Input-to-State Stability of Time-Delay Systems with Delay-Dependent Impulses

Xinzhi Liu and Kexue Zhang

Abstract—This paper studies input-to-state stability (ISS) of general nonlinear time-delay systems subject to delay-dependent impulse effects. Sufficient conditions for ISS are constructed by using the method of Lyapunov functionals. It is shown that, when the continuous dynamics are ISS but the discrete dynamics governing the delay-dependent impulses are not, the impulsive system as a whole is ISS provided the destabilizing impulses do not occur too frequently. On the contrary, when the discrete dynamics are ISS but the continuous dynamics are not, the delayed impulses must occur frequently enough to overcome the destabilizing effects of the continuous dynamics so that the ISS can be achieved for the impulsive system. Particularly, when the discrete dynamics are ISS and the continuous dynamics are also ISS or just stable for the zero input, the impulsive system is ISS for arbitrary impulse time sequences. Compared with the existing results on impulsive time-delay systems, the obtained ISS criteria are more general in the sense that these results are applicable to systems with delay dependent impulses while the existing ones are not. Moreover, when consider time-delay systems with delay-free impulses, our result for systems with unstable continuous dynamics and stabilizing impulses is less conservative than the existing ones, as a weaker condition on the upper bound of impulsive intervals is obtained. To demonstrate the theoretical results, we provide two examples with numerical simulations, in which distributed delays and discrete delays in the impulses are considered respectively.

Index Terms—Impulsive systems, time-delay systems, input-to-state stability, delay-dependent impulses.

I. INTRODUCTION

IMPULSIVE system is a dynamical system (modeled by impulsive differential equations) that combines continuous-time evolution and abrupt state changes or resets at a sequence of times. Due to their wide applications in consensus of multi-agent systems [1], synchronization of dynamical networks and applications to secure communication [2], infection disease control [3], spacecraft maneuvers [4], etc., extensive studies have been done on stability analysis and control problems of impulsive systems (see, e.g., [5]–[8]). Since time-delay is unavoidable in sampling, processing, and transmission of the impulse information in the system, recent years have witnessed a rapid progress in stability analysis and control of dynamical systems with delay-dependent impulses (see, e.g., [9]–[12]).

The notion of input-to-state stability (ISS), introduced in [13], has been proved very useful in stability analysis and control of dynamical systems. This notion characterize the effects of external inputs to the stability of control systems. The ISS properties have been investigated for various types of control systems (see, e.g., [14]–[22]). In terms of dynamical systems with impulse effects, the ISS notion is extended to impulsive systems (without time-delay) in [23], and then to impulsive systems with time-delay in the continuous time dynamics in [24]. More recently, the work of [25] generalizes the ISS notion to impulsive time-delay systems with switchings, and sufficient conditions for ISS are obtained using the method of Lyapunov functionals. In [12], the effect of delayed impulses on the ISS property of nonlinear delay-free systems is studied by the Lyapunov method. Up to now, numerous researchers have investigated the ISS properties of impulsive systems with time-delay (see, e.g., [12], [23]–[30]). However, to our best knowledge, time-delay in only considered in either the continuous dynamics of impulsive systems or just the impulses in the existing literature, and very few ISS results has been reported for systems with time-delay effects in both the continuous dynamics and the impulses. Due to the ubiquity of time-delay, examples of such systems can be easily found in impulsive stabilization of neural networks [10], impulsive consensus of networked multi-agent systems [31], impulsive master-slave synchronization [32], etc.

Motivated by the above discussion, we focus on nonlinear time-delay systems with delay-dependent impulses. The objective of this paper is to construct ISS criteria for such systems using the method of Lyapunov functionals. Compared with the existing results, especially the representative results in [12], [23]–[26], the main contribution is that our ISS results are constructed for nonlinear systems with time-delay effects in both the continuous dynamics and the impulses, while the results in [23]–[26] cannot be applied to systems with delay-dependent impulses and the ISS results in [12], [23] is not applicable to impulsive systems with time-delay in the continuous evolution. Furthermore, our ISS result for systems with unstable continuous dynamics and ISS impulses is less conservative than the ones in [25], [26] in the sense that our result can be applied to systems with a larger class of impulse time sequences.

The rest of the paper is organized as follows. Section II gives the basic notation and definitions, and formulate a general nonlinear impulsive systems with time-delay and external input. Sufficient conditions for ISS of the impulsive time-delay systems are constructed in Section III by using the method of Lyapunov functionals. Two numerical examples are provided in Section IV to demonstrate the main results. The paper is concluded by Section IV, where some future research directions are pointed out.

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II. PRELIMINARIES

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{R}$ the set of real numbers, $\mathbb{R}^+$ the set of nonnegative reals, and $\mathbb{R}^n$ the n-dimensional real space equipped with the Euclidean norm denoted by $\|\cdot\|$. For $a, b \in \mathbb{R}$ with $b > a$, denote $\mathcal{PC}([a,b],\mathbb{R}^n)$ the set of piecewise right continuous functions $\varphi : [a,b] \to \mathbb{R}^n$, and $\mathcal{PC}([a,\infty),\mathbb{R}^n)$ the set of functions $\phi : [a,\infty) \to \mathbb{R}^n$ satisfying $\phi|_{[a,b]} \in \mathcal{PC}([a,b],\mathbb{R}^n)$ for all $b > a$, where $\phi|_{[a,b]}$ is a restriction of ϕ on interval $[a,b]$. Given $r > 0$, the linear space $\mathcal{PC}([-r,0],\mathbb{R}^n)$ is equipped with a norm defined by $\|\varphi\|_r := \sup_{s \in [-r,0]} \|\varphi(s)\|$ for $\varphi \in \mathcal{PC}([-r,0],\mathbb{R}^n)$. For simplicity, we use $\mathcal{PC}$ to represent $\mathcal{PC}([-r,0],\mathbb{R}^n)$. Given $x \in \mathcal{PC}([-r,\infty),\mathbb{R}^n)$ and for each $t \in \mathbb{R}^+$, we define $x_t \in \mathcal{PC}$ as $x_t(s) := x(t+s)$ for $s \in [-r,0]$.

Consider the following nonlinear time-delay impulsive system:

$$\begin{cases} \dot{x}(t) = f(t, x_t, w(t)), & t \neq t_k, \ k \in \mathbb{N} \\ \Delta x(t) = I_k(t, x_{t^-}, w(t^-)), & t = t_k, \ k \in \mathbb{N} \\ x_{t_0} = \varphi, \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the system state; $w \in \mathcal{PC}([t_0,\infty),\mathcal{R}^m)$ is the input function; $\varphi \in \mathcal{PC}$ is the initial function; $f, I_k : \mathbb{R}^+ \times \mathcal{PC} \times \mathbb{R}^m \to \mathbb{R}^n$ satisfy $f(t,0,0) = I_k(t,0,0) = 0$ for all $k \in \mathbb{N}$; $\{t_1, t_2, t_3, ...\}$ is a strictly increasing sequence such that $t_k \to \infty$ as $t \to \infty$; $\Delta x(t) := x(t^+) - x(t^-)$ where $x(t^+) = \lim_{s \to t^+} x(s)$ and $x(t^-) = \lim_{s \to t^-} x(s)$ (similarly, $w(t^-) = \lim_{s \to t^-} w(s)$); x_{t^-} is defined as $x_{t^-}(s) = x(t+s)$ if $s \in [-r,0)$ and $x_{t^-}(s) = x(t^-)$ if $s = 0$. Given $w \in \mathcal{PC}([t_0,\infty),\mathcal{R}^m)$, define $g(t,\phi) = f(t,\phi,w(t))$ and assume g satisfies all the necessary conditions in [33] so that, for any initial condition $\varphi \in \mathcal{PC}$, system (1) has a unique solution $x(t,t_0,\varphi)$ that exists in a maximal interval $[t_0 - r, t_0 + \beta)$, where $0 < \beta \leq \infty$.

Before giving the formal ISS definition for system (1), we introduce the following function classes. A continuous function $\alpha : \mathbb{R}^+ \to \mathbb{R}$ is said to be of class $\mathcal{K}$ and we write $\alpha \in \mathcal{K}$, if α is strictly increasing and $\alpha(0) = 0$. If α also unbounded, we say that α is of class $\mathcal{K}_\infty$ and we write $\alpha \in \mathcal{K}_\infty$. A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be of class $\mathcal{KL}$ and we write $\beta \in \mathcal{KL}$, if $\beta(\cdot,t) \in \mathcal{K}$ for each $t \in \mathbb{R}^+$ and $\beta(s,t)$ decreases to 0 as $t \to \infty$ for each $s \in \mathbb{R}^+$. Now we are in the position to state the ISS definition for system (1).

Definition 1. *System* (1) *is said to be uniformly input-to-state stable (ISS) over a certain class ℓ of admissible impulse time sequences, if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$, independent of the choice of the sequences in ℓ, such that, for each initial condition $\varphi \in \mathcal{PC}$ and input function $w \in \mathcal{PC}([t_0,\infty),\mathbb{R}^m)$, the corresponding solution to (1) exists globally and satisfies*

$$\|x(t)\| \leq \beta(\|\varphi\|_r, t-t_0) + \gamma\left(\sup_{s \in [t_0,t]} \|w(s)\|\right), \ \text{for all } t \geq t_0.$$

To study the ISS properties of system (1), we next introduce two function classes related to the Lyapunov functional candidates. A function $v : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ is said to be of class ν_0 and we write $v \in \nu_0$, if, for each $x \in \mathcal{PC}(\mathbb{R}^+,\mathbb{R}^n)$, the composite function $t \mapsto v(t,x(t))$ is also in $\mathcal{PC}(\mathbb{R}^+,\mathbb{R}^n)$

and can be discontinuous at some $t' \in \mathbb{R}^+$ only when t' is a discontinuity point of x. A functional $v : \mathbb{R}^+ \times \mathcal{PC} \to \mathbb{R}^n$ is said to be of class ν_0^* and we write $v \in \nu_0^*$, if, for each function $x \in \mathcal{PC}([-r,\infty),\mathbb{R}^n)$, the composite function $t \mapsto v(t,x_t)$ is continuous in t for all $t \geq 0$. To analyze the continuous dynamics of system (1), we introduce the upper right-hand derivative of the Lyapunov functional candidate $V(t,x_t)$ with respect to system (1):

$$D^+V(t,\phi) = \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, x_{t+h}(t,\phi)) - V(t,\phi)],$$

where $x(t,\phi)$ is a solution to (1) satisfying $x_t = \phi$. , that is, $x(s) := x(t,\phi)(s)$ is a solution to the initial value problem:
$$\begin{cases} \dot{x}(s) = f(s, x_s, w(s)) \\ x_{s_0} = \varphi \end{cases} \quad \text{for } s \in [t, t+h) \text{ and } s_0 = t, \text{ where}$$
h is a small enough positive number such that no impulse time lies in the interval $(t, t+h)$.

III. SUFFICIENT CONDITIONS FOR ISS

In this section, we establish several ISS results for system (1) over the following three types of impulse time sequences, respectively: (i) $\ell_{\inf}(\delta)$, the class of impulse time sequences satisfying $\inf_{k \in \mathbb{N}}\{t_k - t_{k-1}\} \geq \delta$; (ii) $\ell_{\sup}(\delta)$, the class of impulse time sequences satisfying $\sup_{k \in \mathbb{N}}\{t_k - t_{k-1}\} \leq \delta$; (iii) ℓ_{all}, the set containing all the possible impulse time sequences. We first introduce two results for ISS of system (1) with stable continuous dynamics and unstable discrete dynamics.

Theorem 1. *Assume that there exist $V_1 \in \nu_0$, $V_2 \in \nu_0^*$, functions $\alpha_1, \alpha_2, \alpha_3, \chi \in \mathcal{K}_\infty$ and constants $\mu > 0$, $\rho_1 \geq 1$ and $\rho_2 \geq 0$, such that, for all $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $\phi \in \mathcal{PC}$,*

(i) $\alpha_1(\|x\|) \leq V_1(t,x) \leq \alpha_2(\|x\|)$ *and* $0 \leq V_2(t,\phi) \leq \alpha_3(\|\phi\|_r)$;
(ii) $D^+V(t,\phi) \leq -\mu V(t,\phi) + \chi(\|w(t)\|)$, *where* $V(t,\phi) = V_1(t,\phi(0)) + V_2(t,\phi)$;
(iii) $V_1(t, \phi(0) + I_k(t,\phi,y)) \leq \rho_1 V_1(t^-, \phi(0)) + \rho_2 \sup_{s \in [-r,0]}\{V_1(t^- + s, \phi(s))\} + \chi(\|y\|)$;
(iv) $\ln \rho < \mu\delta$ *where* $\rho := \rho_1 + \rho_2 e^{\mu r}$.
Then system (1) is uniformly ISS over $\ell_{\inf}(\delta)$.

Proof. Condition (iv) implies that there exists a small enough constant $\lambda > 0$ so that $\ln \rho \leq (\mu - \lambda)\delta$. Then we can choose a large enough constant $c > 0$ such that $c\rho e^{-\mu\delta} + \rho/\mu \leq c$ and $-\mu + 1/c + \lambda \leq 0$. Let x be a solution of (1), and set $v_1(t) := V_1(t,x(t))$, $v_2(t) := V_2(t,x_t)$, and $v(t) := v_1(t) + v_2(t)$. By mathematical induction, we shall prove that

$$\begin{aligned} v(t)e^{\lambda(t-t_0)} \leq & \alpha(\|\varphi\|_r) + ce^{\lambda(t-t_0)}\bar{\chi}(t) \\ & + \sum_{t_0 < t_k \leq t} e^{\lambda(t_k-t_0)}\chi(\|w(t_k^-)\|), \quad (2) \end{aligned}$$

where $\alpha(\|\varphi\|_r) := \alpha_2(\|\varphi(0)\|) + \alpha_3(\|\varphi\|_r)$ and $\bar{\chi}(t) = \chi(\sup_{s \in [t_0,t]} \|w(s)\|)$. For convenience, denote the RHS of (2) as $u(t)$.

For $k \geq 1$, condition (ii) on $[t_k, t_{k+1})$ implies $e^{\mu t}v(t) - e^{\mu t_k}v(t_k) \leq \int_{t_k}^t e^{\mu s}\chi(\|w(s)\|)ds$, which gives

$$v(t) \leq e^{-\mu(t-t_k)}v(t_k) + \frac{1}{\mu}\bar{\chi}(t), \quad (3)$$

for $t \in [t_k, t_{k+1})$ and $k \in \mathbb{Z}^+$. For $t \in [t_0, t_1)$, multiplying both sides of (3) with $e^{\lambda(t-t_0)}$ and using condition (i) and the fact that $1/\mu < c$, we conclude that $v(t)e^{\lambda(t-t_0)} \leq \alpha(\|\varphi\|) + ce^{\lambda(t-t_0)}\bar{\chi}(t)$ which shows that (2) holds for $t \in [t_0, t_1)$.

Now suppose that (2) holds for $t \in [t_0, t_m]$ where $m \geq 1$. We shall prove (2) is true on $[t_m, t_{m+1})$. To do this, we firstly conduct the following estimation:

$$
\begin{aligned}
&\rho v(t_m^-) e^{\lambda(t_m - t_0)} \\
\leq & \rho e^{-(\mu-\lambda)(t_m - t_{m-1})} \alpha(\|\varphi\|) \\
& + [c\rho e^{-\mu(t_m - t_{m-1})} + \frac{\rho}{\mu}] e^{\lambda(t_m - t_0)} \bar{\chi}(t_m^-) \\
& + \rho e^{-(\mu-\lambda)(t_m - t_{m-1})} \sum_{t_0 < t_k \leq t_{m-1}} e^{\lambda(t_k - t_0)} \chi(\|w(t_k^-)\|) \\
\leq & u(t_m^-).
\end{aligned}
\tag{4}
$$

For the first inequality of (4), we used (3) and then (2). For the second inequality of (4), we used the facts that $t_m - t_{m-1} \geq \delta$, $\rho e^{-(\mu-\lambda)\delta} \leq 1$ and $c\rho e^{-\mu\delta} + \rho/\mu \leq c$. Next, we will show that (2) is true for $t = t_m$. We start with making the claim that for $s \in [-r, 0]$, we have

$$
\rho v(t_m^- + s) e^{\lambda(t_m + s - t_0)} \leq e^{(\mu-\lambda)r} u(t_m^-).
\tag{5}
$$

Without loss of generality, suppose $t_m + s \geq t_0$ for all $s \in [-r, 0]$, then, for a given $s \in [-r, 0]$, there exists an integer j $(0 \leq j \leq m-1)$ such that $t_m + s \in [t_j, t_{j+1})$ and

$$
\begin{aligned}
&\rho v(t_m^- + s) e^{\lambda(t_m + s - t_0)} \\
\leq & \rho e^{-(\mu-\lambda)(t_m + s - t_j)} u(t_j) + \frac{\rho}{\mu} e^{\lambda(t_m + s - t_0)} \bar{\chi}(t_m^- + s) \\
\leq & e^{(\mu-\lambda)r} \rho e^{-(\mu-\lambda)(t_{j+1} - t_j)} u(t_j) + \frac{\rho}{\mu} e^{\lambda(t_m + s - t_0)} \bar{\chi}(t_m^- + s) \\
\leq & e^{(\mu-\lambda)r} u(t_m^-),
\end{aligned}
\tag{6}
$$

which implies (5) is true for all $s \in [-r, 0]$. Combining this with condition (iii) and the fact that $\rho_1 \geq 1$, we conclude that

$$
\begin{aligned}
&v(t_m) e^{\lambda(t_m - t_0)} \\
\leq & [\rho_1 v_1(t_m^-) + \rho_2 \sup_{-r \leq s \leq 0} \{v_1(t_m^- + s)\} + \chi(\|w(t_m^-)\|) \\
& + v_2(t_m^-)] e^{\lambda(t_m - t_0)} \\
\leq & [\rho_1 v(t_m^-) + \rho_2 \sup_{-r \leq s \leq 0} \{v(t_m^- + s)\} + \chi(\|w(t_m^-)\|)] \\
& \times e^{\lambda(t_m - t_0)} \\
\leq & \rho_1 v(t_m^-) e^{\lambda(t_m - t_0)} + \rho_2 e^{\lambda r} \sup_{-r \leq s \leq 0} \{v(t_m^- + s) e^{\lambda(t_m + s - t_0)}\} \\
& + \chi(\|w(t_m^-)\|) e^{\lambda(t_m - t_0)} \\
\leq & \frac{\rho_1 + \rho_2 e^{\mu r}}{\rho} u(t_m^-) + \chi(\|w(t_m^-)\|) e^{\lambda(t_m - t_0)} \\
= & u(t_m),
\end{aligned}
\tag{7}
$$

which implies (2) holds for $t = t_m$. We now show that (2) is true on (t_m, t_{m+1}) by contradiction. Suppose that there exists a $t \in (t_m, t_{m+1})$ such that $v(t)e^{\lambda(t-t_0)} > u(t)$, and then define $t^* = \inf\{t \in (t_m, t_{m+1}) \mid v(t)e^{\lambda(t-t_0)} > u(t)\}$. Combining (7) and the continuities of $v(t)$ and $u(t)$ on (t_m, t_{m+1}), we conclude that $v(t^*)e^{\lambda(t^*-t_0)} = u(t^*)$, which, by the definition of $u(t)$, implies $v(t^*) > c\bar{\chi}(t^*)$. In view of

this, condition (ii) and the fact that $-\mu + \frac{1}{c} + \lambda \leq 0$, it follows that

$$
\begin{aligned}
&D^+[v(t^*)e^{\lambda(t^*-t_0)}] \\
\leq & [-\mu v(t^*) + \chi(\|w(t^*)\|) + \lambda v(t^*)] e^{\lambda(t^*-t_0)} \\
< & (-\mu + \frac{1}{c} + \lambda) v(t^*) e^{\lambda(t^*-t_0)} \leq 0,
\end{aligned}
\tag{8}
$$

which means $v(t)e^{\lambda(t-t_0)}$ is strictly decreasing at $t = t^*$. This contradicts how t^* is defined. Hence, (2) holds on (t_m, t_{m+1}). By induction, we conclude that (2) is true for all $t \geq t_0$. The ISS estimation can be conducted from (2) by standard arguments. The details are the same as that in [25] and thus omitted. Boundedness of the solution to (1) follows from this estimate, which then implies the solution's global existence (see [33]). □

Remark 1. *Compared with Theorem 3.1 in [25] for system (1), the main difference is that the time-delay is considered in the impulses in our result. To be more specific, at the impulse time $t = t_k$, state $x(t_k^-)$ depends on $x_{t_k^-}$ which characterizes the states at some history moments. Therefore, in condition (iii) of Theorem 1, the impulse effects on V_1 are described by a delay-independent part and a delay-dependent part which are associated with parameters ρ_1 and ρ_2, respectively. It is worth noting that $\rho_1 \geq 1$ implies the delay-independent impulse part destabilizes the system. On the other hand, if the state jumps are independent of the time-delay (e.g., $\Delta x(t_k) = I_k(t_k, x(t_k^-), w(t_k^-))$), then V_1 depends only on state $x(t_k^-)$ (that is, V_1 is independent of the delayed states, which implies $\rho_2 = 0$) and Theorem 1 reduces to Theorem 3.1 in [25] for system (1).*

The second result is concerned with system (1) in the case when the impulses are potentially destabilizing but the delay-independent impulse parts are stabilizing (i.e., $\rho_1 < 1$).

Theorem 2. *Assume that there exist $V_1 \in \nu_0$, $V_2 \in \nu_0^*$, functions $\alpha_1, \alpha_2, \alpha_3, \chi \in \mathcal{K}_\infty$ and constants $\mu > 0$, $1 > \rho_1 \geq 0$ and $\rho_2 \geq 0$, such that, for all $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $\phi \in \mathcal{PC}$, conditions (i), (ii) and (iii) of Theorem 1 are satisfied, and there exists a positive constant κ such that $V_2(t, \phi) \leq \kappa \sup_{s \in [-r, 0]} \{V_1(t + s, \phi(s))\}$. If we further assume that condition (iv) of Theorem 1 holds with $\rho := \rho_1 + [\rho_2 + (1-\rho_1)\kappa]e^{\mu r}$ and $\rho_1 + \rho_2 + (1-\rho_1)\kappa \geq 1$, then system (1) is uniformly ISS over $\ell_{inf}(\delta)$.*

Proof. The proof is essentially identical to that of Theorem 1. The main difference is to replaced the following estimate of $v(t_m)$ in (7):

$$
\begin{aligned}
&v(t_m) \\
\leq & \rho_1 v_1(t_m^-) + \rho_2 \sup_{-r \leq s \leq 0} \{v_1(t_m^- + s)\} + \chi(\|w(t_m^-)\|) \\
& + (\rho_1 + 1 - \rho_1) v_2(t_m^-) \\
\leq & \rho_1 v(t_m^-) + [\rho_2 + (1-\rho_1)\kappa] \sup_{-r \leq s \leq 0} \{v(t_m^- + s)\} \\
& + \chi(\|w(t_m^-)\|).
\end{aligned}
$$

□

We next introduce an ISS result for system (1) for the case when the impulses are stabilizing but the continuous dynamics can be unstable.

Theorem 3. *Assume that there exist $V_1 \in \nu_0$, $V_2 \in \nu_0^*$, functions $\alpha_1, \alpha_2, \alpha_3, \chi \in \mathcal{K}_\infty$ and constants $\mu > 0$, $\kappa > 0$, $1 > \rho_1 \geq 0$ and $\rho_2 \geq 0$, such that, for all $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$, and $\phi \in \mathcal{PC}$,*

  (i) *$\alpha_1(\|x\|) \leq V_1(t,x) \leq \alpha_2(\|x\|)$ and $0 \leq V_2(t,\phi) \leq \alpha_3(\|\phi\|_r)$;*
  (ii) *$D^+V(t,\phi) \leq \mu V(t,\phi) + \chi(\|w(t)\|)$, where $V(t,\phi) = V_1(t,\phi(0)) + V_2(t,\phi)$;*
  (iii) *$V_1(t,\phi(0) + I_k(t,\phi,w(t^-))) \leq \rho_1 V_1(t^-,\phi(0)) + \rho_2 \sup_{s\in[-r,0]}\{V_1(t^-+s,\phi(s))\}+\chi(\sup_{s\in[-r,0]}\|w(t^-+s)\|)$;*
  (iv) *$V_2(t,\phi) \leq \kappa \sup_{s\in[-r,0]}\{V_1(t+s,\phi(s))\}$;*
  (v) *$\ln[\rho_1 + \rho_2 + (1-\rho_1)\kappa] < -\mu\delta$.*

Then system (1) is uniformly ISS over $\ell_{sup}(\delta)$.

Proof. We conclude from condition (v) that there exists a small enough constant $\lambda > 0$ so that $\ln(\rho_1 + [\rho_2 + (1-\rho_1)\kappa]e^{\lambda r}) \leq -(\mu+\lambda)\delta$. Denote $\rho := \rho_1 + [\rho_2 + (1-\rho_1)\kappa]e^{\lambda r}$, then we have $\rho e^{(\mu+\lambda)\delta} \leq 1$, which implies $\rho e^{\mu\delta} < 1$. This further implies that there exists a large enough constant $c > 0$ such that $c\rho e^{\mu\delta} + e^{\mu\delta}/\mu \leq c$. Let $M := e^{(\mu+\lambda)\delta}$, and we shall show that

$$v(t)e^{\lambda(t-t_0)} \leq M\alpha(\|\varphi\|_r) + ce^{\lambda(t-t_0)}\bar\chi(t) + M\sum_{t_0 < t_k \leq t} e^{\lambda(t_k-t_0)}\hat\chi(t_k^-), \quad (9)$$

where α and $\bar\chi$ are the same as in the proof of Theorem 1, and $\hat\chi(t_k^-) := \chi(\sup_{s\in[-r,0]}\|w(t_k^-+s)\|)$. For convenience, let $u(t)$ represent the RHS of (9). Similar to the estimate of (3), we conclude from condition (ii) that

$$v(t) \leq e^{\mu(t-t_k)}v(t_k) + \frac{e^{\mu\delta}}{\mu}\bar\chi(t), \quad (10)$$

for $t \in [t_k, t_{k+1})$ and $k \geq 1$. Then, on $[t_0, t_1)$, we have

$$v(t)e^{\lambda(t-t_0)}$$
$$\leq e^{(\mu+\lambda)(t-t_0)}v(t_0) + \frac{e^{\mu\delta}}{\mu}e^{\lambda(t-t_0)}\bar\chi(t)$$
$$\leq M\alpha(\|\varphi\|) + ce^{\lambda(t-t_0)}\bar\chi(t), \quad (11)$$

which means (9) holds on $[t_0, t_1)$. Here, we used (10) in the first inequality of (11) and the fact that $e^{\mu\delta}/\mu \leq c$ in the second inequality. Now suppose (9) is true for $t \in [t_0, t_m)$ with $m \geq 1$. We shall prove that (9) holds on $[t_m, t_{m+1})$. To do this, we start with proving that (9) is true for $t = t_m$:

$$v(t_m)e^{\lambda(t_m-t_0)}$$
$$\leq \{\rho_1 v(t_m^-) + [\rho_2 + (1-\rho_1)\kappa]\sup_{s\in[-r,0]}\{v_1(t_m^-+s)\} + \hat\chi(t_m^-)\}e^{\lambda(t_m-t_0)}$$
$$\leq \rho_1 u(t_m^-) + e^{\lambda(t_m-t_0)}\hat\chi(t_m^-) + [\rho_2 + (1-\rho_1)\kappa]e^{\lambda r}\sup_{s\in[-r,0]}\{v(t_m^-+s)e^{\lambda(t_m+s-t_0)}\}$$
$$\leq \rho u(t_m^-) + e^{\lambda(t_m-t_0)}\hat\chi(t_m^-)$$
$$\leq u(t_m^-). \quad (12)$$

Here, we used conditions (iii) and (iv) in the first inequality of (12), and (9) with the fact that $u(t_m^- + s) \leq u(t_m^-)$ for all $s \in [-r, 0]$ in the estimate of the third inequality. For $t \in (t_m, t_{m+1})$, we conclude from (10) and the third inequality of (12) that

$$v(t)e^{\lambda(t-t_0)}$$
$$\leq \rho e^{(\mu+\lambda)(t-t_m)}u(t_m^-) + e^{(\mu+\lambda)(t-t_m)}e^{\lambda(t_m-t_0)}\chi(\|w(t_m^-)\|)$$
$$+ \frac{e^{\mu\delta}}{\mu}e^{\lambda(t-t_0)}\hat\chi(t_m^-). \quad (13)$$

In view of (9), we then get from (13) that

$$v(t)e^{\lambda(t-t_0)}$$
$$\leq \rho e^{(\mu+\lambda)\delta}M\alpha(\|\varphi\|) + \left(c\rho e^{\mu\delta} + \frac{e^{\mu\delta}}{\mu}\right)e^{\lambda(t-t_0)}\bar\chi(t)$$
$$+ \rho e^{(\mu+\lambda)\delta}M\sum_{t_0 < t_k < t_m} e^{\lambda(t_k-t_0)}\hat\chi(\|w(t_k^-)\|)$$
$$+ e^{(\mu+\lambda)\delta}Me^{\lambda(t_m-t_0)}\hat\chi(\|w(t_m^-)\|)$$
$$\leq u(t), \quad (14)$$

i.e., (9) hold on (t_m, t_{m+1}). By the method of induction, we conclude that (9) is true for all $t \geq t_0$. The ISS estimate from (9) is similar to that in the proof of Theorem 1, and global existence of the solution to (1) follows from this estimate. $\square$

Remark 2. *Theorem 3 provides sufficient conditions for ISS of system (1) with delay-dependent impulses. However, ISS analysis of this type of systems cannot be conducted based on the existing results in [24]–[26] due to the existence of time-delay in the impulses. Furthermore, for system (1) with delay-free impulses, our result with $\rho_2 = 0$ is less conservative in the sense that the upper bound of δ is required to be smaller than $\frac{1}{\mu}\ln(\frac{1}{\rho_1+(1-\rho_1)\kappa})$, while Theorem 1 in [26] requires $\delta < \frac{1}{\mu}\ln(\frac{1}{\rho_1+\kappa})$.*

Remark 3. *Compared with Condition (iii) of Theorem 1, Condition (iii) of Theorem 3 is a weaker requirement on the Lyapunov function at each impulse moment. Therefore, the conclusion of Theorem 3 still holds if we replace this condition with condition (iii) of Theorem 1, that is, replace $\chi(\sup_{s\in[-r,0]}\|w(t^-+s)\|)$ with $\chi(\|w(t^-)\|)$ in the condition. However, Theorem 3 is more applicable to systems in the form of (1) with impulses only depending on states at history moments rather than the states at the impulse times. This issue will be demonstrated with Example 2. Though the discussion in this remark is also applicable to Theorem 1, condition (iii) of Theorem 3 is not necessary for Theorem 1 (see Example 1 for illustration).*

Remark 4. *Specify the class $\mathcal{K}_\infty$ functions in condition (i) of the above theorem: $\alpha_1(t) = w_1 t^p$, $\alpha_2(t) = w_2 t^p$, and $\alpha_3(t) = w_3 t^p$ with positive constants w_1, w_2, w_3, and p, then condition (iv) of Theorem 3 holds with $\kappa = \frac{w_3}{w_1}$. If we further assume that there is no external input to system (1) (i.e., $w(t) = 0$), then Theorem 3 serves as an global exponential stability result for system (1) with zero input and can be applied to study the delay-dependent impulsive control problems of time-delay systems (see Example 2 in Section IV*

for demonstrations). Our requirement on the upper bound of δ in condition (v) of Theorem 3 is less conservative than the one of $\delta < \frac{1}{\mu}\ln(\frac{1}{\rho_1 + \rho_2 + \kappa})$ in Theorem 3.1 of [10].

The last ISS result is for system (1) with both stable continuous and discrete dynamics, which shows that system (1) is ISS for arbitrary impulse sequences.

Theorem 4. *Assume that there exist $V_1 \in \nu_0$, $V_2 \in \nu_0^*$, functions $\alpha_1, \alpha_2, \alpha_3, \chi \in \mathcal{K}_\infty$ and constants $\mu \geq 0$, $\kappa > 0$, $1 > \rho_1 \geq 0$ and $\rho_2 \geq 0$, such that, for all $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $\phi \in \mathcal{PC}$, conditions (i), (ii) and (iii) of Theorem 1 and condition (iv) of Theorem 3 are satisfied and $\rho_1 + \rho_2 + (1 - \rho_1)\kappa < 1$. Then system (1) is uniformly ISS over ℓ_{all}.*

Proof. First, we consider the scenario of $\mu > 0$. Since $\rho_1 + \rho_2 < 1$, there exists a small enough constant $\lambda > 0$ such that $\rho_1 + [\rho_2 + (1 - \rho_1)\kappa]e^{\lambda r} \leq 1$ and $\lambda < \mu$. Then we can pick a large enough constant $c > 0$ such that $-\mu + 1/c + \lambda \leq 0$. By mathematical induction, we will prove (2) holds for all $t \geq t_0$. Similar to the proof of Theorem 1, we can show that (2) is true on $[t_0, t_1)$. Then, suppose (2) holds on $[t_0, t_m)$ with $m \geq 1$. We will show (2) is true for $t \in [t_m, t_{m+1})$. When $t = t_m$, we can revise the estimate in (7) as what we did in (12) with $\hat{\chi}(t_m^-)$ replaced with $\chi(\|w(t_m^-)\|)$. Identical to the discussion in the proof of Theorem 1, we can show (7) is true for $t \in (t_m, t_{m+1})$ by contradiction. The rest of the proof are essentially the same as that of Theorem 1 and thus omitted.

If $\mu = 0$, then condition (ii) of Theorem 3 is true, i.e., $D^+V(t,\phi) \leq \varepsilon V(t,\phi) + \chi(\|w(t)\|)$ for any $\varepsilon > 0$. We can conclude from Theorem 3 that system (1) is uniformly ISS over $\ell_{\sup}(\delta)$ with

$$\delta < \frac{\ln[\rho_1 + \rho_2 + (1 - \rho_1)\kappa]}{-\varepsilon}.$$

Let $\varepsilon \to 0$ in the above inequality and we can see that system (1) is uniformly ISS over ℓ_{all}. $\square$

Remark 5. *In the above theorems, Lyapunov candidate V comprises a function V_1 and a functional V_2. Since V_2 is indifferent to the impulses, it is necessary to incorporate the function V_1 into the Lyapunov candidate to capture the impulse effects on it. Condition (iii) of Theorem 1 (or Theorem 3) characterizes the effects of the state jumps on V_1. If $\rho_1 < 1$, then condition (iv) in Theorem 3 is necessary to quantify the growth or decay of the Lyapunov candidate V at each impulse moment. Condition (ii) of Theorem 1 (or Theorem 3) describes the unstable (or stable) continuous-time dynamics. Condition (iv) of Theorem 1 (or condition (v) of Theorem 3) balances the continuous-time dynamics and the impulse effects to guarantee the convergence of V in the ISS sense, and then the convergence of the state x can be derived by using the positive definite and decrescent properties of V_1 and V_2 in condition (i). It is worth mentioning that system (1) is in the general form of nonlinear impulsive time-delay systems and it has been studied in [24]–[27] for the ISS properties. However, the results in [24]–[26] are generally applicable to systems with delay-independent impulses (e.g., $\Delta x(t_k) = I_k(t_k, x(t_k^-), w(t_k^-))$ for $k \in \mathbb{N}$). The reason is that* the condition $V_1(t, \phi(0) + I_k(t, \phi, w(t^-))) \leq \rho_1 V_1(t^-, \phi(0))$ in [24]–[26] actually is only verifiable for non-delayed impulses. The Lyapunov-Krasovskii-type result obtained in [27] requires an explicit relationship between $V(t_k)$ and $V(t_k^-)$ which is difficult to quantify. Because the impulses can only affect the function part V_1 not the functional part V_2. We overcome this difficulty by analyzing the impulse effects on the function part and requiring certain comparison between V_1 and V_2 when it is necessary, so that the impulse effects on the whole Lyapunov candidate can be obtained.

IV. EXAMPLES

In this section, we present two examples with numerical simulations. To demonstrate the advantage of Theorem 1 over the existing results for time-delay systems with delay-independent impulse effects, the first example is slighted adopted from [24], [25] with distributed time-delay considered in the impulses.

Example 1. *Consider the following time-delay system with distributed-delay dependent impulses:*

$$\dot{x}(t) = -\text{sat}(x(t)) + a\text{sat}(x(t - \tau)) + b\text{sat}(w(t)), \ t \neq t_k,$$
(15a)

$$\Delta x(t) = \frac{1}{4}\text{sat}(\int_{t-\tau}^t x(s)\mathrm{d}s) + \frac{1}{4}\text{sat}(w(t^-)), \ t = t_k, \ (15b)$$

where $a = 0.2$, $b = 0.1$, $\tau = 1$, and $\text{sat}(x)$ is the saturation function defined as $\text{sat}(x) = \frac{1}{2}(|x + 1| - |x - 1|)$.

To study the ISS property of system (15), we choose Lyapunov functional $V(t, \phi) = V_1(t, \phi(0)) + V_2(t, \phi)$ with

$$V_1(t, x) = \begin{cases} x^2, & |x| \leq 1, \\ e^{2(|x|-1)}, & |x| > 1, \end{cases}$$

$$V_2(t, \phi) = |a| \int_\tau^0 \text{sat}^2(\phi(s))\left(\epsilon + 1 + \frac{\epsilon s}{\tau}\right)\mathrm{d}s,$$

where $\epsilon > 0$. Clearly, condition (i) of Theorem 1 holds. Similar to the discussion in Example 4.1 of [25], we have $D^+V(t,\phi) \leq -\mu V(t,\phi) + \frac{|b|^2}{2}w^2(t)$ with $\mu = \min\{2 - (\epsilon + 2)|a| - 2|b|, \frac{\epsilon}{(\epsilon+1)\tau}\}$. Therefore, condition (ii) of Theorem 1 is satisfied. Next, we verify condition (iii) of Theorem 1 at each impulse time. To to this, we consider the following two scenarios:

If $|x(t_k)| \leq 1$, then $V_1(t, x(t_k)) = x^2(t_k) \leq 3x^2(t_k^-) + \frac{3}{16}\text{sat}^2(\int_{t_k-\tau}^{t_k} x(s)\mathrm{d}s) + \frac{3}{16}\text{sat}^2(w(t_k^-))$. Furthermore, if $|x(t_k^-)| \leq 1$, then $x^2(t_k^-) = V_1(t_k^-, x(t_k^-))$; if $|x(t_k^-)| > 1$, then $x^2(t_k^-) \leq e^{2(|x(t_k^-)|-1)} = V_1(t_k^-, x(t_k^-))$. Thus, $x^2(t_k^-) \leq V_1(t_k^-, x(t_k^-))$, and similarly, we can obtain $x^2(t_k^- + s) \leq V_1(t_k^- + s, x(t_k^- + s))$ for any $s \in [-\tau, 0]$, which then implies $\text{sat}(\int_{t_k-\tau}^{t_k} x(s)\mathrm{d}s) \leq (\int_{t_k-\tau}^{t_k} x(s)\mathrm{d}s)^2 \leq \tau \int_{t_k-\tau}^{t_k} x^2(s)\mathrm{d}s \leq \tau \sup_{s \in [-\tau, 0]}\{V_1(t_k^- + s, x(t_k^- + s))\}$. Based on the above discussion, we can have $V_1(t_k, x(t_k)) \leq 3V_1(t_k^-, x(t_k^-)) + \frac{3\tau}{16}\sup_{s \in [-\tau, 0]}\{V_1(t_k^- + s, x(t_k^- + s))\} + \frac{3}{16}\text{sat}^2(w(t_k^-))$.
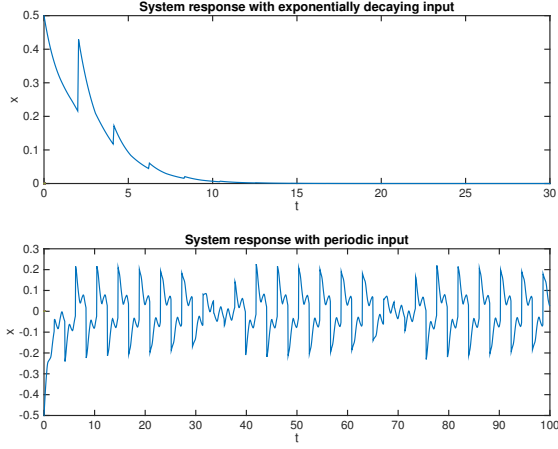
Fig. 1. Simulation results for system (15) with exponentially decaying input $w(t) = 5e^{-t}$ and periodic input $w(t) = 2\sin(14\pi t)$.

If $|x(t_k)| > 1$, then

$$V_1(t_k, x(t_k)) = e^{2(|x(t_k)|-1)}$$

$$\leq e^{2(|x(t_k^-)| + \frac{1}{4}|\mathrm{sat}(\int_{t_k-\tau}^{t_k} x(s)\mathrm{d}s)| + \frac{1}{4}|\mathrm{sat}(w(t_k^-))|-1)}$$

$$\leq e^{2|x(t_k^-)|-1}$$

$$\leq \begin{cases} ee^{2|x(t_k^-)|-2} = eV_1(t_k^-, x(t_k^-)), & |x(t_k^-)| > 1, \\ 2e|x(t_k^-)|^2 = 2eV_1(t_k^-, x(t_k^-)), & \frac{1}{2} < |x(t_k^-)| \leq 1, \end{cases}$$

where we used the fact that $e^{2x-2} \leq 2x^2$ for $x \in (\frac{1}{2}, 1]$.

In either of the scenario, we have shown that condition (iii) holds with $\rho_1 = 2e$ and $\rho_2 = \frac{3\tau}{16}$. According to Theorem 1, if $\delta > \ln(\rho_1 + \rho_2 e^{\mu\tau})/\mu$, then system (15) is uniformly ISS over $\ell_{\inf}(\delta)$. For illustration, choosing $\epsilon = 5$, we can compute $\mu = 0.4$ and then $\ln(\rho_1 + \rho_2 e^{\mu\tau})/\mu = 2.06$, which gives the condition: $\delta > 2.06$. Simulation results for system (15) with the given parameters and $\delta = 2.1$ are shown in Fig. 1.

The second example studies the effects of disturbance inputs on impulsive synchronization of time-delay systems. Applications of the analysis conducted in this example can be extended to impulsive synchronization and stabilization of complex dynamical networks with time-delay (see, e.g., [10], [32]).

Example 2. *Consider the following two time-delay systems:*

$$\dot{x}(t) = Ax(t) + g(x(t-r)), \tag{16}$$

and

$$\begin{cases} \dot{y}(t) = Ay(t) + g(y(t-r)) + Bw(t), & t \neq t_k, \\ \Delta y(t) = C[y(t-d) - x(t-d)] + Dw(t^-), & t = t_k, \end{cases} \tag{17}$$

where $x, y \in \mathbb{R}^n$, $A, C \in \mathbb{R}^{n\times n}$, and $B, D \in \mathbb{R}^{n\times m}$; r and d represent the delays in the continuous and discrete dynamics, respectively; $w(t)$ is the disturbance input; f satisfies Lipschitz condition, that is, there exists a positive constant L such that $\|g(x) - g(y)\| \leq L\|x - y\|$ for any $x, y \in \mathbb{R}^n$. Without loss of generality, we assume that $t_k - t_{k-1} = \delta$ for all $k \in \mathbb{N}$.

Denote the error state $e(t) := y(t) - x(t)$, the dynamics of which can be described by the following impulsive time-delay system:

$$\begin{cases} \dot{e}(t) = Ay(t) + \hat{g}(e(t-r)) + Bw(t), & t \neq t_k, \\ \Delta e(t) = Ce(t-d) + Dw(t^-), & t = t_k, \end{cases} \tag{18}$$

where $\hat{g}(e(t-r)) = g(y(t-r)) - g(x(t-r))$. The objective of this example is to study the synchronization between systems (16) and (17) in the sense that $\lim_{t\to\infty} e(t) = 0$, and the impact of the disturbance input $w(t)$ on this system synchronization, i.e., ISS properties of error system (18). To do so, consider a Lyapunov functional candidate $V(t, e_t) = V_1(t, e(t)) + V_2(t, e_t)$ with $V_1(t, e(t)) = e^T(t)e(t)$ and $V_2(t, e_t) = \varepsilon L \int_{t-r}^t e^T(s)e(s)\mathrm{d}s$, then both conditions (i) and (iv) of Theorem 3 are satisfied with the functions $\alpha_1, \alpha_2, \alpha_3$ specified in Remark 4 and $p = 2$, $w_1 = w_2 = 1$, $w_3 = \kappa = \varepsilon r L$. For simplicity, we denote $v(t) = V(t, e_t)$, $v_1(t) = V_1(t, e(t))$, and $v_2(t) = V_2(t, e_t)$. Similar to the estimation in the proof of Theorem 4.1 from [10], we have

$$\dot{v}(t) \leq \mu v(t) + \chi_{\epsilon_1}(\|w(t)\|), \ t \neq t_k,$$

$$v_1(t_k) \leq \rho_1 v_1(t_k^-) + \rho_2 \sup_{s\in[-\tau,0]} \{v_1(t_k^- + s)\}$$

$$+ \chi_{\epsilon_2}(\sup_{s\in[-\tau,0]} \|w(t_k^- + s)\|), \ k \in \mathbb{N},$$

where $\mu = \lambda_{max}(A + A^T) + L(\varepsilon + \varepsilon^{-1}) + \epsilon_1$, $\rho_1 = (1+\xi)\|I + C\|^2$, and $\rho_2 = (1 + \xi^{-1})[\epsilon_2 d\|C\|(\|A\| + L) + \zeta\|C\|^2]^2$ with $\epsilon_1 > 0$, $\epsilon_2 > 1$, and $\xi > 0$; ζ denotes the number of impulses on the interval $(t_k - d, t_k)$; χ_{ϵ_1} and χ_{ϵ_2} are $\mathcal{K}_\infty$ class functions which depend on ϵ_2 and ϵ_2, respectively. It can be seen that conditions (ii) and (iii) of Theorem 3 are satisfied. Minimizing $\rho_1 + \rho_2 + (1-\rho_1)\kappa$ with $\xi > 0$, we have $\min_{\xi>0}\{\rho_1 + \rho_2 + (1-\rho_1)\kappa\} = [\sqrt{1-\kappa}\|I + C\| + \epsilon_2 d\|C\|(\|A\| + L) + \zeta\|C\|^2]^2 + \kappa$. If

$$\ln\{[\sqrt{1-\kappa}\|I + C\| + d\|C\|(\|A\| + L) + \zeta\|C\|^2]^2 + \kappa\}$$
$$< -[\lambda_{max}(A + A^T) + L(\varepsilon + \varepsilon^{-1})]\delta \tag{19}$$

holds, then there exists positive constants ϵ_1 close to 0 and $\epsilon_2 > 1$ close to 1 such that condition (v) of Theorem 3 is satisfied. Therefore, the conclusion of Theorem 3 implies that error system (18) is uniformly ISS, provided (19) holds. As a numerical example, we take $A = \begin{bmatrix} -18/7 & 9 & 0 \\ 1 & -1 & 1 \\ 0 & -100/7 & 0 \end{bmatrix}$,

$g(x) = \mathrm{sat}(x_1)\begin{bmatrix} 27/7 \\ 0 \\ 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1/7 \\ 1/7 \end{bmatrix}$, $C = -0.2I$, $D = \begin{bmatrix} 2/7 \\ 0 \\ 0 \end{bmatrix}$, and $r = 2d = 0.02$. It can be verified that (19) is satisfied with $\varepsilon = 1$, $L = 27/7$, $\delta = 0.01$, and $\zeta = 0$. According to Theorem 3, system (18) is uniformly ISS. With the given parameters, system (16) is a delayed Chua's circuit which exhibits chaotic behaviors as shown in [25]. Simulation results for error system (18) are shown in Fig. 2. Synchronization between systems (16) and (17) is demonstrated in Fig. 2(a) with $w(t) = 0$, and the ISS properties of error system
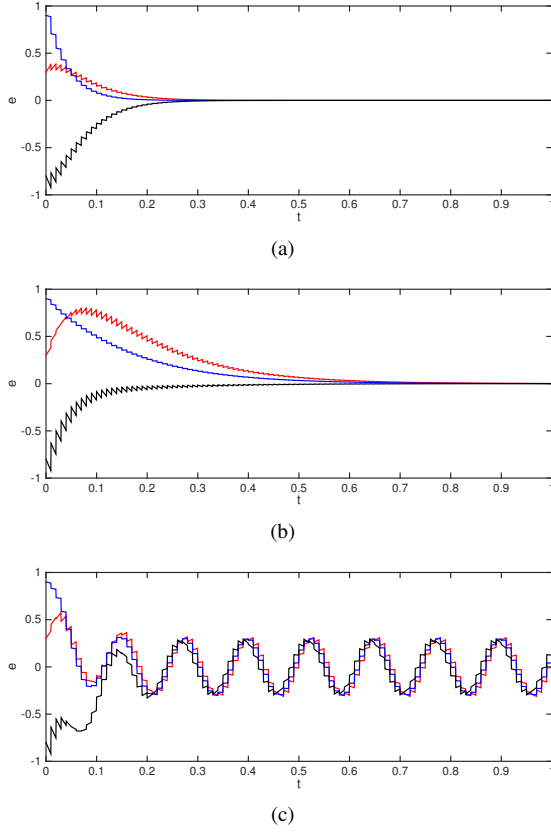
Fig. 2. Simulation results for Example 2 with (a) zero input $w(t) = 0$; (b) exponentially decaying input $w(t) = e^{-7t}$; (c) periodic input $w(t) = \cos(-16\pi t)$.

(18) are illustrated in Fig. 2(b) and 2(c) with an exponentially decaying input and a periodic input, respectively.

Remark 6. *The derivation of ρ_1 and ρ_2 is based on the relationship between states $e(t_k^-)$ and $e(t_k - d)$ which is obtained by integrating both sides of (18) from $t_k - d$ to t_k^- (see [10] for more details). Therefore, the external input over the time interval $[t_k - d, t_k)$ is brought into the estimation of V_1 at the impulse time t_k as discussed in Remark 3. On the one hand, the delay-dependent impulses synchronize the systems in the sense of ISS. This positive effects of the delayed state $e(t_k - d)$ in the impulses is captured through parameter ρ_1. On the other hand, the existence of time-delay in the impulses leads to smaller upper bound for δ (i.e., $\delta < \frac{-\ln[\rho_1 + \rho_2 + (1-\rho_1)\kappa]}{\mu}$ with positive ρ_2), which is a drawback of the delayed states quantified by parameter ρ_2. Actually, the positive effect of the delay-dependent impulses is balanced with this disadvantage in condition (v) of Theorem 3 so that the synchronization can be achieved.*

V. Conclusions

The method of Lyapunov functionals has been used to investigate ISS property of time-delay systems with delay-dependent impulses. Sufficient conditions have been constructed for systems with ISS continuous dynamics and destabilizing discrete dynamics, unstable continuous dynamics and

ISS discrete dynamics, and stable continuous dynamics with zero control input and ISS discrete dynamics, respectively.

Extension of our results to hybrid systems with both switching and impulse effects can be conducted by using the method of multiple Lyapunov-Krasovskii functionals along the line of [25], [26], which is a topic for future research. Another topic is to investigate integral-input-to-state stability (iISS) of nonlinear systems with delay-dependent impulses. Weaker conditions on the Lyapunov functional candidates are expected since iISS is a relaxed form of ISS as discussed in [24]–[26]. Improvements of the current results is also a direction of the future work. For example, less conservative results on dwell time δ might be available by following the line of [20], and the results could be applicable to systems with a larger class of impulse sequences by using the average impulsive interval approach as discussed in [27], [28].

References

[1] I.-C. Morarescu, A. Martin, A. Girard, and A. Muller-Gueudin, Coordination in networks of linear impulsive agents, IEEE Transactions on Automatic Control, vol. 61, no. 9, pp. 2402-2415, 2016.

[2] T. Yang and L.O. Chua, Impulsive control and synchronization of nonlinear dynamical systems and application to secure communication, International Journal of Bifurcation and Chaos, vol. 07, no. 03, pp. 645-664, 1997.

[3] S. Gao, L. Chen, J.J. Nieto, and A Torres, Analysis of a delayed epidemic model with pulse vaccination and saturation incidence, Vaccine, vol. 24, no. 35-36, pp. 6037-6045, 2006.

[4] T. Carter, Optimal impulsive space trajectories based on linear equations, Journal of Optimization Theory and Applications, vol. 70, pp. 277-297, 1991.

[5] B.M. Miller and E.Y. Rubinovich, Impulsive control in continuous and discrete-continuous systems, Springer Science & Business Media, 2012.

[6] N. Perestyuk and A.M. Samoylenko, Impulsive Differential Equations, World Scientific, 1995.

[7] W.M. Haddad, V.S. Chellaboina, and S.G. Nersesov, Impulsive and hybrid dynamical systems: stability, dissipativity, and control, Princeton University Press, 2006.

[8] S.T. Zavalishchin and A.N. Sesekin, Dynamic impulsive systems: theory and applications, Springer Netherlands, 1997.

[9] X. Liu and K. Zhang, Stabilization of nonlinear time-delay systems: distributed-delay dependent impulsive control, Systems & Control Letters, vol. 120, pp. 17-23, 2018.

[10] X. Liu, K. Zhang, and W.C. Xie, Stabilization of time-delay neural networks via delayed pinning impulses, Chaos, Solitons & Fractals, vol. 93, pp. 223-234, 2016.

[11] A. Khadra, X. Liu, and X Shen, Analyzing the robustness of impulsive synchronization coupled by linear delayed impulses, IEEE Transactions on Automatic Control, vol. 54, no. 4, pp. 923-928, 2009.

[12] X. Li, X. Zhang, and S. Song, Effect of delayed impulses on input-to-state stability of nonlinear systems, Automatica, vol. 76, pp. 378-382, 2017.

[13] E.D. Sontag, Smooth stabilization implies coprime factorization, IEEE Transactions on Automatic Control, vol. 34, no. 4, pp. 435-443, 1989.

[14] Z.-P. Jiang and Y. Wang, Input-to-state stability for discrete-time nonlinear systems, Automatic, vol. 37, no. 6. pp. 857-869, 2001.

[15] L. Vu, D. Chatterjee, and D. Liberzon, Input-to-state stability of switched systems and switching adaptive control, Automatica, vol. 43, no. 4, pp. 639-646, 2007.

[16] D. Nešić and A.R. Teel, Input-to-state stability of networked control systems, Automatica, vol. 40, no. 12, pp. 2121-2128, 2004.

[17] A.R. Teel, L. Moreau, and D. Nesic, A unified framework for input-to-state stability in systems with two time scales, IEEE Transactions on Automatic Control, vol. 48, no. 9, pp. 1526-1544, 2003.

[18] M. Arcak and A.R. Teel, Input-to-state stability for a class of Lurie systems, Automatica, vol. 38, no. 11, pp. 1945-1949, 2012.

[19] M. Lazar, W.P. Maurice H. Heemels, and A.R. Teel, Lyapunov functions, stability and input-to-State stability subtleties for discrete-time discontinuous systems, IEEE Transactions on Automatic Control, vol. 54, no. 10, pp. 2421-2425, 2009.

[20] S. Dashkovskiy and A. Mironchenko, Input-to-State Stability of Nonlinear Impulsive Systems, SIAM Journal on Control and Optimization, vol. 51, no. 3, pp. 1962-1987, 2013.

[21] N. Yeganefar, P. Pepe, and M. Dambrine, Input-to-State Stability of Time-Delay Systems: A Link With Exponential Stability, IEEE Transactions on Automatic Control, vol. 53, no. 6, pp. 1526-1531, 2008.

[22] E.N. Sanchez and J.P. Perez, Input-to-state stability (ISS) analysis for dynamic neural networks, IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications, vol. 46, no. 11, pp. 1395-1398, 1999.

[23] J.P. Hespanha, D. Liberzon, and A.R. Teel, Lyapunov conditions for input-to-state stability of impulsive systems, Automatica, vol. 44, pp. 2735-2744, 2008.

[24] W.H. Chen and W.X. Zheng, Input-to-state stability and integral input-to-state stability of nonlinear impulsive systems with delays, Automatica, vol. 45, pp. 1481-1488, 2009.

[25] J. Liu, X. Liu, and W.C. Xie, Input-to-state stability of impulsive and switching hybrid systems with time-delay, Automatica, vol. 47, pp. 899-908, 2011.

[26] X.M. Sun and W. Wang, Integral input-to-state stability for hybrid delayed systems with unstable continuous dynamics, Automatica, vol. 48, pp. 2359-2364, 2012.

[27] S. Dashkovskiya, M. Kosmykov, A. Mironchenko, and L. Naujok, Stability of interconnected impulsive systems with and without time delays, using Lyapunov methods, Nonlinear Analysis: Hybrid Systems, vol. 6, no. 3, pp. 899-915, 2012.

[28] X. Wu, Y. Tang, and W. Zhang, Input-to-state stability of impulsive stochastic delayed systems under linear assumptions, Automatica, vol. 66, pp. 195-204, 2016.

[29] Y. Wang, R. Wang, and J. Zhao, Input-to-state stability of non-linear impulsive and switched delay systems, IET Control Theory & Applications, vol. 7, no. 8, pp. 1179-1185, 2013.

[30] X. Zhang and X. Li, Input-to-state stability of non-linear systems with distributed-delayed impulses, IET Control Theory & Applications, vol. 11, no. 1, pp. 81-89, 2016.

[31] X. Liu, K. Zhang, and W.C. Xie, Impulsive consensus of networked multi-agent systems with distributed delays in agent dynamics and impulsive protocols, Journal of Dynamic Systems, Measurement, and Control, vol. 141, no. 1, 011008 (8 pages), 2019.

[32] J. Sun, Q.-L. Han, X. Jiang, Impulsive control of time-delay systems using delayed impulse and its application to impulsive master-slave synchronization, Physics Letters A, vol 372, no. 42, pp. 6375-6380, 2008.

[33] G. Ballinger and X. Liu, Existence and uniqueness results for impulsive delay differential equations, Dynamics of Continuous, Discrete & Impulsive Systems, vol. 5, pp. 579-591, 1999.