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Input-to-State Stability of a Clamped-Free Damped String in the Presence of Distributed and Boundary Disturbances

Hugo Lhachemi, David Saussié, Guchuan Zhu, Robert Shorten

Abstract—**This note establishes the Exponential Input-to-State Stability (EISS) property for a clamped-free damped string with respect to distributed and boundary disturbances. While efficient methods for establishing ISS properties for distributed parameter systems with respect to distributed disturbances have been developed during the last decades, establishing ISS properties with respect to boundary disturbances remains challenging. One of the well-known methods for well-posedness analysis of systems with boundary inputs is the use of a lifting operator for transferring the boundary disturbance to a distributed one. However, the resulting distributed disturbance involves time derivatives of the boundary perturbation. Thus, the subsequent ISS estimate depends on its amplitude, and may not be expressed in the strict form of ISS properties. To solve this problem, we show for a clamped-free damped string equation that the projection of the original system trajectories in an adequate Riesz basis can be used to establish the desired EISS property.**

Index Terms—**Distributed parameter systems, Boundary disturbance, Input-to-state stability.**

I. INTRODUCTION

Originally introduced by Sontag for finite dimensional systems [37], Input-to-State Stability (ISS) is one of the central notions in the modern theory of robust control. Specifically, ISS aims at ensuring that disturbances can only induce, in the worst case, a proportional perturbation of the magnitude of the system trajectory. While this notion has been widely studied for finite dimensional systems, its extension to Partial Differential Equations (PDEs), and more generally to infinite dimensional systems, remains challenging [15], [21], [33], [34].

For systems described by PDEs, there exist essentially two types of perturbations. The first type includes distributed (or in-domain) perturbations, i.e., perturbations acting over the domain. The second type concerns boundary perturbations, i.e., perturbations acting on the boundary of the domain. This second type of perturbation naturally appears in numerous boundary control problems such as heat equations [5], transport equations [22], diffusion or diffusive equations [2], and vibration of structures [5] with practical applications, e.g., in robotics [9], [14], aerospace engineering [3], [25], [26], and additive manufacturing [8], [13].

In the recent literature, many results have been reported regarding the ISS property with respect to distributed disturbances [1], [6], [7], [29]–[31], [36]. In contrast, the literature dealing with the establishment of ISS properties with respect to boundary disturbances is less developed [2], [18], [19], [22], [32]. The main difficulty relies in the fact that boundary disturbances are generally transferred into distributed disturbances by means of a lifting operator in the framework of the boundary control systems [5]. By doing so, the original system with boundary perturbations is made equivalent to a system with exclusively distributed perturbations, for which efficient tools for analyzing the ISS properties exist. However, the resulting distributed perturbation, and consequently the subsequent ISS estimate, involve time derivatives of the boundary perturbation [5]. Thus, it is paramount to obtain an ISS property compliant with the original definition of ISS, which is exclusively expressed in terms of the amplitude of the disturbances.

A possible approach for establishing ISS properties of PDEs with respect to boundary disturbances consists in resorting to an adequate Lyapunov function [2], [38], [40]–[42]. While very efficient, such an approach relies on the practical capability to construct an adequate Lyapunov function, which is generally challenging and highly case-dependent. Alternative approaches relying on functional analysis tools were investigated in [15], [16], [22]. In [15], [16], the ISS property for disturbances evaluated in the uniform norm is obtained for a class of analytic semigroups. The problem is embedded into the extrapolation space while invoking admissible conditions for returning to the original state-space. A different approach that avoids the incursion into the extrapolation space was developed in [22] for the analysis of 1-D parabolic equations. One of the key ideas was to take advantage of the intrinsic properties of the underlying disturbance-free operator. Indeed, as its opposite belongs to the class of Sturm-Liouville operator, it is self-adjoint, and an adequate selection of a sequence of its eigenvectors provides a Hilbert basis of the underlying Hilbert space. Then, by projecting the system trajectories onto this Hilbert basis and taking advantage of the self-adjoint nature of the disturbance-free operator, it was shown that the analysis of the system trajectories reduces to the study of a countably infinite number of Ordinary Differential Equations (ODEs). Each of these ODEs describes the time domain evolution of one coefficient of the system trajectory in the aforementioned Hilbert basis. The ISS property was finally obtained by solving these ODEs and by resorting to Parseval's identity.

The first motivation of this note is to establish the Exponential Input-to-State (EISS) property of a clamped-free damped string in the presence of both distributed and boundary disturbances. Also known as wave equation, the underlying second-order hyperbolic linear PDE occurs in many fields such as mechanics, acoustics, and fluid dynamics. For this reason, its study in the disturbance free case has attracted a lot of attention [10], [11], [27], [39]. The assessment of the finite asymptotic gain of a clamped damped string in both spatial L^2 and sup norms has been reported very recently in [17] for boundary disturbances of class $\mathcal{C}^4$. The EISS property in spatial sup norm of a similar clamped configuration was also obtained in [20] via stability analysis of an equivalent hyperbolic–parabolic PDE loop by means of a small-gain approach. The result presented in this note differs from [17] as we study the EISS property of a clamped-free configuration (that corresponds to a different set of boundary conditions) in the state-space norm, for disturbance signals evaluated in both uniform and L^2 norms, and under the weaker regularity assumptions that the boundary disturbances are of class $\mathcal{C}^2$.

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The second motivation is to show that the approach relying on functional analysis tools employed in [22] can be extended to the problem considered in this work. Specifically, the desired EISS property of the clamped-free damped string is obtained through three main steps. First, the well-posedness of the distributed parameter system is assessed in the framework of boundary control systems. Second, the properties of the underlying disturbance-free operator are studied. Unlike the problem studied in [22], the disturbance-free operator is not self-adjoint and its eigenvectors do not form a Hilbert basis of the underlying Hilbert space. However, it is a Riesz-spectral operator, implying in particular that its eigenvectors form a Riesz basis [4], which is an important generalization of the concept of Hilbert basis. In particular, even if the Parseval's identity does not hold for Riesz bases, a connexion still exists between the norm of a vector and its coefficients in the Riesz basis. Thus, taking finally advantage of the projection of system trajectories over this Riesz basis, as well as the connection between the eigenstructures of a Riesz-spectral operator and its adjoint operator, we show that the analysis of system trajectories reduces to the study of a countably infinite number of ODEs. By doing so, the EISS property can be derived directly from the original system, which allows avoiding the occurrence of the time derivative of the boundary perturbation.

The remainder of this note is organized as follows. Notations and definitions are introduced in Section II. The considered clamped-free damped string model and its well-posedness analysis are presented in Section III. The detailed study of the properties of the underlying disturbance-free operator is completed in Section IV. Section V is devoted to the establishment of the EISS property of the system in presence of both distributed and boundary perturbations. Finally, some concluding remarks are provided in Section VI.

II. Notations and definitions

The sets of non-negative integers, integers, real, non-negative real, positive real, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_+^*$, and $\mathbb{C}$, respectively. For any $z \in \mathbb{C}$, $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real part and the imaginary part of z, respectively. For any integer $k \in \mathbb{Z}$, we define $\tilde{k} \triangleq k + 1/2$. We define for $N \in \mathbb{N}$ the following sets:

$$\mathcal{I}_N = \{(k,\epsilon) \,:\, 0 \leq k \leq N,\, \epsilon \in \{-1,+1\}\},$$
$$\mathcal{I}_\infty = \mathbb{N} \times \{-1,+1\}.$$

For an interval $I \subset \mathbb{R}$ and a normed space $(E, \|\cdot\|_E)$, $\mathcal{C}^n(I;E)$ (simply $\mathcal{C}^n(I)$ when $E = \mathbb{C}$ endowed with the absolute value) denotes the set of functions $f : I \to E$ that are n times continuously differentiable. For any $a < b$, we endowed $\mathcal{C}^0([a,b];E)$ with the usual norm $\|\cdot\|_{\mathcal{C}^0([a,b];E)}$ defined for any $f \in \mathcal{C}^0([a,b];E)$ by

$$\|f\|_{\mathcal{C}^0([a,b];E)} = \sup_{t\in[a,b]} \|f(t)\|_E.$$

The set of square-integrable functions (w.r.t. the Lebesgue measure) over an interval $(a,b) \subset \mathbb{R}$ is denoted by $L^2(a,b)$ and is endowed with its natural inner product $\langle f,g\rangle_{L^2(a,b)} = \int_a^b f(\xi)\overline{g(\xi)}\,\mathrm{d}\xi$, providing a structure of Hilbert space. Denoting by f', when it exists, the weak derivative of $f \in L^2(a,b)$, we consider the Sobolev space $H^1(a,b) \triangleq \{f \in L^2(a,b) \,:\, f' \in L^2(a,b)\}$. Finally, $H^1_L(a,b) \triangleq \{f \in H^1(a,b) \,:\, f(a) = 0\}$ is endowed with the inner product $\langle f,g\rangle_{H^1_L(a,b)} \triangleq \langle f',g'\rangle_{L^2(a,b)}$, providing a structure of Hilbert space.

For a given linear operator L, $\mathrm{R}(L)$, $\ker(L)$, and $\rho(L)$ denote its range, its kernel, and its resolvent set, respectively. The set of linear bounded operators $L : E \to E$ is denoted by $\mathcal{L}(E)$. The time derivative of a complex-valued differentiable function $f : I \to \mathbb{C}$ is denoted by $\dot{f}$. Denoting by $(\mathcal{H}, \langle\cdot,\cdot\rangle_\mathcal{H})$ a $\mathbb{C}$-Hilbert space, the time derivative of a $\mathcal{H}$-valued differentiable function $f : I \to \mathcal{H}$ is denoted by $\mathrm{d}f/\mathrm{d}t$.

Finally, we introduce the following classical definitions.

Definition 2.1 (Riesz basis [4]): A sequence $\Phi = \{\varphi_k,\ k \in \mathbb{N}\}$ of vectors of $\mathcal{H}$ is a Riesz basis if 1) Φ is maximal: $\overline{\operatorname{span}_\mathbb{C}(\Phi)} = \mathcal{H}$, i.e., the closure of the vector space spanned by Φ coincides with the whole space $\mathcal{H}$; 2) there exist $m_R, M_R \in \mathbb{R}_+^*$ such that for any $N \in \mathbb{N}$ and any $a_k \in \mathbb{C}$,

$$m_R \sum_{0\leq k\leq N} |a_k|^2 \leq \left\| \sum_{0\leq k\leq N} a_k\varphi_k \right\|^2_\mathcal{H} \leq M_R \sum_{0\leq k\leq N} |a_k|^2. \quad (1)$$

Definition 2.2 (Riesz spectral operator [5]): Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be a linear and closed operator with simple eigenvalues λ_n and corresponding eigenvectors $\varphi_n \in D(A)$, $n \in \mathbb{N}$. Operator A is a Riesz-spectral operator if 1) $\{\varphi_n,\ n \in \mathbb{N}\}$ is a Riesz basis; 2) the closure of $\{\lambda_n,\ n \in \mathbb{N}\}$ is totally disconnected, i.e., for any two distinct $a,b \in \overline{\{\lambda_n,\ n \in \mathbb{N}\}}$, $[a,b] \not\subset \overline{\{\lambda_n,\ n \in \mathbb{N}\}}$.

III. Problem Description and Main Result

A. Problem setting

Consider a string with Kelvin-Voigt damping [10], [27], [39] and clamped-free boundary conditions described by:

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x}\left(\alpha\frac{\partial y}{\partial x} + \beta\frac{\partial^2 y}{\partial t\partial x}\right) = u, \qquad \text{in } \mathbb{R}_+ \times (0,1) \quad (2a)$$
$$y(t,0) = 0, \qquad t \in \mathbb{R}_+ \quad (2b)$$
$$\left(\alpha\frac{\partial y}{\partial x} + \beta\frac{\partial^2 y}{\partial t\partial x}\right)(t,1) = d(t), \qquad t \in \mathbb{R}_+ \quad (2c)$$
$$y(0,x) = y_0(x), \qquad x \in (0,1) \quad (2d)$$
$$\frac{\partial y}{\partial t}(0,x) = y_{t0}(x), \qquad x \in (0,1) \quad (2e)$$

where $\alpha, \beta \in \mathbb{R}_+^*$ are constant parameters. Functions $u \in \mathcal{C}^1(\mathbb{R}_+; L^2(0,1))$ and $d \in \mathcal{C}^2(\mathbb{R}_+)$ represent distributed and boundary disturbances, respectively. Functions $y_0 \in H^1_L(0,1)$ and $y_{t0} \in L^2(0,1)$ are the initial conditions.

Throughout the paper, we assume that the following assumption holds. Its introduction is motivated by the properties of the underlying operators, as it will be shown in the subsequent developments.

Assumption 3.1: The coefficients $\alpha, \beta \in \mathbb{R}_+^*$ in the system (2a-2e) are such that

$$\frac{2\sqrt{\alpha}}{\pi\beta} - \frac{1}{2} \notin \mathbb{N}.$$

To study (2a-2e), we introduce the functional space:

$$\mathcal{H} = H^1_L(0,1) \times L^2(0,1),$$

which is a Hilbert space when endowed with the inner product defined for all $(x_1,x_2), (\hat{x}_1,\hat{x}_2) \in \mathcal{H}$ by

$$\langle (x_1,x_2), (\hat{x}_1,\hat{x}_2)\rangle_\mathcal{H} = \int_0^1 \alpha x_1'(\xi)\overline{\hat{x}_1'(\xi)} + x_2(\xi)\overline{\hat{x}_2(\xi)}\,\mathrm{d}\xi.$$

Let the operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ be defined by

$$\mathcal{A}(x_1,x_2) = (x_2, (\alpha x_1' + \beta x_2')')$$

over the domain

$$D(\mathcal{A}) = \{(x_1,x_2) \in \mathcal{H} \,:\, x_2 \in H^1_L(0,1), \\ (\alpha x_1' + \beta x_2') \in H^1(0,1)\}.$$

Let $\mathcal{B} : D(\mathcal{B}) \to \mathbb{C}$ be the boundary operator defined by

$$\mathcal{B}(x_1,x_2) = (\alpha x_1' + \beta x_2')(1)$$

with $D(\mathcal{B}) = D(\mathcal{A})$. Finally, introducing $U = (0, u) \in \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$, (2a-2e) can be written under the following abstract form [5]:

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}X(t) + U(t) & , \ t \geq 0 \\ \mathcal{B}X(t) = d(t) & , \ t \geq 0 \\ X(0) = X_0 \end{cases} \tag{3}$$

with the state $X(t) = (y(t, \cdot), y_t(t, \cdot))$ and the initial condition $X_0 = (y_0, y_{t0})$.

B. Well-posedness

We introduce the disturbance-free operator $\mathcal{A}_0$ defined over the domain $D(\mathcal{A}_0) \triangleq D(\mathcal{A}) \cap \ker(\mathcal{B})$ by $\mathcal{A}_0 \triangleq \mathcal{A}|_{D(\mathcal{A}_0)}$. Straightforward computations show that $\mathcal{A}_0$ generates a C_0-semigroup of contractions. Indeed, $\mathcal{A}_0$ is dissipative since a simple integration by parts yields for any $(x_1, x_2) \in D(\mathcal{A}_0)$,

$$\mathrm{Re}\left(\langle \mathcal{A}_0(x_1, x_2), (x_1, x_2) \rangle_{\mathcal{H}} \right) = -\beta \int_0^1 |x_2'(\xi)|^2 \,\mathrm{d}\xi \leq 0.$$

Furthermore, direct computations show that $\mathcal{A}_0$ is invertible and is defined for any $(x_1, x_2) \in \mathcal{H}$ by

$$\mathcal{A}_0^{-1}(x_1, x_2) = \left(-\frac{\beta}{\alpha} x_1 - \frac{1}{\alpha} \int_0^{(\cdot)} \int_{\xi_1}^1 x_2(\xi_2) \,\mathrm{d}\xi_2 \,\mathrm{d}\xi_1, x_1 \right).$$

Finally, Poincaré and Cauchy-Schwarz inequalities [12] imply that $\mathcal{A}_0^{-1} \in \mathcal{L}(\mathcal{H})$. The application of the Lumer-Phillips theorem [28], [35] yields the desired result, i.e., $\mathcal{A}_0$ generates a C_0-semigroup of contractions T.

In order to conclude on the well-posedness of the abstract system (3) for an initial condition $X_0 \in D(\mathcal{A})$ and perturbations $u \in \mathcal{C}^1(\mathbb{R}_+; L^2(0,1))$ and $d \in \mathcal{C}^2(\mathbb{R}_+)$ such that $\mathcal{B}X_0 = d(0)$, it is sufficient to check that the abstract system satisfies the definition of a *Boundary control system* [5, Def. 3.3.2]. Introducing the lifting operator $B : \mathbb{C} \to \mathcal{H}$ defined for any $d \in \mathbb{C}$ by $Bd = (f_d, 0)$ with $f_d(x) \triangleq (d/\alpha)x$ for all $x \in [0,1]$, we have $\mathrm{R}(B) \subset D(\mathcal{A})$, $\mathcal{A}B = 0_{\mathcal{L}(\mathbb{C},\mathcal{H})}$, and $\mathcal{B}B = I_{\mathbb{C}}$. Thus the abstract system (3) is well-posed for any $X_0 \in D(\mathcal{A})$, $u \in \mathcal{C}^1(\mathbb{R}_+; L^2(0,1))$, and $d \in \mathcal{C}^2(\mathbb{R}_+)$ such that $\mathcal{B}X_0 = d(0)$ [5, Th 3.1.3 ; Th. 3.3.3]. Furthermore, $X \in \mathcal{C}^0(\mathbb{R}_+; D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$ is the classical solution[1] of (3) if and only if $V = X - Bd \in \mathcal{C}^0(\mathbb{R}_+; D(\mathcal{A}_0)) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$ is the classical solution of the following abstract system:

$$\begin{cases} \dfrac{\mathrm{d}V}{\mathrm{d}t}(t) = \mathcal{A}_0 V(t) - B\dot{d}(t) + U(t), & t \geq 0 \\ V(0) = V_0 \end{cases}$$

where $V_0 = X_0 - Bd(0) \in D(\mathcal{A}_0)$. Thus, by direct integration, the solution of (3) is given for $t \geq 0$ by

$$\begin{aligned} X(t) &= T(t)\left(X_0 - Bd(0)\right) + Bd(t) \\ &+ \int_0^t T(t-\tau) \left\{ -B\dot{d}(\tau) + U(\tau) \right\} \mathrm{d}\tau. \end{aligned} \tag{4}$$

Remark 3.2: It is pointed out that a weak version of the ISS property, implying the norm of time derivative $\dot{d}$ of the boundary disturbance d, can be easily obtained from (4). However, the traditional definition of ISS is stronger because it is only limited to the amplitude of the boundary disturbance, and not its time derivatives. The objective of this paper is to establish such an ISS estimate for (2a-2e), only with respect to the magnitude of the perturbations d and u.

[1]The function X is a classical solution of (3) if $X \in \mathcal{C}^0(\mathbb{R}_+; D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$ and satisfies (3) for all $t \geq 0$.

C. Main result

Throughout the paper, let $k_0 \in \mathbb{N}$ be defined by

$$k_0 \triangleq \left\lceil \frac{2\sqrt{\alpha}}{\pi\beta} - \frac{1}{2} \right\rceil \geq 0, \tag{5}$$

where $\lceil \cdot \rceil$ denotes the ceiling function.

The main result of the paper regarding the ISS property of the trajectories of the abstract system (3) is stated in the following theorem.

Theorem 3.3: For any initial condition $X_0 \in D(\mathcal{A})$ and any disturbances $u \in \mathcal{C}^1(\mathbb{R}_+; L^2(0,1))$ and $d \in \mathcal{C}^2(\mathbb{R}_+)$ such that $\mathcal{B}X_0 = d(0)$, the abstract system (3) has a unique classical solution $X \in \mathcal{C}^0(\mathbb{R}_+; D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$. Furthermore, under Assumption 3.1, the system is EISS with respect to disturbances in both uniform and L^2 norms in the sense that there exist constants $C_0, C_1, C_2, C_3, C_4 \in \mathbb{R}_+^*$, independent of X_0, u, and d, such that for all $t \geq 0$,

$$\begin{aligned} \|X(t)\|_{\mathcal{H}} \leq& C_0 e^{-\kappa_0 t} \|X_0\|_{\mathcal{H}} + C_1 \|d\|_{\mathcal{C}^0([0,t])} \\ &+ C_2 \|u\|_{\mathcal{C}^0([0,t];L^2(0,1))}, \end{aligned} \tag{6}$$

and

$$\begin{aligned} \|X(t)\|_{\mathcal{H}} \leq& C_0 e^{-\kappa_0 t} \|X_0\|_{\mathcal{H}} + C_3 \|d\|_{L^2(0,t)} \\ &+ C_4 \|u\|_{L^2((0,t)\times(0,1))}, \end{aligned} \tag{7}$$

where

$$\kappa_0 = \begin{cases} \min\left(\dfrac{\beta\pi^2}{8}, \dfrac{\alpha}{\beta} \right) & \text{if } k_0 \geq 1; \\ \dfrac{\alpha}{\beta} & \text{if } k_0 = 0, \end{cases} \tag{8}$$

and is such that $\omega_0 = -\kappa_0 < 0$ is the growth bound of $T(t)$.

The proof of Theorem 3.3 is presented in Section V after studying the spectral properties of $\mathcal{A}_0$ in Section IV.

Remark 3.4: As $\omega_0 = -\kappa_0$ is the growth bound of T, the convergence rate of the exponential term in (6-7) is tight in the sense that κ_0 cannot be replaced in (6) or (7) by any $\kappa > \kappa_0$ such that the ISS estimate still holds true.

IV. Study of the properties of $\mathcal{A}_0$

The objective of this section is to demonstrate that $\mathcal{A}_0$ is a Riesz-spectral operator while characterizing the eigenvalues and eigenfunctions of both $\mathcal{A}_0$ and $\mathcal{A}_0^*$.

A. Characterization of the spectral properties of $\mathcal{A}_0$

Lemma 4.1: The eigenvalues of $\mathcal{A}_0$ are simple and are given by $\{\lambda_{k,\epsilon},\ k \in \mathbb{N},\ \epsilon \in \{-1,+1\}\}$ where

$$\lambda_{k,\epsilon} = \begin{cases} -\dfrac{\tilde{k}^2\beta\pi^2}{2} + \epsilon i \dfrac{\tilde{k}\pi\sqrt{4\alpha - \tilde{k}^2\beta^2\pi^2}}{2}, & 0 \leq k \leq k_0 - 1; \\ -\dfrac{\tilde{k}^2\beta\pi^2}{2} + \epsilon \dfrac{\tilde{k}\pi\sqrt{\tilde{k}^2\beta^2\pi^2 - 4\alpha}}{2}, & k \geq k_0, \end{cases} \tag{9}$$

with $\tilde{k} \triangleq k + 1/2$. Furthermore, the associated eigenspaces are given by $\ker(\mathcal{A}_0 - \lambda_{k,\epsilon} I_{\mathcal{H}}) = \mathrm{span}_{\mathbb{C}}(\phi_{k,\epsilon})$ with

$$\phi_{k,\epsilon} = \frac{1}{\lambda_{k,\epsilon}} \left(\sin(\tilde{k}\pi\cdot), \lambda_{k,\epsilon} \sin(\tilde{k}\pi\cdot) \right). \tag{10}$$

Proof. Let $\lambda \in \mathbb{C}$ and $(x_1, x_2) \in D(\mathcal{A}_0) \backslash \{0\}$ be such that $\mathcal{A}_0(x_1, x_2) = \lambda(x_1, x_2)$, i.e., $x_1(0) = x_2(0) = (\alpha x_1' + \beta x_2')(1) = 0$ with $x_2 = \lambda x_1$ and $(\alpha x_1' + \beta x_2')' = \lambda x_2$. As $\lambda \neq -\alpha/\beta$ (otherwise we would have $x_1 = x_2 = 0$), we deduce that

$$x_2 = \lambda x_1, \qquad x_1'' = \frac{\lambda^2}{(\alpha + \lambda\beta)} x_1,$$

with $x_1'(1)=0$. Denoting by $r(\lambda)\in\mathbb{C}$ one of the two distinct[2] square-roots of $\lambda^2/(\alpha+\lambda\beta)$, there exist $a,b\in\mathbb{C}$ such that

$$x_1(\xi)=ae^{r(\lambda)\xi}+be^{-r(\lambda)\xi}.$$

From the boundary conditions we get $b=-a$ and $ar(\lambda)(e^{r(\lambda)}+e^{-r(\lambda)})=0$. As $\lambda\neq 0$ and because we are looking for non-trivial solutions (i.e., such that $(x_1,x_2)\neq 0$), $ar(\lambda)\neq 0$ whence $e^{2r(\lambda)}=-1$. We obtain that $2r(\lambda)\equiv i\pi\ (2i\pi)$, i.e., there exists $k\in\mathbb{Z}$ such that $r(\lambda)=i\tilde{k}\pi$. From the definition of $r(\lambda)$, we deduce that

$$r(\lambda)^2=\frac{\lambda^2}{(\alpha+\lambda\beta)}=-\tilde{k}^2\pi^2.$$

Thus $P_k(\lambda)=0$ where $P_k=X^2+\tilde{k}^2\beta\pi^2X+\tilde{k}^2\alpha\pi^2\in\mathbb{R}[X]$. As $P_{-k-1}=P_k$, the study for $k\in\mathbb{Z}$ reduces to $k\in\mathbb{N}$. The discriminant of P_k is given by $\mathrm{disc}(P_k)=\tilde{k}^2\pi^2(\tilde{k}^2\beta^2\pi^2-4\alpha)$. Based on Assumption 3.1, $\mathrm{disc}(P_k)\neq 0$, thus $\mathrm{disc}(P_k)>0$ for $k\geq k_0$ and $\mathrm{disc}(P_k)<0$ for $0\leq k\leq k_0-1$, providing the eigenvalues $\lambda_{k,\epsilon}$ given by (9). Finally the associated eigenvectors are characterized by

$$x_1(\xi)=a(e^{i\tilde{k}\pi\xi}-e^{-i\tilde{k}\pi\xi})=2ai\sin(\tilde{k}\pi\xi),$$

and $x_2=\lambda_{k,\epsilon}x_1$, providing (10). □

In order to work with unitary eigenvectors, we introduce $\Phi_{k,\epsilon}\triangleq\phi_{k,\epsilon}/\|\phi_{k,\epsilon}\|_{\mathcal{H}}$, where a straightforward integration shows that

$$\|\phi_{k,\epsilon}\|_{\mathcal{H}}=\frac{1}{\sqrt{2}}\sqrt{1+\frac{\tilde{k}^2\alpha\pi^2}{|\lambda_{k,\epsilon}|^2}}. \tag{11}$$

Finally, we denote $\Phi=\{\Phi_{k,\epsilon},\ k\in\mathbb{N},\ \epsilon\in\{-1,+1\}\}$.

For the upcoming developments, we establish certain equalities, inequalities, and asymptotic behaviours for the eigenvalues of $\mathcal{A}_0$. First, note that

$$\forall k\geq 0,\ \ \lambda_{k,-1}\lambda_{k,+1}=\tilde{k}^2\alpha\pi^2, \tag{12}$$

$$\forall 0\leq k\leq k_0-1,\ \forall\epsilon\in\{-1,+1\},\ \ \mathrm{Re}\,\lambda_{k,\epsilon}\leq-\frac{\beta\pi^2}{8}. \tag{13}$$

Furthermore, as $\sqrt{1+x}\leq 1+x/2$ for all $x\geq -1$,

$$\forall k\geq k_0,\ \ \lambda_{k,-1}\leq\lambda_{k,+1}\leq-\frac{\alpha}{\beta}. \tag{14}$$

To study the asymptotic behaviours, we consider $k\geq k_0$, giving

$$\lambda_{k,-1}\underset{k\to+\infty}{\sim}-k^2\beta\pi^2, \tag{15}$$

and

$$\begin{aligned}\lambda_{k,+1}&=\frac{\tilde{k}^2\beta\pi^2}{2}\left(-1+\sqrt{1-\frac{4\alpha}{\tilde{k}^2\beta^2\pi^2}}\right)\\&=\frac{\tilde{k}^2\beta\pi^2}{2}\left(-1+\left\{1-\frac{2\alpha}{\tilde{k}^2\beta^2\pi^2}+o(k^{-2})\right\}\right)\\&=-\alpha/\beta+o(1)\\&\xrightarrow[k\to+\infty]{}-\alpha/\beta.\end{aligned} \tag{16}$$

B. Characterization and properties of $\mathcal{A}_0^*$

In the subsequent developments, we will use the adjoint operator $\mathcal{A}_0^*$ and in particular the connections between the eigenstructures of $\mathcal{A}_0$ and $\mathcal{A}_0^*$. This is motivated by the fact that Φ is not a Hilbert basis for $\mathcal{H}$ (more details provided latter in Subsection IV-C).

Lemma 4.2: The adjoint operator $\mathcal{A}_0^*$ is defined over the domain

$$\begin{aligned}D(\mathcal{A}_0^*)=\{(x_1,x_2)\in\mathcal{H}\ :\ &x_2\in H_L^1(0,1),\\&(\alpha x_1'-\beta x_2')\in H^1(0,1),\\&(\alpha x_1'-\beta x_2')(1)=0\},\end{aligned}$$

by

$$\mathcal{A}_0^*(x_1,x_2)=\left(-x_2,-(\alpha x_1'-\beta x_2')'\right).$$

Proof. As $\mathcal{A}_0^{-1}\in\mathcal{L}(\mathcal{H})$, $(\mathcal{A}_0^{-1})^*\in\mathcal{L}(\mathcal{H})$ and $(\mathcal{A}_0^*)^{-1}=(\mathcal{A}_0^{-1})^*$ [23, Th. III.5.30]. Integration by parts and application of Fubini theorem yields for any $(x_1,x_2)\in\mathcal{H}$,

$$(\mathcal{A}_0^{-1})^*(x_1,x_2)=\left(-\frac{\beta}{\alpha}x_1+\frac{1}{\alpha}\int_0^{(\cdot)}\int_{\xi_1}^1x_2(\xi_2)\,\mathrm{d}\xi_2\,\mathrm{d}\xi_1,-x_1\right).$$

The inversion of $(\mathcal{A}_0^{-1})^*$ gives the claimed result. □

Lemma 4.3: The eigenvalues of $\mathcal{A}_0^*$ are given by $\{\mu_{k,\epsilon},\ k\in\mathbb{N},\ \epsilon\in\{-1,+1\}\}$ where $\mu_{k,\epsilon}=\overline{\lambda_{k,\epsilon}}$. Furthermore, the associated eigenspaces are given by $\ker(\mathcal{A}_0^*-\mu_{k,\epsilon}I_{\mathcal{H}})=\mathrm{span}_{\mathbb{C}}(\psi_{k,\epsilon})$ with

$$\psi_{k,\epsilon}=\frac{1}{\mu_{k,\epsilon}}\left(-\sin(\tilde{k}\pi\cdot),\mu_{k,\epsilon}\sin(\tilde{k}\pi\cdot)\right). \tag{17}$$

Proof. Let $\mu\in\mathbb{C}$ and $(x_1,x_2)\in D(\mathcal{A}_0^*)\backslash\{0\}$ be such that $\mathcal{A}_0^*(x_1,x_2)=\mu(x_1,x_2)$, i.e., $x_1(0)=x_2(0)=(\alpha x_1'-\beta x_2')(1)=0$ with $-x_2=\mu x_1$ and $-(\alpha x_1'-\beta x_2')'=\mu x_2$. We deduce that

$$x_2=-\mu x_1,\qquad x_1''=\frac{\mu^2}{(\alpha+\mu\beta)}x_1,$$

with $x_1'(1)=0$. Therefore x_1 satisfies the same differential equation as the one in the proof of Lemma 4.1 where $x_2=\lambda x_1$ is replaced by $x_2=-\mu x_1$. Thus the claimed conclusion follows from the proof of Lemma 4.1. □

For any $(k_1,\epsilon_1)\neq(k_2,\epsilon_2)$,

$$\begin{aligned}\lambda_{k_1,\epsilon_1}\left\langle\Phi_{k_1,\epsilon_1},\psi_{k_2,\epsilon_2}\right\rangle_{\mathcal{H}}&=\left\langle\mathcal{A}_0\Phi_{k_1,\epsilon_1},\psi_{k_2,\epsilon_2}\right\rangle_{\mathcal{H}}\\&=\left\langle\Phi_{k_1,\epsilon_1},\mathcal{A}_0^*\psi_{k_2,\epsilon_2}\right\rangle_{\mathcal{H}}\\&=\lambda_{k_2,\epsilon_2}\left\langle\Phi_{k_1,\epsilon_1},\psi_{k_2,\epsilon_2}\right\rangle_{\mathcal{H}}.\end{aligned}$$

Thus, as[3] $\lambda_{k_1,\epsilon_1}\neq\lambda_{k_2,\epsilon_2}$, $\left\langle\Phi_{k_1,\epsilon_1},\psi_{k_2,\epsilon_2}\right\rangle_{\mathcal{H}}=0$. Furthermore, based on $\lambda_{k,\epsilon}\overline{\mu_{k,\epsilon}}=\lambda_{k,\epsilon}^2$ and (12), a direct integration yields

$$\left\langle\Phi_{k,\epsilon},\psi_{k,\epsilon}\right\rangle_{\mathcal{H}}=\frac{1}{2\|\phi_{k,\epsilon}\|_{\mathcal{H}}}\left(1-\frac{\lambda_{k,-\epsilon}}{\lambda_{k,\epsilon}}\right)\neq 0,$$

because Assumption 3.1 implies $\lambda_{k,\epsilon}\neq\lambda_{k,-\epsilon}$. Thus, introducing

$$\Psi_{k,\epsilon}\triangleq\frac{1}{\overline{\left\langle\Phi_{k,\epsilon},\psi_{k,\epsilon}\right\rangle_{\mathcal{H}}}}\psi_{k,\epsilon},$$

and letting $\Psi=\{\Psi_{k,\epsilon},\ k\in\mathbb{N},\ \epsilon\in\{-1,+1\}\}$, the set of eigenvectors Φ of $\mathcal{A}_0$ is biorthogonal to the set of eigenvectors Ψ of $\mathcal{A}_0^*$ in the sense that $\left\langle\Phi_{k_1,\epsilon_1},\Psi_{k_2,\epsilon_2}\right\rangle_{\mathcal{H}}=\delta_{(k_1,\epsilon_1),(k_2,\epsilon_2)}$.

C. $\mathcal{A}_0$ is a Riesz-Spectral Operator

We show that Φ is a Riesz basis of $\mathcal{H}$ (see Definition 2.1) and $\mathcal{A}_0$ is a Riesz-spectral operator (see Definition 2.2).

1) Φ is maximal: Let us first introduce the following technical lemma whose proof is provided in Appendix.

Lemma 4.4: Both $\{\cos(\tilde{k}\pi\cdot),\ k\in\mathbb{N}\}$ and $\{\sin(\tilde{k}\pi\cdot),\ k\in\mathbb{N}\}$ are maximal in $L^2(0,1)$.

Then, the following result holds true.

Lemma 4.5: Φ is maximal in $\mathcal{H}$.

Proof. Let $z=(z_1,z_2)\in\mathcal{H}$ be such that $\left\langle\Phi_{k,\epsilon},z\right\rangle_{\mathcal{H}}=0$ for all $k\in\mathbb{N}$ and $\epsilon\in\{-1,+1\}$. Then,

$$\alpha\tilde{k}\pi\left\langle\cos(\tilde{k}\pi\cdot),z_1'\right\rangle_{L^2(0,1)}+\lambda_{k,\epsilon}\left\langle\sin(\tilde{k}\pi\cdot),z_2\right\rangle_{L^2(0,1)}=0,$$

[2]Because $\lambda^2/(\alpha+\lambda\beta)=0$ implies $\lambda=0$ which yields $x_1=x_2=0$.

[3]It directly follows from Assumption 3.1 and the fact that $\lambda_{k,\epsilon}^2/(\alpha+\lambda_{k,\epsilon}\beta)=-\tilde{k}^2\pi^2$.

from which we obtain that, for all $k \in \mathbb{N}$,

$$\begin{bmatrix} \alpha\tilde{k}\pi & \lambda_{k,-1} \\ \alpha\tilde{k}\pi & \lambda_{k,+1} \end{bmatrix} \begin{bmatrix} \left\langle \cos(\tilde{k}\pi\cdot), z_1' \right\rangle_{L^2(0,1)} \\ \left\langle \sin(\tilde{k}\pi\cdot), z_2 \right\rangle_{L^2(0,1)} \end{bmatrix} = 0.$$

Based on Assumption 3.1, $\alpha\tilde{k}\pi(\lambda_{k,+1} - \lambda_{k,-1}) \neq 0$, which implies the invertibility of the 2×2 matrix. Therefore, we have

$$\forall k \in \mathbb{N}, \qquad \left\langle \cos(\tilde{k}\pi\cdot), z_1' \right\rangle_{L^2(0,1)} = \left\langle \sin(\tilde{k}\pi\cdot), z_2 \right\rangle_{L^2(0,1)} = 0.$$

Hence, Lemma 4.4 ensures that $z_1' = z_2 = 0$. As $z_1(0) = 0$, we conclude that $z = 0$. □

2) Φ is a Riesz basis: Direct integrations show that for any non negative integers $k_1 \neq k_2$ and $\epsilon_1, \epsilon_2 \in \{-1, +1\}$,

$$\langle \Phi_{k_1,\epsilon_1}, \Phi_{k_2,\epsilon_2} \rangle_{\mathcal{H}} = 0. \tag{18}$$

Nevertheless, Φ is not a Hilbert basis because for any $k \in \mathbb{N}$ and $\epsilon \in \{-1, +1\}$,

$$\langle \Phi_{k,\epsilon}, \Phi_{k,-\epsilon} \rangle_{\mathcal{H}} = \frac{1}{2\|\phi_{k,\epsilon}\|_{\mathcal{H}} \|\phi_{k,-\epsilon}\|_{\mathcal{H}}} \left(1 + \frac{\tilde{k}^2\alpha\pi^2}{\lambda_{k,\epsilon}\overline{\lambda_{k,-\epsilon}}} \right) \neq 0.$$

However, we have the following result.

Lemma 4.6: Φ is a Riesz basis.

Proof. Based on Lemma 4.5 it is sufficient to show (1). For any $N \in \mathbb{N}$ and any $a_{k,\epsilon} \in \mathbb{C}$, we infer from (18) that

$$\begin{aligned} &\left\| \sum_{(k,\epsilon)\in\mathcal{I}_N} a_{k,\epsilon}\Phi_{k,\epsilon} \right\|_{\mathcal{H}}^2 \\ &= \sum_{(k_1,\epsilon_1)\in\mathcal{I}_N} \sum_{(k_2,\epsilon_2)\in\mathcal{I}_N} a_{k_1,\epsilon_1}\overline{a_{k_2,\epsilon_2}} \langle \Phi_{k_1,\epsilon_1}, \Phi_{k_2,\epsilon_2} \rangle_{\mathcal{H}} \\ &= \sum_{0\leq k\leq N} S_k, \end{aligned} \tag{19}$$

where

$$S_k = |a_{k,-1}|^2 + |a_{k,+1}|^2 + 2\operatorname{Re}\left(a_{k,-1}\overline{a_{k,+1}} \langle \Phi_{k,-1}, \Phi_{k,+1} \rangle_{\mathcal{H}} \right).$$

We evaluate the term $\langle \Phi_{k,-1}, \Phi_{k,+1} \rangle_{\mathcal{H}}$ as follows:

$$\langle \Phi_{k,-1}, \Phi_{k,+1} \rangle_{\mathcal{H}} = \frac{1 + \dfrac{\tilde{k}^2\alpha\pi^2}{\lambda_{k,-1}\overline{\lambda_{k,+1}}}}{\sqrt{1 + \dfrac{\tilde{k}^2\alpha\pi^2}{|\lambda_{k,-1}|^2}}\sqrt{1 + \dfrac{\tilde{k}^2\alpha\pi^2}{|\lambda_{k,+1}|^2}}}.$$

We first consider the case $k \geq k_0$. As $\lambda_{k,-1}\overline{\lambda_{k,+1}} = \lambda_{k,-1}\lambda_{k,+1} = \tilde{k}^2\alpha\pi^2$, we have that

$$\langle \Phi_{k,-1}, \Phi_{k,+1} \rangle_{\mathcal{H}} = \frac{2}{\sqrt{2 + \dfrac{|\lambda_{k,-1}|^2 + |\lambda_{k,+1}|^2}{\tilde{k}^2\alpha\pi^2}}}.$$

Based on (9),

$$\frac{|\lambda_{k,-1}|^2 + |\lambda_{k,+1}|^2}{\tilde{k}^2\alpha\pi^2} = \frac{\tilde{k}^2\beta^2\pi^2}{\alpha} - 2 \geq \frac{\tilde{k}_0^2\beta^2\pi^2}{\alpha} - 2,$$

yielding for all $k \geq k_0$,

$$\left| \langle \Phi_{k,-1}, \Phi_{k,+1} \rangle_{\mathcal{H}} \right| \leq \frac{4\sqrt{\alpha}}{(2k_0+1)\beta\pi} < 1,$$

where the last inequality holds true because, based on the definition (5) of k_0 and Assumption 3.1, $k_0 > 2\sqrt{\alpha}/(\beta\pi) - 1/2$.

We now consider the case $0 \leq k \leq k_0 - 1$ when $k_0 \geq 1$. A similar computation shows that

$$\left| \langle \Phi_{k,-1}, \Phi_{k,+1} \rangle_{\mathcal{H}} \right| = \frac{\tilde{k}\beta\pi}{2\sqrt{\alpha}} \leq \frac{(2k_0-1)\beta\pi}{4\sqrt{\alpha}} < 1,$$

where the last inequality holds true because, based on the definition (5) of k_0 and Assumption 3.1, $k_0 < 2\sqrt{\alpha}/(\beta\pi) + 1/2$.

Thus, introducing

$$C \triangleq \begin{cases} \max\left(\dfrac{4\sqrt{\alpha}}{(2k_0+1)\beta\pi}, \dfrac{(2k_0-1)\beta\pi}{4\sqrt{\alpha}} \right) & \text{if } k_0 \geq 1; \\ \dfrac{4\sqrt{\alpha}}{\beta\pi} & \text{if } k_0 = 0, \end{cases}$$

we obtain that $C \in (0,1)$ and $\left| \langle \Phi_{k,-1}, \Phi_{k,+1} \rangle_{\mathcal{H}} \right| \leq C$ for all $k \geq 0$. This yields

$$\left| \operatorname{Re}\left(a_{k,-1}\overline{a_{k,+1}} \langle \Phi_{k,-1}, \Phi_{k,+1} \rangle_{\mathcal{H}} \right) \right| \leq C|a_{k,-1}||a_{k,+1}|.$$

Consequently, we have

$$\begin{aligned} S_k &\leq |a_{k,-1}|^2 + |a_{k,+1}|^2 + 2C|a_{k,-1}||a_{k,+1}| \\ &\leq (1-C)\left(|a_{k,-1}|^2 + |a_{k,+1}|^2 \right) + C\left(|a_{k,-1}| + |a_{k,+1}| \right)^2 \\ &\leq (1+C)\left(|a_{k,-1}|^2 + |a_{k,+1}|^2 \right), \end{aligned}$$

and

$$\begin{aligned} S_k &\geq |a_{k,-1}|^2 + |a_{k,+1}|^2 - 2C|a_{k,-1}||a_{k,+1}| \\ &\geq (1-C)\left(|a_{k,-1}|^2 + |a_{k,+1}|^2 \right) + C\left(|a_{k,-1}| - |a_{k,+1}| \right)^2 \\ &\geq (1-C)\left(|a_{k,-1}|^2 + |a_{k,+1}|^2 \right). \end{aligned}$$

Combining the two inequalities above with (19), we obtain the desired result:

$$\begin{aligned} m_R \sum_{(k,\epsilon)\in\mathcal{I}_N} |a_{k,\epsilon}|^2 &\leq \left\| \sum_{(k,\epsilon)\in\mathcal{I}_N} a_{k,\epsilon}\Phi_{k,\epsilon} \right\|_{\mathcal{H}}^2 \\ &\leq M_R \sum_{(k,\epsilon)\in\mathcal{I}_N} |a_{k,\epsilon}|^2 \end{aligned} \tag{20}$$

with $m_R = 1 - C > 0$ and $M_R = 1 + C > 0$. As m_R and M_R are constants independent of $N \in \mathbb{N}$ and $a_k \in \mathbb{C}$, the claimed conclusion holds true. □

Remark 4.7: The constants $m_R = 1 - C$ and $M_R = 1 + C$ provide a tight version of (20). Indeed, it follows from the proof that there exists $k \in \{k_0 - 1, k_0\}$ such that $\left| \langle \Phi_{k,-1}, \Phi_{k,+1} \rangle_{\mathcal{H}} \right| = C$. Thus $\langle \Phi_{k,-1}, \Phi_{k,+1} \rangle_{\mathcal{H}} = Ce^{i\theta}$ for some $\theta \in [0, 2\pi)$. Considering $a_{k,-1} = \overline{a_{k,+1}} = e^{-i\theta/2}$ we obtain $S_k = 2(1+C) = (1+C)\left(|a_{k,-1}|^2 + |a_{k,+1}|^2 \right)$. Conversely, with $a_{k,-1} = -\overline{a_{k,+1}} = e^{-i\theta/2}$ we obtain $S_k = 2(1-C) = (1-C)\left(|a_{k,-1}|^2 + |a_{k,+1}|^2 \right)$.

As Φ is a Riesz basis biorthogonal to Ψ, we obtain from the general theory on Riesz basis [4] that for all $z \in \mathcal{H}$,

$$z = \sum_{(k,\epsilon)\in\mathcal{I}_\infty} \langle z, \Psi_{k,\epsilon} \rangle_{\mathcal{H}} \Phi_{k,\epsilon}, \tag{21}$$

and

$$\begin{aligned} (1-C) \sum_{(k,\epsilon)\in\mathcal{I}_\infty} \left| \langle z, \Psi_{k,\epsilon} \rangle_{\mathcal{H}} \right|^2 &\leq \|z\|_{\mathcal{H}}^2 \\ &\leq (1+C) \sum_{(k,\epsilon)\in\mathcal{I}_\infty} \left| \langle z, \Psi_{k,\epsilon} \rangle_{\mathcal{H}} \right|^2. \end{aligned} \tag{22}$$

3) $\mathcal{A}_0$ is a Riesz-Spectral Operator: We can now state the main result of this section.

Lemma 4.8: The operator $\mathcal{A}_0$ is a Riesz-spectral operator generating an exponentially stable C_0-semigroup with growth $\omega_0 = -\kappa_0 < 0$ where κ_0 is given by (8)

Proof. We directly deduce from the fact that $\mathcal{A}_0$ generates a C_0-semigroup and from Lemmas 4.1 and 4.6 that $\mathcal{A}_0$ is a Riesz-spectral operator. Thus, its growth bound ω_0 satisfies [5, Th. 2.3.5]:

$$\omega_0 = \sup_{(k,\epsilon)\in\mathcal{I}_\infty} \operatorname{Re}\lambda_{k,\epsilon}.$$

Based on (13-14), $\operatorname{Re}\lambda_{k,\epsilon} \leq -\kappa_0$ where κ_0 is given by (8). If $k_0 \geq 1$, (13) becomes an equality for $k = 0$. Furthermore, as (16) holds, this yields $\omega_0 = -\kappa_0$. □

V. Proof of the EISS property

We can now prove the main result of this note.

Proof of Theorem 3.3. Let $X_0 \in D(\mathcal{A})$, $u \in \mathcal{C}^1(\mathbb{R}_+; L^2(0,1))$, and $d \in \mathcal{C}^2(\mathbb{R}_+)$ such that $\mathcal{B}X_0 = d(0)$. Let $X = (x_1, x_2) \in \mathcal{C}^0(\mathbb{R}_+; D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$ be the unique classical solution of the abstract system (3). Based on (21-22),

$$\forall t \geq 0, \ \ \|X(t)\|_{\mathcal{H}}^2 \leq (1+C) \sum_{(k,\epsilon)\in\mathcal{I}_\infty} |c_{k,\epsilon}(t)|^2, \tag{23}$$

where $c_{k,\epsilon} \triangleq \langle X, \Psi_{k,\epsilon}\rangle_{\mathcal{H}} \in \mathcal{C}^1(\mathbb{R}_+)$. With $\Psi_{k,\epsilon} \triangleq (\Psi_{k,\epsilon}^1, \Psi_{k,\epsilon}^2)$, we have for all $t \geq 0$,

$$\begin{aligned}
\dot{c}_{k,\epsilon}(t) &= \left\langle \frac{\mathrm{d}X}{\mathrm{d}t}(t), \Psi_{k,\epsilon} \right\rangle_{\mathcal{H}} \\
&= \langle \mathcal{A}X(t) + U(t), \Psi_{k,\epsilon}\rangle_{\mathcal{H}} \\
&= \left\langle (x_2(t), (\alpha x_1' + \beta x_2')'(t) + u(t)), (\Psi_{k,\epsilon}^1, \Psi_{k,\epsilon}^2) \right\rangle_{\mathcal{H}} \\
&= \int_0^1 \alpha x_2'(t)\overline{\Psi_{k,\epsilon}^{1'}} + \{(\alpha x_1' + \beta x_2')'(t) + u(t)\}\overline{\Psi_{k,\epsilon}^2}\,\mathrm{d}\xi \\
&= \int_0^1 \alpha x_2'(t)\overline{\Psi_{k,\epsilon}^{1'}}\,\mathrm{d}\xi + \left[(\alpha x_1' + \beta x_2')(t)\overline{\Psi_{k,\epsilon}^2}\right]_{\xi=0}^{\xi=1} \\
&\quad - \int_0^1 (\alpha x_1' + \beta x_2')(t)\overline{\Psi_{k,\epsilon}^{2'}}\,\mathrm{d}\xi + \int_0^1 u(t)\overline{\Psi_{k,\epsilon}^2}\,\mathrm{d}\xi \\
&= \int_0^1 \alpha x_1'(t)\overline{(-\Psi_{k,\epsilon}^2)'}\,\mathrm{d}\xi + \int_0^1 x_2'(t)\overline{\{\alpha\Psi_{k,\epsilon}^{1'} - \beta\Psi_{k,\epsilon}^{2'}\}}\,\mathrm{d}\xi \\
&\quad + d(t)\overline{\Psi_{k,\epsilon}^2(1)} + \int_0^1 u(t)\overline{\Psi_{k,\epsilon}^2}\,\mathrm{d}\xi \\
&= \int_0^1 \alpha x_1'(t)\overline{(-\Psi_{k,\epsilon}^2)'}\,\mathrm{d}\xi + \left[x_2(t)\overline{\{\alpha\Psi_{k,\epsilon}^{1'} - \beta\Psi_{k,\epsilon}^{2'}\}}\right]_{\xi=0}^{\xi=1} \\
&\quad - \int_0^1 x_2(t)\overline{\{\alpha\Psi_{k,\epsilon}^{1'} - \beta\Psi_{k,\epsilon}^{2'}\}'}\,\mathrm{d}\xi + d(t)\overline{\Psi_{k,\epsilon}^2(1)} \\
&\quad + \int_0^1 u(t)\overline{\Psi_{k,\epsilon}^2}\,\mathrm{d}\xi \\
&= \langle X(t), \mathcal{A}_0^*\Psi_{k,\epsilon}\rangle_{\mathcal{H}} + d(t)\overline{\Psi_{k,\epsilon}^2(1)} + \int_0^1 u(t)\overline{\Psi_{k,\epsilon}^2}\,\mathrm{d}\xi.
\end{aligned} \tag{24}$$

As $\mathcal{A}_0^*\Psi_{k,\epsilon} = \mu_{k,\epsilon}\Psi_{k,\epsilon} = \overline{\lambda_{k,\epsilon}}\Psi_{k,\epsilon}$, this yields for any $t \geq 0$,

$$\dot{c}_{k,\epsilon}(t) = \lambda_{k,\epsilon}c_{k,\epsilon}(t) + d(t)\overline{\Psi_{k,\epsilon}^2(1)} + \int_0^1 u(t)\overline{\Psi_{k,\epsilon}^2}\,\mathrm{d}\xi,$$

which gives after integration:

$$\begin{aligned}
c_{k,\epsilon}(t) &= e^{\lambda_{k,\epsilon}t}c_{k,\epsilon}(0) + \int_0^t e^{\lambda_{k,\epsilon}(t-\tau)}d(\tau)\overline{\Psi_{k,\epsilon}^2(1)}\,\mathrm{d}\tau \\
&\quad + \int_0^t e^{\lambda_{k,\epsilon}(t-\tau)}\int_0^1 u(\tau)\overline{\Psi_{k,\epsilon}^2}\,\mathrm{d}\xi\,\mathrm{d}\tau.
\end{aligned} \tag{25}$$

We estimate the three terms on the right-hand side of (25) as follows. First,

$$\left|e^{\lambda_{k,\epsilon}t}c_{k,\epsilon}(0)\right| \leq e^{\operatorname{Re}\lambda_{k,\epsilon}t}\,|c_{k,\epsilon}(0)| \leq e^{-\kappa_0 t}\,|c_{k,\epsilon}(0)|\,. \tag{26}$$

Second, introducing $\gamma_{k,\epsilon} \triangleq |\Psi_{k,\epsilon}^2(1)/\operatorname{Re}\lambda_{k,\epsilon}|$,

$$\begin{aligned}
&\left|\int_0^t e^{\lambda_{k,\epsilon}(t-\tau)}d(\tau)\overline{\Psi_{k,\epsilon}^2(1)}\,\mathrm{d}\tau\right| \\
&\qquad \leq \gamma_{k,\epsilon}\int_0^t -\operatorname{Re}\lambda_{k,\epsilon}e^{\operatorname{Re}\lambda_{k,\epsilon}(t-\tau)}\,\mathrm{d}\tau\,\|d\|_{\mathcal{C}^0([0,t])} \\
&\qquad \leq \gamma_{k,\epsilon}\left(1 - e^{\operatorname{Re}\lambda_{k,\epsilon}t}\right)\|d\|_{\mathcal{C}^0([0,t])} \\
&\qquad \leq \gamma_{k,\epsilon}\|d\|_{\mathcal{C}^0([0,t])},
\end{aligned} \tag{27}$$

with

$$\begin{aligned}
\gamma_{k,\epsilon} &= \left|\frac{\psi_{k,\epsilon}^2(1)}{\operatorname{Re}(\lambda_{k,\epsilon})\,\langle\Phi_{k,\epsilon},\psi_{k,\epsilon}\rangle_{\mathcal{H}}}\right| \\
&= \frac{2\|\phi_{k,\epsilon}\|_{\mathcal{H}}}{\left|\operatorname{Re}(\lambda_{k,\epsilon})\left(1 - \dfrac{\lambda_{k,-\epsilon}}{\lambda_{k,\epsilon}}\right)\right|}.
\end{aligned}$$

Finally, by using Cauchy-Schwarz inequality,

$$\begin{aligned}
&\left|\int_0^t e^{\lambda_{k,\epsilon}(t-\tau)}\int_0^1 u(\tau)\overline{\Psi_{k,\epsilon}^2}\,\mathrm{d}\xi\,\mathrm{d}\tau\right| \\
\leq &\frac{1}{|\operatorname{Re}\lambda_{k,\epsilon}|}\int_0^t -\operatorname{Re}\lambda_{k,\epsilon}e^{\operatorname{Re}\lambda_{k,\epsilon}(t-\tau)}\int_0^1\left|u(\tau)\overline{\Psi_{k,\epsilon}^2}\right|\mathrm{d}\xi\,\mathrm{d}\tau \\
\leq &\frac{\|\Psi_{k,\epsilon}^2\|_{L^2(0,1)}}{|\operatorname{Re}\lambda_{k,\epsilon}|}\int_0^t -\operatorname{Re}\lambda_{k,\epsilon}e^{\operatorname{Re}\lambda_{k,\epsilon}(t-\tau)}\|u(\tau)\|_{L^2(0,1)}\,\mathrm{d}\tau \\
\leq &\frac{\|\Psi_{k,\epsilon}^2\|_{L^2(0,1)}}{|\operatorname{Re}\lambda_{k,\epsilon}|}\left(1 - e^{\operatorname{Re}\lambda_{k,\epsilon}t}\right)\|u\|_{\mathcal{C}^0([0,t];L^2(0,1))} \\
\leq &\frac{\|\Psi_{k,\epsilon}^2\|_{L^2(0,1)}}{|\operatorname{Re}\lambda_{k,\epsilon}|}\|u\|_{\mathcal{C}^0([0,t];L^2(0,1))},
\end{aligned}$$

and as

$$\begin{aligned}
\|\Psi_{k,\epsilon}^2\|_{L^2(0,1)} &= \frac{1}{|\langle\Phi_{k,\epsilon},\psi_{k,\epsilon}\rangle_{\mathcal{H}}|}\sqrt{\int_0^1 \sin^2(\tilde{k}\pi\xi)\,\mathrm{d}\xi} \\
&= \frac{\sqrt{2}\|\phi_{k,\epsilon}\|_{\mathcal{H}}}{\left|1 - \dfrac{\lambda_{k,-\epsilon}}{\lambda_{k,\epsilon}}\right|} \\
&= \gamma_{k,\epsilon}|\operatorname{Re}\lambda_{k,\epsilon}|/\sqrt{2},
\end{aligned}$$

this yields

$$\begin{aligned}
&\left|\int_0^t e^{\lambda_{k,\epsilon}(t-\tau)}\int_0^1 u(\tau)\overline{\Psi_{k,\epsilon}^2}\,\mathrm{d}\xi\,\mathrm{d}\tau\right| \\
&\qquad \leq \frac{\sqrt{2}}{2}\gamma_{k,\epsilon}\|u\|_{\mathcal{C}^0([0,t];L^2(0,1))}.
\end{aligned} \tag{28}$$

Putting together (25) with the inequalities (26-28), this yields for all $t \geq 0$,

$$\begin{aligned}
|c_{k,\epsilon}(t)| \leq &e^{-\kappa_0 t}\,|c_{k,\epsilon}(0)| + \gamma_{k,\epsilon}\|d\|_{\mathcal{C}^0([0,t])} \\
&+ \frac{\sqrt{2}}{2}\gamma_{k,\epsilon}\|u\|_{\mathcal{C}^0([0,t];L^2(0,1))}.
\end{aligned}$$

As $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ for all $a,b,c \in \mathbb{R}$,

$$\begin{aligned}
|c_{k,\epsilon}(t)|^2 \leq &3e^{-2\kappa_0 t}\,|c_{k,\epsilon}(0)|^2 + 3\gamma_{k,\epsilon}^2\|d\|_{\mathcal{C}^0([0,t])}^2 \\
&+ \frac{3}{2}\gamma_{k,\epsilon}^2\|u\|_{\mathcal{C}^0([0,t];L^2(0,1))}^2.
\end{aligned} \tag{29}$$

We need to check that $\gamma_{k,\epsilon}^2$ is a summable sequence. To do so, considering $k \geq k_0$, $\operatorname{Re}\lambda_{k,\epsilon} = \lambda_{k,\epsilon}$, which gives along with (9) and (11)

$$\gamma_{k,\epsilon} = \frac{\sqrt{2}}{\tilde{k}\pi\sqrt{\tilde{k}^2\beta^2\pi^2 - 4\alpha}}\sqrt{1 + \frac{\tilde{k}^2\alpha\pi^2}{|\lambda_{k,\epsilon}|^2}}.$$

Based on (15-16), the following asymptotic behaviours hold

$$\gamma_{k,+1} \underset{k\to+\infty}{\sim} \frac{1}{k\pi}\sqrt{\frac{2}{\alpha}}, \qquad \gamma_{k,-1} \underset{k\to+\infty}{\sim} \frac{\sqrt{2}}{k^2\beta\pi^2},$$

assessing that $\gamma_{k,\epsilon}$ is a square summable sequence. Therefore, we can define the constant $\gamma \in \mathbb{R}_+$ by

$$\gamma^2 \triangleq \sum_{(k,\epsilon)\in\mathcal{I}_\infty} \gamma_{k,\epsilon}^2 < \infty.$$

Noting that, based on (21-22),

$$\sum_{(k,\epsilon)\in\mathcal{I}_\infty} |c_{k,\epsilon}(0)|^2 \leq \frac{1}{1-C}\|X_0\|_{\mathcal{H}}^2,$$

we obtain by using (29) into (23) that

$$\begin{aligned}\|X(t)\|_{\mathcal{H}}^2 \leq & 3\frac{1+C}{1-C}e^{-2\kappa_0 t}\|X_0\|_{\mathcal{H}}^2 + 3(1+C)\gamma^2\|d\|_{\mathcal{C}^0([0,t])}^2 \\ & + \frac{3}{2}(1+C)\gamma^2\|u\|_{\mathcal{C}^0([0,t];L^2(0,1))}^2.\end{aligned}$$

To conclude, it is sufficient to note that $\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$ for all $a,b\in\mathbb{R}_+$, which yields

$$\begin{aligned}\|X(t)\|_{\mathcal{H}} \leq & \sqrt{3\frac{1+C}{1-C}}e^{-\kappa_0 t}\|X_0\|_{\mathcal{H}} + \gamma\sqrt{3(1+C)}\|d\|_{\mathcal{C}^0([0,t])} \\ & + \gamma\sqrt{\frac{3}{2}(1+C)}\|u\|_{\mathcal{C}^0([0,t];L^2(0,1))}.\end{aligned}$$

Thus, the claimed ISS estimate (6) holds with

$$C_0 = \sqrt{3\frac{1+C}{1-C}},\ C_1 = \gamma\sqrt{3(1+C)},\ C_2 = \gamma\sqrt{\frac{3}{2}(1+C)}.$$

To prove the second ISS estimate (7), we substitute the estimations (27-28) with

$$\left|\int_0^t e^{\lambda_{k,\epsilon}(t-\tau)} d(\tau)\overline{\Psi_{k,\epsilon}^2(1)}\,\mathrm{d}\tau\right| \leq \sqrt{\frac{|\operatorname{Re}\lambda_{k,\epsilon}|}{2}}\gamma_{k,\epsilon}\|d\|_{L^2(0,t)},$$

and

$$\begin{aligned}&\left|\int_0^t e^{\lambda_{k,\epsilon}(t-\tau)}\int_0^1 u(\tau)\overline{\Psi_{k,\epsilon}^2}\,\mathrm{d}\xi\,\mathrm{d}\tau\right| \\ &\qquad \leq \frac{1}{2}\sqrt{|\operatorname{Re}\lambda_{k,\epsilon}|}\gamma_{k,\epsilon}\|u\|_{L^2((0,t)\times(0,1))},\end{aligned}$$

where the Cauchy-Schwartz inequality has been used. This yields for all $t\geq 0$,

$$\begin{aligned}|c_{k,\epsilon}(t)| \leq & e^{-\kappa_0 t}|c_{k,\epsilon}(0)| + \sqrt{\frac{|\operatorname{Re}\lambda_{k,\epsilon}|}{2}}\gamma_{k,\epsilon}\|d\|_{L^2(0,t)} \\ & + \frac{1}{2}\sqrt{|\operatorname{Re}\lambda_{k,\epsilon}|}\gamma_{k,\epsilon}\|u\|_{L^2((0,t)\times(0,1))}.\end{aligned}$$

Noting that, for any $\epsilon\in\{-1,+1\}$,

$$\sqrt{|\operatorname{Re}\lambda_{k,\epsilon}|}\gamma_{k,\epsilon} \underset{k\to+\infty}{\sim} \frac{1}{k\pi}\sqrt{\frac{2}{\beta}},$$

$\sqrt{|\operatorname{Re}\lambda_{k,\epsilon}|}\gamma_{k,\epsilon}$ is a square summable sequence and we can define the constant $\gamma'\in\mathbb{R}_+$ by

$$\gamma'^2 \triangleq \sum_{(k,\epsilon)\in\mathcal{I}_\infty} |\operatorname{Re}\lambda_{k,\epsilon}|\gamma_{k,\epsilon}^2 < \infty.$$

Following the same procedure used above to demonstrate the ISS estimate (6), we obtain for all $t\geq 0$,

$$\begin{aligned}\|X(t)\|_{\mathcal{H}} \leq & \sqrt{3\frac{1+C}{1-C}}e^{-\kappa_0 t}\|X_0\|_{\mathcal{H}} + \gamma'\sqrt{\frac{3}{2}(1+C)}\|d\|_{L^2(0,t)} \\ & + \frac{\gamma'}{2}\sqrt{3(1+C)}\|u\|_{L^2((0,t)\times(0,1))}.\end{aligned}$$

Thus, introducing the constants $C_3, C_4\in\mathbb{R}_+^*$ defined by

$$C_3 = \gamma'\sqrt{\frac{3}{2}(1+C)},\ C_4 = \frac{\gamma'}{2}\sqrt{3(1+C)},$$

the second claimed ISS estimate (7) holds. □

Remark 5.1: The key idea in the proof of the main result lies in the computation of (24). In the disturbance free case, i.e., $d=0$ and $u=0$, one has $X\in\mathcal{C}^0(\mathbb{R}_+;D(\mathcal{A}_0))\cap\mathcal{C}^1(\mathbb{R}_+;\mathcal{H})$. Then, because $\Psi_{k,\epsilon}\in D(\mathcal{A}_0^*)$, we obtain as a direct consequence of the definition of the adjoint operator that

$$\begin{aligned}\dot{c}_{k,\epsilon}(t) = \left\langle \frac{\mathrm{d}X}{\mathrm{d}t}(t),\Psi_{k,\epsilon}\right\rangle_{\mathcal{H}} &= \langle\mathcal{A}_0X(t),\Psi_{k,\epsilon}\rangle_{\mathcal{H}} \\ &= \langle X(t),\mathcal{A}_0^*\Psi_{k,\epsilon}\rangle_{\mathcal{H}},\end{aligned}$$

which coincides with (24) in the disturbance free case. In the disturbed case, the computation (24) is nothing but the heuristic computation of the adjoint operator while letting appear 1) an extra non zero boundary condition, via the integrations by parts, due to the boundary disturbance d ; 2) an integral term due to the distributed disturbance U.

Remark 5.2: Putting together (21) and (25), one can get an explicit formula of the system trajectory X in function of X_0, d, U, and the eigenstructures of operators $\mathcal{A}_0$ and $\mathcal{A}_0^*$.

We deduce, as a direct consequence of the ISS estimates (6-7) and of the semigroup property of (3), the following asymptotic behaviour.

Corollary 5.2.1: Under the notations and assumptions of Theorem 3.3, assume that one of the two following conditions holds:

- the perturbations are vanishing in the sense that $|d(t)| \underset{t\to+\infty}{\longrightarrow} 0$ and $\|u(t)\|_{L^2(0,1)} \underset{t\to+\infty}{\longrightarrow} 0$;
- the perturbations are of finite energy, i.e., $d\in L^2(\mathbb{R}_+)$ and $u\in L^2(\mathbb{R}_+;L^2(0,1))\cong L^2(\mathbb{R}_+\times(0,1))$,

then $\|X(t)\|_{\mathcal{H}} \underset{t\to+\infty}{\longrightarrow} 0$.

VI. Conclusion

This paper established the property of Exponential Input-to-State Stability (EISS) for a clamped-free damped string with respect to distributed and boundary disturbances. The adopted approach does not rely on the construction of an adequate Lyapunov function but takes advantage of functional analysis tools. Specifically, by projecting the system trajectories onto a Riesz basis of the underlying Hilbert space formed by the eigenvectors of the disturbance-free operator, the EISS property was derived directly on the original system, avoiding the appearance of the time derivative of the boundary perturbation.

Appendix
Proof of Lemma 4.4

From the Fourier series theory [24], the set of functions $\{e^{ik\pi\cdot},\ k\in\mathbb{Z}\}$ is a Hilbert basis of $L^2(-1,1)$ endowed with $\langle f,g\rangle_{L^2(-1,1)} = \frac{1}{2}\int_{-1}^1 f(\xi)\overline{g(\xi)}\,\mathrm{d}\xi$. Let $f\in L^2(-1,1)$ and consider $\hat{f}=e^{-i\pi\cdot/2}f\in L^2(-1,1)$. As

$$\hat{f} = \sum_{k\in\mathbb{Z}}\left\langle \hat{f},e^{ik\pi\cdot}\right\rangle_{L^2(-1,1)} e^{ik\pi\cdot},$$

and $|e^{i\pi\cdot/2}|=1$, then

$$f = e^{i\pi\cdot/2}\hat{f} = \sum_{k\in\mathbb{Z}}\left\langle \hat{f},e^{ik\pi\cdot}\right\rangle_{L^2(-1,1)} e^{i\tilde{k}\pi\cdot}.$$

Furthermore, with

$$a_k \triangleq \left\langle \hat{f},e^{ik\pi\cdot}\right\rangle_{L^2(-1,1)} = \frac{1}{2}\int_{-1}^1 f(\xi)e^{-i\tilde{k}\pi\xi}\,\mathrm{d}\xi,$$

and $\widetilde{(-k-1)} = -\tilde{k}$, we obtain that

$$f = \sum_{k\in\mathbb{N}} a_k e^{i\tilde{k}\pi\cdot} + a_{-k-1}e^{-i\tilde{k}\pi\cdot}.$$

Let an arbitrary function $g \in L^2(0,1)$ be given and consider the functions $f_{\text{even}}, f_{\text{odd}} \in L^2(-1,1)$ defined by

$$f_{\text{even}}(x) = \begin{cases} g(x) \text{ if } x \geq 0; \\ g(-x) \text{ if } x < 0. \end{cases} \quad f_{\text{odd}}(x) = \begin{cases} g(x) \text{ if } x \geq 0; \\ -g(-x) \text{ if } x < 0. \end{cases}$$

As f_{even} is an even function, we have,

$$a_k = \frac{1}{2}\int_{-1}^{1} f_{\text{even}}(\xi)\cos(\tilde{k}\pi\xi)\,\mathrm{d}\xi = a_{-k-1},$$

and thus

$$f_{\text{even}} = 2\sum_{k\in\mathbb{N}} a_k \cos(\tilde{k}\pi\cdot).$$

As the above equality holds in $L^2(-1,1)$, it also holds in $L^2(0,1)$. Noting that $f_{\text{even}}|_{(0,1)} = g$, we conclude that $\{\cos(\tilde{k}\pi\cdot),\ k \in \mathbb{N}\}$ is maximal in $L^2(0,1)$.

Similarly, as f_{odd} is an odd function,

$$a_k = -\frac{i}{2}\int_{-1}^{1} f_{\text{odd}}(\xi)\sin(\tilde{k}\pi\xi)\,\mathrm{d}\xi = -a_{-k-1},$$

and thus

$$f = 2i\sum_{k\in\mathbb{N}} a_k \sin(\tilde{k}\pi\cdot).$$

Applying the same argument as above, we conclude that $\{\sin(\tilde{k}\pi\cdot),\ k \in \mathbb{N}\}$ is maximal in $L^2(0,1)$. $\square$

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