On the stability analysis of mixed traffic with vehicles under car-following and bilateral control

On the stability analysis of mixed traffic with vehicles under car-following and bilateral control

Liang Wang[a] and Berthold K.P. Horn[b*]

Abstract—**In this paper, we study a mixed traffic flow in which some cars are under car-following control and others are under bilateral control. We also provide the necessary (modular string) stability condition for this type of mixed traffic, which can be viewed as an extension of the condition for pure car-following control based traffic. This necessary stability condition provides some indication of how the introduction of self-driving cars (under bilateral control) will affect today's traffic.**

Index Terms—**bilateral control, car-following model, mixed traffic, stability analysis, adaptive cruise control (ACC).**

I. Introduction and related work

We are all familiar with "stop-and-go" traffic and so-called "phantom traffic jams." These are "emergent behaviors" of a dynamic system comprised of human-driven cars according to the so-called "car following" model (CFM). Models of this type of traffic flow have been studied since the 1930s [1], and several ways have been proposed to explain the origin of this highly undesirable phenomenon [2]–[4]. Some suggestions have also been made about how to suppress such traffic flow instabilities. One well-known approach is known as the platoon [5]–[8]. In brief, the platoon controller tries to bind successive cars together and force them to move in lock-step like "carriages" in a train. A single lead vehicle controls a whole vehicular chain and plays the same role as a locomotive in a train. New platoon models, e.g., decentralized platoon, bi-directional platoon, multi-neighbor platoon, are continuing to be proposed [9]–[17]. See also [18], [19] for theoretical analyses of various platoon models.

Another approach is to let vehicles themselves solve the traffic flow problems by designing new adaptive cruise control (ACC) systems. In order to make vehicles under such modified ACC systems move independently like cars, rather than like "carriages" in a train, global-control parameters are *not* allowed: *No* preset desired speed for the whole traffic, *or* preset desired space between cars (i.e., preset relative position in the traffic), and the control system (including control commands) in one car is *not* accessible to ACC systems in other cars. The ACC system's input comes from the vehicle's on-board sensors, and control of the vehicle is based entirely on the outputs of its own sensors. One such new ACC system is known as bilateral control model (BCM) [20]–[23], in which the vehicle is controlled to stay as far from the leading car as from the following car. See also [24]–[26] for previous efforts involving use of bi-directional information flow.

Although the traffic flow purely under car-following control, or purely under bilateral control, is similar in some respects to special cases of platooning, in particular a decentralized platoon with infinite boundaries, here, we should mention that both car-following control and bilateral control are applied to individual vehicles, and thus, there is *no* requirement that all vehicles use the same control strategy. One result is that cars under car-following control and vehicles under bilateral control can run independently and coexist in the same traffic flow. Realistically, not all cars will be converted to bilateral control at once, so the question arises as to how a mixture of cars under car following control and cars under bilateral control will behave. That is, will they "play well together?"

In this paper, we study such mixed traffic in which some vehicles are under car-following control and others are under bilateral control. We focus on a particular type of mixed traffic in which K successive cars under car-following control and L successive cars under bilateral control appear in the traffic flow alternately. We provide the necessary condition for the modular string stability of such mixed traffic, which is an extension of the condition for pure car-following control traffic. The stability analysis also provides some indication on how self-driving cars (under BCM) will affect traffic. In brief, one important cause of the "stop-and-go" traffic instabilities is the tailgating behavior of human drivers. By adding bilateral control cars, the impact of such tailgating behavior can be weakened, and thus constraints on control-system gains for stability can be reduced. How much such stability condition can be relaxed depends on 1). the percentage of bilateral-control vehicles in the mixed traffic and 2). the distribution of bilateral-control cars (i.e., concentrated or dispersed). The conditions developed in Section IV provide more details.

II. Car-following control and bilateral control

Let $y_n(t)$ be the position of the $n-$th car, and $v_n(t) = \dot{y}_n(t)$ be its velocity[1]. The pair $\{y_n(t), v_n(t)\}$ gives the state of the $n-$th car, which is adjusted through the acceleration $a_n(t) = \ddot{y}_n(t)$ commanded by the control system. In this paper, control of car n is provided by a simple linear feedback system[2]

$$a_n = k_d(d_n - s_n) + k_v(r_n - u_n), \tag{1}$$

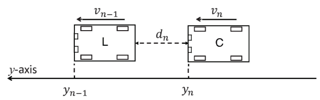
[a] School of Electronics and Communication Engineering, Sun Yat-sen University (SYSU), Guangzhou, Guangdong, 510275, China, e-mail: wangliang@csail.mit.edu. This work is partly done when Liang worked in MIT.

[b] Department of Electrical Engineering and Computer Science, MIT, Cambridge, MA 02139, USA, e-mail: bkph@csail.mit.edu

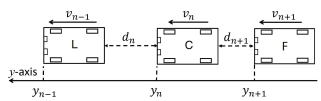
* The corresponding author is Berthold K.P. Horn. Sponsored by Toyota Research Institute (LP-C000765-SR) and SYSU Fund (76150-18841217).

[1] The positive direction is chosen as the direction in which cars are moving, thus, $y_{n-1} - y_n > 0$ (see Figure 1).

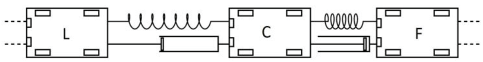
[2] The simplified "decision-making module" (1) of the ACC system is widely used for traffic-flow analysis. Delay and non-linearity can also be considered to build more accurate and complicated car models [27].



(a) Car-following model



(b) Bilateral control model



(c) A physical analog of the BCM traffic

Fig. 1. The car-following model (CFM) and bilateral control model (BCM). The blocks with "L", "C" and "F" denote the leading car, current car and following car. (a) CFM is based *only* on the state of the leading car "L". (b) BCM uses the states of *both* leading car "L" and following car "F". (c) A physical analog of the traffic flow under bilateral control is a big "spring-damper-mass" system.

where $d_n = y_{n-1} - y_n - \ell$ denotes the space between the current car and its leading car (with car length ℓ), and $r_n = v_{n-1} - v_n$ denotes the relative velocity between the current car and its leading car. The proportional gain k_d and derivative gain k_v are both *positive*. The desired space s_n and desired speed difference u_n are specified differently in different vehicle-control models as explained next.

In this paper, the car-following model (CFM) implements "constant time headway" control [2], [4], [27], i.e.,

$$s_n = v_n T \qquad \text{and} \qquad u_n = 0, \tag{2}$$

where T is known as the *reaction time*. In this model, control of car n is based *only* on the relative position and relative velocity of car $n-1$ immediately ahead.

For self-driving cars, a second pair of sensors can be used to measure space and speed difference between the current car and the car following. These two new measurements d_{n+1} and r_{n+1} can then be used for control. For instance, we can set

$$s_n = d_{n+1} \qquad \text{and} \qquad u_n = r_{n+1}. \tag{3}$$

Then, eq. (1) becomes

$$a_n = k_d\big(d_n - d_{n+1}\big) + k_v\big(r_n - r_{n+1}\big). \tag{4}$$

We call this new control strategy the bilateral control model (BCM). Here, control of car n is based on the relative positions and relative velocities of *both* car $n-1$ ahead and car $n+1$ behind. The control objective of BCM is to stay in the middle between the "front and back" neighbors, and run at the average speed of these two neighbors. Fig. 1 shows the car-following control model and bilateral control model. See [20], [21] for more details about BCM implementation.

III. Stability condition for pure traffic

In the equilibrium state, all the cars are spaced the safe distance $s = v_0 T$ apart and move at the same speed v_0. In this case, all the accelerations a_n in eqs. (1) and (4) are zero[3], and the traffic flow continues in the *equilibrium state*. The important question then is whether this equilibrium is stable, metastable, or unstable. If there is a small perturbation in $x_n(t)$ or $v_n(t)$, will the traffic system return to the equilibrium state or will there be increasing departures from the equilibrium state, which ultimately lead to a traffic jam?

For the traffic flow purely under constant-time headway CFM, the stability (and string stability) requires[4]

$$\frac{1}{2}k_d T^2 + k_v T > 1. \tag{5}$$

In consideration of passengers' comfort, in general, k_d and k_v are chosen to be relatively small. If most of the drivers try to tailgate, i.e., effectively choosing smaller values for T, then (5) will be easily violated, and traffic jams will appear. When an ACC system is used, even though sensor's response can be much faster than that of human beings, still, relatively large values for T (e.g., $T = 1.5$ sec. [18]) should be used.

Traffic flow under bilateral control *is* stable for all $k_d > 0$ and $k_v > 0$ [20]–[22]. A physical analog of a line of traffic under BCM is a big "spring-damper-mass" system shown in Fig. 1(c) [20]. Intuitively, a perturbation will lead to damped waves travelling outward in *both* directions from the point of perturbation, and the amplitude of these waves will decay as they travel [21]. Ref. [22] also proves that BCM traffic is stable under any and all of the various boundary conditions[5]: infinite line, circular boundaries, fixed-fixed boundaries, free-free boundaries and fixed-free boundaries. Thus, traffic flow instabilities can be suppressed by automated control systems in individual vehicles without global control.

As bilateral control is introduced, the mixed traffic — in which some cars are under CFM control while others are under BCM control — will be of interest.

IV. Stability analysis of mixed traffic

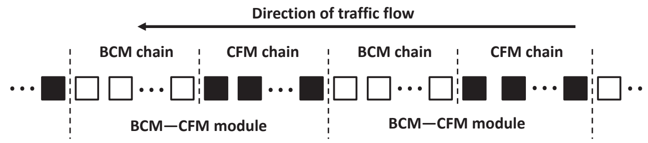
Intuitively, mixed traffic flow can be cut into "single mode segments." In each segment, the vehicles are either all under CFM (called a CFM chain) or all under BCM (called a BCM chain). The BCM chains in themselves are always stable, while the CFM chains are not stable if (5) is not satisfied. However, the CFM chains and BCM chains interact with one another, thus, this simple intuition does not provide a stability statement about traffic flow as a whole. Moreover, the CFM chains and BCM chains could be very short. In an extreme case, such chains may consist of single cars, i.e., CFM vehicles and BCM vehicles alternating. What is the stability requirement for this particular form of mixed traffic and how does it generalize?

In this paper, we focus on a special type of mixed traffic shown in Fig. 2(a). The CFM chains and BCM chains appear alternately in the traffic flow. Suppose each CFM chain contains K vehicles and each BCM chain contains L vehicles. In such mixed traffic, we can combine the BCM chain and its successive CFM chain into a BCM-CFM module (containing $L+K$ cars). Then the whole traffic is a cascade of such BCM-CFM modules (See Fig. 2(b)). For each BCM-CFM module

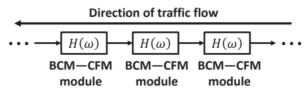
[3] In eq. (4), $a_n = 0$ for all n indicates that $d_n = d_{n+1}$ and $r_n = r_{n+1}$ for all n. Let $r_n = r$ for all n, then $v_n = v_0 - n\,r$ and $d_n(t) = r\,t + d_n(0)$. In practice, v_n and $d_n(t)$ are bounded (for all n and t). Thus, $r = 0$.

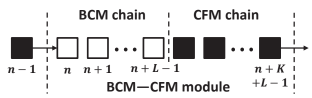
[4] The string stability condition and stability condition might be different [13]. For instance, eigenvalue decomposition might not work due to some boundary conditions [22]. For infinite boundaries, both stability analysis and string stability analysis provide the same condition (5) [21], [22].

[5] The boundaries in platooning are used to control the desired states of all vehicles in the platoon.The boundary condition in BCM is just to design the ACC system such that the car can run on the road alone.



(a) The mixed traffic with BCM and CFM chains.





(b) Cascade of BCM-CFM modules (c) Input/Output of each module

Fig. 2. The mixed traffic studied in this paper. The white “blocks” denote cars under BCM, and the black “blocks” denote cars under CFM. (a). The CFM chains and BCM chains appear alternatively in the traffic. (b). The BCM chain and its successive CFM chain are combined as a BCM-CFM module. (c). For each BCM-CFM module, the input is the state of the car immediately ahead of this BCM-CFM module, and the output is the state of the last car in this BCM-CFM module.

(e.g., car n, car $n+1, \cdots$, car $n+L+K-1$), the input is the state of the last vehicle in the BCM-CFM module ahead of it (i.e., car $n-1$), and the output is the state of the last car in this BCM-CFM module (i.e. car $n+L+K-1$). The output of this BCM-CFM module will become the input to the next BCM-CFM module (See Fig. 2(c)).

Breaking the overall traffic flow into BCM-CFM modules has the advantage that the resulting traffic flow can be analyzed using tools similar to those used for analysing pure CFM traffic, because information can only flow “upstream,” since the CFM vehicle at the tail end of the BCM-CFM module only considers information about the car ahead of it.

A. the “B-C” module

First, let’s analyze a special case where CFM vehicles and BCM vehicles appear alternately, i.e.,

$$\cdots BC, BC, BC, BC, BC, \cdots$$

That is, $K=1$ and $L=1$. The BCM-CFM module in Fig. 2 becomes a “B-C” module. Each “B-C” module provides two ordinary differential equations (ODEs):

$$\begin{aligned}\ddot{x}_n &= k_d(x_{n+1}+x_{n-1}-2x_n)\\ &\quad + k_v(\dot{x}_{n+1}+\dot{x}_{n-1}-2\dot{x}_n), &(6)\\ \ddot{x}_{n+1} &= k_d(x_n-x_{n+1})+k_v\dot{x}_n-(k_v+k_dT)\dot{x}_{n+1}, &(7)\end{aligned}$$

where $x_n = y_n - n\ell$. By Fourier analysis, i.e., analyzing the response $X_n(\omega)e^{j\omega t}$ to pure oscillatory input, we find the following two linear equations:

$$c(\omega)X_{n+1}+d(\omega)X_n+c(\omega)X_{n-1}=0, \tag{8}$$

$$a(\omega)X_{n+1}+b(\omega)X_n=0. \tag{9}$$

The four coefficients are: $a(\omega) = -\omega^2+(k_v+k_dT)j\omega+k_d$, $b(\omega)=c(\omega)=-jk_v\omega-k_d$ and $d(\omega)=-\omega^2+2jk_v\omega+2k_d$. By solving eqs. (8) and (9), we can find

$$X_{n+1}=H(\omega)X_{n-1}. \tag{10}$$

Definition 1: For the mixed traffic flow system, if

$$\|H(\omega)\|^2<1 \qquad (\text{for all } |\omega|>0), \tag{11}$$

then the system is called *modular string stable* [6].

Note that $H(0)=1$ when $\omega=0$. Rewrite $H(\omega)$ as

$$H(\omega)=\frac{M(\omega)}{N(\omega)}, \tag{12}$$

where both $M(\omega)$ and $N(\omega)$ are polynomial functions of ω. Thus, the function

$$D(\omega)=\|M(\omega)\|^2-\|N(\omega)\|^2 \tag{13}$$

has the following form:

$$D(\omega)=w_1|\omega|^2+w_2|\omega|^4+\cdots \tag{14}$$

Definition 2: The necessary condition for the modular string stability of the mixed traffic system is $w_1<0$. That is, low-frequency perturbations will be suppressed.

In this paper, we focus on the necessary (modular string) stability condition, i.e., $w_1<0$, for the mixed traffic flow.

Theorem 1: The necessary (modular string) stability condition for the cascaded “B-C” modules is:

$$k_vT+k_dT^2>\frac{3}{2}. \tag{15}$$

Proof of Theorem 1: By solving (8) and (9), we find

$$H(\omega)=\frac{b(\omega)c(\omega)}{a(\omega)d(\omega)-b(\omega)c(\omega)}. \tag{16}$$

Thus, $D(\omega)=\|b(\omega)c(\omega)\|^2-\|a(\omega)d(\omega)-b(\omega)c(\omega)\|^2$. By tedious calculation, we find

$$w_1=2k_d^3(-2k_dT^2-2k_vT+3). \tag{17}$$

The necessary stability condition $w_1<0$ provides (15). □

The feedback gains used in BCM and the ones used in CFM need not be the same. Let τ be the ratio of the gains used in BCM to the gains used in CFM, that is, replace k_d and k_v in (6) by τk_d and τk_v respectively[7]. We then find:

Proposition 1: The necessary stability condition of the traffic line of cascaded B-C modules is:

$$k_vT+k_dT^2>\left(1+\frac{1}{2\tau}\right). \tag{18}$$

Proof of Proposition 1: the same analysis as the proof of Theorem 1 by changing the 4 coefficients to

$$a(\omega)=-\omega^2+(k_v+k_dT)j\omega+k_d, \tag{19}$$

$$b(\omega)=-jk_v\omega-k_d, \tag{20}$$

$$c(\omega)=-j\tau k_v\omega-\tau k_d, \tag{21}$$

$$d(\omega)=-\omega^2+j2\tau k_v\omega+2\tau k_d. \tag{22}$$

[6]Modular string stability indicates that the perturbations on the tail-end vehicle of each BCM-CFM module are *bounded*. Each BCM-CFM module contains a finite number of vehicles. Thus, perturbations on each vehicle in the BCM-CFM module will also be bounded.

[7]The ratio used for k_d and the one used for k_v could be different. In this paper, in order to emphasize that the architectures of ACC systems for CFM and BCM are almost the same, we just scale the ACC system’s output, i.e., desired acceleration/deceleration, by τ, rather than redesign new feedback gains for BCM. Thus, the same τ is used for the two gains k_d and k_v.

Correspondingly, the coefficient w_1 in $D(\omega) = \|b(\omega)c(\omega)\|^2 - \|a(\omega)d(\omega) - b(\omega)c(\omega)\|^2$ becomes

$$w_1 = 2k_d^3\tau^2(-2k_dT^2 - 2k_vT + 2 + 1/\tau). \tag{23}$$

The necessary stability condition, i.e., $w_1 < 0$, provides (18).□

Comparing (18) to (5), the stability condition for pure CFM traffic can be relaxed by adding BCM vehicles, only if

$$\tau k_d > \frac{1}{T^2}. \tag{24}$$

The larger τ is, the easier (24) can be satisfied, and the more (5) is relaxed. Thus, larger τ, e.g., $\tau = 2$ or 3, should be used. We will see later that (18) is a special case of the necessary stability condition in (83) when $K = 1$ and $L = 1$.

B. the "B-C-⋯-C" module

Now, let's extend the "B-C" module to a somewhat more general case: in each BCM-CFM module, the first vehicle is under BCM and the next K vehicles are under CFM, i.e.

$$\cdots BC \cdots C, BC \cdots C, BC \cdots C, BC \cdots C, \cdots$$

The necessary (modular string) stability condition for such "B-C-⋯-C" module case can be analyzed using a similar approach to that in the proof of Theorem 1. Now, the $K+1$ cars in each module gives $K+1$ linear equations, i.e.,

$$\mathbf{T}\mathbf{X}_n = \mathbf{C}. \tag{25}$$

The $(K+1) \times (K+1)$ matrix $\mathbf{T}$ is

$$\mathbf{T} = \begin{pmatrix} d & c & & & \\ b & a & & & \\ & b & a & & \\ & & \ddots & \ddots & \\ & & & b & a \end{pmatrix}, \tag{26}$$

where a b c and d are in (19) to (22). The two vectors are

$$\mathbf{X}_n = (X_n, X_{n+1}, \cdots, X_{n+K})^T, \tag{27}$$
$$\mathbf{C} = (-cX_{n-1}, 0, \cdots, 0)^T. \tag{28}$$

Linear equations (8) and (9) are the special case when $K = 1$. By backward substitution, i.e., starting from the last equation and working backward to the first one [28], we find

$$X_{n+K} = H(\omega)X_{n-1}, \tag{29}$$

where

$$H(\omega) = \frac{b(\omega)c(\omega)}{a(\omega)d(\omega) - b(\omega)c(\omega)} \left(-\frac{b(\omega)}{a(\omega)}\right)^{K-1}. \tag{30}$$

By tedious calculation, we find the necessary (modular string) stability condition for several different K, i.e.,

$$\begin{aligned} K=1: &\quad k_dT^2 + k_vT > 1 + 1/(2\tau), \\ K=2: &\quad (5/6)k_dT^2 + k_vT > 1 + 1/(3\tau), \\ K=3: &\quad (3/4)k_dT^2 + k_vT > 1 + 1/(4\tau), \\ K=4: &\quad (7/10)k_dT^2 + k_vT > 1 + 1/(5\tau), \\ K=5: &\quad (2/3)k_dT^2 + k_vT > 1 + 1/(6\tau). \end{aligned} \tag{31}$$

We can find the following pattern:

Theorem 2: the necessary (modular string) stability condition for the cascaded B-C-⋯-C modules (i.e., one BCM car followed by K CFM cars in each module) is

$$\frac{1}{2}k_dT^2 + k_vT > 1 - \frac{1}{K+1}\left(k_dT^2 - \frac{1}{\tau}\right). \tag{32}$$

Proof of Theorem 2: Now, the $D(\omega)$ in (13) is

$$D(\omega) = \|c\|^2\left(\|b\|^2\right)^K - \|ad - bc\|^2\left(\|a\|^2\right)^{K-1}. \tag{33}$$

Note that $D(0) = 0$ (see eq. (30)), thus $D(\omega)$ must have the form in (14). The necessary (modular string) stability condition is a straightforward result of (34) in Lemma 1. □

Lemma 1: The coefficient $w_1 = S_K$ for the $D(\omega)$ in (33), where S_K has the following expression:

$$\begin{aligned} S_K = &k_d^{2K+1}\tau^2 \times \\ &\left((K+1)\big(2 - k_dT^2 - 2k_vT\big) - 2k_dT^2 + \frac{2}{\tau}\right). \end{aligned} \tag{34}$$

Proof of Lemma 1: From (19) to (22) we find

$$\|a\|^2 = k_d^2 + \left((k_v + k_dT)^2 - 2k_d\right)\omega^2 + \omega^4, \tag{35}$$
$$\|b\|^2 = k_d^2 + k_v^2\omega^2. \tag{36}$$

By tedious calculation, we find $\|bc\|^2 = k_d^4\tau^2 + 2k_d^2k_v^2\tau^2\omega^2 + k_v^4\tau^2\omega^4$ and $\|ad-bc\|^2 = k_d^4\tau^2 + o(\omega^2) + 2k_d^2\tau(k_v^2\tau - 2k_d\tau - k_d + 2T^2k_d^2\tau + 2Tk_dk_v\tau)\omega^2$.

Prove by induction. When $K = 1$, we find $\|bc\|^2 - \|ad - bc\|^2 = 2k_d^3\tau(2\tau + 1 - 2T^2k_d\tau - 2Tk_v\tau)\omega^2 + o(\omega^2)$. The coefficient w_1 matches S_1 in (34) (by substituting $K = 1$). Suppose that $w_1 = S_n$ when $K = n$. That is, $\|c\|^2\left(\|b\|^2\right)^n = A_n + B_n\omega^2 + o(\omega^2)$ and $\|ad-bc\|^2\left(\|a\|^2\right)^{n-1} = A_n + (B_n - S_n)\omega^2 + o(\omega^2)$, where $A_n = k_d^{2(n+1)}\tau^2$, i.e., the constant term in $\|c\|^2\left(\|b\|^2\right)^n$, can be calculated directly. Then we can calculate $\|c\|^2\left(\|b\|^2\right)^{n+1} - \|ad - bc\|^2\left(\|a\|^2\right)^n$, That is $A_n(\|b\|^2 - \|a\|^2) + S_n\|b\|^2\omega^2 + o(\omega^2)$. Thus, we find

$$w_1 = k_d^{2n+3}\tau^2(2 - 2k_vT - k_dT^2) + S_nk_d^2 = S_{n+1}, \tag{37}$$

which matches the expression in (34) when $K = n+1$. □

C. the "B-⋯-B-C" module

Let's consider another special case. Only one vehicle in each CFM chain and L vehicles in each BCM chain, i.e.

$$\cdots B \cdots BC, B \cdots BC, B \cdots BC, B \cdots BC, \cdots$$

By similar analysis used above, the $L+1$ vehicles in the "B-⋯-B-C" module provide $L+1$ linear equations in (25). Now, $\mathbf{T}$ becomes a $(L+1)\times(L+1)$ *tri-diagonal matrix*, i.e.,

$$\mathbf{T} = \begin{pmatrix} d & c & & & \\ c & d & c & & \\ & \ddots & \ddots & \ddots & \\ & & c & d & c \\ & & & b & a \end{pmatrix}. \tag{38}$$

Again, we can solve the system by backward substitution [28]. From the last (i.e., the $(L+1)$-th) equation, we find

$$X_{n+L-1}=\frac{-a}{b}X_{n+L}=f_1X_{n+L}, \tag{39}$$

then from the L-th equation, we find $X_{n+L-2} = -\left(\frac{d}{c}X_{n+L-1}+X_{n+L}\right) = -\left(\frac{d}{c}f_1+1\right)X_{n+L} = f_2X_{n+L}$, then go to the $(L-1)$-th equation, and we find $X_{n+L-3} - \left(\frac{d}{c}X_{n+L-2}+X_{n+L-1}\right) = -\left(\frac{d}{c}f_2+f_1\right)X_{n+L} = f_3X_{n+L}$, and so on. We then find the following recursive formula:

$$f_l=-\left(\frac{d}{c}f_{l-1}+f_{l-2}\right), \quad (\text{for } l=2,3,\cdots L+1) \tag{40}$$

with two initial values $f_0=1$ and $f_1=-a/b$. And finally,

$$f_{L+1}X_{n+L}=X_{n-1}. \tag{41}$$

Thus, the transfer function is:

$$H(\omega)=\frac{1}{f_{L+1}(\omega)}. \tag{42}$$

Note that $f_1(0)=-a(0)/b(0)=1$, and $-d(0)/c(0)=2$ (see eq. (19) to (22)), thus, the recursive formula (40) provides $f_l(0)=1$ (for all $l=2,3,\cdots L+1$). That is, $H(0)=1$. In order to figure out the necessary stability condition, we need to find polynomial functions $N_{l+1}(\omega)$ and $M_{l+1}(\omega)$ such that

$$f_{l+1}(\omega)=\frac{N_{l+1}(\omega)}{M_{l+1}(\omega)}. \tag{43}$$

From the recursive definition of f_l in (40), we can do some calculations. First, let $p=-d/c$, then we find $f_l = pf_{l-1} - f_{l-2} = p(pf_{l-2}-f_{l-3}) - f_{l-2} = (p^2-1)f_{l-2} - pf_{l-3} = (p^3-2p)f_{l-3} - (p^2-1)f_{l-4} = \cdots$. That is,

$$f_l=q_k(p)f_{l-k}-q_{k-1}(p)f_{l-k-1}, \tag{44}$$

where $q_k(p)$ is a k-th order polynomial of p, i.e.,

$$q_k(p)=\alpha_{k,0}p^k+\alpha_{k,1}p^{k-1}+\cdots+\alpha_{k,k-1}p+\alpha_{k,k}. \tag{45}$$

Lemma 2: the polynomial $q_k(p)$ in (44) satisfies the recursive form:

$$q_{k+1}(p)=pq_k(p)-q_{k-1}(p). \tag{46}$$

Proof of Lemma 2: Substitute $f_{l-k}=pf_{l-k-1}-f_{l-k-2}$ (see (40)) into (44), we find the expression of q_{k+1} in $f_l=q_{k+1}(p)f_{l-k-1}-q_k(p)f_{l-k-2}$, which is exact (46). □

Thus, we find the closed form of $f_{L+1}(\omega)$ by (44), i.e.,

$$f_{L+1}=q_L(p)f_1-q_{L-1}(p)f_0=-\frac{a}{b}q_L(p)-q_{L-1}(p). \tag{47}$$

Then, we can figure out $M_{l+1}(\omega)$ and $N_{l+1}(\omega)$. That is[8],

$$M_{l+1}(\omega)=-b(-c)^l, \tag{48}$$

$$N_{l+1}(\omega)=\sum_{k=0}^{l}\left(a\alpha_{l,k}+b\alpha_{l-1,k-1}\right)d^{l-k}(-c)^k. \tag{49}$$

[8]Here, we define $\alpha_{k,m}=0$ for both $m<0$ and $m>k$. That is, the coefficients for the terms p^{k+1}, p^{k+2}, $p^{k+2}, \cdots$, and p^{-1}, p^{-2}, $p^{-3}, \cdots$ are all zeros, thus, these terms will not appear in $q_k(p)$ in (45).

By tedious calculation, we find the necessary (modular string) stability condition for several different L, i.e.,

$$\begin{aligned} L=1: &\quad k_dT^2+k_vT-1-1/(2\tau)>0,\\ L=2: &\quad (3/2)k_dT^2+k_vT-1-1/\tau>0,\\ L=3: &\quad 2k_dT^2+k_vT-1-3/(2\tau)>0,\\ L=4: &\quad (5/2)k_dT^2+k_vT-1-2/\tau>0,\\ L=5: &\quad 3k_dT^2+k_vT-1-5/(2\tau)>0. \end{aligned} \tag{50}$$

It's easy to find the following "patten":

Theorem 3: the necessary (modular string) stability condition for the cascaded B-$\cdots$-B-C modules (i.e., L BCM cars followed by one CFM car in each module) is

$$\frac{1}{2}k_dT^2+k_vT>1-\frac{L}{2}\left(k_dT^2-\frac{1}{\tau}\right). \tag{51}$$

Proof of Theorem 3: First, note that

$$\|M_{l+1}(\omega)\|^2=\tau^{2l}k_d^{2(l+1)}+(l+1)\tau^{2l}k_d^{2l}k_v^2\omega^2+o(\omega^2). \tag{52}$$

Then, we need to write $N_l(\omega)$ as the following form

$$N_{l+1}(\omega)=P_{l+1}+Q_{l+1}j\omega+R_{l+1}\omega^2+o(\omega^2). \tag{53}$$

Then, we find

$$\|N_{l+1}(\omega)\|^2=P_{l+1}^2+\left(2P_{l+1}R_{l+1}+Q_{l+1}^2\right)\omega^2+o(\omega^2). \tag{54}$$

Note that $f_{l+1}(0)=1$ and thus $N_{l+1}(0)=M_{l+1}(0)=P_{l+1}$. From eqs. (48), (20) and (21), we can calculate $M_{l+1}(0)$. Thus, we find (by using eqs. (19) and (22)):

$$P_{l+1}=\tau^lk_d^{l+1}. \tag{55}$$

We first calculate $\{\alpha_{l,k}\}$ for several l and k, and then we further find the following pattern:

$$Q_{l+1}=(l+1)k_d^l\tau^l(k_v+Tk_d), \tag{56}$$

$$\begin{aligned} R_{l+1}=&-\frac{l+1}{2}(k_d\tau)^{l-1}\times\\ &\left(l\tau k_v^2+2lTk_d\tau k_v+lk_d+2k_d\tau\right). \end{aligned} \tag{57}$$

We will prove the above two conjectures (56) and (57) in the following Theorem 4 and Theorem 5, respectively. Substitute eqs. (52), (54), (56) and (57) into (13), we find

$$\begin{aligned} D(\omega)=&\|M_{l+1}(\omega)\|^2-\|N_{l+1}(\omega)\|^2=o(\omega^2)-\\ &(L+1)k_d^{2L+1}\tau^{2L}\left(2k_vT+(L+1)k_dT^2-\frac{L}{\tau}-2\right)\omega^2. \end{aligned} \tag{58}$$

Thus, the necessary stability condition provides (51). □

In order to prove the conjectures (56) and (57), let's first calculate the P_{l+1} in (55) directly. The constant term P_{l+1} in $N_{l+1}(\omega)$ can be computed from (49) by substituting a,b,c,d as $a(0)$, $b(0)$, $c(0)$ and $d(0)$, respectively. That is,

$$P_{l+1}=\left(\sum_{k=0}^{l}(\alpha_{l,k}-\alpha_{l-1,k-1})2^{l-k}\right)k_d^{l+1}\tau^l \tag{59}$$

$$=\left(\sum_{k=0}^{l}\alpha_{l,k}2^{l-k}-\sum_{k=0}^{l-1}\alpha_{l-1,k}2^{l-1-k}\right)k_d^{l+1}\tau^l \tag{60}$$

$$=\left(q_l(2)-q_{l-1}(2)\right)k_d^{l+1}\tau^l. \tag{61}$$

Substitute $p=2$ into (46), we find

$$q_{l+1}(2)=2q_l(2)-q_{l-1}(2). \tag{62}$$

Thus, $q_l(2)-q_{l-1}(2)=q_{l-1}(2)-q_{l-2}(2)$. Repeat this process, we finally find $q_l(2)-q_{l-1}(2)=\cdots=q_1(2)-q_0(2)=1$. Note that $\{q_l(2)\}$ forms an *arithmetic sequence* with common difference 1 and initial value $q_0(2)=1$. Thus,

$$q_l(2)=\sum_{k=0}^{l}\alpha_{l,k}2^{l-k}=l+1. \tag{63}$$

The above analysis provides us with some suggestion for proving the two conjectures in eqs. (56) and (57) by using the recursive relation (46). The details are summarized in the Theorem 4 and Theorem 5.

Theorem 4: The coefficient Q_{l+1} corresponding to the $j\omega$ term in $N_{l+1}(\omega)$ is as shown in (56).

Proof of Theorem 4: First, note that $d^{l-k}=(2\tau k_d)^{l-k}+(l-k)(2\tau)^{l-k}k_d^{l-k-1}k_v j\omega+o(\omega^2)-(l-k)(2\tau)^{l-k-1}k_d^{l-k-2}\big(k_d+(l-k-1)\tau k_v^2\big)\omega^2$ and $(-c)^k=\tau^k k_d^k+k\tau^k k_d^{k-1}k_v j\omega-\frac{k(k-1)}{2}\tau^k k_d^{k-2}k_v^2\omega^2+o(\omega^2)$. Thus,

$$\begin{aligned}d^{l-k}(-c)^k=&\,2^{l-k}\tau^l k_d^l+l2^{l-k}\tau^l k_d^{l-1}k_v j\omega\\&-2^{l-k-1}k_d^{l-2}\tau^{l-1}\big((l-k)k_d+(l^2-l)\tau k_v^2\big)\omega^2+o(\omega^2).\end{aligned} \tag{64}$$

Thus, the coefficient corresponding to the term $j\omega$ in $ad^{l-k}(-c)^k$ is $2^{l-k}\tau^l k_d^l((l+1)k_v+k_dT)$, and the coefficient to the term $j\omega$ in $bd^{l-k}(-c)^k$ is $-2^{l-k}\tau^l k_d^l(l+1)k_v$. Finally, we find the expression of Q_{l+1}, i.e., $Q_{l+1}=\widetilde{Q}_{l+1}\tau^l k_d^l$, with

$$\begin{aligned}\widetilde{Q}_{l+1}=&\big(k_v(l+1)+k_dT\big)\sum_{k=0}^{l}\alpha_{l,k}2^{l-k}\\&-k_v(l+1)\sum_{k=0}^{l-1}\alpha_{l-1,k}2^{l-k-1}.\end{aligned} \tag{65}$$

Substituting (63), we find

$$\begin{aligned}\widetilde{Q}_{l+1}&=k_dTq_l(2)+k_v(l+1)\big(q_l(2)-q_{l-1}(2)\big)\\&=(k_v+k_dT)(l+1).\end{aligned} \tag{66}$$

Thus, $Q_{l+1}=\widetilde{Q}_{l+1}\tau^l k_d^l$ is exact (56). □

Before calculating the R_{l+1} in (57), let us first prove the following result:

Lemma 3: The coefficients $\{\alpha_{l,k}\}$ (in the polynomial $q_l(p)$) satisfy the following constraint:

$$\sum_{k=0}^{l-1}(l-k)\alpha_{l,k}2^{l-k-1}=\frac{1}{6}l^3+\frac{1}{2}l^2+\frac{1}{3}l. \tag{67}$$

Proof of Lemma 3: Take the derivative of $q_l(p)$ in (45) (over p), we find:

$$\dot{q}_l(p)=\sum_{k=0}^{l-1}(l-k)\alpha_{l,k}p^{l-k-1}. \tag{68}$$

Note that the left hand side term in (67) is exact $\dot{q}_l(2)$. By taking the derivative of the recursive formula (46), we find

$$\dot{q}_{l+1}(p)-p\dot{q}_l(p)+\dot{q}_{l-1}(p)=q_l(p). \tag{69}$$

Substituting $p=2$ and eq. (63), we find

$$\dot{q}_{l+1}(2)-2\dot{q}_l(2)+\dot{q}_{l-1}(2)=l+1. \tag{70}$$

It's well-known how to solve such inhomogeneous (second order) linear difference equation (70) (e.g., using z-transform) [29]. By using the initial conditions $\dot{q}_0(2)=0$ and $\dot{q}_1(2)=1$, the solution is exact (67). □

From (67), we can further calculate:

$$\dot{q}_l(2)-\dot{q}_{l-1}(2)=\frac{1}{2}l(l+1). \tag{71}$$

Now, let's prove the following result, i.e.,

Theorem 5: The coefficient R_{l+1} corresponding to the ω^2 term in $N_{l+1}(\omega)$ is as shown in (57).

Proof of Theorem 5: From (64), we can further compute the coefficients $U_{l+1,k}$ of the term ω^2 in $ad^{l-k}(-c)^k$ and the coefficients $V_{l+1,k}$ of the term ω^2 in $bd^{l-k}(-c)^k$. By tedious calculation, we find

$$\begin{aligned}U_{l+1,k}=&-2^{l-k-1}(k_d\tau)^{l-1}\times\\&\big((l^2+l)\tau k_v^2+2T\tau k_dk_vl+(l-k)k_d+2\tau k_d\big)\end{aligned} \tag{72}$$

$$V_{l+1,k}=2^{l-k-1}(k_d\tau)^{l-1}\big(\tau(l^2+l)k_v^2+(l-k)k_d\big) \tag{73}$$

Substituting $U_{l+1,k}$ and $V_{l+1,k}$ into $N_{l+1}(\omega)$ in (49), i.e.,

$$R_{l+1}=\sum_{k=0}^{l}\alpha_{l,k}U_{l+1,k}+\sum_{k=0}^{l-1}\alpha_{l-1,k}V_{l+1,k+1}, \tag{74}$$

we find the expression of R_{l+1}, i.e.

$$\begin{aligned}R_{l+1}=&-(k_d\tau)^{l-1}k_d\big(\dot{q}_l(2)-\dot{q}_{l-1}(2)\big)\\&-\frac{(k_d\tau)^{l-1}}{2}\tau(l^2+l)k_v^2\big(q_l(2)-q_{l-1}(2)\big)\\&-\frac{(k_d\tau)^{l-1}}{2}\big(2T\tau k_dk_vl+2\tau k_d\big)q_l(2).\end{aligned} \tag{75}$$

Substituting (63) and (71), we finally find the expression of R_{l+1}, which is exactly the same as (57). □

D. the "B-$\cdots$-B-C-$\cdots$-C" module

Now, we are ready to analyze the general case shown in Fig. 2. Each CFM chain contains K vehicles and each BCM chain contains L vehicles, i.e.,

$$\cdots B\cdots BC\cdots C,\ B\cdots BC\cdots C,\ B\cdots BC\cdots C,\ \cdots$$

By similar analysis to that used above, the $L+K$ vehicles in the "B-$\cdots$-B-C-$\cdots$-C" module provide $L+K$ linear equations in (25). Now, $\mathbf{T}$ becomes a $(L+K)\times(L+K)$ matrix, i.e.,

$$\mathbf{T}=\left(\begin{array}{cccccccc}d&c&&&&&&\\c&d&c&&&&&\\&\ddots&\ddots&\ddots&&&&\\&&c&d&c&&&\\&&&b&a&&&\\&&&&b&a&&\\&&&&&\ddots&\ddots&\\&&&&&&b&a\end{array}\right) \tag{76}$$

Again, we can solve the system by backward substitution [28]. First, note that the bottom-right $K \times K$ block in $\mathbf{T}$ is *lower triangular* with only two diagonals. Thus, we find

$$X_{n+L+K-1} = H_1(\omega) X_{n+L}, \tag{77}$$

with transfer function

$$H_1(\omega) = \left(-\frac{b}{a}\right)^{K-1}. \tag{78}$$

The top-left $(L+1) \times (L+1)$ block in $\mathbf{T}$ is the same as the matrix in (38), thus,

$$X_{n+L} = H_2(\omega) X_{n-1}, \tag{79}$$

with transfer function

$$H_2(\omega) = \frac{1}{f_{L+1}(\omega)} = \frac{M_{L+1}(\omega)}{N_{L+1}(\omega)} \tag{80}$$

obtained by the recursive iteration of $f_{L+1}(\omega)$ in (44). Thus, the transfer function of the whole module is

$$H(\omega) = H_1(\omega) H_2(\omega) = \frac{M_{L+1}(\omega)}{N_{L+1}(\omega)} \left(-\frac{b}{a}\right)^{K-1}. \tag{81}$$

Then, the $D(\omega)$ in (13) is:

$$\|M_{L+1}(\omega)\|^2 \|b\|^{2(K-1)} - \|N_{L+1}(\omega)\|^2 \|a\|^{2(K-1)}. \tag{82}$$

Note that $H_1(0) = 1$ and $H_2(0) = 1$. Thus, $H(0) = 1$. That is, $D(0) = 0$. Finally, we find:

Theorem 6: The necessary stability condition for the cascaded "B-$\cdots$-B-C-$\cdots$-C" modules (i.e., L BCM cars followed by K CFM cars in each module) is:

$$\frac{1}{2} k_d T^2 + k_v T > 1 - \frac{L(L+1)}{2(L+K)} \left(k_d T^2 - \frac{1}{\tau}\right). \tag{83}$$

Proof of Theorem 6: From (35) and (36), we find

$$\begin{aligned} \|a\|^{2(K-1)} =& k_d^{2(K-1)} + o(\omega^2) \\ &+ (K-1) k_d^{2(K-2)} \left((k_v + k_d T)^2 - 2k_d\right) \omega^2, \end{aligned} \tag{84}$$

$$\|b\|^{2(K-1)} = k_d^{2(K-1)} + (K-1) k_d^{2(K-2)} k_v^2 \omega^2 + o(\omega^2) \tag{85}$$

From (52) and (54), we can then calculate w_1 in (82), i.e.,

$$\begin{aligned} w_1 =& -P_{L+1}^2 (K-1) k_d^{2K-3} \left(2k_v T + k_d T^2 - 2\right) + \\ & - k_d^{2(K-1)} \left(2P_{L+1} R_{L+1} + Q_{L+1}^2 - (L+1)\tau^{2L} k_d^{2L} k_v^2\right). \end{aligned} \tag{86}$$

By substituting (55), (56) and (57), finally, we find

$$\begin{aligned} w_1 =& -k_d^{2(K+L)-1} \tau^{2L} \big[2(K+L) k_v T - 2(L+K) \\ & + (L^2 + 2L + K) k_d T^2 - (L^2 + L)/\tau\big]. \end{aligned} \tag{87}$$

The necessary stability condition $w_1 < 0$ provides (83). $\square$

Note that (32) in Thm. 2 and (51) in Thm. 3 are exactly two special cases of (83) when $L = 1$ and $K = 1$, respectively. When $L = 0$, Fig. 2 becomes pure CFM based highway traffic, and (83) becomes the well-known stability condition (5) correspondingly. When $K = 0$, Fig. 2 becomes the pure BCM traffic. Note that the topology of pure BCM traffic is *not* a one-directional cascade of control systems of successive cars [20]. Thus, (83) can not be used directly by simply substituting $K = 0$. That is, (83) is correct for $K \geq 1$ and $L \geq 0$.

Let $\rho = L/(L+K)$ be the percentage of BCM cars in the mixed traffic, then condition (83) becomes:

$$\frac{1}{2} k_d T^2 + k_v T > 1 - \rho \frac{L+1}{2} \left(k_d T^2 - \frac{1}{\tau}\right). \tag{88}$$

In [30], we provide some numerical simulations (and MATLAB codes) to verify the theoretical analysis. Eq. (88) also provides suggestion for designing ACC system.

V. Suggestion for ACC-system design

Comparing (88) to (5), only if $k_d T^2 > 1/\tau$, the stability condition can be relaxed by involving BCM cars in the traffic also. Thus, the ACC system should first be designed such that

$$k_d T^2 = \frac{1}{\tau} + \epsilon, \qquad (\text{with } \epsilon > 0). \tag{89}$$

Then the necessary stability condition (88) becomes

$$\frac{1}{2} k_d T^2 + k_v T > 1 - \frac{L+1}{2} \rho \epsilon. \tag{90}$$

How much (5) can be relaxed (by involving BCM vehicles) depends not only on the percentage ρ, but also on the topology. The longer the BCM chain is (with the same ρ), the more the necessary stability condition can be relaxed. Thus, even if the percentage of BCM cars is small, e.g., only 20%, if L is not too small, e.g., $L = 20$, some noticeable improvement can still be achieved (in which the requirement for $\frac{1}{2} k_d T^2 + k_v T$ reduces form 1 to $1 - 2.1\epsilon$). On the other hand, even if L is small, e.g., $L = 5$, if the percentage of BCM cars is not small (e.g., $\rho = 50\%$), some noticeable improvement can also be achieved (in which the requirement for $\frac{1}{2} k_d T^2 + k_v T$ reduces form 1 to $1 - 1.5\epsilon$). Substituting (89) into (90), we find that if we choose $\epsilon = \frac{2 - 1/\tau}{(L+1)\rho + 1}$, then (90) can be satisfied by any $k_v > 0$. If L is large enough, e.g., $L > 100$, and ρ is not too small, e.g., $\rho > 50\%$, then such ϵ will be very small, e.g., less than 0.04. Thus, we can set a small threshold ϵ_0, e.g., $\epsilon_0 = 0.05$, and choose ϵ in (89) as

$$\epsilon = \min\left(\frac{2 - 1/\tau}{(L+1)\rho + 1}, \epsilon_0\right). \tag{91}$$

We can make k_d in (89) smaller by choosing larger τ. Note that the gains in BCM is τk_d and τk_v, and the coefficient

$$\tau k_d = \frac{1 + \tau\epsilon}{T^2} \tag{92}$$

increases with the increase of τ. Thus, τ can not be arbitrarily large. However, since ϵ is small, the increase in τk_d (for BCM) is negligible comparing to the decrease in k_d (for CFM) by choosing relatively large τ, e.g., $\tau = 2$ or 3. One reasonable choice is to control the value of $\tau\epsilon$ in (92). That is, $\tau\epsilon = \eta\epsilon_0$, or equivalently $\tau = \eta\epsilon_0/\epsilon$, with some preset η, e.g., $1 < \eta \leq 2$.

Speed-difference sensors are considerably more expensive than simple range sensors. If only range sensors are used by the ACC system, i.e., $k_v = 0$, then (5) becomes $k_d > 2/T^2$ for the traffic purely under CFM. For mixed traffic, when $\epsilon < \epsilon_0$, the gain requirement for CFM cars is $k_d > 1/(\tau T^2)$, and the one for BCM cars is $\tau k_d > 1/T^2$. We can see the obvious advantage of introducing BCM vehicles.

VI. Conclusion

In today's traffic, drivers generally focus only on the car ahead of them. This results in traffic flow instabilities including alternating "stop-and-go" driving conditions. Such traffic flow instabilities can be suppressed effectively if the vehicle also take into account of the state of its following. Different from human drivers, ACC system equipped with suitable sensors can implement BCM easily. Thus, we can expect smooth traffic in the future when BCM vehicles are widely used.

Between today's pure CFM traffic and pure BCM traffic in the future, there is a transition period of mixed traffic in which some vehicles are under car-following control while others are under bilateral control. In this study, we analyze a particular type of such mixed traffic. This is where CFM chains and BCM chains alternate in mixed traffic, with each CFM chain containing K vehicles and each BCM chain containing L vehicles. We provide a necessary (modular string) stability condition for such mixed traffic, which can be viewed as an extension of the one for the pure CFM traffic. Unsurprisingly, how much the stability condition for pure CFM traffic can be relaxed by adding BCM vehicles depends on 1). the percentage of BCM vehicles, and 2). the distribution of BCM vehicles. The necessary stability condition (88) provides detailed mathematical descriptions. Moreover, this condition (88) also provides suggestions for the design of extended ACC system. The two most important rules are:

- Use a large value for the "reaction time" T, e.g., $T = 1.5$ sec. [18]. (Even if the real "reaction time" needed by the sensor based control system can be much smaller.)
- Use larger gains for BCM than CFM by setting $\tau > 1$.

In this paper, we focus on the mixed traffic flow with CFM cars and BCM cars. Some ACC system may use asymmetric weights, the approaches provided in this paper can also be used. Moreover, the CFM cars and BCM cars may not be mixed evenly as the special case studied in this paper, or may even be mixed randomly. In this case, we cannot expect to find closed form formulae such as (83) for the stability analysis, and thus can study mixed traffic flow only using numerical simulations. Another interesting problem is to explore the sufficient conditions for stability analysis. Theorem 6 is just a preliminary attempt to explore mixed traffic flow, in which the ideally simplified linear feedback control (1) is used for analysis. To build more accurate car model, delay and non-linearity should also be considered [27]. The analysis of the corresponding traffic flow will be topics for future work.

Acknowledgment

We want to thank the good suggestions from the reviewers.

References

[1] B. Greenshields *et al.*, "A study of traffic capacity," in *Highway research board proceedings*, vol. 35. National Research Council (USA), 1935.

[2] M. J. Lighthill and G. B. Whitham, "On kinematic waves ii. a theory of traffic flow on long crowded roads," *Proc. R. Soc. Lond. A*, vol. 229, no. 1178, pp. 317–345, 1955.

[3] P. I. Richards, "Shock waves on the highway," *Operations research*, vol. 4, no. 1, pp. 42–51, 1956.

[4] T. Nagatani, "Traffic jams induced by fluctuation of a leading car," *Physical Review E*, vol. 61, no. 4, p. 3534, 2000.

[5] W. Levine and M. Athans, "On the optimal error regulation of a string of moving vehicles," *IEEE Transactions on Automatic Control*, vol. 11, no. 3, pp. 355–361, 1966.

[6] P. Varaiya, "Smart cars on smart roads: problems of control," *IEEE Transactions on automatic control*, vol. 38, no. 2, pp. 195–207, 1993.

[7] D. N. Godbole and J. Lygeros, "Longitudinal control of the lead car of a platoon," *IEEE Transactions on Vehicular Technology*, vol. 43, no. 4, pp. 1125–1135, 1994.

[8] S. S. Stankovic, M. J. Stanojevic, and D. D. Siljak, "Decentralized overlapping control of a platoon of vehicles," *IEEE Transactions on Control Systems Technology*, vol. 8, no. 5, pp. 816–832, 2000.

[9] P. Barooah, P. G. Mehta, and J. P. Hespanha, "Mistuning-based control design to improve closed-loop stability margin of vehicular platoons," *IEEE Trans. on Automatic Control*, vol. 54, no. 9, pp. 2100–2113, 2009.

[10] R. H. Middleton and J. H. Braslavsky, "String instability in classes of linear time invariant formation control with limited communication range," *IEEE Transactions on Automatic Control*, vol. 55, no. 7, pp. 1519–1530, 2010.

[11] F. Lin, M. Fardad, and M. R. Jovanovic, "Optimal control of vehicular formations with nearest neighbor interactions," *IEEE Transactions on Automatic Control*, vol. 57, no. 9, pp. 2203–2218, 2012.

[12] H. Hao, P. Barooah, and P. G. Mehta, "Stability margin scaling laws for distributed formation control as a function of network structure," *IEEE Transactions on Automatic Control*, vol. 56, no. 4, pp. 923–929, 2011.

[13] Y. Zheng, S. E. Li, J. Wang, D. Cao, and K. Li, "Stability and scalability of homogeneous vehicular platoon: Study on the influence of information flow topologies," *IEEE Transactions on Intelligent Transportation Systems*, vol. 17, no. 1, pp. 14–26, 2016.

[14] Y. Zheng, S. E. Li, K. Li, and L.-Y. Wang, "Stability margin improvement of vehicular platoon considering undirected topology and asymmetric control," *IEEE Transactions on Control Systems Technology*, vol. 24, no. 4, pp. 1253–1265, 2016.

[15] S. E. Li, X. Qin, K. Li, J. Wang, and B. Xie, "Robustness analysis and controller synthesis of homogeneous vehicular platoons with bounded parameter uncertainty," *IEEE/ASME Transactions on Mechatronics*, vol. 22, no. 2, pp. 1014–1025, 2017.

[16] Y. Zheng, S. E. Li, K. Li, and W. Ren, "Platooning of connected vehicles with undirected topologies: Robustness analysis and distributed h-infinity controller synthesis," *IEEE Transactions on Intelligent Transportation Systems*, vol. 19, no. 5, pp. 1353–1364, 2018.

[17] H. Hao and P. Barooah, "On achieving size-independent stability margin of vehicular lattice formations with distributed control," *IEEE Transactions on Automatic Control*, vol. 57, no. 10, pp. 2688–2694, 2012.

[18] J. Ploeg, N. Van De Wouw, and H. Nijmeijer, "Lp string stability of cascaded systems: Application to vehicle platooning," *IEEE Transactions on Control Systems Technology*, vol. 22, no. 2, pp. 786–793, 2014.

[19] E. Shaw and J. K. Hedrick, "String stability analysis for heterogeneous vehicle strings," in *American Control Conference, 2007. ACC'07*. IEEE, 2007, pp. 3118–3125.

[20] B. K. Horn, "Suppressing traffic flow instabilities," in *Intelligent Transportation Systems-(ITSC), 2013 16th International IEEE Conference on*. IEEE, 2013, pp. 13–20.

[21] B. K. Horn and L. Wang, "Wave equation of suppressed traffic flow instabilities," *IEEE Transactions on Intelligent Transportation Systems*, vol. 19, no. 9, pp. 2955–2964, 2018.

[22] L. Wang, B. K. Horn, and G. Strang, "Eigenvalue and eigenvector analysis of stability for a line of traffic," *Studies in Applied Mathematics*, vol. 138, no. 1, pp. 103–132, 2017.

[23] T. A. Baran and B. K. Horn, "A robust signal-flow architecture for cooperative vehicle density control." in *ICASSP*, 2013, pp. 2790–2794.

[24] A. Nakayama, Y. Sugiyama, and K. Hasebe, "Effect of looking at the car that follows in an optimal velocity model of traffic flow," *Physical Review E*, vol. 65, no. 1, p. 016112, 2001.

[25] M. Treiber and D. Helbing, "Hamilton-like statistics in onedimensional driven dissipative many-particle systems," *The European Physical Journal B*, vol. 68, no. 4, pp. 607–618, 2009.

[26] P. Ioannou and C. Chien, "Autonomous intelligent cruise control," *IEEE Trans. on Vehicular technology*, vol. 42, no. 4, pp. 657–672, 1993.

[27] P. Ioannou and Z. Xu, "Throttle and brake control systems for automatic vehicle following," *IVHS Journal*, vol. 1, no. 4, pp. 345–377, 1994.

[28] G. Strang, *Introduction to linear algebra*. Wellesley-Cambridge Press, Wellesle, MA, 2016.

[29] G. Strang, *Computational science and engineering*.Wellesley-Cambridge Press, Wellesle, MA, 2007.

[30] B. K.P. Horn and L. Wang, *Mixed traffic flow* [Online], Oct. 2018. Available: http://people.csail.mit.edu/bkph/TrafficFiles/mixedtraffic.html