust Stability Condition

The necessity of Theorem 1 will be proved by using the contrapositive argument. Given condition (12) violated, we will construct a perturbed plant $\tilde{P}$, a perturbed controller $\tilde{C}$ and a series of uncertainties $\{\Delta_k\}_{k=1}^{l}$, which destabilize the NCS.

The stability of the feedback system is closely related to the minimum angle between the graphs of the plant and controller. The construction of the worst-case uncertainties involved in $\tilde{P}$ and $\tilde{C}$ is based on [8, Theorem 2]. The construction of the worst-case uncertainty quartets $\{\Delta_k\}_{k=1}^{l}$ is based on geometric properties of finite dimensional subspaces involving the rotation of vectors between subspaces.

A. Grassmann Manifold and Rotation of Subspaces

This subsection develops technical results in finite dimensional subspaces, which help construct the worst-case uncertainty quartets. For detailed background, see [35]–[37].

The set of all m-dimensional subspaces in $\mathbb{F}^n$, denoted by $\mathcal{G}_{m,n}$, is called the Grassmann manifold. For $\mathcal{U} \in \mathcal{G}_{m,n}$, its orthogonal complement is denoted by $\mathcal{U}_\perp$. Let $U \in \mathbb{F}^{n\times m}$ and $U_\perp \in \mathbb{F}^{n\times(n-m)}$ be the isometries onto $\mathcal{U}$ and $\mathcal{U}_\perp$, respectively. Similar notation applies to another subspace $\mathcal{V} \in \mathcal{G}_{m,n}$. The principal angles between $\mathcal{U}$ and $\mathcal{V}$ are defined as [36], [37]

$$\theta_i(\mathcal{U}, \mathcal{V}) := \arccos \sigma_{m+1-i}(U^* V)$$

in a nonincreasing order for $i = 1, 2, \ldots, m$. Denote

$$\theta(\mathcal{U}, \mathcal{V}) := [\ \theta_1(\mathcal{U}, \mathcal{V}) \ \cdots \ \theta_m(\mathcal{U}, \mathcal{V})\ ]^T$$

as the vector composed of principal angles. Denote the largest angle as $\bar{\theta}(\mathcal{U}, \mathcal{V}) := \theta_1(\mathcal{U}, \mathcal{V})$ and the smallest one as $\underline{\theta}(\mathcal{U}, \mathcal{V}) := \theta_m(\mathcal{U}, \mathcal{V})$.

The gap between $\mathcal{U}$ and $\mathcal{V}$, as defined in (2), can now be obtained in various ways as follows [10], [36]:

$$\begin{aligned}\gamma(\mathcal{U}, \mathcal{V}) &= \|U^* V_\perp\| = \min_{Q\in\mathbb{F}^{m\times m}} \|U - VQ\| \\ &= \sin \bar{\theta}(\mathcal{U}, \mathcal{V}).\end{aligned} \tag{13}$$

Without loss of generality, assume $2m \leq n$. Applying the C-S decomposition [35] to unitary matrix $[U\ U_\perp]^* [V\ V_\perp]$ yields

$$[U\ U_\perp]^* [V\ V_\perp] = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix}^*,$$

where $X_1, Y_1 \in \mathbb{F}^{m\times m}$ and $X_2, Y_2 \in \mathbb{F}^{(n-m)\times(n-m)}$ are unitary matrices, and

$$C = \mathrm{diag}[\cos\theta_1(\mathcal{U}, \mathcal{V}), \cos\theta_2(\mathcal{U}, \mathcal{V}), \ldots, \cos\theta_m(\mathcal{U}, \mathcal{V})],$$
$$S = \mathrm{diag}[\sin\theta_1(\mathcal{U}, \mathcal{V}), \sin\theta_2(\mathcal{U}, \mathcal{V}), \ldots, \sin\theta_m(\mathcal{U}, \mathcal{V})]$$

are diagonal matrices. We now construct the principal bases of $\mathcal{U}$, $\mathcal{U}_\perp$, $\mathcal{V}$ and $\mathcal{V}_\perp$, respectively, as

$$\begin{bmatrix} \hat{U}\ \hat{U}_\perp \end{bmatrix} = [U\ U_\perp] \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix},$$
$$\begin{bmatrix} \hat{V}\ \hat{V}_\perp \end{bmatrix} = [V\ V_\perp] \begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix}.$$

Then there holds equality

$$\begin{bmatrix} \hat{U}\ \hat{U}_\perp \end{bmatrix}^* \begin{bmatrix} \hat{V}\ \hat{V}_\perp \end{bmatrix} = \begin{bmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & I \end{bmatrix} = \exp \begin{bmatrix} 0 & -\Theta & 0 \\ \Theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\Theta = \mathrm{diag}[\theta(\mathcal{U}, \mathcal{V})]$ is a diagonal matrix of the ordered principal angles. We next construct a set of unitary operators parametrized by $\lambda \in [0, 1]$ as follows:

$$\begin{aligned}\Phi_\lambda : \mathcal{G}_{m,n} \to \mathcal{G}_{m,n} &= \mathcal{X} \mapsto \Phi_\lambda \mathcal{X} \\ &= \begin{bmatrix} \hat{U}\ \hat{U}_\perp \end{bmatrix} \exp \begin{bmatrix} 0 & -\lambda\Theta & 0 \\ \lambda\Theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{U}\ \hat{U}_\perp \end{bmatrix}^*.\end{aligned} \tag{14}$$

It is straightforward to verify that

$$\Phi_0 \begin{bmatrix} \hat{U}\ \hat{U}_\perp \end{bmatrix} = \begin{bmatrix} \hat{U}\ \hat{U}_\perp \end{bmatrix} \text{ and } \Phi_1 \begin{bmatrix} \hat{U}\ \hat{U}_\perp \end{bmatrix} = \begin{bmatrix} \hat{V}\ \hat{V}_\perp \end{bmatrix}.$$

As a consequence, $\Phi_\lambda \mathcal{U}$ for $\lambda \in [0, 1]$ is a curve (geodesic) from $\mathcal{U}$ to $\mathcal{V}$ [36]. Here, Φ_1 is called the direct rotation from $\mathcal{U}$ to $\mathcal{V}$.

The principal angles, especially the smallest one, can be defined for a pair of subspaces with possibly different dimensions. Let $\mathcal{U} \in \mathcal{G}_{m,n}$ and $\mathcal{W} \in \mathcal{G}_{p,n}$. The smallest principal angle between them is defined as

$$\begin{aligned}\underline{\theta}(\mathcal{U}, \mathcal{W}) :&= \arccos \bar{\sigma}(U^* W) \\ &= \min_{\substack{u\in\mathcal{U}, \|u\|=1, \\ w\in\mathcal{W}, \|w\|=1}} \arccos \mathrm{Re}(u^* w),\end{aligned} \tag{15}$$

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where $W \in \mathbb{F}^{n\times p}$ is an isometry onto $\mathcal{W}$ and the second equality is simply from the singular value decomposition (SVD) on U^*W. Let the minimum in (15) be attained at unit vectors $\hat{u} \in \mathcal{U}$ and $\hat{w} \in \mathcal{W}$. Then $\hat{u}$ and $\hat{w}$ form a pair of principal vectors corresponding to the smallest principal angle $\underline{\theta}(\mathcal{U},\mathcal{W})$, in the sense that

$$\hat{u}^*\hat{w} = \cos\underline{\theta}(\mathcal{U},\mathcal{W}). \tag{16}$$

Based on the rotation of subspaces, we introduce the following result on how to construct matrices $\{\Delta_k\}_{k=1}^l$ so that the rotated subspace $(I+\Delta_l)\cdots(I+\Delta_1)\mathcal{U}$ intersects $\mathcal{W}$ nontrivially.

Proposition 1. *Let $\mathcal{U} \in \mathcal{G}_{m,n}$ and $\mathcal{W} \in \mathcal{G}_{p,n}$, with $r = \sin\underline{\theta}(\mathcal{U},\mathcal{W}) \in [0,1)$. Then for $r_k \in [0,1)$, subject to $\sum_{k=1}^l \arcsin r_k = \arcsin r$, there exist a series of matrices $\Delta_k \in \mathbb{F}^{n\times n}$, satisfying $\|\Delta_k\| = r_k,\ k=1,2,\ldots,l$, such that*

$$\underline{\theta}\left[(I+\Delta_l)\cdots(I+\Delta_1)\mathcal{U},\mathcal{W}\right] = 0.$$

Proof. We prove the proposition by construction. Let $\hat{u}$ and $\hat{w}$ be defined in (16), and find a matrix $\hat{U}_0$ such that $\begin{bmatrix}\hat{u} & \hat{U}_0\end{bmatrix} \in \mathbb{F}^{n\times m}$ be an isometry onto $\mathcal{U}$. Construct an m-dimensional subspace $\mathcal{V} := \mathcal{R}\left(\begin{bmatrix}\hat{w} & \hat{U}_0\end{bmatrix}\right)$, and it is clear that $\theta(\mathcal{U},\mathcal{V}) = \begin{bmatrix}\underline{\theta}(\mathcal{U},\mathcal{W}) & 0 & \cdots & 0\end{bmatrix}^T$ and $\underline{\theta}(\mathcal{V},\mathcal{W}) = 0$. Hence the only nonzero (the largest) angle between $\mathcal{U}$ and $\mathcal{V}$ satisfies that

$$\bar{\theta}(\mathcal{U},\mathcal{V}) = \underline{\theta}(\mathcal{U},\mathcal{W}) = \arcsin r := \psi.$$

Let Φ_λ be defined in (14). Clearly, the principal bases of $\mathcal{U}$ and $\mathcal{V}$ are $\begin{bmatrix}\hat{u} & \hat{U}_0\end{bmatrix}$ and $\begin{bmatrix}\hat{w} & \hat{U}_0\end{bmatrix}$ respectively. We set

$$\lambda_k = \frac{\sum_{j=1}^k \arcsin r_j}{\arcsin r}, \quad k = 0,1,\ldots,l.$$

Then $\lambda_0 = 0$ and $\lambda_l = 1$. Denote by $\hat{V}_k := \Phi_{\lambda_k}\hat{U}$. Hence, $\hat{V}_k = \begin{bmatrix}\hat{v}_k & \hat{U}_0\end{bmatrix}$ for $k = 0,1,\ldots,l$, with $\hat{v}_0 = \hat{u}$ and $\hat{v}_l = \hat{w}$. By the properties of the direct rotation [37], we know that the only nonzero (the largest) principal angle between $\mathcal{R}(\hat{V}_{k-1})$ and $\mathcal{R}(\hat{V}_k)$ is

$$\arccos(\hat{v}_{k-1}^*\hat{v}_k) = \arcsin r_k := \psi_k,\ k = 1,2,\ldots,l.$$

Next for each integer $k \in [1,l]$, we construct

$$\Delta_k = (\hat{v}_k\cos\psi_k - \hat{v}_{k-1})\,\hat{v}_{k-1}^*. \tag{17}$$

Based on simple plane geometry, we know

$$\begin{aligned}\|\Delta_k\| &= \left\|(\hat{v}_k\cos\psi_k - \hat{v}_{k-1})\,\hat{v}_{k-1}^*\right\| \\ &= \|\hat{v}_k\cos\psi_k - \hat{v}_{k-1}\| \\ &= \sin\psi_k = r_k.\end{aligned}$$

We claim

$$\mathcal{V} = (I+\Delta_l)\cdots(I+\Delta_1)\mathcal{U}, \tag{18}$$

which can be deduced from that

$$\begin{aligned}(I+\Delta_1)\hat{U} &= \hat{U} + (\hat{v}_1\cos\psi_1 - \hat{u})\hat{u}^*\begin{bmatrix}\hat{u} & \hat{U}_0\end{bmatrix} \\ &= \begin{bmatrix}\hat{u} & \hat{U}_0\end{bmatrix} + (\hat{v}_1\cos\psi_1 - \hat{u})\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix} \\ &= \begin{bmatrix}\hat{v}_1\cos\psi_1 & \hat{U}_0\end{bmatrix},\end{aligned}$$

and that each integer $k \in [1,l-1]$,

$$\begin{aligned}&(I+\Delta_{k+1})\begin{bmatrix}\hat{v}_k\cos\psi_1\cdots\cos\psi_k & \hat{U}_0\end{bmatrix} \\ &= \begin{bmatrix}\hat{v}_k\cos\psi_1\cdots\cos\psi_k & \hat{U}_0\end{bmatrix} \\ &\quad + (\hat{v}_{k+1}\cos\psi_{k+1} - \hat{v}_k)\begin{bmatrix}\cos\psi_1\cdots\cos\psi_k & 0 & \cdots & 0\end{bmatrix} \\ &= \begin{bmatrix}\hat{v}_{k+1}\cos\psi_1\cdots\cos\psi_{k+1} & \hat{U}_0\end{bmatrix}.\end{aligned}$$

Therefore, we have

$$\underline{\theta}\left[(I+\Delta_l)\cdots(I+\Delta_1)\mathcal{U},\mathcal{W}\right] = \underline{\theta}\left(\mathcal{V},\mathcal{W}\right) = 0,$$

which completes the proof. □

B. Proof for the Necessity Part

In this subsection, we show the necessity of the robust stability condition in Theorem 1. Based on Lemma 3, we know the stability of the two-port NCS relies on the stability of the equivalent closed-loop systems $[\tilde{P}_k,\tilde{C}_k]$ for $k = 0,1,\ldots,l$. Moreover, we intend to relate the stability to what we have introduced about the subspaces, based on the following graphical interpretation of the closed-loop stability. The stability of a closed-loop system $[P,C]$ is equivalent to the complementarity of the graphs of the plant and controller [8], i.e.,

$$\mathcal{G}_P \oplus \mathcal{G}'_C = \mathcal{H}_2^m \oplus \mathcal{H}_2^p,$$

where symbol $\oplus$ denotes the direct sum of two linear subspaces. Consequently, a necessary condition for $[P,C]$ to be stable is given in [10, Proposition 19] as

$$\begin{aligned}&\|\mathrm{GoF}(P,C)\|_\infty^{-1} \\ &\quad= \min_{\omega\in\mathbb{R}\cup\{\infty\}} \sin\underline{\theta}[\mathcal{G}(P)(j\omega),\mathcal{G}'(C)(j\omega)] > 0,\end{aligned} \tag{19}$$

where
$$\mathcal{G}(P)(s) := \mathcal{R}\left(\begin{bmatrix}M(s)\\ N(s)\end{bmatrix}\right)$$

is defined as the pointwise graph of plant $P = NM^{-1}$ for each s on the closed right half plane, and similarly, $\mathcal{G}'(C)(s)$ as the inverse pointwise graph of controller C.

Note that the contents of Subsection V-A are about rotations of finite dimensional subspaces through a series of multiplicative operators in the form of $I+\Delta_k$, where $\{\Delta_k\}_{k=1}^l$ are well bounded. To connect the static matrices with transfer matrices, we state the following lemma on interpolating a complex-valued matrix with a stable real rational transfer matrix.

Lemma 5. *For matrix $\Delta \in \mathbb{C}^{n\times n}$ and some frequency $\bar{\omega} \in \mathbb{R}\cup\{\infty\}$, there exists a transfer matrix $\tilde{\Delta} \in \mathcal{RH}_\infty$ that interpolates Δ at $s = j\bar{\omega}$, i.e., $\tilde{\Delta}(j\bar{\omega}) = \Delta$, and satisfies*

$$\|\tilde{\Delta}\|_\infty = \|\Delta\|.$$

This lemma can be proved through the standard boundary interpolation [12], [14].

We are now ready to prove the necessity part of Theorem 1. Using the contrapositive argument, we assume that

$$\arcsin r_p + \arcsin r_c + \sum_{k=1}^l \arcsin r_k \geq \arcsin\|\mathrm{GoF}(P,C)\|_\infty^{-1}.$$

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Next we will construct the worst-case uncertainties such that the two-port NCS is unstable. Specifically, we only need to show that for a certain stage k, say $k = l$, the equivalent closed-loop system $[\tilde{P}_l,\ \tilde{C}]$ is unstable. From [8, Theorem 2], there holds

$$\begin{aligned}\min\left\{\arcsin\left\|\mathrm{GoF}(\tilde{P},\tilde{C})\right\|_\infty^{-1} : \tilde{P}\in\mathcal{B}(P,r_p),\tilde{C}\in\mathcal{B}(C,r_c)\right\}\\ =\arcsin\|\mathrm{GoF}(P,C)\|_\infty^{-1}-\arcsin r_p-\arcsin r_c\\ \leq\sum_{k=1}^{l}\arcsin r_k.\end{aligned}\tag{20}$$

Let a particular closed-loop system $[\tilde{P},\tilde{C}]$ satisfy

$$\left\|\mathrm{GoF}(\tilde{P},\tilde{C})\right\|_\infty^{-1}\leq\sin\left(\sum_{k=1}^{l}\arcsin r_k\right)$$

with $\tilde{P}\in\mathcal{B}(P,r_p)$ and $\tilde{C}\in\mathcal{B}(C,r_c)$. If

$$\left\|\mathrm{GoF}(\tilde{P},\tilde{C})\right\|_\infty^{-1}<\sin\left(\sum_{k=1}^{l}\arcsin r_k\right),$$

we can always choose $\bar{r}_k\in(0,r_k]$ for $k=1,2,\ldots,l$ such that $\|\mathrm{GoF}(\tilde{P},\tilde{C})\|_\infty^{-1}=\sin\left(\sum_{k=1}^{l}\arcsin\bar{r}_k\right)$. Hence, without loss of generality, it is assumed that

$$\left\|\mathrm{GoF}(\tilde{P},\tilde{C})\right\|_\infty^{-1}=\sin\left(\sum_{k=1}^{l}\arcsin r_k\right)\tag{21}$$

holds for some $\tilde{P}\in\mathcal{B}(P,r_p)$ and $\tilde{C}\in\mathcal{B}(C,r_c)$.

The above completes the construction of the worst-case perturbed plant $\tilde{P}$ and controller $\tilde{C}$. Next we will construct uncertainty quartets $\{\Delta_k\}_{k=1}^{l}$ in order to complete the proof.

It can be deduced from the stability of $[\tilde{P},\tilde{C}]$ and equation (19) that the most "vulnerable" frequency is given by

$$\bar{\omega}:=\operatorname*{argmin}_{\omega\in\mathbb{R}\cup\{\infty\}}\sin\underline{\theta}\left[\mathcal{G}(\tilde{P})(j\omega),\mathcal{G}'(\tilde{C})(j\omega)\right].\tag{22}$$

Noting $\tilde{P}\in\mathcal{P}^{p\times m}$ and $\tilde{C}\in\mathcal{P}^{m\times p}$, we let $n=m+p$, and denote subspaces $\mathcal{U}=\mathcal{G}(\tilde{P})(j\bar{\omega})\in\mathcal{G}_{m,n}$ and $\mathcal{W}=\mathcal{G}'(\tilde{C})(j\bar{\omega})\in\mathcal{G}_{p,n}$. It follows from (19) that

$$\begin{aligned}\underline{\theta}(\mathcal{U},\mathcal{W})=\underline{\theta}\left[\mathcal{G}(\tilde{P})(j\bar{\omega}),\mathcal{G}'(\tilde{C})(j\bar{\omega})\right]\\ =\arcsin\left\|\mathrm{GoF}(\tilde{P},\tilde{C})\right\|_\infty^{-1}.\end{aligned}\tag{23}$$

Following from Proposition 1 with $\mathbb{F}=\mathbb{C}$ and $\{r_k\}_{k=1}^{l}$ subject to (21), there exist matrices $\Delta_k\in\mathbb{C}^{n\times n}$ satisfying $\|\Delta_k\|=r_k$ such that

$$\underline{\theta}\left[(I+\Delta_l)\cdots(I+\Delta_1)\mathcal{U},\mathcal{W}\right]=0.\tag{24}$$

Lemma 5 implies the existence of transfer matrix (uncertainty quartet) $\tilde{\Delta}_k\in\mathcal{RH}_\infty$ that interpolates Δ_k for each integer $k\in[1,l]$ in the sense that

$$\tilde{\Delta}_k(j\bar{\omega})=\Delta_k \text{ and } \|\tilde{\Delta}_k\|_\infty=\|\Delta_k\|.$$

Let the perturbed plant $\tilde{P}_l=N_lM_l^{-1}$ be determined by

$$\begin{bmatrix}M_l\\N_l\end{bmatrix}=\left(I+\tilde{\Delta}_l\right)\cdots\left(I+\tilde{\Delta}_1\right)\begin{bmatrix}\tilde{M}\\\tilde{N}\end{bmatrix},$$

where $\tilde{N}\tilde{M}^{-1}$ is a RCF of $\tilde{P}$. As a result, it is clear that

$$\mathcal{G}(\tilde{P}_l)(j\bar{\omega})=(I+\Delta_l)\cdots(I+\Delta_1)\mathcal{G}(\tilde{P})(j\bar{\omega}).$$

From equation (24), we conclude that

$$\underline{\theta}\left[\mathcal{G}(\tilde{P}_l)(j\bar{\omega}),\mathcal{G}'(\tilde{C})(j\bar{\omega})\right]=0.$$

Hence, the necessary condition for stability (19) is violated as

$$\min_{\omega\in\mathbb{R}\cup\{\infty\}}\sin\underline{\theta}\left[\mathcal{G}(\tilde{P}_l)(j\omega),\mathcal{G}'(\tilde{C})(j\omega)\right]=\sin 0=0,$$

and $[\tilde{P}_l,\ \tilde{C}]$ must be unstable. This completes the proof for the necessity part of Theorem 1.

VI. Extension to Frequency-wise Stability Condition

In retrospect, we have assumed that the uncertainty quartets $\Delta_k\in\mathcal{RH}_\infty$ for $1\leq k\leq l$ are measured by their $\mathcal{H}_\infty$ norms, and the model uncertainties in the plant and controller are measured by the gap metric. In this section, we will study the stabilization of the two-port NCS with respect to frequency-wise bounded uncertainties. Additionally, we use the ν-gap metric [12] to describe the plant and controller uncertainties so to further reduce the conservatism of the robustness analysis.

In terms of practical systems, we may have some prior knowledge of the uncertainties, which is more precise than uniform bounds over all frequencies. For instance, the uncertainties in mechanical systems can be estimated by analyzing the mechanical structures, the uncertainties in wireless communication channels can be estimated by analyzing the transmission medium and the distribution of obstacles, and so on. Through frequency-wise bounds, more precise characterizations in terms of the weighting functions can be adopted to describe the uncertainties, in contrast to the previously studied ones with uniform $\mathcal{H}_\infty$-norm bounds or the gap-type bounds. Furthermore, appropriate weightings on the uncertainties and systems are often employed to solve the synthesis problem for $\mathcal{H}_\infty$ control, which provide the flexibility on fine tuning the feedback controllers. These weighting functions become more important in complex networks involving multiple dynamical systems and uncertainties, such as satellite networks, wireless sensor networks and two-port NCSs.

A. Stability Condition

We start with the definitions of weighting functions. The frequency-wise bound for the uncertainty quartet Δ is described by

$$\bar{\sigma}[\Delta(j\omega)]\leq|W(j\omega)|,\quad\forall\ \omega\in\mathbb{R},\tag{25}$$

where $W\in\mathcal{RH}_\infty$ is a scalar weighting function. Without loss of generality, $W^{-1}\in\mathcal{RH}_\infty$ is assumed. Then equation (25) can be equivalently expressed as $\|W^{-1}\Delta\|_\infty\leq 1$. If $W(j\omega)$ is a constant function, we return to the unweighted case.

The frequency-wise bound for the gap-type uncertainty is related to the ν-gap metric [11], [12]. For a closed-loop system $[P,C]$, denote by $\eta[P,C]$ the number of its closed-loop poles on the open right half complex plane. Recall that $\gamma(\cdot,\cdot)$ is the gap metric as defined in (2). Then the ν-gap metric between P_1 and P_2 is defined as

$$\delta_\nu(P_1,P_2) := \sup_{\omega\in\mathbb{R}} \gamma\left[\mathcal{G}(P_1)(j\omega),\mathcal{G}(P_2)(j\omega)\right]$$

if $\eta[P_2,-P_1^\sim] = \eta[P_1,-P_1^\sim]$, and $\delta_\nu(P_1,P_2)=1$ otherwise. Given an LTI system $P\in\mathcal{P}$ and $W, W^{-1}\in\mathcal{RH}_\infty$, we define the weighted ν-gap ball as [12, Chapter 3]

$$\begin{aligned}\mathcal{B}_\nu(P,W) := \Big\{ &\tilde{P}\in\mathcal{P}:\ \delta_\nu(P,\tilde{P})<1,\\ &\gamma\left[\mathcal{G}(P)(j\omega),\mathcal{G}(\tilde{P})(j\omega)\right] \le |W(j\omega)|,\ \forall\omega\in\mathbb{R}\Big\},\end{aligned} \tag{26}$$

where W is a scalar weighting function. When $W(j\omega)$ is a constant function, the above set reduces to the ν-gap ball. The requirement on $\delta_\nu(P,\tilde{P})<1$ guarantees that we can continuously perturb P to $\tilde{P}$ [12, Chapter 3].

The following result extends robust stability condition (12) in Theorem 1 to cover frequency-wise bounded uncertainties.

Theorem 2. *Assume that the nominal system $[P,C]$ is stable. Let $W_p, W_c, W_k, W_k^{-1}\in\mathcal{RH}_\infty$ satisfy $\|W_p\|_\infty<1$, $\|W_c\|_\infty<1$, and $\|W_k\|_\infty<1$. The NCS in Fig. 2 is stable for all $\tilde{P}\in\mathcal{B}_\nu(P,W_p)$, $\tilde{C}\in\mathcal{B}_\nu(C,W_c)$, and $\Delta_k\in\mathcal{RH}_\infty$ with $\|W_k^{-1}\Delta_k\|_\infty\le 1$, $k=1,2,\dots,l$, if and only if for all $\omega\in\mathbb{R}$, it holds uniformly*

$$\begin{aligned}\arcsin|W_p(j\omega)| + \arcsin|W_c(j\omega)| + \sum_{k=1}^{l}\arcsin|W_k(j\omega)|\\ < \arcsin\{\bar{\sigma}[\mathrm{GoF}(P,C)(j\omega)]^{-1}\}.\end{aligned} \tag{27}$$

Theorem 2 precisely quantifies the frequency-wise bounded uncertainties that the two-port NCS can tolerate while the closed-loop stability is maintained. Under the circumstances that all weighting functions are constant, i.e., $W_p=r_p$, $W_c=r_c$, and $W_k=r_k\in[0,1)$, $k=1,2,\dots,l$, Theorem 2 virtually reduces to Theorem 1 except that the gap metric in Theorem 1 is replaced by the ν-gap metric. On the other hand, this shows that Theorem 1 holds true when the model uncertainties are measured by the ν-gap metric as well.

An earlier version of Theorem 2 was announced in conference paper [26] without proof, involving only the frequency-wise bounded uncertainties in two-port networks. The new version here is more general as the model uncertainties are incorporated, and its complete proof is given in Appendix A.

B. Synthesis with Frequency-wise Uncertainties

Regarding the robust stability condition in Theorem 2, one may ask naturally what an optimal robust controller is, and how we obtain it efficiently. Suppose we are given a nominal plant P and uncertainty bounds W_p, W_c and W_k, $k=1,2,\dots,l$ as in Theorem 2. A feasible controller C should stabilize the nominal closed-loop system $[P,C]$ and satisfy condition (27). Denote a composition of the weighting functions via

$$\begin{aligned}|W(j\omega)| := \sin\Bigg(&\arcsin|W_p(j\omega)| + \arcsin|W_c(j\omega)|\\ &+\sum_{k=1}^{l}\arcsin|W_k(j\omega)|\Bigg).\end{aligned} \tag{28}$$

Equivalently, the controller satisfies that

$$\bar{\sigma}[W(j\omega)\mathrm{GoF}(P,C)(j\omega)]<1,$$

for all $\omega\in\mathbb{R}$ uniformly, or equivalently

$$\|W\mathrm{GoF}(P,C)\|_{\mathcal{L}_\infty}<1, \tag{29}$$

where $\|\cdot\|_{\mathcal{L}_\infty}$ denotes the $\mathcal{L}_\infty$ norm. If such controllers exist, they are not necessarily unique. Naturally, the optimal robust controller at which we target should be given by

$$C_{\text{opt}} = \arg\min_{C}\|W\mathrm{GoF}(P,C)\|_{\mathcal{L}_\infty}. \tag{30}$$

If $W(j\omega)$ is a nonzero constant, then the problem in (30) reduces to the previously introduced synthesis problem (4), which has been elegantly solved in [13]. If $W(j\omega)$ can be extended to the complex plane such that $W, W^{-1}\in\mathcal{RH}_\infty$, the optimal robust controller can be obtained by solving an $\mathcal{H}_\infty$ control problem. Essentially, it is a two-block $\mathcal{H}_\infty$ model matching problem, which will be shown in details momentarily. However, in general, there does not exist such an extension since what we have involves transcendental functions of "sine" and "arcsine". In this case, solving the problem directly is difficult and may result in infinite order controllers. In practice, we may attempt to approximate $W(j\omega)$ by $\hat{W}(j\omega)$ subject to $\hat{W},\hat{W}^{-1}\in\mathcal{RH}_\infty$. Since only the magnitude response of $W(j\omega)$ is known, we propose the following approximation method based on finite samples from $|W(j\omega)|$, which is tailored from an algorithm in [38].

Algorithm 1 Real Rational Approximation

1. Set sampling period as T_s and number of samples as N.
2. Construct the sampled transfer function of $|W(j\omega)|$ as
$$\left|W_z(e^{j\omega})\right| := \left|W\left(j\frac{2}{T_s}\tan\frac{\omega}{2}\right)\right|.$$
3. For $0\le k\le N-1$, compute data sample sequence
$$\Phi_k = \left|W_z(e^{j2\pi k/N})\right|^2,$$
and for $0\le i\le N-1$, compute N-point inverse discrete Fourier transformation of $\{\Phi_k\}$ according to
$$\phi_i = \frac{1}{N}\sum_{k=0}^{N-1}\Phi_k e^{j2\pi ik/N}.$$
4. Compute window function w_i according to
$$w_i = 1-\frac{|i|}{N}$$
for $|i|\le N$ and form the two-sided polynomial
$$R(z) = \sum_{i=-N}^{N} w_i\phi_i z^{-i}.$$
5. Apply the balanced stochastic realization (BSR) algorithm in [38] on $R(z)$, and obtain a discrete-time system $\hat{W}_z(z)$ with prescribed order r, and the approximation is
$$\hat{W}(s) := \hat{W}_z\left(\frac{2+T_s s}{2-T_s s}\right).$$

It can be shown that the approximation error can be made arbitrarily small by taking N and r sufficiently large. To proceed with the controller design, let the normalized coprime factorizations be $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$. Then there exist $U, V, \tilde{U}, \tilde{V} \in \mathcal{RH}_\infty$ [4, Chapter 11], [7] such that

$$\begin{bmatrix} \tilde{U} & -\tilde{V} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & V \\ N & U \end{bmatrix} = \begin{bmatrix} M & V \\ N & U \end{bmatrix} \begin{bmatrix} \tilde{U} & -\tilde{V} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

It is well-known from the Youla parametrization [39] that all controllers stabilizing P can be parametrized in the form

$$C = (V + MQ)(U + NQ)^{-1} = (\tilde{U} + Q\tilde{N})^{-1}(\tilde{V} + Q\tilde{M}),$$

where $Q \in \mathcal{RH}_\infty$. Let $\hat{W}$ be obtained from Algorithm 1. Substituting W with $\hat{W}$ into (30), we target at minimizing

$$\|\hat{W}\mathrm{GoF}(P,C)\|_{\mathcal{L}_\infty} = \|\hat{W}\mathrm{GoF}(P,C)\|_\infty.$$

This is intuitively a four-block $\mathcal{H}_\infty$ model matching problem due to the four-block structure of the Gang of Four transfer matrix. However, we can reduce it to a two-block problem by noting [7, Theorem 1]. It follows that

$$\|\hat{W}\mathrm{GoF}(P,C)\|_\infty = \left\| \begin{bmatrix} \hat{W}(M^\sim V + N^\sim U) + \hat{W}Q \\ \hat{W}I \end{bmatrix} \right\|_{\mathcal{L}_\infty}. \quad (31)$$

Hence, by substituting Q with $\hat{W}^{-1}Q$, we obtain the optimal robust controller

$$C_{\text{opt}} = (V + \hat{W}^{-1}MQ_{\text{opt}})(U + \hat{W}^{-1}NQ_{\text{opt}})^{-1},$$

where Q_{opt} is given by

$$Q_{\text{opt}} = \underset{Q \in \mathcal{RH}_\infty}{\operatorname{argmin}} \left\| \begin{bmatrix} \hat{W}(M^\sim V + N^\sim U) + Q \\ \hat{W}I \end{bmatrix} \right\|_{\mathcal{L}_\infty}.$$

Precisely speaking, Q_{opt} solves a special two-block $\mathcal{H}_\infty$ model matching problem, which can be approached by standard methods; See, for instance, the γ-iteration algorithm in [40] and [41, Chapter 8]. The order of the optimal robust controller is no larger than the sum of the McMillan degrees of P and $\hat{W}$. An illustrative example will be presented in the next section.

VII. Examples and Simulations

In this section, we first present an analytic example of a special two-port NCS containing two stages, illustrating the sufficiency and necessity of the main robust stability condition in Theorem 1. Then we simulate a two-port NCS in the presence of time delays and quantization errors induced by two-port communication channels, which shows, regarding Theorem 1, its capability of dealing with the nonlinear perturbations. Finally, we show an illustrative example about designing an optimal robust controller given frequency-wise bounded uncertainties regarding Theorem 2.

A. Analytic Example

Assume that plant P and the corresponding optimal robust controller C are given respectively by

$$P(s) = \frac{\sqrt{3}}{\sqrt{3}s + 1}, \qquad C(s) = -\frac{1}{\sqrt{3}}.$$

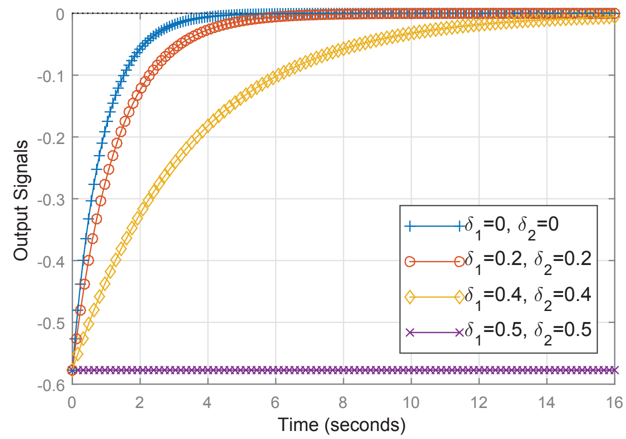

Fig. 7: Impulse responses for different choices of δ_1 and δ_2.

The associated robust stability margin is obtained as

$$\|\mathrm{GoF}(P,C)\|_\infty^{-1} = \frac{\sqrt{3}}{2}.$$

See [42, Chapter 9.5] for details. The two-port NCS contains two stages, represented by the following uncertainty quartets

$$\Delta_1 = \frac{\delta_1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ 0 & 0 \end{bmatrix} \text{ and } \Delta_2 = \frac{\delta_2}{2}\begin{bmatrix} 0 & -\sqrt{3} \\ 0 & -1 \end{bmatrix},$$

where $\delta_1 > 0$ and $\delta_2 > 0$ are the parameters of the uncertainties satisfying $\|\Delta_1\|_\infty = \delta_1 \le r_1$ and $\|\Delta_2\|_\infty = \delta_2 \le r_2$.

We begin with the stability analysis for the equivalent closed-loop system $[\tilde{P}_2, \tilde{C}_2]$. Similarly, we can analyze the other one, namely $[\tilde{P}_1, \tilde{C}_1]$, which should produce the same result. From simple derivation based on equation (9), we have

$$\mathcal{G}_{\tilde{P}_2} = \frac{\begin{bmatrix} (1 - \frac{1}{2}\delta_1)s + \frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}}\delta_1 - \frac{\sqrt{3}}{2}\delta_2 \\ 1 - \frac{1}{2}\delta_2 \end{bmatrix}}{s + \frac{2}{\sqrt{3}}}\mathcal{H}_2.$$

Thus the characteristic polynomial of the closed-loop system $[\tilde{P}_2, \tilde{C}_2]$ can be determined as

$$c(s) = \sqrt{3}(1 - \delta_1/2)s + 2 - 2(\delta_1 + \delta_2).$$

It is clear that $[\tilde{P}_2, \tilde{C}_2]$ is stable if and only if all the roots of $c(s)$ are in the open left half complex plane, or equivalently $\delta_1 + \delta_2 < 1$. Based on Theorem 1, the robust stability condition in (12) for this particular example can be obtained as

$$\arcsin r_1 + \arcsin r_2 < \arcsin \|\mathrm{GoF}(P,C)\|_\infty^{-1} = \frac{\pi}{3}. \quad (32)$$

The above inequality implies that $\delta_1 + \delta_2 \le r_1 + r_2 < 1$, and guarantees the stability of the closed-loop system. Hence (32) is a sufficient condition for the feedback stability of the two-port NCS. On the other hand, by taking $r_1 = r_2 = 0.5$, which violates (32), we can construct matrices Δ_k satisfying $\|\Delta_k\|_\infty \le r_k$ for $k = 1, 2$, such that the two-port NCS is unstable by setting parameters $\delta_1 = \delta_2 = 0.5$. Hence (32) is also a necessary condition. Fig. 7 shows the impulse responses of the sensitivity function $\tilde{S}_2 = (1 + \tilde{P}_2\tilde{C}_2)^{-1}$ corresponding

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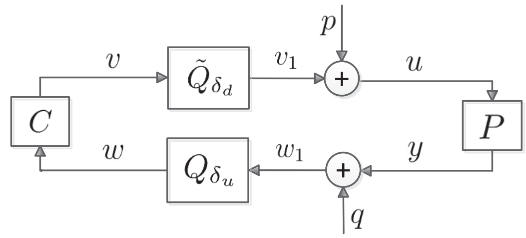

Fig. 8: System diagram with logarithmic quantizers.

to different choices of δ_1 and δ_2. When $\delta_1 = \delta_2 \leq 0.4$, which satisfies condition (32), the output signals are energy-bounded. When $\delta_1 = \delta_2 = 0.5$, which violates condition (32) critically in the sense that Δ_1 and Δ_2 are the worst-case uncertainties, the output signal is a nonzero constant, implying that the two-port NCS is unstable.

B. Simulation with Nonlinearity and Time Delay

Assume that plant P and the corresponding optimal robust controller C are given respectively by

$$P(s) = \frac{1}{s^2}, \qquad C(s) = -\frac{(1+\sqrt{2})s+1}{s+1+\sqrt{2}}.$$

The associated robust stability margin is obtained as

$$\|\mathrm{GoF}(P,C)\|_\infty^{-1} = \left(4+2\sqrt{2}\right)^{-1/2}.$$

See [42, Chapter 9.5] for details.

Shown in Fig. 8, we assume that the two-port network suffers from nonlinear perturbations. Specifically the plant and controller communicate through a bidirectional digital channel equipped with time-invariant logarithmic quantizers. The detailed definitions of the logarithmic quantizer $Q_\delta : \mathbb{R} \to \mathbb{R}$ and the alternative logarithmic quantizer $\tilde{Q}_\delta : \mathbb{R} \to \mathbb{R}$ can be referred to [43], [44]. In the two-port network, we only consider multiplicative and inverse multiplicative channel uncertainties, determined by the "downlink" quantizer

$$Q_{\delta_d} : v(t) \mapsto v_1(t), t \in [0,\infty), \text{ satisfying } \frac{\|v_1 - v\|_2}{\|v\|_2} \leq \delta_d$$

and the "uplink" quantizer

$$\tilde{Q}_{\delta_u} : w_1(t) \mapsto w(t), t \in [0,\infty), \text{ satisfying } \frac{\|w - w_1\|_2}{\|w\|_2} \leq \delta_u.$$

Additionally, we consider the case when the communication through the two-port network involves transmission delay [45]. The plant P is thus replaced by the delayed model $Pe^{-2\tau s}$ with τ being the time delay for one-way transmission.

Let $\delta_d = 0.2$, $\delta_u = 0.25$ and $\tau = 0.1$[sec]. We have $\delta_1 = \max\{\delta_d, \delta_u\} = 0.25$ and $\delta_p = \delta(P, Pe^{-2\tau s}) < 0.1141$, estimated by Padé approximation [46]. Hence,

$$\begin{aligned}\arcsin\delta_1 + \arcsin\delta_p = 0.3671 < 0.3927 \\ = \arcsin\|\mathrm{GoF}(P,C)\|_\infty^{-1},\end{aligned}$$

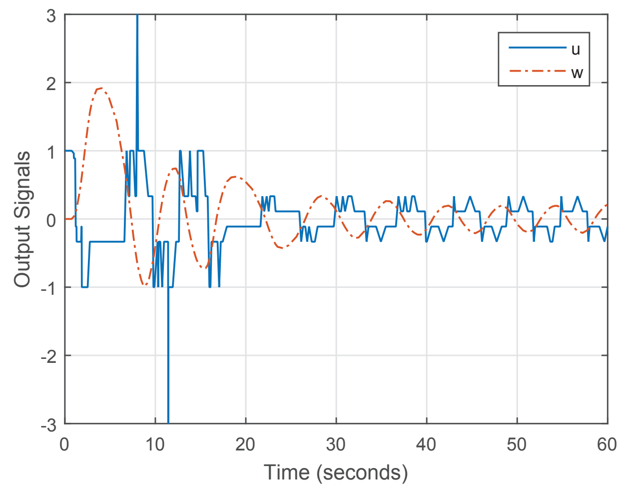

Fig. 9: Output evolutions when condition (12) violated.

which satisfies condition (12). It can be shown from the simulation that the two-port NCS is stable in this case. We comment that the nonlinear perturbations in the two-port channel due to quantization and time delay in this specific example are not the worst case in most situations. In other words, the stability may be maintained even if condition (12) is violated. However, the system will eventually turn to be unstable if the uncertainties are kept increasing. Indeed, when the parameters of uncertainties increase to $\delta_u = 0.5$, $\delta_d = 0.6$ and $\tau = 0.75$[sec],

$$\begin{aligned}\arcsin\delta_1 + \arcsin\delta_p = 1.067 > 0.3927 \\ = \arcsin\|\mathrm{GoF}(P,C)\|_\infty^{-1}.\end{aligned}$$

The corresponding output signals fluctuate persistently as time $t \to \infty$, as shown in Fig. 9.

C. Synthesis Example with Frequency-wise Uncertainties

Consider a two-port NCS containing two stages. Assume that the plant P, and frequency-wise uncertainty bounds W_1 and W_2 are given respectively by

$$P(s) = \frac{1}{s^2}, \quad W_1(s) = \frac{50}{800+s}, \quad W_2(s) = \frac{0.3s}{1200+s}.$$

Here, W_1 describes two-port uncertainty in the first channel concentrated on low frequency ($|\omega| < 800$[rad/sec]), and W_2 describes uncertainty in the second channel on high frequency ($|\omega| > 1200$[rad/sec]). The composition of weighting functions is given by

$$|W(j\omega)| = \sin\left(\arcsin\left|\frac{50}{800+j\omega}\right| + \arcsin\left|\frac{0.3j\omega}{1200+j\omega}\right|\right).$$

We apply Algorithm 1 with sampling number $N = 39$ and period $T_s = 3.5$[msec], and obtain a 2nd order LTI system

$$\hat{W}(s) = \frac{0.2983s^2 + 59.18s + 1673}{s^2 + 736.4s + 2.659 \times 10^4}$$

as the approximation for $W(j\omega)$; See Fig. 10 for the magnitude Bode plots. Clearly, $\hat{W}$ is stable and minimum phase, i.e.,

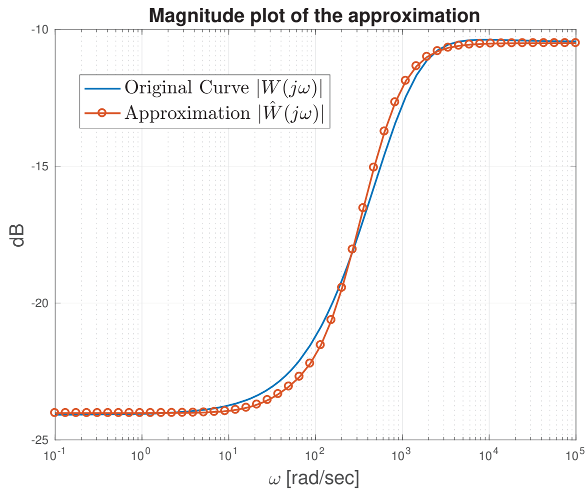

Fig. 10: The weighting function $W(j\omega)$ is approximated by a 2nd order LTI system $\hat{W}(j\omega)$. The magnitudes of $W(j\omega)$ and $\hat{W}(j\omega)$ are plotted in the unit of decibel (dB), i.e., $20\log(\cdot)$.

$\hat{W}, \hat{W}^{-1} \in \mathcal{RH}_\infty$. We ignore the phase difference between the weighting functions as it will not affect the controller design in terms of (30). Solving the two-block $\mathcal{H}_\infty$ model matching problem (31), we obtain a strictly proper 4th order optimal robust controller

$$C_{\text{opt}}(s) = \frac{-0.1908s^3 - 599.4s^2 - 2.289 \times 10^4 s - 8350}{s^4 + 201.7s^3 + 6276s^2 + (1.975s + 2.941) \times 10^4}.$$

Our numerical calculation results in

$$\|\hat{W}\text{GoF}(P, C_{\text{opt}})\|_\infty = 0.3058 < 1.$$

This shows that the two-port NCS is stabilized by controller C_{opt} and that inequality (29) is satisfied with a large "stability margin" $1 - 0.3058 \approx 0.7$. Therefore, the approximation error between $|\hat{W}(j\omega)|$ and $|W(j\omega)|$ (less than 0.3[dB]) can be compensated by this margin, and the two-port NCS will remain stable when more uncertainties appear, as long as inequality (29) remains true. Finally, we would like to mention that the order of C_{opt} equals to the sum of the orders of P and $\hat{W}$.

VIII. Conclusion

An uncertainty model for the NCS with cascaded two-port connections is proposed and analyzed. The model uncertainties in the plant and controller are measured by the gap metric. The communication uncertainties involved in the two-port networks at the kth stage are measured by the $\mathcal{H}_\infty$ norm of uncertainty quartet $\Delta_k \in \mathcal{RH}_\infty$ for $k = 1, 2, \ldots, l$. The architecture is applicable to many practical scenarios, especially those involving noisy communications with the cascaded structure in the feedback system. A necessary and sufficient condition for the robust stability of the NCS is presented, which is determined by an "arcsine" inequality. The sufficiency is mainly derived from the triangular inequality of the angular metric. The necessity is mainly attributed to the tightness of the angular metric and the techniques in handling the rotations of subspaces. The stability condition is later generalized to the case where all the uncertainties are frequency-wise bounded.

Further generalization of the problem setup and the two-port approach may work if we model the network uncertainties to be nonlinear time-varying or stochastic processes, motivated by the practical channel conditions, equipment with quantizer and attack patterns of potential enemies, such as the simulation example in Subsection VII-B. Some attempts have been made recently on extending uncertainty quartets to be nonlinear, which can be found in [47].

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Appendix A
Proof for Theorem 2

Proof. We first show the sufficiency. Fixing an arbitrary stage $k \in [0, l]$, we show that equivalent closed-loop system $[\tilde{P}_k, \tilde{C}_k]$ is stable, where $\tilde{P}_k$ and $\tilde{C}_k$ are given by (9) and (10) with P replaced by $\tilde{P}$ and C by $\tilde{C}$, respectively. At each frequency $\omega \in \mathbb{R}$, there hold the following inequalities:

$$\begin{aligned}
&\arcsin\left\{\bar{\sigma}\left[\text{GoF}(\tilde{P}, \tilde{C})(j\omega)\right]^{-1}\right\} \\
&\geq \arcsin\left\{\bar{\sigma}[\text{GoF}(P, C)(j\omega)]^{-1}\right\} \\
&\quad - \gamma\left[\mathcal{G}(P)(j\omega), \mathcal{G}(\tilde{P})(j\omega)\right] - \gamma\left[\mathcal{G}'(C)(j\omega), \mathcal{G}'(\tilde{C})(j\omega)\right] \\
&\geq \arcsin\left\{\bar{\sigma}[\text{GoF}(P, C)(j\omega)]^{-1}\right\} \\
&\qquad\qquad - \arcsin|W_p(j\omega)| - \arcsin|W_c(j\omega)|.
\end{aligned}$$

The first inequality follows from [8, Theorem 2]. Lemma 4 and the triangular inequality of the angular metric imply that

$$\arcsin\gamma\left[\mathcal{G}(\tilde{P}_k)(j\omega), \mathcal{G}(\tilde{P})(j\omega)\right] \leq \sum_{i=1}^{k} \arcsin|W_i(j\omega)|,$$

$$\arcsin\gamma\left[\mathcal{G}'(\tilde{C}_k)(j\omega), \mathcal{G}'(\tilde{C})(j\omega)\right] \leq \sum_{i=k+1}^{l} \arcsin|W_i(j\omega)|.$$

Again from [8, Theorem 2], we know

$$\begin{aligned}
&\arcsin\left\{\bar{\sigma}\left[\text{GoF}(\tilde{P}_k, \tilde{C}_k)(j\omega)\right]^{-1}\right\} \\
&\geq \arcsin\left\{\bar{\sigma}\left[\text{GoF}(\tilde{P}, \tilde{C})(j\omega)\right]^{-1}\right\} - \sum_{k=1}^{l} \arcsin|W_k(j\omega)| \\
&\geq \arcsin\left\{\bar{\sigma}[\text{GoF}(P, C)(j\omega)]^{-1}\right\} - \arcsin|W_p(j\omega)| \\
&\qquad - \arcsin|W_c(j\omega)| - \sum_{k=1}^{l} \arcsin|W_k(j\omega)| > 0,
\end{aligned}$$

where the last inequality follows from condition (27). It holds that $\bar{\sigma}\left[\text{GoF}(\tilde{P}_k, \tilde{C}_k)(j\omega)\right] < \infty$ uniformly for every

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$\omega \in \mathbb{R}$. As a result, $\mathrm{GoF}(\tilde{P}_k, \tilde{C}_k) \in \mathcal{L}_\infty$. To further show that $\mathrm{GoF}(\tilde{P}_k, \tilde{C}_k) \in \mathcal{H}_\infty$, we utilize a continuity argument as follows.

Replacing Δ_k by $\Delta_k^{[\epsilon]} := \epsilon\Delta_k$ for parameter $\epsilon \in [0,1]$ and $k = 1, 2, \ldots, l$ in the two-port NCS, we obtain a series of new equivalent closed-loop systems $\left[\tilde{P}_k^{[\epsilon]}, \tilde{C}_k^{[\epsilon]}\right]$. Define a mapping $\boldsymbol{F}(\epsilon) : \epsilon \mapsto \mathrm{GoF}\left(\tilde{P}_k^{[\epsilon]}, \tilde{C}_k^{[\epsilon]}\right)$ with $\epsilon \in [0,1]$. Since $0 \le \epsilon \le 1$, it holds $\bar{\sigma}\left[\Delta_k^{[\epsilon]}(j\omega)\right] \le \bar{\sigma}[\Delta_k(j\omega)]$ for every $\omega \in \mathbb{R}$, and it is clear $\mathrm{GoF}\left(\tilde{P}_k^{[\epsilon]}, \tilde{C}_k^{[\epsilon]}\right) \in \mathcal{L}_\infty$ by the same arguments as used earlier. Clearly, $\boldsymbol{F}(\epsilon)$ is a continuous function of ϵ under the $\mathcal{L}_\infty$ norm. In addition, each of the poles of $\mathrm{GoF}\left(\tilde{P}_k^{[\epsilon]}, \tilde{C}_k^{[\epsilon]}\right)$ is a continuous function of ϵ. For $\epsilon = 0$, the hypothesis on $\delta_\nu(P, \tilde{P}) < 1$ and $\delta_\nu(C, \tilde{C}) < 1$ implies that $\boldsymbol{F}(0) = \mathrm{GoF}(\tilde{P}, \tilde{C}) \in \mathcal{H}_\infty$, in light of [12, Theorem 3.17]. Since $\boldsymbol{F}(\epsilon) \in \mathcal{L}_\infty$ for all $\epsilon \in [0,1]$, when ϵ changes from 0 to 1 continuously, the poles of $\boldsymbol{F}(\epsilon)$ cannot cross the imaginary axis due to their continuity with respect to ϵ. We thus conclude that $\boldsymbol{F}(\epsilon) \in \mathcal{H}_\infty$ for all $\epsilon \in [0,\ 1]$. In particular, at $\epsilon = 1$, $\boldsymbol{F}(1) = \mathrm{GoF}(\tilde{P}_k, \tilde{C}_k) \in \mathcal{H}_\infty$. An application of the stability criterion in Lemma 3 validates the stability of the NCS.

In the rest, we prove the necessity using the contrapositive argument. If condition (27) does not hold, then there exists some $\bar{\omega} \in \mathbb{R} \cup \{\infty\}$ such that

$$\begin{aligned}\arcsin|W_p(j\bar{\omega})| + \arcsin|W_c(j\bar{\omega})| + \sum_{k=1}^{l} \arcsin|W_k(j\bar{\omega})| \\ \ge \arcsin\left\{\bar{\sigma}[\mathrm{GoF}(P,C)(j\bar{\omega})]^{-1}\right\}. \quad (33)\end{aligned}$$

From [8, Theorem 2], we know that

$$\begin{aligned}&\min\Big\{\arcsin\left\{\bar{\sigma}[\mathrm{GoF}(\tilde{P},\tilde{C})(j\bar{\omega})]^{-1}\right\} : \\ &\qquad \gamma\left[\mathcal{G}(P)(j\bar{\omega}), \mathcal{G}(\tilde{P})(j\bar{\omega})\right] \le |W_p(j\bar{\omega})|, \\ &\qquad\qquad \gamma\left[\mathcal{G}'(C)(j\bar{\omega}), \mathcal{G}'(\tilde{C})(j\bar{\omega})\right] \le |W_c(j\bar{\omega})|\Big\} \\ &= \arcsin\left\{\bar{\sigma}[\mathrm{GoF}(P,C)(j\bar{\omega})]^{-1}\right\} \\ &\qquad\qquad - \arcsin|W_p(j\bar{\omega})| - \arcsin|W_c(j\bar{\omega})| \\ &\le \sum_{k=1}^{l} \arcsin|W_k(j\bar{\omega})|.\end{aligned}$$

Without loss of generality, we assume that $\tilde{P} \in \mathcal{B}_\nu(P, W_p)$ and $\tilde{C} \in \mathcal{B}_\nu(C, W_c)$ satisfying

$$\arcsin\left\{\bar{\sigma}\left[\mathrm{GoF}(\tilde{P},\tilde{C})(j\bar{\omega})\right]^{-1}\right\} = \sum_{k=1}^{l} \arcsin|W_k(j\bar{\omega})|.$$

Consider the closed-loop system $[\tilde{P}_l, \tilde{C}]$ defined by (9) with P replaced by $\tilde{P}$. We will show that there exists $\tilde{\Delta}_k \in \mathcal{RH}_\infty$, subject to $\|W_k^{-1}\tilde{\Delta}_k\|_\infty \le 1$ for $k = 1, 2, \ldots, l$ such that $[\tilde{P}_l, \tilde{C}]$ is unstable.

Noting $\tilde{P} \in \mathcal{P}^{p\times m}$ and $\tilde{C} \in \mathcal{P}^{m\times p}$, we let $n = m + p$, and denote subspaces $\mathcal{U} = \mathcal{G}(\tilde{P})(j\bar{\omega}) \in \mathcal{G}_{m,n}$ and $\mathcal{W} = \mathcal{G}'(\tilde{C})(j\bar{\omega}) \in \mathcal{G}_{p,n}$. Then from Proposition 1, there exists $\Delta_k \in \mathbb{C}^{n\times n}$ satisfying $\bar{\sigma}(\Delta_k) \le |W_k(j\bar{\omega})|$ such that

$$\underline{\theta}\left[(I + \Delta_l)\cdots(I + \Delta_1)\mathcal{U}, \mathcal{W}\right] = 0.$$

In light of Lemma 5, there exists $\hat{\Delta}_k \in \mathcal{RH}_\infty$ for which $\hat{\Delta}_k(j\bar{\omega}) = \Delta_k$ and $\|\hat{\Delta}_k\|_\infty = \bar{\sigma}(\Delta_k)$. We next construct

$$\tilde{\Delta}_k(s) = \frac{W_k(s)\hat{\Delta}_k(s)}{|W_k(j\bar{\omega})|} \in \mathcal{RH}_\infty,\ k = 1, 2, \ldots, l,$$

which satisfies that $\bar{\sigma}\left[\tilde{\Delta}_k(j\omega)\right] \le |W_k(j\omega)|$ for all $\omega \in \mathbb{R}$. Let $\tilde{P}_l$ be determined by its graph, i.e.,

$$\mathcal{G}_{\tilde{P}_l} = \left(I + \tilde{\Delta}_l\right)\cdots\left(I + \tilde{\Delta}_1\right)\mathcal{G}_{\tilde{P}}.$$

Then it is clear that

$$\begin{aligned}&\bar{\sigma}\left[\mathrm{GoF}(\tilde{P}_l,\tilde{C})(j\bar{\omega})\right]^{-1} \\ &\qquad = \sin\underline{\theta}\left[(I + \Delta_l)\cdots(I + \Delta_1)\mathcal{U}, \mathcal{W}\right] = 0.\end{aligned}$$

As a result, it holds $\|\mathrm{GoF}(\tilde{P}_l, \tilde{C})\|_\infty = \infty$, i.e., closed-loop system $[\tilde{P}_l, \tilde{C}]$ is unstable, and so is the two-port NCS. □

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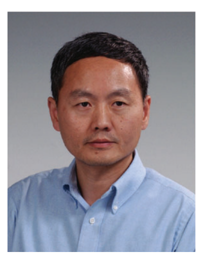

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