STABILIZATION OF NETWORKED CONTROL SYSTEMS UNDER DOS ATTACKS AND OUTPUT QUANTIZATION*

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Abstract. This paper addresses quantized output feedback stabilization under Denial-of-Service (DoS) attacks. First, assuming that the duration and frequency of DoS attacks are averagely bounded and that an initial bound of the plant state is known, we propose an output encoding scheme that achieves exponential convergence with finite data rates. Next we show that a suitable state transformation allows us to remove the assumption on the DoS frequency. Finally, we discuss the derivation of state bounds under DoS attacks and obtain sufficient conditions on the bounds of DoS duration and frequency for achieving Lyapunov stability of the closed-loop system.

Key words. Networked control systems, quantized control, denial-of-service attacks.

1. Introduction. Recent advances in computer and communication technology contribute to the efficiency of data transmission in control systems. However, control systems become also vulnerable to cyber attacks. For instance, it was reported that attackers can adversarially control cars [8] and unmanned aerial vehicles [22]. Malicious attacks are a major concern for the deployment of networked control systems, and enhancing the resilience to cyber attacks is an important issue.

There are many possible attacks for control systems. Recent results such as [10, 15,37] focus on the scenario where measurement data obtained from some sensors can be manipulated by malicious attackers. Another line of research [30,45] investigates control under replay attacks, which maliciously repeat transmitted data. Denial-of-Service (DoS) attacks destroy the data availability by inducing packet losses. DoS attacks are launched by malicious routers [2] and jammers [33], which can be set up without detailed knowledge on the structure of targeted systems. Hence, even attackers with little information on control systems can create a security threat by DoS attacks.

In this paper, we consider networked control systems in which the plant output is sent through a communication channel and DoS attacks are launched to block the transmission of the output data over this channel. Probabilistic models such as the Bernoulli model has been used for nonmalicious packet losses caused by network traffic congestion and packet transmission failures; see the survey papers [18, 44]. However, attackers may not launch DoS attacks based on such probabilistic models. The effect of DoS attacks has been recently investigated in several studies [1, 3, 5, 6, 9, 11–13, 16, 17, 23, 28, 29, 36]. To deal with the uncertainty of DoS, the previous studies [5, 6, 11, 12, 16, 17, 23, 29, 36] characterized DoS attacks by the average duration and frequency of packet losses.

Data transmission through digital channels requires signal quantization. Although a plenty of communication bandwidth is available in modern applications, many devices compete for this bandwidth in complex systems. Moreover, it is theoretically interesting to solve the problem of how much information is needed to achieve

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a given control objective. From this point of view, data rate limitations for stabilization have been extensively studied; see the survey papers [20, 31] for details. The so-called zooming-in and zooming-out method developed in [4] also yields a quantizer that achieves asymptotic stabilization with finite-data rates. This method was first applied to linear time-invariant systems and then was extended to a wide class of systems such as nonlinear systems [24, 27] and switched systems [26, 40].

Despite the above active research on control problems with limited information, quantized control under cyber attacks does not seem to have received much attention so far. In this paper, we extend the zooming-in and zooming-out method to achieve output feedback stabilization under DoS attacks. Our objective is to develop output encoding schemes that guarantee closed-loop stability under DoS attacks. The proposed encoding schemes generally require more than minimal data rates for stabilization but relatively modest computational resources of the coders. In contrast, data rate limitations for state feedback stabilization under DoS attacks have been recently studied in [17]. The authors of [9] have proposed a design method of eventtriggered controllers for stabilization under quantization and DoS attacks. However, static logarithmic quantizers with infinitely many quantization levels are used in [9], which would remove most of the difficulties arising from quantization in our problem formulation.

First, we assume that an initial bound of the plant state is given and design an output encoding scheme that achieves exponential convergence with finite data rates in the presence of DoS. The difficulty here is to switch an update rule of the coders depending on DoS. In the absence of DoS, the coders can decrease their quantization ranges and make quantization errors small, by using the plant model and the transmitted measurements. However, if DoS attacks are launched, then the decoder at the controller side cannot receive the measurements. As a result, the worst-case estimation error of the plant output, which is used for quantization, becomes large. Therefore, the coders should increase their quantization range so that the plant output can be captured in the quantization region. This switching of the update rule of the coders makes it difficult to analyze the stability of the closed-loop system.

We adopt a general model that constrains DoS attacks only in terms of duration and frequency, as in [5, 6, 11, 12, 16, 17, 23, 29, 36]. In particular, the assumption we make for DoS attacks is that their duration and frequency are averagely bounded. Hence we can deal with a wide class of packet losses. We first propose an encoding scheme that generally needs the assumption both on DoS duration and frequency. Next we show that the frequency condition can be removed, by applying a suitable state transformation. An invertible matrix for the state transformation is a design parameter, and we can choose it in various ways. In the section of a numerical example, this matrix is chosen so that the closed-loop system allows longer DoS duration under low DoS frequency.

Next, we develop methods to derive initial state bounds under DoS attacks. In the absence of packet losses [25], state bounds can be obtained from consecutive output data. In our setting, output data may not be received consecutively due to DoS attacks. Hence we need to construct state bounds from intermittent output data. In the case without DoS, it is easy to find state bounds from finitely many measurements. The difficulty of the case with DoS is that we may not obtain a state bound using even an infinite number of intermittent measurements. This is because there exist time-steps at which the output does not contribute to the construction of state bounds. This problem is related to basic questions on how many samples are needed to obtain state estimates. Such questions have also been addressed in the context of sampled-data control under irregular sampling; see, e.g., [21, 32, 34, 42, 43].

We provide several sufficient conditions on DoS duration and frequency for the derivation of initial state bounds under DoS attacks. In the first approach, we analyze the generalized observability matrix by exploiting a periodic property of the eigenvalues of the system matrix. Next, we design coders that construct initial state bounds only from consecutive measurements. Finally, applying the results in [21], we see that if the lengths of DoS periods are bounded, then the problem of whether or not a state bound can be constructed is decidable. All of these approaches provide initial state bounds in finite time. Consequently, the proposed encoding schemes achieve Lyapunov stability if the bounds of DoS duration and frequency are sufficiently small.

The remainder of this paper is organized as follows. The networked control system we consider and assumptions on DoS attacks are introduced in Section II. In Section III, we propose output encoding schemes that achieve exponential convergence of the state and its estimate under DoS attacks. Section IV is devoted to the derivation of initial state bounds in the presence of DoS. We present a numerical example in Section V.

The results in Section III partially appeared in our conference paper [39]. Here we provide complete proofs not included in the conference version and make significant structural improvements. Moreover, the present paper has additional results on the derivation of initial state bounds and Lyapunov stability.

Notation. The set of non-negative integers is denoted by \mathbb{Z}_+ . We denote by $\varrho(P)$ the spectral radius of $P \in \mathbb{C}^{n \times n}$. Let us denote by A^* the complex conjugate transpose of $A \in \mathbb{C}^{m \times n}$. For a vector $v \in \mathbb{C}^n$ with ℓ th element v_ℓ , its maximum norm is $|v|_{\infty} := \max\{|v_1|, \ldots, |v_n|\}$, and the corresponding induced norm of $A \in \mathbb{C}^{m \times n}$ with (ℓ, j) th element $A_{\ell j}$ is given by $||A||_{\infty} = \max\{\sum_{j=1}^n |A_{\ell j}| : 1 \leq \ell \leq m\}$. We denote by diag $(\Lambda_1, \ldots, \Lambda_n)$ a block diagonal matrix with diagonal blocks $\Lambda_1, \ldots, \Lambda_n$. For a full column rank matrix $A \in \mathbb{C}^{m \times n}$, its left inverse is denoted by $A^{\dagger} = (A^*A)^{-1}A^*$. A square matrix in $\mathbb{C}^{n \times n}$ is said to be *Schur stable* if all its eigenvalues lie in the unit disc.

2. Networked control system and DoS attack. In this section, the networked control system we consider and assumptions on DoS attacks are introduced.

2.1. Networked control system. Consider the following discrete-time linear time-invariant system:

$$(1a) x_{k+1} = Ax_k + Bu_k$$

(1b)
$$y_k = C x_k$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$, and $y_k \in \mathbb{R}^{n_y}$ are the state, the input, and the output of the plant, respectively. The output y_k is encoded and then transmitted through a communication channel subject to DoS. In contrast, we assume that the input u_k is not affected by any network phenomena, i.e., u_k goes through the ideal channel.

The decoder sends an acknowledgment to the plant side without delays when it receives the output data. If the encoder does not receive the acknowledgment, then it can detect the DoS attack. The acknowledgment-based protocol was used in the previous study [16] on control without quantization under DoS attacks and also has been commonly employed in networked control under nonmalicious packet losses; see, e.g., [19, 38]. Fig. 1 illustrates the networked control system we study.

The system matrix A is assumed not to be Schur stable. This is because if A is Schur stable, then the zero control input $u_k = 0$ ($k \in \mathbb{Z}_+$) achieves the closed-loop



Fig. 1: Networked control system under DoS attacks.

stability for arbitrary DoS attacks, and hence the stabilization problem we consider would be trivial.

2.2. DoS attack. Let us denote by $\Phi_d(k)$ the number of time-steps when DoS attacks are launched on the interval [0, k). As in [5, 6, 11, 12, 16, 17, 23, 29, 36], we assume that the duration of DoS attacks grows linearly with the length of the interval.

ASSUMPTION 2.1 (Duration of DoS attacks). There exist $\Pi_d \ge 0$ and $\nu_d \in [0, 1]$ such that for every $k \in \mathbb{Z}_+$, the DoS duration $\Phi_d(k)$ satisfies

(2)
$$\Phi_d(k) \le \Pi_d + \nu_d k.$$

We call ν_d the DoS duration bound.

The condition (2) implies that at most $\Pi_d + \nu_d k$ packets are affected by DoS attacks on the interval [0, k). The DoS duration bound ν_d is an upper bound of the limit superior of the DoS duration per time-step.

Next, let us denote by $\Phi_f(k)$ the number of consecutive DoS attacks on the interval [0, k).

ASSUMPTION 2.2 (Frequency of DoS attacks). There exist $\Pi_f \geq 0$ and $\nu_f \in [0, 0.5]$ such that for every $k \in \mathbb{Z}_+$, the DoS frequency $\Phi_f(k)$ satisfies

(3)
$$\Phi_f(k) \le \Pi_f + \nu_f k.$$

We call ν_f the DoS frequency bound.

The DoS frequency bound ν_f is an upper bound of the limit superior of DoS occurrences per time-step. High-frequency DoS attacks satisfy (3) with large values of ν_f .

REMARK 2.3. The authors of [11, 12, 16, 17, 29, 36] placed stronger conditions than (2) and (3) such as

(4)
$$\Phi_d(k,k+\tau) \le \Pi_d + \nu_d \tau \qquad \forall k,\tau \in \mathbb{Z}_+,$$

where $\Phi_d(k, k + \tau)$ is the number of time-steps when DoS attacks are launched on the interval $[k, k + \tau)$. The major reason to place such stronger conditions is that systems

with disturbances and noise were considered. Although we also consider networked control systems with quantization noise, quantization noise decreases under a certain condition on DoS attacks. This is the reason why we use the weaker conditions (2) and (3).

3. Exponential convergence under DoS. In this section, we present an encoding and decoding scheme to achieve the exponential convergence of the state under the assumption that an initial state bound is known. The proposed schemes are extensions of the zooming-in method developed in [25] to the case under DoS attacks.

We impose the following assumptions throughout this section:

ASSUMPTION 3.1 (Stabilizability and detectability). The pairs (A, B) and (C, A) are stabilizable and detectable, respectively. Matrices $K \in \mathbb{R}^{n_u \times n_x}$ and $L \in \mathbb{R}^{n_x \times n_y}$ are chosen so that A - BK and A - LC are Schur stable.

ASSUMPTION 3.2 (Initial state bound). A constant $E_0 > 0$ satisfying $|x_0|_{\infty} \leq E_0$ is known.

An initial bound E_0 in Assumption 3.2 may be given in advance or may be obtained from prior measurements via the zooming-out method; see Section IV for the derivation of initial state bounds.

3.1. Observer-based controller. To achieve the exponential convergence of the state, we use a controller that consists of a Luenberger observer and a feedback gain. Observer-based controllers update the estimate of the plant state, by using the output data. However, when an attack occurs, the controller cannot receive the output data. Hence, if DoS occurs, then the controller updates the estimate in the open-loop form. Define

(5)
$$L_k := \begin{cases} 0 & \text{if DoS occurs at } k \\ L & \text{if DoS does not occur at } k \end{cases}$$

The dynamics of the controller is given by

(6a)
$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L_k(q_k - \hat{y}_k)$$

(6b)
$$u_k = -K\hat{x}_k$$

(6c) $\hat{y}_k = C\hat{x}_k,$

where $\hat{x}_k \in \mathbb{R}^{n_x}$, $\hat{y} \in \mathbb{R}^{n_y}$, and $q_k \in \mathbb{R}^{n_y}$ are the state estimate, the output estimate, and the quantized value of y_k , respectively. We will provide the details of how to generate the quantized output q_k in the next subsection. We set an initial state estimate \hat{x}_0 to be $\hat{x}_0 = 0$.

3.2. Basic encoding and decoding scheme. Define the error $e_k \in \mathbb{R}^{n_x}$ of the state estimation by $e_k := x_k - \hat{x}_k$. Using an invertible matrix $R \in \mathbb{C}^{n_x \times n_x}$, we also define the transformed error $e_{R,k} \in \mathbb{C}^{n_x}$ by $e_{R,k} := Re_k$. The invertible matrix R is a design parameter, and we fix the matrix R arbitrarily in this and next subsections. Section 3.5 includes the discussion on how to choose the matrix R. In particular, we show there that if we choose the matrix R that transforms A - LC into its Jordan canonical form, then the assumption on the DoS frequency can be removed. For this reason, the matrix R is complex-valued.

Let $E_{R,k} \ge 0$ satisfy

$$(7) |e_{R,k}|_{\infty} \le E_{R,k}$$

The estimation error of the output is given by

$$y_k - \hat{y}_k = Ce_k = CR^{-1}e_{R,k} \qquad \forall k \ge \mathbb{Z}_+.$$

If the error bound $E_{R,k}$ satisfies (7), then

$$|y_k - \hat{y}_k|_{\infty} \le ||CR^{-1}||_{\infty} E_{R,k}.$$

We partition the hypercube

(8)
$$\left\{ y \in \mathbb{R}^{n_y} : |y - \hat{y}_k|_{\infty} \le \|CR^{-1}\|_{\infty} E_{R,k} \right\}$$

into N^{n_y} equal boxes. An index in $\{1, \ldots, N^{n_y}\}$ is assigned to each partitioned box by a certain one-to-one mapping for all $k \in \mathbb{Z}_+$. The encoder sends to the decoder the index q_k^{ind} of the partitioned box containing y_k . Then the decoder generates q_k equal to the center of the box having the index q_k^{ind} . If y_k lies on the boundary of several boxes, then we can choose any one of them. The quantization error $|y_k - q_k|_{\infty}$ of this encoding scheme satisfies

(9)
$$|y_k - q_k|_{\infty} \le \frac{\|CR^{-1}\|_{\infty}}{N} E_{R,k}.$$

In the next subsection, we will design a sequence $\{E_{R,k} : k \in \mathbb{Z}_+\}$ of error bounds that achieves (7) for every $k \in \mathbb{Z}_+$ and exponentially decreases to zero.

3.3. Main result on exponential convergence. Before stating the main result, we first introduce the notion of exponential convergence.

DEFINITION 3.3 (Exponential convergence). The feedback system with the plant (1) and the controller (6) achieves exponential convergence under Assumption 3.2 if there exist $\Omega \geq 1$ and $\gamma \in (0, 1)$, independent of E_0 , such that

$$|x_k|_{\infty}, \ |\hat{x}_k|_{\infty} \le \Omega E_0 \gamma^k \qquad \forall k \in \mathbb{Z}_+$$

for every initial state $x_0 \in \mathbb{R}^{n_x}$ satisfying $|x_0|_{\infty} \leq E_0$.

Let us introduce an update rule of $\{E_{R,k} : k \in \mathbb{Z}_+\}$ we study here. Fix an invertible matrix $R \in \mathbb{C}^{n_x \times n_x}$, and choose $M_0 \ge 1$, $M \ge ||RL||_{\infty}$, and $\rho \in (0, 1)$ satisfying

(10a)
$$||R(A - LC)^{\ell}R^{-1}||_{\infty} \le M_0 \rho^{\ell} \qquad \forall \ell \ge 0$$

(10b)
$$||R(A - LC)^{\ell}L||_{\infty} \le M\rho^{\ell} \qquad \forall \ell \ge 0.$$

Define constants $\theta_a, \theta_0, \theta > 0$ by

(11a)
$$\theta_a := \|RAR^{-1}\|_{\infty}$$

(11b)
$$\theta_0 := M_0 \rho + \frac{M \|CR^{-1}\|_{\infty}}{N}$$

(11c)
$$\theta := \rho + \frac{M \|CR^{-1}\|_{\infty}}{N}.$$

Using these constants, we set the error bound $\{E_{R,k} : k \in \mathbb{Z}_+\}$ to be

(12)
$$E_{R,k+1} := \begin{cases} \theta_a E_{R,k} & \text{if DoS occurs at } k \\ \theta_0 E_{R,k} & \text{else if } k = 0 \text{ or DoS occurs at } k - 1 \\ \theta E_{R,k} & \text{otherwise} \end{cases}$$

for all $k \in \mathbb{Z}_+$. In terms of the initial value $E_{R,0}$, we have from Assumption 3.2 that

$$|e_{R,0}|_{\infty} = |Rx_0|_{\infty} \le ||R||_{\infty} E_0 =: E_{R,0},$$

where we used $\hat{x}_0 = 0$.

The following theorem shows that the encoding scheme with the above error bound $\{E_{R,k} : k \in \mathbb{Z}_+\}$ achieves exponential convergence.

THEOREM 3.4. Suppose that Assumptions 2.1, 2.2, 3.1, and 3.2 hold. If the number of quantization levels N and the DoS duration and frequency bounds ν_d and ν_f satisfy

(13a)
$$N > \frac{M \|CR^{-1}\|_{\infty}}{1-\rho}$$

(13b)
$$\nu_d < \frac{\log(1/\theta)}{\log(\theta_a/\theta)} - \frac{\log(\theta_0/\theta)}{\log(\theta_a/\theta)}\nu_f$$

then the feedback system achieves exponential convergence under the encoding scheme with the error bound $\{E_{R,k} : k \in \mathbb{Z}_+\}$ constructed by the update rule (12).

The proof of this theorem is provided in the next subsection.

3.4. Proof of Theorem 3.4. We begin by showing that (7) holds in the absence of DoS attacks. To this end, we use the technique developed in [41].

The following lemma provides a useful representation of $\{E_{R,k} : k \in \mathbb{Z}_+\}$ in the case without DoS.

LEMMA 3.5. For $\ell \in \mathbb{Z}_+$, set

(14)
$$E_{R,k+\ell+1} := \begin{cases} \theta_0 E_{R,k} & \text{if } \ell = 0\\ \theta E_{R,k+\ell} & \text{otherwise} \end{cases}$$

where θ_0 and θ are defined by (11b) and (11c). Then

(15)
$$E_{R,k+\ell} = M_0 \rho^{\ell} E_{R,k} + \frac{M \|CR^{-1}\|_{\infty}}{N} \sum_{j=0}^{\ell-1} \rho^{\ell-j-1} E_{R,k+j}$$

for every $\ell \in \mathbb{N}$.

Proof. If $\ell = 1$, then (15) holds by the definition of θ_0 . The general case follows by induction. Define $\Delta := M \|CR^{-1}\|_{\infty}/N$. If (15) holds with $\ell = \ell_0 \in \mathbb{N}$, then

$$E_{R,k+\ell_0+1} = \theta E_{R,k+\ell_0} = \rho E_{R,k+\ell_0} + \Delta E_{R,k+\ell_0}$$
$$= M_0 \rho^{\ell_0+1} E_{R,k} + \Delta \sum_{j=0}^{\ell_0} \rho^{\ell_0-j} E_{R,k+j}.$$

Thus, we obtain (15) holds with $\ell = \ell_0 + 1$.

Using the representation of $E_{R,k+\ell}$ in (15), we show that (7) is satisfied in the case without DoS attacks.

LEMMA 3.6. Consider the feedback system in the absence of DoS, that is, $L_{k+\ell} = L$ in (6) for every $\ell \in \mathbb{Z}_+$. Assume that $|e_{R,k}|_{\infty} \leq E_{R,k}$, and set $\{E_{R,k+\ell} : \ell \in \mathbb{Z}_+\}$ as in Lemma 3.5. Then $|e_{R,k+\ell}|_{\infty} \leq E_{R,k+\ell}$ for all $\ell \in \mathbb{Z}_+$.

Proof. We see from (1) and (6) that the state estimation error e_k satisfies

$$e_{k+1} = (A - LC)e_k + L(y_k - q_k).$$

Since $e_{R,k} = Re_k$, it follows that

(16)
$$e_{R,k+1} = R(A - LC)R^{-1}e_{R,k} + RL(y_k - q_k).$$

Applying induction to (16), we obtain

$$e_{R,k+\ell} = R(A - LC)^{\ell} R^{-1} e_{R,k} + \sum_{j=0}^{\ell-1} R(A - LC)^{\ell-j-1} L(y_{k+j} - q_{k+j})$$

for every $\ell \in \mathbb{N}$. It follows from (9) that

$$|e_{R,k+\ell}|_{\infty} \leq ||R(A - LC)^{\ell}R^{-1}||_{\infty}E_{R,k} + \sum_{j=0}^{\ell-1} ||R(A - LC)^{\ell-j-1}L||_{\infty} \frac{||CR^{-1}||_{\infty}}{N}E_{R,k+j}$$

for every $\ell \in \mathbb{N}$. Using the norm condition (10), we further have

(17)
$$|e_{R,k+\ell}|_{\infty} \le M_0 \rho^{\ell} E_{R,k} + \frac{M ||CR^{-1}||_{\infty}}{N} \sum_{j=0}^{\ell-1} \rho^{\ell-j-1} E_{R,k+j}$$

for every $\ell \in \mathbb{N}$. By Lemma 3.5, we obtain $|e_{R,k+\ell}|_{\infty} \leq E_{R,k+\ell}$ for all $\ell \in \mathbb{Z}_+$.

Next we investigate the error bound in the presence of DoS attacks.

LEMMA 3.7. Consider the closed-loop system in the presence of DoS, that is, $L_{k+\ell} = 0$ in (6) for every $\ell \in \mathbb{Z}_+$. Assume that $|e_{R,k}|_{\infty} \leq E_{R,k}$, and set

(18)
$$E_{R,k+\ell+1} := \theta_a E_{R,k+\ell} \qquad \forall \ell \in \mathbb{Z}_+,$$

where θ_a is defined as in (11a). Then $|e_{R,k+\ell}|_{\infty} \leq E_{R,k+\ell}$ for all $\ell \in \mathbb{Z}_+$.

Proof. The estimation error e_k satisfies $e_{k+1} = Ae_k$, and hence

(19)
$$e_{R,k+1} = RAR^{-1}e_{R,k}.$$

This yields

$$|e_{R,k+1}|_{\infty} \le ||RAR^{-1}||_{\infty} \cdot |e_{R,k}|_{\infty} \le ||RAR^{-1}||_{\infty} E_{R,k}.$$

By induction, we obtain $|e_{R,k+\ell}|_{\infty} \leq E_{R,k+\ell}$ for every $\ell \in \mathbb{Z}_+$.

We immediately obtain the following result from Lemmas 3.6 and 3.7:

LEMMA 3.8. For the transformed estimation error $e_{R,k}$, the error bound $\{E_{R,k} : k \in \mathbb{Z}_+\}$ defined by (12) satisfies

(20)
$$|e_{R,k}|_{\infty} \leq E_{R,k} \quad \forall k \in \mathbb{Z}_+.$$

Next, we show that the error bound $\{E_{R,k} : k \in \mathbb{Z}_+\}$ in (12) converges to zero under the condition (13).

LEMMA 3.9. Under the same hypotheses of Theorem 3.4, there exist $\Omega \geq 1$ and $\gamma \in (0,1)$ such that

(21)
$$E_{R,k} \leq \Omega E_{R,0} \gamma^k \quad \forall k \in \mathbb{Z}_+.$$

Proof. Choose $k_e \in \mathbb{N}$ arbitrarily, and assume that DoS attacks are launched at

$$k = k_m, \dots, k_m + \tau_m - 1 \qquad \forall m = 1, \dots, p$$

on the interval $[0, k_e)$, where $k_m \in \mathbb{Z}_+$, $\tau_m \in \mathbb{N}$ for every $m = 1, \ldots, p$ and

$$k_m + \tau_m < k_{m+1} \qquad \forall m = 1, \dots, p-1.$$

Namely, k_m and τ_m are the beginning time and the length of the *m*th DoS interval. Here $\sum_{m=1}^{p} \tau_m$ is the total duration of DoS attacks on $[0, k_e)$, and p is the total number of consecutive DoS attacks on $[0, k_e)$. Therefore, $\sum_{m=1}^{p} \tau_m = \Phi_d(k_e)$ and $p = \Phi_f(k_e)$.

In what follows, we assume that $k_1 > 0$ and $k_p + \tau_p < k_e$ for simplicity. In the case where $k_1 = 0$ or $k_p + \tau_p = k_e$, one can prove the convergence of the error bound (21) in a similar way.

Define

$$r_1 := k_1 - 1, \quad r_{p+1} := k_e - k_p - \tau_p - 1$$

$$r_m := k_m - k_{m-1} - \tau_{m-1} - 1 \qquad \forall m = 2, \dots, p.$$

Then $r_m \ge 0$ for every $m = 1, \ldots, p + 1$. Since DoS attacks are not launched on the interval $[0, k_1)$, it follows that

$$E_{R,k_1} = \theta^{r_1} \theta_0 E_{R,0}.$$

On the other hand, DoS occurs on the interval $[k_1, \ldots, k_1 + \tau_1)$, and hence

$$E_{R,k_1+\tau_1} = \theta_a^{\tau_1} E_{R,k_1} = \theta_a^{\tau_1} \theta^{r_1} \theta_0 E_{R,0}.$$

Continuing in this way, we see that the error bound E_{R,k_e} at the time $k = k_e$ satisfies

$$E_{R,k_e} = \theta^{r_{p+1}} \theta_0 E_{R,k_p+\tau_p} = \theta^{\sum_{m=1}^{p+1} r_m} \cdot \theta_0^{p+1} \cdot \theta_a^{\sum_{m=1}^{p} \tau_m} E_{R,0}.$$

By definition,

(22)

(23)
$$\sum_{m=1}^{p+1} r_m = k_e - (p+1) - \sum_{m=1}^{p} \tau_m.$$

Moreover, it follows from Assumptions 2.1 and 2.2 that

(24)
$$\sum_{m=1}^{p} \tau_m \le \Pi_d + \nu_d k_e, \quad p \le \Pi_f + \nu_f k_e.$$

Substituting (23) and (24) into (22), we obtain

$$E_{R,k_e} = \theta^{k_e} \cdot \left(\frac{\theta_0}{\theta}\right)^{p+1} \cdot \left(\frac{\theta_a}{\theta}\right)^{\sum_{m=1}^{p} \tau_m} E_{R,0}$$
$$\leq \frac{\theta_0^{\Pi_f + 1} \cdot \theta_a^{\Pi_d}}{\theta^{\Pi_f + \Pi_d + 1}} \left(\theta \cdot \left(\frac{\theta_0}{\theta}\right)^{\nu_f} \cdot \left(\frac{\theta_a}{\theta}\right)^{\nu_d}\right)^{k_e} E_{R,0}$$

Since the inequality (13b) is equivalent to

$$\theta \cdot \left(\frac{\theta_0}{\theta}\right)^{\nu_f} \cdot \left(\frac{\theta_a}{\theta}\right)^{\nu_d} < 1,$$

the exponential convergence of the error bound (21) is established.

We are now in a position to prove Theorem 3.4.

Proof of Theorem 3.4. The state x_k satisfies

$$x_{k+1} = (A - BK)^{k+1} x_0 + \sum_{\ell=0}^{k} (A - BK)^{k-\ell} BKR^{-1} e_{R,\ell}.$$

Therefore,

(25)

$$|x_{k+1}|_{\infty} \le \|(A - BK)^{k+1}\|_{\infty} \cdot |x_0|_{\infty} + \sum_{\ell=0}^{k} \|(A - BK)^{k-\ell}\|_{\infty} \cdot \|BKR^{-1}\|_{\infty} \cdot |e_{R,\ell}|_{\infty}.$$

By Lemmas 3.8 and 3.9, there exist $\Omega \geq 1$ and $\gamma \in (0, 1)$ such that

(26)
$$|e_{R,\ell}|_{\infty} \leq \Omega E_{R,0} \gamma^{\ell} \quad \forall \ell \in \mathbb{Z}_+.$$

Moreover, since A - BK is Schur stable by Assumption 3.1, there exist $\Omega_K \ge 1$ and $\tilde{\gamma} \in [\gamma, 1)$ such that

(27)
$$\|(A - BK)^{\ell}\|_{\infty} \le \Omega_K \tilde{\gamma}^{\ell} \qquad \forall \ell \in \mathbb{Z}_+$$

Substituting (26) and (27) into (25), we obtain

(28)
$$|x_{k+1}|_{\infty} \leq \Omega_K \tilde{\gamma}^{k+1} |x_0|_{\infty} + \Omega \Omega_K E_{R,0} ||BKR^{-1}||_{\infty} (k+1) \tilde{\gamma}^k.$$

For every $\varepsilon > 0$, there exists a constant $\alpha \ge 1$ such that $k\tilde{\gamma}^k \le \alpha(\tilde{\gamma} + \varepsilon)^k$ for all $k \in \mathbb{Z}_+$. Thus (28) leads to the exponential convergence of the state. Additionally, since $\hat{x}_k = x_k - R^{-1}e_{R,k}$, it follows that \hat{x}_k also exponentially converges to zero. This completes the proof.

REMARK 3.10. In the previous studies [11, 12, 16, 17, 29, 36], the assumption on the frequency of DoS attacks is used in a different way. The above studies consider continuous-time attacks, and hence the frequency at which DoS attacks are launched must be smaller than the sampling rate. Therefore, in the discrete-time case [5], the assumption on the DoS frequency bound is not used. However, the output encoding scheme in Theorem 3.4 increases the error bound, $E_{R,k+1} = \theta_0 E_{R,k}$, at the first time-step after DoS attacks occur. For this reason, we here employ the frequency assumption to obtain a less conservative sufficient condition.

3.5. Choice of invertible matrix **R**. In this subsection, we provide a guideline for choosing the invertible matrix $R \in \mathbb{C}^{n_x \times n_x}$. We show that if the matrix R is chosen appropriately, then the encoding scheme in Theorem 3.4 does not need the assumption of the DoS frequency. To this end, we first provide a basic fact of the maximum norm.

PROPOSITION 3.11. For a matrix $\Xi \in \mathbb{C}^{n \times n}$ and a scalar $\varepsilon > 0$, take an invertible matrix $R \in \mathbb{C}^{n \times n}$ satisfying

(29)
$$\frac{1}{\varepsilon}R\Xi R^{-1} = J,$$

where J is the Jordan canonical form of Ξ/ε . Then the matrix R satisfies

(30)
$$||R\Xi R^{-1}||_{\infty} \le \varrho(\Xi) + \varepsilon.$$

Proof. Let us denote the eigenvalues of Ξ by $\lambda_1, \ldots, \lambda_n$ (including multiplicity). Then the diagonal part of the Jordan canonical form J consists of $\lambda_1/\varepsilon, \ldots, \lambda_n/\varepsilon$. Therefore,

(31)
$$\|\varepsilon J\|_{\infty} \leq \max_{j=1,\dots,n} |\lambda_j| + \varepsilon = \varrho(\Xi) + \varepsilon.$$

By (29) and (31), we obtain the desired conclusion.

In particular, if the matrix Ξ is Schur stable in Proposition 3.11, then we obtain the following result by choosing a sufficiently small $\varepsilon > 0$.

COROLLARY 3.12. For every Schur stable matrix $\Xi \in \mathbb{C}^{n \times n}$, there exists an invertible matrix $R \in \mathbb{C}^{n \times n}$ such that $\|R\Xi R^{-1}\|_{\infty} < 1$.

Let an invertible matrix $R \in \mathbb{C}^{n_x \times n_x}$ satisfy

(32)
$$||R(A - LC)R^{-1}||_{\infty} < 1.$$

Corollary 3.12 shows that such a matrix R always exists under Assumption 3.1. We set the error bound $\{E_{R,k} : k \in \mathbb{Z}_+\}$ to be

(33)
$$E_{R,k+1} := \begin{cases} \vartheta_a E_{R,k} & \text{if DoS occurs at } k \\ \vartheta E_{R,k} & \text{otherwise,} \end{cases}$$

where

(34a)
$$\vartheta_a := \|RAR^{-1}\|_{\infty}$$

(34b)
$$\vartheta := \|R(A - LC)R^{-1}\|_{\infty} + \frac{\|RL\|_{\infty} \cdot \|CR^{-1}\|_{\infty}}{N}$$

The following result, which is a corollary of Theorem 3.4, shows that the encoding scheme with the error bound $\{E_{R,k} : k \in \mathbb{Z}_+\}$ updated by (33) achieves exponential convergence without any DoS frequency assumptions.

COROLLARY 3.13. Suppose that Assumptions 2.1, 3.1, and 3.2 hold. Assume that an invertible matrix $R \in \mathbb{C}^{n_x \times n_x}$ satisfies (32). If the number of quantization levels N and the DoS duration bound ν_d satisfy

(35a)
$$N > \frac{\|RL\|_{\infty} \cdot \|CR^{-1}\|_{\infty}}{1 - \|R(A - LC)R^{-1}\|_{\infty}}$$

(35b)
$$\nu_d < \frac{\log(1/\vartheta)}{\log(\vartheta_a/\vartheta)},$$

then the feedback system achieves exponential convergence under the encoding scheme with the error bound $\{E_{R,k} : k \in \mathbb{Z}_+\}$ constructed by the update rule (33).

Proof. We can set the constants ρ, M_0, M in (10) to be

$$\rho = \|R(A - LC)R^{-1}\|_{\infty}, \quad M_0 = 1, \quad M = \|RL\|_{\infty}.$$

Then θ_0 and θ defined in (11b), (11c) are equal to ϑ in (34b). By definition, $\theta_a = \vartheta_a$. Since $\log(\theta_0/\theta) = 0$, the conditions (35) on N and ν_d are the same as the conditions (13). Thus, the desired result follows from Theorem 3.4.

REMARK 3.14. If the pair (C, A) is observable, then there exists a deadbeat gain $L \in \mathbb{R}^{n_x \times n_y}$ such that $\varrho(A - LC) = 0$. Proposition 3.11 shows that for every $\varepsilon > 0$, there exists an invertible matrix $R \in \mathbb{C}^{n_x \times n_x}$ such that

$$||R(A - LC)R^{-1}||_{\infty} < \varepsilon.$$

In this case, since ϑ in (34b) satisfies $\lim_{N\to\infty} \vartheta \leq \varepsilon$, Corollary 3.13 shows that for every DoS duration bound $\nu_d \in [0,1)$, there exists $N \in \mathbb{N}$ such that exponential convergence is achieved, which is consistent with Theorem 2 of [16].

REMARK 3.15. In Corollary 3.13, we choose the matrix R so that the growth rate θ_0 in (11b) of $E_{R,k}$ is less than one. Another choice of the matrix R is to reduce the other growth rate θ_a in (11a). In fact, Proposition 3.11 shows that for every $\varepsilon > 0$, we can obtain an invertible matrix $R \in \mathbb{C}^{n_x \times n_x}$ satisfying

$$\varrho(A) \le \theta_a = \|RAR^{-1}\|_{\infty} \le \varrho(A) + \varepsilon$$

As seen in the numerical example of Section V, if the DoS frequency bound ν_f is sufficiently small, then the latter choice can lead to the update rule (12) that allows longer DoS duration than the update rule (33).

REMARK 3.16. The encoding scheme of Theorem 3.4 has a freedom in the choice of the matrix R, whereas the matrix R in Corollary 3.13 needs to satisfy (32) but allows stability analysis without any assumption on the DoS frequency. In general, we cannot say which encoding scheme is better with respect to the quantization level N and the DoS duration bound ν_d . Moreover, it is difficult to design the matrix Rsatisfying given conditions on the quantization level N and the DoS duration bound ν_d without employing metaheuristics such as genetic algorithms. We leave this issue for future investigation.

3.6. Encoding scheme with center at origin. In Sections 3.2–3.5, we have considered the encoding scheme that uses the output estimate as the quantization center. Here we propose encoding schemes with center at the origin. In such an encoding scheme, the encoder does not need to compute the output estimate. Therefore, we can encode the output with less computational resources.

Define

$$z_k := \begin{bmatrix} x_k \\ e_k \end{bmatrix} \qquad C_{\rm cl} := \begin{bmatrix} C & 0 \end{bmatrix}.$$

Using an invertible matrix $R_{\rm cl} \in \mathbb{C}^{2n_x \times 2n_x}$, we define the transformed closed-loop state $z_{R,k} \in \mathbb{C}^{2n_x}$ by $z_{R,k} := R_{\rm cl} z_k$. Let $E_{R,k}^z \ge 0$ satisfy

$$(36) |z_{R,k}|_{\infty} \le E_{R,k}^z$$

Then $|y_k|_{\infty} = |C_{cl}R_{cl}^{-1}z_{R,k}|_{\infty} \le ||C_{cl}R_{cl}^{-1}||_{\infty}E_{R,k}^z$. The only difference from the encoding and decoding scheme in Section III-B is that we here employ the hypercube with center at the origin

$$\{y \in \mathbb{R}^{n_y} : |y|_{\infty} \le ||C_{\rm cl}R_{\rm cl}^{-1}||_{\infty}E_{R,k}^z\},\$$

and partition it into N^{n_y} equal boxes, instead of the hypercube with center at the output estimate (8). The quantization error $|y_k - q_k|_{\infty}$ of this encoding scheme satisfies

$$|y_k - q_k|_{\infty} \le \frac{\|C_{\rm cl} R_{\rm cl}^{-1}\|_{\infty}}{N} E_{R,k}^z.$$

To achieve the exponential convergence of the closed-loop state z_k , we aim at designing a sequence $\{E_{R,k}^z : k \in \mathbb{Z}_+\}$ of state bounds that satisfies (36) for every $k \in \mathbb{Z}_+$ and exponentially decreases to zero. We start with the dynamics of the transformed closed-loop state $z_{R,k}$. Define

$$A_{\rm cl} := \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}, \quad A_{\rm op} := \begin{bmatrix} A - BK & BK \\ 0 & A \end{bmatrix}, \quad L_{\rm cl} := \begin{bmatrix} 0 \\ L \end{bmatrix}.$$

If DoS does not occur at time k, then

$$z_{R,k+1} = R_{cl}A_{cl}R_{cl}^{-1}z_{R,k} + R_{cl}L_{cl}(y_k - q_k);$$

otherwise

$$z_{R,k+1} = R_{\rm cl}A_{\rm op}R_{\rm cl}^{-1}z_{R,k}$$

The dynamics of the closed-loop state $z_{R,k}$ has the same structure as that of the error $e_{R,k}$ in (16) and (19). Therefore, we apply the discussion in Sections 3.3–3.5 to the sequence $\{E_{R,k}^z : k \in \mathbb{Z}_+\}$ of state bounds with minor modifications.

First, we introduce the counterpart of the encoding scheme in Theorem 3.4. Choose $M_0^z \ge 1$, $M^z \ge ||R_{\rm cl}L_{\rm cl}||_{\infty}$, and $\rho_{\rm cl} \in (0,1)$ satisfying

(37a)
$$\|R_{\rm cl}A^{\ell}_{\rm cl}R^{-1}_{\rm cl}\|_{\infty} \le M_0^z \rho_{\rm cl}^{\ell} \qquad \forall \ell \ge 0$$

(37b)
$$\|R_{\rm cl}A^{\ell}_{\rm cl}L_{\rm cl}\|_{\infty} \le M^z \rho^{\ell}_{\rm cl} \qquad \forall \ell \ge 0.$$

Define constants $\phi_a, \phi_0, \phi > 0$ by

$$\begin{split} \phi_{a} &:= \|R_{\rm cl}A_{\rm op}R_{\rm cl}^{-1}\|_{\infty} \\ \phi_{0} &:= M_{0}^{z}\rho_{\rm cl} + \frac{M^{z}\|C_{\rm cl}R_{\rm cl}^{-1}\|_{\infty}}{N} \\ \phi &:= \rho_{\rm cl} + \frac{M^{z}\|C_{\rm cl}R_{\rm cl}^{-1}\|_{\infty}}{N}. \end{split}$$

Using these constants, we set the sequence $\{E_{R,k}^z : k \in \mathbb{Z}_+\}$ of state bounds to be

(38)
$$E_{R,k+1}^{z} := \begin{cases} \phi_{a} E_{R,k}^{z} & \text{if DoS occurs at } k \\ \phi_{0} E_{R,k}^{z} & \text{else if } k = 0 \text{ or DoS occurs at } k - 1 \\ \phi E_{R,k}^{z} & \text{otherwise} \end{cases}$$

for all $k \in \mathbb{Z}_+$. Since $\hat{x}_0 = 0$, it follows that $|z_0|_{\infty} \leq E_0$ under Assumption 3.2. Therefore,

$$|z_{R,0}|_{\infty} = |R_{\rm cl} z_0|_{\infty} \le ||R_{\rm cl}||_{\infty} E_0 =: E_{R,0}^z.$$

THEOREM 3.17. Suppose that Assumptions 2.1, 2.2, 3.1, and 3.2 hold. If the number of quantization levels N and the DoS duration and frequency bounds ν_d and ν_f satisfy

(39a)
$$N > \frac{M^z \|C_{cl} R_{cl}^{-1}\|_{\infty}}{1 - \rho_{cl}}$$

(39b)
$$\nu_d < \frac{\log(1/\phi)}{\log(\phi_a/\phi)} - \frac{\log(\phi_0/\phi)}{\log(\phi_a/\phi)}\nu_f,$$

then the feedback system achieves exponential convergence under the encoding scheme with the state bound $\{E_{R,k}^z : k \in \mathbb{Z}_+\}$ constructed by the update rule (38).

Proof. As in Lemmas 3.8 and 3.9, one can show that (36) holds for every $k \in \mathbb{Z}_+$ and that there exist $\Omega \geq 1$ and $\gamma \in (0, 1)$ such that

$$E_{R,k}^z \le \Omega E_{R,0}^z \gamma^k \qquad \forall k \in \mathbb{Z}_+.$$

By the definition of $z_{R,k}$, these facts yield the exponential convergence of the closed-loop system.

The decay rate ρ in (10) depends only on A - LC and satisfies $\rho \geq \rho(A - LC)$. In contrast, the counterpart ρ_{cl} in (37) depends on A - BK as well as A - LC, and $\rho_{cl} \geq \max\{\rho(A - BK), \rho(A - LC)\}$ holds. Therefore, when $\rho(A - LC)$ is small but $\rho(A - BK)$ is large, the encoding scheme with center at the origin decreases quantization errors slowly.

Next we present the counterpart of the encoding scheme in Corollary 3.13. We set the state bound $\{E_{R,k}^z : k \in \mathbb{Z}_+\}$ to be

(40)
$$E_{R,k+1}^{z} := \begin{cases} \varphi_{a} E_{R,k}^{z} & \text{if DoS occurs at } k \\ \varphi E_{R,k}^{z} & \text{otherwise,} \end{cases}$$

where

$$\varphi_a := \|R_{\rm cl}A_{\rm op}R_{\rm cl}^{-1}\|_{\infty}$$
$$\varphi := \|R_{\rm cl}A_{\rm cl}R_{\rm cl}^{-1}\|_{\infty} + \frac{\|R_{\rm cl}L_{\rm cl}\|_{\infty} \cdot \|C_{\rm cl}R_{\rm cl}^{-1}\|_{\infty}}{N}.$$

We obtain the following corollary of Theorem 3.17 from the same argument as in Corollary 3.13.

COROLLARY 3.18. Suppose that Assumptions 2.1, 3.1, and 3.2 hold. Assume that an invertible matrix $R \in \mathbb{C}^{2n_x \times 2n_x}$ satisfies

$$\|R_{cl}A_{cl}R_{cl}^{-1}\|_{\infty} < 1$$

If the number of quantization levels N and the DoS duration bound ν_d satisfy

(41a)
$$N > \frac{\|R_{cl}L_{cl}\|_{\infty} \cdot \|C_{cl}R_{cl}^{-1}\|_{\infty}}{1 - \|R_{cl}A_{cl}R_{cl}^{-1}\|_{\infty}}$$

(41b)
$$\nu_d < \frac{\log(1/\varphi)}{\log(\varphi_a/\varphi)},$$

then the feedback system achieves exponential convergence under the encoding scheme with the state bound $\{E_{R,k}^z : k \in \mathbb{Z}_+\}$ constructed by the update rule (40).

4. Derivation of initial state bounds under DoS. In this section, we present an encoding and decoding scheme to obtain an initial state bound E_0 under DoS attacks. To this end, we extend the zooming-out method proposed in [25] to the case under DoS attacks. Due to the attacks and the missing output data, it can be difficult to obtain correct state estimates on the decoder side, which is a basic question related to observability.

We place the following assumptions in this section:

Assumption 4.1 (Observability). The pair (C, A) is observable.

ASSUMPTION 4.2 (Odd quantization level). The quantization level N is an odd number.

The derivation of state bounds requires observability rather than detectability. If the quantization number N is odd, then the quantized value q_k is zero for a sufficiently small output y_k . We use this property for Lyapunov stability in Section 4.3.

4.1. Basic encoding and decoding scheme. We set the control input u_k to be $u_k = 0$ until we get a state bound. For a given increasing sequence $\{E_k^y \ge 0 : k \in \mathbb{Z}_+\}$, define the binary function $Q_k : \mathbb{R}^{n_y} \to \{0, 1\}$ by

$$Q_k(y) := \begin{cases} 0 & \text{if } |y|_{\infty} \le E_k^y \\ 1 & \text{otherwise.} \end{cases}$$

For $s_0, s_1, \ldots, s_{\chi} \in \mathbb{Z}_+$ satisfying $s_0 < s_1 < \cdots < s_{\chi}$, we define the generalized observability matrix $O(\{s_m\}_{m=0}^{\chi})$ by

(42)
$$O\left(\{s_m\}_{m=0}^{\chi}\right) := \begin{bmatrix} C \\ CA^{s_1-s_0} \\ \vdots \\ CA^{s_{\chi}-s_0} \end{bmatrix}.$$

Assume that there exists $\{s_m\}_{m=0}^{\chi} \subset \mathbb{Z}_+$ with $s_0 < s_1 < \cdots < s_{\chi}$ such that DoS attacks do not occur at times $k = s_0, \ldots, s_{\chi}$ and the following two conditions hold: (C1) $Q_{s_m}(y_{s_m}) = 0$ holds for every $m = 0, \ldots, \chi$; (C2) The matrix $O(\{s_m\}_{m=0}^{\chi})$ is full column rank.

Under the conditions (C1) and (C2), we can obtain a state bound at $k = s_{\chi} + 1$ as follows. By the condition (C1), the decoder on the controller side knows at time $k = s_{\chi}$ that

$$|y_{s_m}|_{\infty} \le E_{s_m}^y \qquad \forall m = 0, \dots, \chi$$

and hence

$$\left| \begin{bmatrix} y_{s_0} \\ \vdots \\ y_{s_{\chi}} \end{bmatrix} \right|_{\infty} \leq E_{s_{\chi}}^y.$$

By the condition (C2),

(43)
$$x_{s_0} = O\left(\{s_m\}_{m=0}^{\chi}\right)^{\dagger} \begin{bmatrix} y_{s_0} \\ y_{s_1} \\ \vdots \\ y_{s_{\chi}} \end{bmatrix}.$$

Since $x_{s_{\chi}+1} = A^{s_{\chi}-s_0+1}x_{s_0}$, it follows that if the coders set a state bound $E_{s_{\chi}+1}$ at time $k = s_{\chi} + 1$ to be

(44)
$$E_{s_{\chi}+1} := \left\| A^{s_{\chi}-s_{0}+1} O\left(\{s_{m}\}_{m=0}^{\chi}\right)^{\dagger} \right\|_{\infty} E_{s_{\chi}}^{y},$$

then $|x_{s_{\chi}+1}|_{\infty} \leq E_{s_{\chi+1}}$.

Next, we design a sequence $\{E_k^y : k \in \mathbb{Z}_+\}$ that satisfies $Q_k(y_k) = 0$ for every sufficiently large k. Fix a constant $\kappa > 0$ and an initial value $E_0^x > 0$, and define a sequence $\{E_k^x : k \in \mathbb{Z}_+\}$ by

(45)
$$E_{k+1}^x := (1+\kappa) \|A\|_{\infty} E_k^x.$$

Since the growth rate of E_k^x is larger than that of $|x_k|_{\infty}$, there exists $T \in \mathbb{Z}_+$ such that

(46)
$$|y_k|_{\infty} = |Cx_k|_{\infty} \le ||C||_{\infty} E_k^x \qquad \forall k \ge T.$$

If we set $E_k^y := ||C||_{\infty} E_k^x$, then (46) yields $Q_k(y_k) = 0$ for all $k \ge T$. Thus, the condition (C1) is always satisfied if $s_0 \ge T$.

In what follows, we set the time origin to be $T \in \mathbb{Z}_+$ satisfying (46) for simplicity of notation. Note that we can use the DoS conditions (2) and (3) even after shifting the time origin, by changing the constants Π_d and Π_f there to $\Pi_d + \nu_d T$ and $\Pi_f + \nu_f T$, respectively.

ASSUMPTION 4.3 (Capturing output from initial time). For every $k \in \mathbb{Z}_+$, $Q_k(y_k) = 0$.

In the case without DoS attacks [25], if (C, A) is observable, then we can obtain a state bound at time $k = \eta$, where η is the observability index, because

(47)
$$O := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\eta-1} \end{bmatrix}$$

is full column rank. However, the following example shows that for every DoS duration bound $\nu_d \in (0, 1]$, there exists an observable system (C, A) and a corresponding attack strategy such that the condition (C2) does not hold.

EXAMPLE 4.4. Let

$$A = \begin{bmatrix} 0 & I_{n-1} \\ 1 & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{1 \times n}$$

The system is observable. However, if the DoS attacks are periodically launched at times $k = 0, n, 2n, \ldots$, then we cannot construct state bounds, by using even an infinite number of measurements. In fact, $O(\{s_m\}_{m=0}^{\chi})$ is not full column rank for every set of time-steps $\{s_m\}_{m=0}^{\chi}$ satisfying $\{s_m\}_{m=0}^{\chi} \cap \{\ell n : \ell \in \mathbb{Z}_+\} = \emptyset$.

In the following subsections, we see that the condition (C2) is satisfied under certain assumptions on DoS.

4.2. Sufficient condition for (C2) to hold.

4.2.1. Approach to exploit periodic property of eigenvalues. In Example 4.4, A is a circulant matrix, and the eigenvalues of A are given by $e^{i2\pi \frac{j}{n}}$ (j = 0, ..., n-1), all of which are on the unit circle. In this subsection, exploiting this property of the eigenvalues, we provide a sufficient condition for the generalized observability matrix $O(\{s_m\}_{m=0}^{\chi})$ to be full column rank. Assumptions on the plant are given as follows:

ASSUMPTION 4.5 (Periodicity of eigenvalues). For every j = 1, ..., Q, let $\zeta_j \in \mathbb{N}$ be one or a prime number, $\lambda_j \in \mathbb{C}$ be nonzero, and $a_{j,1}, ..., a_{j,n_j} \in \mathbb{Z}$ satisfy $a_{j,\ell_1} \neq a_{j,\ell_2} \pmod{\zeta_j}$ for all $\ell_1, \ell_2 = 1, ..., n_j$ with $\ell_1 \neq \ell_2$. The matrix A is similar to a diagonal matrix $\Lambda := \text{diag}(\Lambda_1, ..., \Lambda_Q)$, where

$$\Lambda_j := \operatorname{diag}\left(\lambda_j e^{i2\pi \frac{a_{j,1}}{\zeta_j}}, \dots, \lambda_j e^{i2\pi \frac{a_{j,n_j}}{\zeta_j}}\right) \quad \forall j \in 1, \dots, Q.$$

Moreover, $(\lambda_{j_1}/\lambda_{j_2})^k \neq 1$ for every $k \in \mathbb{N}$ and for every $j_1, j_2 = 1, \ldots, Q$ with $j_1 \neq j_2$.

ASSUMPTION 4.6 (Single-output system). The plant is a single-output system, that is, $C \in \mathbb{R}^{1 \times n_x}$.

The following theorem shows that if the assumptions above are satisfied and if the DoS duration bound ν_d is sufficiently small, then the condition (C2) holds in finite time.

THEOREM 4.7. Suppose that Assumptions 2.1, 4.1, 4.3, 4.5, and 4.6 hold. Let $\zeta \in \mathbb{N}$ be the least common multiple of ζ_1, \ldots, ζ_Q . If the DoS duration bound ν_d satisfies

(48)
$$\nu_d < \frac{1}{\zeta}$$

then $O(\{s_m\}_{m=0}^{\chi})$ defined in (42) is full column rank by time $k = (\ell_e + 1)\zeta$, where $\ell_e \in \mathbb{Z}_+$ is the maximum integer satisfying

(49)
$$\ell_e \le \frac{\Pi_d + Z(A, C)}{1 - \zeta \nu_d}$$

for some constant $Z(A, C) \in \mathbb{Z}_+$ that depends only on A, C.

We prove this result, using the techniques developed in [32, 34]; see the appendix for details.

REMARK 4.8. Apply Theorem 4.7 to the case where A is diagonalizable and has only positive real eigenvalues with multiplicity 1. Since the least common multiple ζ is one, the matrix $O(\{s_m\}_{m=0}^{\chi})$ is full column rank in finite time for every $\nu_d \in [0,1)$ under Assumptions 2.1, 4.1, and 4.6. The same result follows also from Theorem 1 in [42].

REMARK 4.9. Lemma B in the appendix shows that if Λ consists of only one block, i.e., $\Lambda = \Lambda_1$, then the conditions (48) and (49) can be replaced by less conservative conditions:

$$\nu_d < \frac{\zeta_1 - n_1 + 1}{\zeta_1}, \quad \ell_e \le \frac{\Pi_d}{\zeta_1 - n_1 + 1 - \zeta_1 \nu_d}.$$

4.2.2. Coders using only consecutive data. If the decoder receives η consecutive data, where η is the observability index of (C, A), then we can obtain a state bound as in the case without DoS. Here we design coders that construct state bounds from η consecutive data. In this case, the time-steps s_0, \ldots, s_{χ} in Section 4.1 are given by $\chi = \eta - 1$ and $s_m = s_0 + m$ for all $m = 1, \ldots, \chi$. The advantage of this approach over Theorem 4.7 is that it is also applicable to multi-output systems.

THEOREM 4.10. Let Assumptions 2.1, 2.2, 4.1, and 4.3 hold. If the DoS duration and frequency bounds ν_d and ν_f satisfy

(50)
$$\nu_d < 1 - (\eta - 1)\nu_f,$$

then the decoder receives η consecutive data by time $k = k_e + 1$, where $k_e \in \mathbb{Z}_+$ is the maximum integer satisfying

(51)
$$k_e \le \frac{\Pi_d + (\Pi_f + 1)(\eta - 1)}{1 - \nu_d + (\eta - 1)\nu_f}.$$

Proof. Choose $k_e \in \mathbb{N}$ arbitrarily. Let DoS attacks occur at

$$k = k_m, \dots, k_m + \tau_m - 1 \qquad \forall m = 1, \dots, p$$

on the interval $[0, k_e)$, where $k_m \in \mathbb{Z}_+$, $\tau_m \in \mathbb{N}$ for every $m = 1, \ldots, p$ and

$$k_m + \tau_m < k_{m+1} \qquad \forall m = 1, \dots, p-1.$$

In other words, k_m and τ_m denote the beginning time and the length of *m*th DoS interval. By definition, $\sum_{m=1}^{p} \tau_m = \Phi_d(k_e)$ and $p = \Phi_f(k_e)$. We also define $k_0 := 0$, $\tau_0 := 0$, and $k_{p+1} := k_e$.

Assume, to reach a contradiction, that

(52)
$$k_{m+1} - k_m - \tau_m \le \eta - 1 \quad \forall m = 0, \dots, p,$$

which implies that the decoder receives at most $\eta - 1$ consecutive data on the interval $[0, k_e)$. Applying induction to (52), we obtain

$$k_e = k_{p+1} \le k_p + \tau_p + \eta - 1$$

 $\le \dots \le \sum_{m=0}^p \tau_m + (p+1)(\eta - 1).$

From Assumptions 2.1 and 2.2, it follows that

$$k_e \leq (\Pi_d + \nu_d k_e) + (\Pi_f + \nu_f k_e + 1)(\eta - 1),$$

and hence

(53)
$$(1 - \nu_d - (\eta - 1)\nu_f)k_e \le \Pi_d + (\Pi_f + 1)(\eta - 1).$$

Since $k_e \in \mathbb{N}$ was arbitrary, the condition (50) leads to a contradiction for a sufficiently large $k_e > 0$. Moreover, if (50) holds, then k_e must satisfy the inequality (51). In other words, if k_e is the maximum integer satisfying the inequality (51), then η consecutive data are transmitted successfully by time $k = k_e + 1$. This completes the proof. \Box

In the case $\nu_d < \nu_f$, namely, when DoS attacks are frequently launched, the following proposition is also useful.

PROPOSITION 4.11. Suppose that Assumptions 2.1, 4.1, and 4.3 hold. If the DoS duration bound ν_d satisfies

(54)
$$\nu_d < \frac{1}{\eta}$$

then the decoder receives η consecutive data by time $k = k_e + 1$, where $k_e \in \mathbb{Z}_+$ is the maximum integer satisfying

(55)
$$k_e \le \frac{(\Pi_d + 1)\eta - 1}{1 - \eta\nu_d}$$

Proof. Choose $k_e \in \mathbb{N}$ arbitrarily and let DoS attacks occur at

$$k = t_m \in \mathbb{Z}_+ \qquad \forall m = 1, \dots, p.$$

on the interval $[0, k_e)$, where $t_m < t_{m+1}$ for all $m = 1, \ldots, p$. Then $p = \Phi_d(k_e)$ by definition.

Define $t_{p+1} := k_e$. Assume, to get a contradiction, that

$$t_1 \le \eta - 1, \quad t_{m+1} - t_m \le \eta \quad \forall m = 1, \dots, p - 1.$$

Then we obtain

$$k_e = t_{p+1} \le t_p + \eta \le \dots \le (p+1)\eta - 1.$$

By Assumptions 2.1, $k_e \leq (\Pi_d + \nu_d k_e + 1)\eta - 1$, and hence

$$(1 - \eta \nu_d)k_e \le (\Pi_d + 1)\eta - 1.$$

The rest of the proof follows the same lines as that of Theorem 4.10. Therefore, we shall omit it. $\hfill \Box$

REMARK 4.12. Suppose that the encoder at the plant side redundantly sends the set of output data, $Q_{k-\eta+1}(y_{k-\eta+1}), \ldots, Q_k(y_k)$, at every time $k \ge \eta - 1$. The decoder can obtain an initial state bound if a data set whose values are all zero is successfully transmitted. Thus, the zooming-out procedure by this redundant scheme finishes in finite time for all $\nu_d \in [0, 1)$.

4.2.3. With bounded lengths in DoS periods. We here place the following assumption:

ASSUMPTION 4.13 (Bounded length of DoS period). For a given $\varpi \in \mathbb{N}$, at most ϖ consecutive DoS attacks occur.

If the DoS duration condition (4) holds, then $\varpi \in \mathbb{N}$ has to satisfy $\varpi \leq \Pi_d + \nu_d \varpi$. Therefore, $\varpi \leq \Pi_d/(1-\nu_d)$.

For packet losses including DoS attacks in Assumption 4.13, the earlier study [21] shows that we can check in finite time whether or not there exists $k \in \mathbb{Z}_+$ such that the matrix $O_{\sigma}(k)$ is full column rank, where the binary function $\sigma : \mathbb{Z}_+ \to \{0, 1\}$ is defined by

 $\sigma(k) := \begin{cases} 0 & \text{if packet loss occurs at } k \\ 1 & \text{if packet loss does not occur at } k, \end{cases}$

and

$$O_{\sigma}(k) := \begin{bmatrix} \sigma(0)C\\ \sigma(1)CA\\ \vdots\\ \sigma(k-1)CA^{k-1} \end{bmatrix}$$

Moreover, if such k (the time when $O_{\sigma}(k)$ is full column rank) exists, then k is upper-bounded by a certain value $k_e \in \mathbb{Z}_+$ that depends only on ϖ and (C, A); see Proposition 1, Theorem 2, and Remark 2 in [21]. When we regard $s_0 \in \mathbb{Z}_+$ as $s_0 = 0$ in (42), $O_{\sigma}(k)$ is full column rank if and only if $O(\{s_m\}_{m=0}^{\chi})$ is full column rank. Therefore, we can immediately apply the result in [21]. Thus, we can find in finite time whether or not a full column rank $O(\{s_m\}_{m=0}^{\chi})$ exists. In addition, if it exists, then the decoder can construct a state bound by a certain time that depends on ϖ and (C, A).

4.3. Lyapunov stability. Combining the encoding schemes in Section 3 and this section, we achieve Lyapunov stability.

DEFINITION 4.14 (Lyapunov stability). The feedback system in Section 2 achieves Lyapunov stability if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

(56)
$$|x_0|_{\infty} < \delta \quad \Rightarrow \quad |x_k|_{\infty}, \ |\hat{x}_k|_{\infty} < \varepsilon \quad \forall k \in \mathbb{Z}_+$$

In the following theorem, we use Theorems 3.4 and 4.10, but similar results can be obtained from other combinations such as Corollary 3.13 and Theorem 4.7.

THEOREM 4.15. Suppose that Assumptions 2.1, 2.2, 3.1, 4.1, and 4.2 hold. If the quantization level N and the DoS duration and frequency bounds ν_d , ν_f satisfy (13) and (50), then the closed-loop system achieves Lyapunov stability under the encoding scheme in Sections 3.2, 3.3, and 4.1.

Proof. Suppose that $\delta > 0$ satisfies

$$\delta < E_0^x.$$

Then $Q_k(y_k) = 0$ for every time k during the zooming-out procedure. Let a state bound be obtained at time $k = T_1$, namely, let $s_{\chi} + 1$ in Section 4.1 be equal to T_1 . By Theorem 4.10, T_1 has a certain upper bound $\overline{T}_1 \in \mathbb{Z}_+$. Hence, for the error bound E_{T_1} defined as in Section 4.1, there exists $\overline{E} > 0$ such that $E_{T_1} \leq \overline{E}$.

Theorem 3.4 shows that, in the zooming-in stage, there exist $\Omega \ge 1$ and $\gamma \in (0, 1)$ such that x_k and \hat{x}_k satisfy

$$|x_k|_{\infty}, \ |\hat{x}_k|_{\infty} \le \Omega \gamma^{k-T_1} E_{T_1} \qquad \forall k \ge T_1.$$

We set an integer $T_2 \geq \overline{T}_1$ so that

(58)
$$\Omega \gamma^{T_2 - T_1} \overline{E} < \varepsilon.$$

Then $|x_k|_{\infty}$, $|\hat{x}_k|_{\infty} < \varepsilon$ for every $k \ge T_2$.

Let us next show that

(59)
$$|x_k|_{\infty}, |\hat{x}_k|_{\infty} < \varepsilon \quad \forall k \le T_2.$$

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Define

$$\Upsilon := \min\{ \|A^{s_{\chi}-s_{0}+1}O(\{s_{m}\}_{m=0}^{p})^{\dagger}\|_{\infty} : O(\{s_{m}\}_{m=0}^{\chi}) \text{ is}$$
full column rank and $\{s_{m}\}_{m=0}^{\chi} \subset \{0, \dots, \overline{T}_{1}\}\}.$

By the definition of E_{T_1} in the zooming-out procedure and the update rule of $E_{R,k}$ in the zooming-in procedure, we obtain

$$E_{R,k} \ge \Upsilon \|C\|_{\infty} \cdot \|R\|_{\infty} \theta^{T_2} E_0^x \qquad \forall k \in [T_1, T_2]$$

For each $k \in (T_1, T_2]$, if $q_{\ell} = 0$ for every $\ell \in [T_1, k)$, then $u_k = 0$, and hence $|y_k|_{\infty} \leq ||C||_{\infty} \cdot ||A||_{\infty}^{T_2} \delta$. Moreover, in such a case, if

$$|y_k - \hat{y}_k|_{\infty} = |y_k|_{\infty} \le \frac{\|CR^{-1}\|_{\infty}E_{R,k}}{N},$$

then $q_k=0,$ because the number of quantization levels N is odd. Therefore, if $\delta>0$ satisfies

(60)
$$||A||^{T_2}\delta < \frac{\Upsilon ||CR^{-1}||_{\infty} \cdot ||R||_{\infty} \theta^{T_2} E_0^x}{N},$$

then $q_k = 0$ for every $k \in [T_1, T_2]$. Hence $u_k = 0$ for every $k \leq T_2$. Thus, if $\delta > 0$ additionally satisfies

$$(61) ||A||^{T_2}\delta < \varepsilon,$$

then (59) holds.

In summary, if $\delta > 0$ satisfies (57), (60), and (61), then (56) holds. Thus, Lyapunov stability is achieved.

5. Numerical Examples.

5.1. Plant and controller. A linearized model of the unstable batch reactor studied in [35] is given by $\dot{x}(t) = A_c x(t) + B_c u(t)$ and $y(t) = C_c x(t)$, where

$$\begin{split} A_c &:= \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & -1.343 & -2.104 \end{bmatrix} \\ B_c &:= \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}, \quad C_c &:= \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \end{split}$$

Here we discretize this plant with the sampling period h = 0.2. We use the feedback gain K that is the linear quadratic regulator whose state weighting matrix and input weighting matrix are the identity matrices I_4 and I_2 , respectively. The observer gain L is given by the gain of the steady-state Kalman filter whose covariances of the process noise and measurement noise are I_4 and $0.1 \times I_2$, respectively.



Fig. 2: Relationship between quantization level N and DoS duration bound ν_d under encoding scheme with center at output estimate.

5.2. Relationship between quantization level and DoS duration and frequency. By Corollary 3.13, we obtain a relationship between the quantization level N and the DoS duration bound ν_d for the state convergence. Here we choose the matrix R so that

(62)
$$||R(A - LC)R^{-1}||_{\infty} = \varrho(A - LC).$$

Each circle in Fig. 2 illustrates the minimum integer N satisfying (35b) in Corollary 3.13. This corollary shows that as the quantization level N increases to infinity, the DoS duration bound ν_d goes to

$$\frac{-\log \|R(A - LC)R^{-1}\|_{\infty}}{\log \|RAR^{-1}\|_{\infty} - \log \|R(A - LC)R^{-1}\|_{\infty}} \approx 0.1405$$

Let us next see a relationship between the quantization level N and the DoS duration and frequency bounds ν_d , ν_f . In Theorem 3.4, we choose the matrix R so that

$$\|RAR^{-1}\|_{\infty} = \varrho(A).$$

The surface in Fig. 3a depicts the minimum integer satisfying (13b) in Theorem 3.4 for given DoS duration and frequency bounds. Using Theorem 3.4, we find that as the quantization level N goes to infinity, the DoS duration and frequency bounds ν_d, ν_f get close to the line

$$\nu_d = \frac{-\nu_f \log M_0 - \log \rho}{\log \|RAR^{-1}\|_{\infty} - \log \rho} \approx -1.9080\nu_f + 0.3042.$$

This can be also observed in Fig. 3b.

Finally, we see a relationship between the quantization level N and the DoS duration bound ν_d for the encoding scheme whose quantization center is the origin. Fig. 4 illustrates the minimum integer N satisfying (41b) in Corollary 3.18, where $R_{\rm cl}$ is chosen so that

$$\|R_{\rm cl}A_{\rm cl}R_{\rm cl}^{-1}\|_{\infty} = \varrho(A_{\rm cl}).$$

By Corollary 3.18, the DoS duration bound ν_d converges to

$$\frac{-\log \|R_{\rm cl}A_{\rm cl}R_{\rm cl}^{-1}\|_{\infty}}{\log \|R_{\rm cl}A_{\rm op}R_{\rm cl}^{-1}\|_{\infty} - \log \|R_{\rm cl}A_{\rm cl}R_{\rm cl}^{-1}\|_{\infty}} \approx 0.0736.$$



Fig. 3: Relationship between quantization level N and DoS duration and frequency bounds ν_d, ν_f under encoding scheme with center at output estimate.



Fig. 4: Relationship between quantization level N and DoS duration bound ν_d under encoding scheme with center at origin.

as the quantization level N goes to infinity. We can observe that the encoder with center at the origin needs more data rates in exchange for the reduction of computational resources of the coders.

5.3. Time responses. We present time responses under the encoding schemes with center at the output estimate. Through simulation results, we see how conser-

vative the obtained sufficient conditions are.

5.3.1. DoS attacks. We set the quantization level N to be N = 71. The closed-loop system with the encoding scheme of Corollary 3.13 achieves exponential convergence under the DoS duration

(64)
$$\nu_d < 0.106$$

where we construct the transformation matrix R so that (62) holds. Moreover, Proposition 4.11 shows that an initial state bound is obtained in finite time if $\nu_d < 0.5$.

On the other hand, the encoding scheme of Theorem 3.4 achieves exponential convergence if the DoS duration and frequency bounds ν_d and ν_f satisfy

(65)
$$\nu_d < 0.230 - 2.041 \nu_f$$

where the matrix R is chosen so that (63) is satisfied. By Theorem 4.10, the coders can construct an initial state bound in finite time if $\nu_d < 1 - \nu_f$.

If the frequency of DoS attacks is sufficiently small, then the encoding scheme of Theorem 3.4 allows longer duration of DoS attacks without compromising the closed-loop stability in this example. However, the encoding scheme of Corollary 3.13 can tolerate DoS attacks with large frequency. For instance, if $\nu_d = 0.1$ in (65), then $\nu_f < 0.063$. The encoding scheme of Corollary 3.13 allows the DoS attacks launched at times $k = 10, 20, 30, \ldots$, but that of Theorem 3.4 does not.

In the simulation below, we assume that the attacker knows all information on the closed-loop system, which leads to effective DoS. After a state bound is obtained, i.e., at the zooming-in stage, DoS attacks occur if the following two conditions are both satisfied in addition to the above constraints on the duration and frequency:

(66)
$$|Ae_k|_{\infty} > \alpha_1 |e_k|_{\infty}, \quad |y_k - q_k|_{\infty} > \alpha_2 \frac{\|CR^{-1}\|_{\infty}}{N} E_{R,k},$$

where $\alpha_1 = 1$ and $\alpha_2 = 1/2$. Recall that the maximum quantization error is given by

$$\frac{\|CR^{-1}\|_{\infty}}{N}E_{R,k}$$

after a state bound is derived, as shown in (9). As the constant $\alpha_1 \geq 0$ increases, the estimation error e_k becomes larger due to DoS attacks. As the constant $\alpha_2 \in [0, 1)$ becomes close to one, the second condition leads to a larger quantization error $y_k - q_k$. If α_1 is too large or if α_2 is too close to one, then DoS attacks rarely occur.

REMARK 5.1. A more sophisticated design of DoS attacks was discussed in Example 2.8 of [7], where the attacker decides whether to block data transmissions or not, by solving an optimization problem over a short horizon at each time like model predictive control (MPC). Compared with the rule based on (66), this MPC-like strategy requires computational resources because the attacker has to solve a 0-1 integer programming problem. However, DoS attacks can be effectively launched without tuning parameters.

5.3.2. Simulation results. Let us denote the state x_k and its estimate \hat{x}_k by $x = \begin{bmatrix} x^1 & x^2 & x^3 & x^4 \end{bmatrix}^\top$ and $\hat{x} = \begin{bmatrix} \hat{x}^1 & \hat{x}^2 & \hat{x}^3 & \hat{x}^4 \end{bmatrix}^\top$, respectively. For the computation of time responses, we set the initial state x_0 to be $x_0 = \begin{bmatrix} 0 & 0.5 & 0.5 & 1 \end{bmatrix}^\top$. The parameters E_0 and κ for the encoding scheme to derive an initial state bound are given by $E_0 = 0.01$ and $\kappa = 0.01$.

Figs. 5 and 6 show time responses under the encoding scheme of Corollary 3.13 without assuming any frequency conditions of DoS attacks. Fig. 5 depicts the stable case where DoS attacks satisfy the duration condition (2) with $(\Pi_d, \nu_d) = (2, 0.10)$. DoS attacks occur on the intervals that are colored in gray. Since the DoS duration $\nu_d = 0.10$ satisfies (64), the error bound E_R exponentially decreases, which leads to exponential convergence.

Fig. 6 illustrates the unstable case of the encoding scheme of Corollary 3.13 under DoS attacks with $(\Pi_d, \nu_d) = (2, 0.11)$. Since the DoS duration bound $\nu_d = 0.11$ does not satisfy (64), DoS attacks make the error bound E_R diverge in Fig. 6b. As the error bound E_R increases, the worst-case quantization error becomes larger, which leads to the instability of the closed-loop system as shown in Fig. 6a, although the difference between the threshold 0.106 in (64) and $\nu_d = 0.11$ used in Fig. 6 is small. We see that the sufficient condition (13b) is fairly tight in this example.

Next we compute time responses in stable and unstable cases under the encoding scheme of Theorem 3.4, assuming that the DoS duration and frequency are both averagely bounded. By (65), if the DoS duration bound ν_d is given by $\nu_d = 0.15$, then the frequency bound ν_f should satisfy $\nu_f < 0.0392$. Fig. 7 illustrates the stable case, where DoS attacks satisfy the duration condition (2) with $(\Pi_d, \nu_d) = (2, 0.15)$ and the frequency condition (3) with $(\Pi_f, \nu_f) = (1, 0.035)$. Fig. 7a shows that the closed-loop system achieves exponential convergence despite longer DoS duration than in the case of Fig. 5. This is because the encoding scheme in Theorem 3.4 has a small growth rate $\theta_a = 1.489$ in the presence of DoS attacks, compared with the growth rate $\vartheta_a = 2.901$ of the encoding scheme in Corollary 3.13.

Fig. 8 shows the time response in the unstable case, where $(\Pi_d, \nu_d) = (2, 0.15)$ and $(\Pi_f, \nu_f) = (1, 0.045)$. The DoS frequency bound ν_f is just slightly larger than the threshold 0.0392, but the error bound E_R diverges. Consequently, the closed-loop system is unstable.

In Figs. 7a and 8a, the trajectories of x^1 and \hat{x}^1 oscillate after DoS attacks, which is unique to the case with quantization. These oscillations are caused by large error bounds E_R due to DoS attacks, as shown in Figs. 7b and 8b. Even after DoS attacks, the quantized output q is zero under coarse quantization until the error bound becomes small. Hence the observer does not estimate the plant state correctly.

6. Conclusion. We proposed output encoding schemes resilient to DoS attacks and obtained sufficient conditions on DoS duration and frequency bounds for exponential convergence and Lyupunov stability with finite data rates. The proposed encoding schemes are extensions of the zooming-in and zooming-out method to the case with DoS. Once an initial state bound is derived, the coders decrease the quantization range in the absence of DoS. However, if DoS attacks are detected, then the coders increase the quantization range so that the output at the next time-step falls into the quantization region. Moreover, we discussed how to obtain state bounds under DoS attacks. Future work is to address more general networked control systems by considering network phenomena at communication channels from the controller to the plant.

Appendix A. Proof of Theorem 4.7.

Let us first consider the case where Q = 1, namely, the matrix A is similar to a diagonal matrix $\Lambda := \operatorname{diag}\left(\lambda e^{i2\pi \frac{a_1}{\zeta}}, \ldots, \lambda e^{i2\pi \frac{a_nx}{\zeta}}\right)$, where $\zeta \in \mathbb{N}$ be one or a prime number, $\lambda \in \mathbb{C}$ be nonzero, and $a_1, \ldots, a_{n_x} \in \mathbb{Z}$ satisfy $a_{\ell_1} \not\equiv a_{\ell_2} \pmod{\zeta}$ for all $\ell_1, \ell_2 = 1, \ldots, n_x$ with $\ell_1 \neq \ell_2$.





Fig. 5: Stable case without frequency condition ($\nu_d = 0.1$).

To obtain a sufficient condition for the matrix $O(\{s_m\}_{m=0}^{\chi})$ to be full column rank, we use the following result on a generalized Vandermonde matrix:

LEMMA A (Theorem 6 of [14]). Let ζ be a prime number, and let $a_1, \ldots, a_n \in \mathbb{Z}$ and $b_1, \ldots, b_n \in \mathbb{Z}$ satisfy $a_{\ell_1} \not\equiv a_{\ell_2} \pmod{\zeta}$ and $b_{\ell_1} \not\equiv b_{\ell_2} \pmod{\zeta}$ for all $\ell_1, \ell_2 = 1, \ldots, n$ with $\ell_1 \neq \ell_2$. Then the generalized Vandermonde matrix

$e^{i2\pi \frac{a_1b_1}{\zeta}}$	$e^{i2\pi \frac{a_2b_1}{\zeta}}$		$e^{i2\pi \frac{a_n b_1}{\zeta}}$
$e^{i2\pi \frac{a_1b_2}{\zeta}}$	$e^{i2\pi \frac{a_2b_2}{\zeta}}$		$e^{i2\pi \frac{a_n b_2}{\zeta}}$
:	÷	·	:
$e^{i2\pi \frac{a_1b_n}{\zeta}}$	$e^{i2\pi \frac{a_2b_n}{\zeta}}$		$e^{i2\pi \frac{a_n b_n}{\zeta}}$

is invertible.

LEMMA B. Assume that the matrix A is similar to the diagonal matrix Λ defined above. Suppose that Assumptions 2.1, 4.1, 4.3, and 4.6 hold. If the DoS duration bound ν_d satisfies

(67)
$$\nu_d < \frac{\zeta - n_x + 1}{\zeta}$$



(a) State x^1 and its estimate \hat{x}^1 .



Fig. 6: Unstable case without frequency condition ($\nu_d = 0.11$).

then $O(\{s_m\}_{m=0}^{\chi})$ defined in (42) is full column rank by time $k = (\ell_e + 1)\zeta$, where $\ell_e \in \mathbb{Z}_+$ is the maximum integer satisfying

(68)
$$\ell_e \le \frac{\Pi_d}{\zeta - n_x + 1 - \zeta \nu_d}.$$

Proof. Let $s_0, s_1, \ldots, s_{\chi}$ be the time-steps without DoS on the interval [0, k), and define

(69)
$$b_m := s_{m-1} - s_0 \quad \forall m = 1, \dots, \chi + 1.$$

There exists an invertible matrix $R \in \mathbb{C}^{n_x \times n_x}$ such that $AR = R\Lambda$. Define $C_\Lambda := CR = \begin{bmatrix} c_1 & \cdots & c_{n_x} \end{bmatrix}$ and

$$\mathcal{V} := \begin{bmatrix} e^{i2\pi \frac{a_1b_1}{\zeta}} & e^{i2\pi \frac{a_2b_1}{\zeta}} & \cdots & e^{i2\pi \frac{a_nxb_1}{\zeta}} \\ e^{i2\pi \frac{a_1b_2}{\zeta}} & e^{i2\pi \frac{a_2b_2}{\zeta}} & \cdots & e^{i2\pi \frac{a_nxb_2}{\zeta}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{i2\pi \frac{a_1b_{\chi+1}}{\zeta}} & e^{i2\pi \frac{a_2b_{\chi+1}}{\zeta}} & \cdots & e^{i2\pi \frac{a_nxb_{\chi+1}}{\zeta}} \end{bmatrix}$$





Fig. 7: Stable case with frequency condition ($\nu_d = 0.15$, $\nu_f = 0.035$).

Since (C, A) is observable by Assumption 4.1, it follows that $c_j \neq 0$ for every $j = 1, \ldots, n_x$. We obtain

$$O\left(\{s_m\}_{m=0}^{\chi}\right) = \begin{bmatrix} CA^{b_1} \\ \vdots \\ CA^{b_{\chi+1}} \end{bmatrix} = \begin{bmatrix} C_{\Lambda}\Lambda^{b_1} \\ \vdots \\ C_{\Lambda}\Lambda^{b_{\chi+1}} \end{bmatrix} R^{-1}$$
$$= \operatorname{diag}(\lambda^{b_1}, \dots, \lambda^{b_{\chi+1}}) \cdot \mathcal{V} \cdot \operatorname{diag}(c_1, \dots, c_{n_x})R^{-1}.$$

Therefore, the rank of $O(\{s_m\}_{m=0}^{\chi})$ is equal to the rank of \mathcal{V} . By the assumption on a_1, \ldots, a_{n_x} , Lemma A shows that if there exist $\tilde{b}_1, \ldots, \tilde{b}_{n_x} \in \{b_1, \ldots, b_{\chi+1}\}$ such that $\tilde{b}_{\ell_1} \neq \tilde{b}_{\ell_2} \pmod{\zeta}$ for all $\ell_1, \ell_2 = 1, \ldots, n_x$ with $\ell_1 \neq \ell_2$, then rank $\mathcal{V} = n_x$. This implies that $O(\{s_m\}_{m=0}^{\chi})$ is full column rank.

Suppose, to reach a contradiction, that for every $k = \ell \zeta$ with $\ell \in \mathbb{Z}_+$, there do not exist such

$$\tilde{b}_1, \dots, \tilde{b}_{n_x} \in \{b_1, \dots, b_{\chi+1}\} = \{0, s_1 - s_0, \dots, s_{\chi} - s_0\},\$$

where s_0, \ldots, s_{χ} are the time-steps without DoS on the interval [0, k). Then DoS attacks occur at least $\zeta - n_x + 1$ times during every interval consisting of consecutive



(a) State x^1 and its estimate \hat{x}^1 .



Fig. 8: Unstable case with frequency condition ($\nu_d = 0.15$, $\nu_f = 0.045$).

 ζ time-steps. Hence $\Phi_d(\ell\zeta) \ge \ell(\zeta - n_x + 1)$. By Assumption 2.1, we obtain $\Phi_d(\ell\zeta) \le \Pi_d + \nu_d(\ell\zeta)$. Therefore,

$$\ell(\zeta - n_x + 1) \le \Pi_d + \nu_d(\ell\zeta),$$

which yields $(\zeta - n_x + 1 - \zeta \nu_d)\ell \leq \Pi_d$. If the DoS duration bound ν_d satisfies (67), then we get a contradiction for $\ell \in \mathbb{Z}_+$ larger than the right side of (68). This implies that $O(\{s_m\}_{m=0}^{\chi})$ is full column rank by time $k = (\ell_e + 1)\zeta$, where $\ell_e \in \mathbb{Z}_+$ is the maximum integer satisfying (68). This completes the proof.

Let us next consider the general case. The following lemma provides a useful algebraic fact, which is used to show Theorem 4.7:

LEMMA C (Lemma 26 of [34]). Let $\lambda_1, \ldots, \lambda_Q \in \mathbb{C}$ satisfy $\lambda_j \neq 0$ for all $j = 1, \ldots, Q$ and $(\lambda_{j_1}/\lambda_{j_2})^k \neq 1$ for every $j_1, j_2 = 1, \ldots, Q$ with $j_1 \neq j_2$ and every $k \in \mathbb{N}$. Let $w_1, \ldots, w_Q \in \mathbb{C}$ satisfy $w_j \neq 0$ for some $j = 1, \ldots, Q$. Then, there exist at most finitely many $k \in \mathbb{Z}_+$ such that

$$\sum_{j=1}^{Q} w_j \lambda_j^k = 0.$$

We are now in a position to prove Theorem 4.7.

Proof of Theorem 4.7. Let $s_0, s_1, \ldots, s_{\chi}$ be the time-steps without DoS on the interval [0, k), and define $b_m \in \mathbb{Z}_+$ $(m = 1, \ldots, \chi + 1)$ as in (69). There exists an invertible matrix $R \in \mathbb{C}^{n_x \times n_x}$ such that $AR = R\Lambda$. Define $C_\Lambda := CR = \begin{bmatrix} C_1 & \cdots & C_Q \end{bmatrix}$ with $C_j \in \mathbb{C}^{1 \times n_j}$ for every $j = 1, \ldots, Q$. Since (C, A) is observable by Assumption 4.1, it follows that (C_j, Λ_j) is also observable for every $j = 1, \ldots, Q$. Define

$$O_j := \begin{bmatrix} C_j \Lambda_j^{b_1} \\ \vdots \\ C_j \Lambda_j^{b_{\chi+1}} \end{bmatrix} \qquad \forall j = 1, \dots, Q.$$

Then $O(\{s_m\}_{m=0}^{\chi}) = \begin{bmatrix} O_1 & \cdots & O_Q \end{bmatrix} R^{-1}.$

Assume that $k \geq \tilde{\ell}_{\min}\zeta$, where $\ell_{\min} \in \mathbb{Z}_+$ is the minimum integer satisfying

$$\ell_{\min} > \frac{\Pi_d}{\zeta_j - n_j + 1 - \zeta_j \nu_d} \qquad \forall j = 1, \dots, Q.$$

Lemma B shows that the matrix O_j is full column rank for every $j = 1, \ldots, Q$, because

$$\frac{1}{\zeta} \le \frac{\zeta_j - n_j + 1}{\zeta_j} \qquad \forall j = 1, \dots, Q.$$

Assume further that there exists $v \in \mathbb{C}^{n_x}$ such that

$$O\left(\{s_m\}_{m=0}^{\chi}\right)Rv = \begin{bmatrix} O_1 & \cdots & O_Q \end{bmatrix} v = 0.$$

To prove that $O(\{s_m\}_{m=0}^{\chi})$ is full column rank, it is enough to show that v = 0. Partition v as $v = \begin{bmatrix} v_1^* & \cdots & v_Q^* \end{bmatrix}^*$ with $v_j \in \mathbb{C}^{n_j}$ for every $j = 1, \ldots, Q$. Suppose, to get a contradiction, $v_{j_0} \neq 0$ for some $j_0 = 1, \ldots, Q$. Since O_{j_0} is full column rank, there exists $\tilde{b} \in \{b_1, \ldots, b_{\chi+1}\}$ such that $C_{j_0} \Lambda_{j_0}^{\tilde{b}} v_{j_0} \neq 0$. Let $\tilde{b} \equiv \alpha \pmod{\zeta}$ with $0 \leq \alpha \leq \zeta - 1$ and define $w_j := C_j (\Lambda_j / \lambda_j)^{\alpha} v_j$ for each $j = 1, \ldots, Q$. Note that $w_{j_0} \neq 0$. Lemma C shows that there exists at most a finite number Z_{j_0} of non-negative integers k such that

$$\sum_{j=1}^{Q} w_j \lambda_j^k = 0.$$

Since $\Lambda_j^{\zeta} = \lambda_j^{\zeta} I_{n_j}$ for every $j = 1, \dots, Q$,

$$C_{\Lambda}\Lambda^{\psi\zeta+\alpha}v = \sum_{j=1}^{Q} C_{j}\Lambda_{j}^{\psi\zeta+\alpha}v_{j} = \sum_{j=1}^{Q} w_{j}\lambda_{j}^{\psi\zeta+\alpha} \quad \forall \psi \in \mathbb{Z}_{+}.$$

Therefore, if $\{b_1, \ldots, b_{\chi+1}\}$ contains more than Z_{j_0} elements in $\{\psi\zeta + \alpha : \psi \in \mathbb{Z}_+\}$, then

$$\begin{bmatrix} C_{\Lambda}\Lambda^{b_1}v\\ \vdots\\ C_{\Lambda}\Lambda^{b_{\chi+1}}v \end{bmatrix} = \begin{bmatrix} O_1 & \cdots & O_Q \end{bmatrix} v = 0$$

contradicts the above fact obtained from Lemma C. From the discussion above, it suffices to show that if the DoS duration bound ν_d satisfies (48), then for every $\alpha = 0, \ldots, \zeta - 1$ and every $Z \in \mathbb{Z}_+$, there exists $k \in \mathbb{Z}_+$ such that the set of time-steps $\leq k$ without DoS, $\{s_0, \ldots, s_{\chi}\}$, contains more than Z elements in $\{\psi\zeta + \alpha : \psi \in \mathbb{Z}_+\}$. To this end, we assume by contradiction that for every $k = \ell \zeta$ with $\ell \in \mathbb{Z}_+$, the number of elements in $\{s_0, \ldots, s_{\chi}\} \cup \{\psi\zeta + \alpha : \psi \in \mathbb{Z}_+\}$ does not exceed Z. Then $\Phi_d(\ell\zeta) \ge \ell - Z$. By Assumption 2.1, $\Phi_d(\ell\zeta) \le \Pi_d + \nu_d(\ell\zeta)$. We obtain

$$\ell - Z \le \Pi_d + \nu_d(\ell\zeta)$$

and hence $(1 - \zeta \nu_d)\ell \leq \Pi_d + Z$. By (48),

(70)
$$\ell \le \frac{\Pi_d + Z}{1 - \zeta \nu_d},$$

which contradicts for a sufficiently large $\ell \in \mathbb{Z}_+$. Moreover, $\{s_0, \ldots, s_{\chi}\}$ contains more than Z elements in $\{\psi\zeta + \alpha : \psi \in \mathbb{Z}_+\}$ for $k = (\ell_e + 1)\zeta$, where $\ell_e \in \mathbb{Z}_+$ is the maximum integer that does not exceed the right side of (70). This completes the proof.

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