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Cascades-based Leader-follower Formation-tracking and Stabilization of Multiple Nonholonomic Vehicles

Mohamed Maghenem Antonio Loría Elena Panteley

***Abstract*— It is known after [1] that nonholonomic systems on the plane cannot be stabilized on *arbitrary* leader's trajectories with piece-wise continuous velocities via continuous controllers, even time-varying. In this paper we identify a wide class of smooth bounded reference trajectories (that includes set-points) and we propose a smooth time-varying controller that stabilizes such trajectories, not only for one nonholonomic mobile robot, but also for a network of mobile robots in a leader-follower configuration. We provide new constructions of strict Lyapunov functions with which we establish uniform global asymptotic stability and other strong robustness properties in the formation-error-coordinates space. Thus, our stability statements are unique in the literature and our tools of analysis are original.**

***Index Terms*— Formation control, persistency of excitation, Lyapunov design, nonholonomic systems.**

I. Introduction

In control of nonholonomic vehicles three essential control problems may be identified: that of trajectory tracking, set-point stabilization, and, what we call, robust stabilization.

In the first case the vehicle is required to follow a trajectory generated by a virtual vehicle [2] and such trajectory is imposed not to vanish nor to be equal to zero [3]. In [4], for the first time in the literature, the condition of persistency of excitation (a concept well-known in adaptive control textbooks [5]) was introduced as a condition on the reference angular velocity, in order to achieve the tracking control objective (in that reference uniform global asymptotic stability is proved). Persistency of excitation is now recognized as a fundamental property to achieve tracking for nonholonomic systems [1]; see also [6], [7], [8].

Nonetheless, using a (tracking) controller that relies on persistency of excitation of the reference velocity is futile in the second scenario, that of set-point stabilization. As it is known from [9], apart from non-differentiable set-point controllers, these must be time-varying. In the seminal paper [10] the problem is solved for n-dimensional chain-form systems (for $n = 3$ we recover the kinematics model of the nonholonomic vehicle) using smooth control laws that rely on a function that depends on time and state. This function, roughly speaking, induces a certain excitation in the system that persists as long as the set-point objective is not attained. Inspired by [10], in [11] the concept of δ–persistency-of-excitation and the set-point controllers of the same name were introduced.

The third problem, robust stabilization, covers set-point stabilization and it pertains to the case in which the reference velocities vanish asymptotically. The terminology is motivated by the idea of regarding a vanishing reference as a set-point "perturbed" by a vanishing function. In that regard, robust stabilization is tantamount to establishing robustness of set-point stabilizing controllers with respect to (slowly) vanishing perturbations. Robust stabilization also generalizes the parking control problem, in which it is usually assumed that the reference velocities converge "fast", in the sense that they are assumed to be integrable functions of time —see, *e.g.*, [12] and [13].

Because tracking both persistently-exciting and vanishing trajectories are mutually exclusive scenarios, the controllers in the above cited-references apply only to one case or another only, and this is true for the greater part of the literature. Indeed, after the milestone paper [1], it is known that nonholonomic systems are not asymptotically stabilizable to arbitrary leader's trajectories with piece-wise continuous velocities via continuous controllers, not even if they are time-varying (the possibility of using discontinuous controllers is not analyzed in that reference and we are not aware of any other work for that matter). Therefore, designing a controller that applies indistinctly to tracking and (robust) stabilization is a very challenging problem that, to the best of our knowledge, has only been studied in [14]–[19] for the case of *one* leader and *one* follower and under various more stringent assumptions (such as integrability of vanishing reference velocities).

In this note we solve the simultaneous *formation* tracking-robust stabilization control problem (which includes tracking of both, persistently exciting and non-integrable vanishing trajectories, as well as set-point stabilization) for *multiagent* systems. This problem pertains to the case in which a swarm of autonomous unicycles are required to follow a trajectory generated by a virtual leader, while keeping a formation.

To the best of our knowledge, the only article in which the simultaneous tracking and robust stabilization control problem has been addressed for multiagent systems is [20]. The control design method in the latter reference follows the framework of [21] and establishes that the formation-errors converge to an arbitrarily small compact ball centered at the origin. Furthermore, the controller from [20] is centralized; hence, it is assumed that the leader's velocities are accessible to *all* the agents in the network.

In this note we adopt an efficient strategy of leader-follower formation-tracking control in which each vehicle follows only one leader, but each agent can have multiple followers. This

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approach leads naturally to a directed spanning-tree topology. Our controller is inspired by a clever idea introduced in [14] (albeit for one leader and one follower only), which consists in combining two control laws, one for tracking and one for parking (a particular case of robust stabilization where the reference velocities are assumed to be integrable). Although our main results are established under fairly relaxed conditions on the reference velocities (for instance no need of integrability), beyond the controller itself, our primary contribution is to establish strong stability and robustness properties, such as *integral input-to-state stability* (iISS) and *small input-to-state stability* (small ISS) —see [22]–[24] for definitions and statements. Significantly, we establish these properties by designing strict Lyapunov functions, based on the *Mazenc construction* —see [25], [26]. Furthermore, for the first time in the literature, we establish uniform global asymptotic stability, specifically, in the multiagent setting.

In the following section we describe the simultaneous tracking-and-robust-stabilization problem. In Section III we present our main results, for the case of two systems and for swarms of vehicles. We conclude with some remarks in Section IV. Due to page constraints much material has been left out of the original manuscript, see [27] for a more complete version of this work.

II. Model and problem formulation

Consider N nonholonomic vehicles moving on the plane with kinematics model

$$\left\{\begin{array}{rcl}\dot{x}_i &=& v_i\cos(\theta_i)\\ \dot{y}_i &=& v_i\sin(\theta_i)\\ \dot{\theta}_i &=& \omega_i,\quad i\in\{1\dots N\}.\end{array}\right. \tag{1}$$

The coordinates $x_i\in\mathbb{R}$ and $y_i\in\mathbb{R}$ determine the Cartesian position of a point on the ith vehicle with respect to a fixed reference frame and $\theta_i\in\mathbb{R}$ denotes the vehicle's orientation. The vehicle moves about with forward velocity $v_i=[\dot{x}_i+\dot{y}_i]^{1/2}$ and angular velocity $\omega_i=\dot{\theta}_i$.

Some times in the literature it is assumed that the vehicle's motion is fully described by the so-called simplified model (1) —see, *e.g.*, [14] and [15]. That is, v_i and ω_i are considered to be control inputs. In a more realistic model, however, the control inputs, which we denote here by $u_i\in\mathbb{R}^2$, are functions of the input torques applied at the steering wheels. In this case, the equations (1) are complemented by velocity-dynamics equations of the generic form

$$\dot{\eta}_i = F_i(t,\eta_i,z_i)+G_i(t,\eta_i,z_i)u_i \tag{2a}$$

$$\eta_i:=[v_i\ \omega_i]^\top,\ z_i:=[x_i\ y_i\ \theta_i]^\top, \tag{2b}$$

where the functions $F_i:\mathbb{R}_{\geq 0}\times\mathbb{R}^2\times\mathbb{R}^3$ and $G_i:\mathbb{R}_{\geq 0}\times\mathbb{R}^2\times\mathbb{R}^3$ may be defined in various ways. Most typically, (2a) are determined by the Euler-Lagrange equations, as for instance in [12], or they are expressed in terms of the system's Hamiltonian — see, *e.g.*, [28]. Our main statements are not restricted to either form; it is only assumed that F_i and G_i satisfy Caratheodory's conditions of local existence and uniqueness of solutions over compact intervals.

Now, it is assumed that the vehicles communicate according to a directed spanning-tree topology. That is, for each $i\leq N$, the ith robot receives the states of exactly one leader, labeled $(i-1)$. For the root-node vehicle, labeled $i=1$, the leader is the virtual reference robot, which is labeled r and whose kinematics model is :

$$\dot{x}_r = v_r\cos(\theta_r) \tag{3a}$$

$$\dot{y}_r = v_r\sin(\theta_r) \tag{3b}$$

$$\dot{\theta}_r = \omega_r. \tag{3c}$$

That is, the vector $z_r:=[x_r,\ y_r,\ \theta_r]^\top$ denotes the position and orientation reference coordinates and v_r, ω_r are given piece-wise continuous functions mapping $\mathbb{R}_{\geq 0}\to\mathbb{R}$ that represent the forward and angular reference velocities respectively.

For such swarm of nonholonomic vehicles the leader-follower formation control problem consists in making the vehicles: (i) acquire and maintain a specified physical formation relative to one another and (ii) follow the reference trajectories, $z_r(t)$, generated through Eqs. (3) after a pair of given reference velocities, (v_r,ω_r). As for the case of *one* pair of vehicles, the task of acquiring and maintaining a formation may be formulated in function of the relative positions, orientations and velocities of all the vehicles. For each $i\leq N$, let d_{xi} and d_{yi} denote given positive numbers and let

$$\begin{array}{rcl}p_{\theta i} &:=& \theta_{i-1}-\theta_i,\\ p_{xi} &:=& x_{i-1}-x_i-d_{xi},\\ p_{yi} &:=& y_{i-1}-y_i-d_{yi};\end{array}$$

by convention, $(\,\cdot\,)_0:=(\,\cdot\,)_r$. That is, the distances d_{xi} and d_{yi} define the position of any leader vehicle with respect to any follower and the swarm leader follows the virtual reference vehicle. Then, as in [2], we transform the error coordinates $(p_{\theta i},\,p_{xi},\,p_{yi})$ for each leader-follower pair from the global coordinate frame to local coordinates, *i.e.*, let

$$\begin{bmatrix}e_{\theta i}\\ e_{xi}\\ e_{yi}\end{bmatrix}:=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta_i) & \sin(\theta_i)\\ 0 & -\sin(\theta_i) & \cos(\theta_i)\end{bmatrix}\begin{bmatrix}p_{\theta i}\\ p_{xi}\\ p_{yi}\end{bmatrix}. \tag{4}$$

In these new coordinates the position errors,

$$e_i:=[e_{\theta i}\ e_{xi}\ e_{yi}]^\top, \tag{5}$$

satisfy the dynamics equations

$$\dot{e}_{\theta i} = \omega_{i-1}-\omega_i \tag{6a}$$

$$\dot{e}_{xi} = \omega_i e_{yi}-v_i+v_{i-1}\cos(e_{\theta i}) \tag{6b}$$

$$\dot{e}_{yi} = -\omega_i e_{xi}+v_{i-1}\sin(e_{\theta i}), \tag{6c}$$

where v_{i-1} and ω_{i-1} are, respectively, the forward and angular velocities of the leader vehicle.

Thus, the leader-follower formation control problem reduces to steering the trajectories of (6) to zero, *i.e.*, ensuring that

$$\lim_{t\to\infty} e_i(t)=0\qquad\forall i\leq N. \tag{7}$$

If the vehicle is considered to be velocity-controlled, this is tantamount to designing a control law $\eta_i^*:=[v_i^*\ \omega_i^*]^\top$ such that, setting $[v_i\ \omega_i]^\top=[v_i^*\ \omega_i^*]^\top$ in (6), the property (7) hold. In the case that the vehicle is force-controlled (as in this note) the leader-follower control problem consists in

designing control inputs $u_i := [u_{i1}\ u_{i2}]^\top$, for all $i \leq N$, such that (7) hold for the system (1)–(2), so η_i^* is considered as a reference trajectory for η_i in (2). More precisely, we address the following (open) problem.

Definition 1 (UGAS leader-follower formation control): Let $\eta_r := [v_r\ \omega_r]^\top$ be a piece-wise continuous function mapping $\mathbb{R}_{\geq 0} \to \mathbb{R}^2$ that generates, through Eqs. (3), trajectories $t \mapsto z_r(t)$. For the system (1)–(2), design a controller $u_i(t, z_{i-1}, \eta_{i-1}, z_i, \eta_i)$, as well as virtual controls v_i^* and ω_i^*, such that, defining

$$\tilde{v}_i := v_i - v_i^*, \quad \tilde{\omega}_i := \omega_i - \omega_i^*, \quad \text{and} \quad \tilde{\eta}_i := [\tilde{v}_i\ \tilde{\omega}_i]^\top, \tag{8}$$

the origin for the closed-loop system, *i.e.*, the equilibrium $\big\{\,(e_i, \tilde{\eta}_i) = (0,0)\,\big\}$, with $i \in \{1 \dots N\}$, is uniformly globally asymptotically stable. □

The UGAS leader-follower formation control problem for arbitrary piece-wise continuous reference velocities is impossible to solve even in the case that $N = 1$ (one-leader-one-follower scenario) via continuous controllers [1]. Nonetheless, below we identify a wide class of stabilizable reference velocities (bounded with bounded derivatives) that includes persistently-exciting, as well as set-points and slowly-vanishing ones.

Let the function $\eta_r : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$, $\eta_r := [v_r\ \omega_r]^\top$ be continuously differentiable and bounded with bounded first derivative. Then, the tracking control and robust stabilization problems evoked in the Introduction are defined as follows.

Tracking control problem: The control objective is to ensure (7) under the generic assumption that $\eta_r \not\equiv 0$. This may be formulated, as in [3], as the condition that

$$\lim_{t\to\infty} |\eta_r(t)| \neq 0, \tag{9}$$

or, as in [4], [6], in terms of requiring that η_r be persistently exciting, that is, that there exist T and $\mu > 0$ such that

$$\int_t^{t+T} |\eta_r(\tau)|^2 d\tau \geq \mu \quad \forall\, t \geq 0. \tag{10}$$

Robust stabilization problem: It is required to guarantee (7) under the assumption that

$$\lim_{t\to\infty} |\eta_r(t)| = 0. \tag{11}$$

This case covers the thoroughly-studied *set-point stabilization* problem as well as the so-called *parking* control problem, in which case the leader vehicle comes to a full stop.

Remark 1: The robust stabilization problem covers the parking problem in which is assumed that the reference velocities converge sufficiently fast, in the sense that there exists $\beta > 0$ such that

$$\int_0^{\infty} |\eta_r(\tau)| d\tau \leq \beta. \tag{12}$$

The parking problem is solved for instance, in [12]–[15]. •

III. Cascades-based leader-follower control

Our control approach relies on the separation of two control loops: one involving the kinematics equations (1) and one involving the dynamics equations (2), whence the term "cascades-based". The controllers are decentralized; for each vehicle we design a local controller that uses measurements of its own states z_i and η_i as well as the states of its leader, z_{i-1} and η_{i-1}.

A. One leader, one follower

Let $i \leq N$ be arbitrary, but fixed. It is required for the ith vehicle to follow its leader, indexed $i-1$ or, equivalently, to guarantee that (7) hold for the system (6). Inspired by the control method proposed in [14] we define

$$\begin{aligned} v_i^* &= v_{i-1}\cos(e_{\theta i}) + k_{xi} e_{xi} && \text{(13a)} \\ \omega_i^* &= \omega_{i-1} + k_{\theta i} e_{\theta i} + k_{yi} e_{yi} v_{i-1} \phi(e_{\theta i}) \\ &\quad + \rho_i(t) k_{yi} p_i(t) |e_{xyi}|, && \text{(13b)} \end{aligned}$$

where $e_{xyi} := [e_{xi}\ e_{yi}]^\top$ and the rest of the variables are defined as follows. The function $p_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is once continuously differentiable, bounded, and with bounded derivative $\dot{p}_i$. The function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ corresponds to the so-called $\mathrm{sinc}(\,\cdot\,)$ function, which is defined by $\phi(x) = \sin(x)/x$, and k_{xi}, k_{yi}, $k_{\theta i}$ are positive constants. Furthermore, we define

$$\rho_i(t) := \exp\left(-\int_0^t F\big(\eta_{i-1}(\tau)\big) d\tau\right) \tag{14}$$

where $F : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ is a piece-wise continuous function that satisfies the following conditions, by construction:

- if (10) holds then there exists $T_1 > 0$ and $\mu_1 > 0$ such that
$$\int_t^{t+T_1} F\big(\eta_r(s)\big)^2 ds \geq \mu_1, \quad \forall t \geq 0; \tag{15}$$
- if (11) holds then there exists $\beta > 0$ such that
$$\int_0^{\infty} F\big(\eta_r(s)\big) ds \leq \beta. \tag{16}$$

The first three terms on the right-hand side of (13b) guarantee the achievement of the tracking control goal of persistently-exciting reference velocities, while the fourth is added to achieve the robust stabilization goal in the case that the leader's velocities converge. That is, the function ρ_i plays the role of a "weighing" function in the sense that if the reference velocities are persistently exciting, $\rho_i \approx 0$ and the action of the third term in (13b), $k_{yi} e_{yi} v_{i-1} \phi(e_{\theta i})$, is enforced over that of the last. If, on the contrary, the leader velocities converge, $\rho_i(t)$ is designed to remain separated from zero so that the term $\rho_i(t) k_{yi} p_i(t) |e_{xyi}|$ compensate the effect of the third term, which is regarded as a vanishing perturbation. In other words, the expression (13b) merges two appropriately weighted control laws that, as we shall prove, are robust with respect to one another.

The role of the function F in the control design is highlighted by the following statement.

Proposition 1: Let $\bar{\eta}_r \geq \sup_{t\geq\infty} |\eta_r(t)|$ and $\alpha \in \mathcal{K}$. Then, the functional

$$F(\eta_r) := \begin{cases} 0 & \text{if } \eta_r \in (0, \frac{\mu}{2T\bar{\eta}_r}] \\ \alpha(|\eta_r|) & \text{otherwise} \end{cases} \tag{17}$$

is persistently exciting (*i.e.* (15) holds) if (10) holds and $F(\eta_r)$ is integrable (*i.e.* (16) holds) if (11) holds. □

Proof. Note that $F(\eta_r(t))$ is integrable if η_r converges since $F(\eta_r(t)) = 0$ for all $\eta_r \leq \frac{\mu}{2T\bar{\eta}_r}$ and (11) holds by assumption.

To prove that $F(\eta_r)$ is persistently exciting under (10) we use [29, Lemma 2], which states that if a function η_r is persistently exciting then, for every $t \geq 0$, there exists a non-null-measure interval

$$I_t := \{\tau \in [t, t+T] : |\eta_r(\tau)| \geq a := \mu/(2T\bar{\eta}_r)\},$$

such that $\text{meas}(I_t) \geq b := T\mu/(2T\bar{\eta}_r^2 - \mu)$. Therefore,

$$\int_t^{t+T} F(\eta_r(s))^2 ds \geq \alpha(a)^2 b > 0.$$

■

The idea of introducing a weighing function depending on the nature of the reference velocities is borrowed from [14]. The controller (13) is reminiscent of the controller in [19], which is restricted to the case of one leader and one follower in the particular scenarios of tracking and parking (integrable reference velocities). In the robust stabilization scenario, the controller (13) may also be compared, to some extent, to the controller in [13]. However, there are several important differences with respect to these references that must be underlined. Firstly, the definition of the "weighting" function ρ_i, in terms of F, gives extra degrees of freedom to the control design, relatively to that in [14] and [19], as shown by Proposition 1 above. On the other hand, our conclusions are more general in the sense that we show integral input-to-state stability —see Proposition 2 below— and uniform global asymptotic stability (UGAS) of the origin —see Corollary 1 and Proposition 3 farther down.

UGAS is fundamental because it guarantees total stability (also known as *local* input-to-state stability). Furthermore, constructive Lyapunov-based proofs, as we provide, permit to establish *global* properties such as strong integral input-to-state stability which, in turn, lead to establishing general statements for the full-dynamics model (1)–(2) —see Corollary 2 and Proposition 3; this is not possible from weaker statements on non-uniform convergence. To the best of our knowledge results of this nature have not been reported in the literature before.

Proposition 2: Let $i \leq N$ be arbitrarily fixed and consider the system (6) with state e_i, inputs ω_i and v_i, and exogenous signal $\eta_{i-1} = [v_{i-1}, \omega_{i-1}]^\top$ such that

$$\max\{|\eta_{i-1}|_\infty, |\dot{\eta}_{i-1}|_\infty\} \leq \bar{\eta}_{i-1}, \tag{18}$$

where $|\varphi|_\infty := \text{ess sup}_{t\geq 0}|\varphi(t)|$. Consider the virtual control laws (v_i^*, ω_i^*) as given by (13), (14)–(16), with the functions p_i and $\dot{p}_i$ being bounded and persistently exciting. Then, if $\tilde{v}_i$ and $\tilde{\omega}_i$ are bounded, the trajectories are forward complete (they exist on $[t_\circ, \infty)$). Moreover,

1) if (10) holds with η_r replaced by η_{i-1}, the closed-loop system is integral input-to-state stable with respect to the input $\tilde{\eta}_i$. Furthermore, if $\tilde{\eta}_i$ tends to zero and is square integrable, the limit in (7) holds.
2) If, alternatively, (11) holds with η_r replaced by η_{i-1} the closed-loop system is small input-to-state stable [24] with respect to the input $\tilde{\eta}_i$. Thus, if $\tilde{\eta}_i$ converges to zero the limit in (7) holds. □

Sketch of proof of Proposition 2 (for a detailed proof see [27]): we use $\omega_i = \tilde{\omega}_i + \omega_i^*$ and $v_i = \tilde{v}_i + v_i^*$ in (6), together with (13) to write the error-dynamics equations as

$$\dot{e}_i = A_{v_{i-1}}(t, e_i)e_i + B_{1i}(t, e_i)\rho_i(t) + B_{2i}(e_i)\tilde{\eta}_i, \tag{19}$$

where

$$A_{v_{i-1}} := \begin{bmatrix} -k_{\theta i} & 0 & -v_{i-1}(t)k_{yi}\phi(e_{\theta i}) \\ 0 & -k_{xi} & \varphi_i(t, e_i) \\ v_{i-1}(t)\phi(e_{\theta i}) & -\varphi_i(t, e_i) & 0 \end{bmatrix},$$

$$B_{1i} := \begin{bmatrix} -k_{yi}p_i(t)|e_{xyi}| \\ k_{yi}p_i(t)|e_{xyi}|e_{yi} \\ -k_{yi}p_i(t)|e_{xyi}|e_{xi} \end{bmatrix}, \quad B_{2i} := \begin{bmatrix} 0 & -1 \\ -1 & e_{yi} \\ 0 & -e_{xi} \end{bmatrix},$$

and $\varphi_i(t, e_i) := \omega_{i-1} + k_{\theta i}e_{\theta i} + k_{yi}e_{yi}v_{i-1}\phi(e_{\theta i})$.

Forward completeness of (19) (*i.e.*, that the solutions exist for all $t \geq 0$) may be established by evaluating the total derivative of the positive-definite function

$$V_{1i}(e_i) := \frac{1}{2}\left[e_{xi}^2 + e_{yi}^2 + \frac{1}{k_{yi}}e_{\theta i}^2\right], \quad i \in \{1, \ldots, N\}$$

to obtain

$$\dot{V}_{1i} = -k_{xi}e_{xi}^2 - \frac{k_{\theta i}}{k_{yi}}e_{\theta i}^2 - p_i\rho_i e_{\theta i}|e_{xyi}| - \frac{1}{k_{yi}}e_{\theta i}\tilde{\omega}_i - e_{xi}\tilde{v}_i.$$

After the uniform boundedness of p_i, ρ_i, $\tilde{\omega}_i$, and $\tilde{v}_i$ and the triangle inequality, it follows that there exist positive constants a_i and b_i such that

$$\dot{V}_{1i}(e_i(t)) \leq a_i V_{1i}(e_i(t)) + b_i$$

which, upon integration from any $t_\circ$ to ∞ leads to the conclusion that the solutions have no finite escape-time.

The rest of the analysis is made separately, for the tracking and robust-stabilization scenarios. In the first case (10), and consequently (15), hold with η_r replaced by η_{i-1} (recall that in this proof η_{i-1} is an exogenous signal). Then, the analysis of the closed-loop equation (19) follows these steps:

1) to design a strict Lyapunov function $V_i(t, e_i)$ for the "nominal" system $\dot{e}_i = A_{v_{i-1}}(t, e_i)e_i$;
2) based on the latter, to construct a strict Lyapunov function $W_i(t, e_i)$ for the system
$$\dot{e}_i = A_{v_{i-1}}(t, e_i)e_i + B_{1i}(t, e_i)\rho_i; \tag{20}$$
3) in turn, to use $W_i(t, e)$ to construct a Lyapunov function $W_{1i}(t, e)$ to establish integral input-to-state stability of (19) with respect to $\tilde{\eta}_i$, as well as the boundedness of the trajectories of (19) under the assumption that $\tilde{\eta}_i \in \mathcal{L}_2$.

Due to space constraints, we only provide the main guidelines to these steps.

Step 1) Uniform global asymptotic stability for

$$\dot{e}_i = A_{v_{i-1}}(t, e_i)e_i \tag{21}$$

is established in [30] via Lyapunov's direct method. Indeed, after [30, Proposition 1], there exists a positive definite radially unbounded function $V_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$,

$$\begin{aligned} V_i(t, e_i) \quad := \quad & P_{[2]}(t, V_{1i})V_{1i}(e_i) - \omega_{i-1}(t)e_{xi}e_{yi} \\ & + v_{i-1}(t)P_{[1]}(t, V_{1i})e_{\theta i}e_{yi}, \end{aligned} \tag{22}$$

satisfying

$$F_{[3]}(V_{1i}(e_i)) \leq V_i(t,e_i) \leq S_{[3]}(V_{1i}(e_i)), \tag{23}$$

where

$$V_{1i}(e_i) := \frac{1}{2}\left[e_{xi}^2 + e_{yi}^2 + \frac{1}{k_{yi}}e_{\theta i}^2\right],$$

$F_{[k]}$, $S_{[k]}$, and $P_{[k]}(t,\cdot)$ are smooth polynomials in V_{1i} with strictly positive coefficients of degree k, and $P_{[k]}(\cdot, V_{1i})$ is uniformly bounded. Furthermore, a direct computation shows that the total derivative of V_{1i} along the trajectories of (21) satisfies

$$\dot{V}_{1i}(e_i) = -k_{xi}e_{xi}^2 - \frac{k_{\theta i}}{k_{yi}}e_{\theta i}^2. \tag{24}$$

Hence, mimicking [30] one finds that the total derivative of $V_i(t,e_i)$ along the trajectories of (21) satisfies

$$\dot{V}_i(t,e_i) \leq -\sigma V_{1i}(e_i) - k_{xi}e_{xi}^2 - \frac{k_{\theta i}}{k_{yi}}e_{\theta i}^2$$

where $\sigma > 0$ is a design parameter that depends on μ and T introduced in (10). Uniform global asymptotic stability follows.

Step 2) Let $Q_{[3]} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be another third-order polynomial in V_{1i} with strictly positive coefficients and define

$$Z_i(t,e_i) := Q_{[3]}(V_{1i})V_{1i}(e_i) + V_i(t,e_i). \tag{25}$$

In view of the fact that $\partial Q_{[3]}/\partial V_{1i} \geq 0$, and after (24), the total derivative of Z_i along the trajectories of (21) yields

$$\begin{aligned}\dot{Z}_i(t,e_i) &\leq -Y_i(e_i)\\ Y_i(e_i) &:= \sigma V_{1i}(e_i) + Q_{[3]}(V_{1i}(e_i))\big[k_{xi}e_{xi}^2 + k_{\theta i}e_{\theta i}^2\big].\end{aligned}$$

Note that Y_i is positive definite and radially unbounded.

On the other hand, from (14) we see that $\dot{\rho}_i = -F(\eta_{i-1}(t))\rho_i$. From this and (15) (in which we replace η_r with η_{i-1}) it follows that $\rho_i \to 0$ exponentially fast (and is uniformly integrable). Therefore, for any $\gamma > 0$, the function

$$G_i(t) := \exp\left(-\gamma \int_0^t \rho_i(s)ds\right) \quad \forall t \geq 0$$

is bounded from above and below. Indeed, defining $G_m := \lim_{t\to\infty} G_i(t) > 0$ and we have $G(t) \in [G_m, 1]$ for all $t \geq 0$. In addition, since $Z_i(t,e_i)$ and $V_i(t,e_i)$ are positive definite radially unbounded —see (23) and (25), so is the function

$$W_i(t,e_i) := G_i(t)Z_i(t,e_i).$$

After somewhat lengthy computations we find that the total derivative of W_i along the trajectories of (20) verifies $\dot{W}(t,e_i) \leq -G_m Y_i(e_i)$ for all $t \geq 0$ and all $e_i \in \mathbb{R}^3$, so uniform global asymptotic stability of the null solution of (20) follows. See [27] for further details.

Step 3) In order to establish iISS of (19) with respect to $\tilde{\eta}_i$ we introduce the positive definite radially unbounded function $W_{1i} : \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$, defined by

$$W_{1i}(t,e_i) := \ln\left(1 + W_i(t,e_i)\right).$$

After long, but straightforward computations one finds that the derivative of W_{1i} along trajectories of (19) satisfies

$$\dot{W}_{1i} \leq -\frac{G_m}{2}\frac{Y_i(e_i)}{1 + Q_{[3]}(V_{1i})V_{1i}} + \frac{\epsilon}{2}\frac{D_{[3]}(V_{1i})}{1 + G_m Q_{[3]}(V_{1i})V_{1i}}\,|\tilde{\eta}_i|^2$$

where $D_{[3]}$ is a third-order polynomial of V_{1i}. We conclude that there exist a constant $c > 0$ and a positive definite function $\alpha : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ such that

$$\dot{W}_{1i} \leq -\alpha(e_i) + c\,|\tilde{\eta}_i|^2 \tag{26}$$

so the statement follows from [31]. This concludes the first part of the proof (in the tracking scenario).

In the stabilization scenario, for any *fixed* i we define $\xi_i^\top := [\eta_{i-1}^\top \;\; \tilde{\eta}_i^\top]$ and we rewrite the closed-loop equation (19) as

$$\dot{e}_i = A_i(t,e_i)e_i + B_i(e_i)\xi_i \tag{27}$$

where

$$A_i := \begin{bmatrix} -k_{\theta i} & -k_{yi}q_i(t)\frac{|e_{xyi}|}{e_{xi}} & -k_{yi}q_i(t)\frac{|e_{xyi}|}{e_{yi}} \\ 0 & -k_{xi} & \pi_i(t,e_i) \\ 0 & -\pi_i(t,e_i) & 0 \end{bmatrix},$$

$$B_i := \begin{bmatrix} -k_{yi}e_{yi}\phi(e_{\theta i}) & 0 & 0 & -1 \\ k_{yi}e_{yi}^2\phi(e_{\theta i}) & e_{yi} & -1 & e_{yi} \\ \sin(e_{\theta i}) - k_{yi}e_{xi}e_{yi}\phi(e_{\theta i}) & -e_{xi} & 0 & -e_{xi} \end{bmatrix},$$

$q_i(t) := \rho_i(t)p_i(t)$, ρ_i is defined in (14), $\pi_i := k_{\theta i}e_{\theta i} + k_{yi}q_i(t)|e_{xyi}|$, and $e_{xyi} = [e_{xi} \;\; e_{yi}]^\top$. Then, we have the following:

Claim 1: The system (27) is small-input-to-state stable respect to ξ_i.

Claim 2: The system (27) is integral-input-to-state stable with respect to ξ_i.

In view of these facts the system (27) is strong integral-input-to-state stable with respect to the input ξ_i, hence the property also holds with respect to the input $\xi_i^\circ := [\eta_{i-1}^\top \; 0]^\top$. By virtue of Lemma 1 in the appendix (with $u = \xi_i^\circ$) and the condition that $\eta_{i-1} \to 0$, which holds by assumption, it follows that the system subject to $\tilde{\eta}_i = 0$ is uniformly globally asymptotically stable. Then, to establish small-input-to-state stability of the system (27) with respect to $\tilde{\eta}_i$, it is left to show that it possesses the so-called *small-input-bounded-state* property with respect to $\tilde{\eta}_i$, for any converging $t \mapsto \eta_{i-1}$. To that end, pick any small $\epsilon > 0$ and let $|\tilde{\eta}_i| \leq \epsilon/2$. Since the system is forward complete and $\eta_{i-1}(t) \to 0$ it follows that there exists a sufficiently large $T_\epsilon > 0$ such that $|\eta_{i-1}(t)| \leq \epsilon/2$ for all $t \geq t_\circ + T_\epsilon$ and $|\xi_i(t)| \leq \epsilon$. On the other hand, the system (27) is small-input-to-state stable with respect to ξ_i hence, the solutions are bounded. This concludes the proof of small-input-to-state stability with respect to $\tilde{\eta}_i$.

The proofs of Claims 1 and 2, on the other hand, rely on the construction of a strict Lyapunov function for the nominal closed-loop system $\dot{e}_i = A_i(t,e_i)$ —*cf.* Eq. (27). Let $\psi_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a twice-continuously-differentiable function, satisfying

$$\dot{\psi}_i = -k_{\theta i}\psi_i + k_{yi}q_i(t) \tag{28}$$

and let $e_{zi} := e_{\theta i} + \psi_i(t)|e_{xyi}|$. Then, the nominal system $\dot{e}_i = A_i(t,e_i)e_i$ becomes

$$\begin{bmatrix}\dot{e}_{xi}\\ \dot{e}_{yi}\end{bmatrix} = \begin{bmatrix} -k_{xi} & \dot{\psi}_i|e_{xyi}| \\ -\dot{\psi}_i|e_{xyi}| & 0 \end{bmatrix}\begin{bmatrix}e_{xi}\\ e_{yi}\end{bmatrix} + e_{zi}\begin{bmatrix}0 & k_{\theta i}\\ -k_{\theta i} & 0\end{bmatrix}\begin{bmatrix}e_{xi}\\ e_{yi}\end{bmatrix} \tag{29a}$$

$$\dot{e}_{zi} = -k_{\theta i}e_{zi} - \psi_i k_{xi}\frac{e_{xi}^2}{|e_{xyi}|} \tag{29b}$$

We stress that, by construction, ρ_i and $\dot{\rho}_i$ are bounded and, by assumption, so are p_i and $\dot{p}_i$. It follows that q_i and $\dot{q}_i$, and in turn ψ_i and $\dot{\psi}_i$, are also bounded. Moreover, since p_i and $\dot{p}_i$ are persistently exciting, so is $\dot{q}_i$ —*cf.* [32] and, consequently, there exist $\psi_M > \psi_m > 0$ such that $\psi_i(t) \in [\psi_m, \psi_M]$ for all $t \geq 0$ —see [33]. Furthermore, since $\dot{q}_i$ is persistently exciting and $\dot{\psi}_i$ satisfies

$$\ddot{\psi}_i = -k_{\theta i}\dot{\psi}_i + \dot{q}_i, \tag{30}$$

it follows that $\dot{\psi}_i$ is also persistently exciting —see [5, Lemma 4.8.3]. Thus, one can show that the following is a strict Lyapunov function for (29):

$$V_{2i} := P_{[1]}(V_{1i})V_{1i} + \Upsilon(t)V_{1i}^2 - \dot{\psi}_i\sqrt{V_{1i}}e_{xi}e_{yi} + Q_{[1]}(V_{1i})e_{zi}^2 \tag{31}$$

where $V_{1i} := e_{xi}^2 + e_{yi}^2$, the function Υ is defined as

$$\Upsilon(t) := 1 + \bar{\psi}_i^2 T - \frac{1}{T}\int_t^{t+T}\int_t^m \dot{\psi}_i(s)^2 ds dm, \tag{32}$$

—*cf* [25], [34], $\bar{\psi}_i \geq \max\{\,|\psi_i|_\infty,\ |\dot{\psi}_i|_\infty,\ |\ddot{\psi}_i|_\infty\}$, and $P_{[1]}$ and $Q_{[1]} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are first-order polynomials of V_{1i}:

$$P_{[1]}(V_{1i}) := \bar{\psi}_i\big[V_{1i} + 1\big] \tag{33}$$

$$Q_{[1]}(V_{1i}) := \frac{P_{[1]}(V_{1i})}{2} + V_{1i}. \tag{34}$$

Note that since $-\dot{\psi}_i\sqrt{V_{1i}}e_{xi}e_{yi} \geq -(\bar{\psi}_i/2)V_{1i}[V_{1i}+1]$ and $\Upsilon(t) \geq 1$, we have

$$V_{2i}(t, e_i) \geq Q_{[1]}(V_{1i})\big[V_{1i} + e_{zi}^2\big] \tag{35}$$

so V_{2i} is positive definite and radially unbounded. Furthermore, mimicking the proof of [13, Proposition 2], one finds that there exists $\sigma > 0$ such that the derivative of V_{2i} along the trajectories of (27) with $\xi_i \equiv 0$ satisfies

$$\dot{V}_{2i}(t, e_i) \leq -\frac{1}{2}k_{\theta i}Q_{[1]}(V_{1i})e_{zi}^2 - \sigma V_{1i}^2 + \frac{\partial V_{2i}}{\partial e_i}B_i(e_i)\xi_i$$

and after lengthy but straightforward computations, one finds that

$$|\xi_i| \leq \min\{c_4,\, c_5,\, \chi(|e_i|)\} \Longrightarrow \dot{V}_{2i} \leq -\frac{\sigma}{4}V_{1i}(e_i)^2$$

where χ is a class-$\mathcal{K}$ function defined as

$$\chi(|e_i|) := \min\left\{c_2 V_{1i}(e_i)^{1/2},\, c_3 V_{1i}(e_i),\, c_4 V_{1i}(e_i)^{3/2}\right\}$$

and c_2, c_3 , and c_4 are positive constants. Small-input-to-state stability with respect to ξ_i follows.

For the proof of Claim 2 we use the positive-definite radially unbounded function $W_{2i} : \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$, defined as

$$W_{2i}(t, e_i) = \ln\left(1 + V_{2i}(t, e_i)\right)$$

which satisfies

$$\dot{W}_2 \leq -\alpha(|e_i|) + c\,|\xi_i|\,.$$

—see [27] for further details. This implies integral input-to-state stability. ■

It is worth to emphasize the following statements that cover others from the literature.

Corollary 1 (UGAS of the kinematics model): Under the conditions of Proposition 2, for the system (6) in closed loop with (13), (14)–(16) and setting $\tilde{\eta}_i = 0$, the origin is uniformly globally asymptotically stable. □

Corollary 2 (Robustness of the full model): Under the conditions of Proposition 2, for any bounded reference trajectories, may they satisfy (10) or (11), the control goal (7) is achieved under the action of *any* controller u_i guaranteeing that

$$\lim_{t\to\infty} |\tilde{\eta}_i(t)| = 0 \tag{36}$$

holds and $\tilde{\eta}_i \in \mathcal{L}_2$. □

Our strongest statement for the case of two vehicles in a leader-follower configuration, and which to the best of our knowledge has no precedent in the literature, is the following.

Proposition 3 (UGAS of the full model): Consider the system (2) under the action of *any* controller u_i guaranteeing uniform global asymptotic stability of the origin $\{\tilde{\eta}_i = 0\}$ with error trajectories satisfying $\tilde{\eta}_i(t) \in \mathcal{L}_2$. Then, under the conditions of Proposition 2, for the system (1)-(2) the origin $\{\,(\tilde{e}_i, \tilde{\eta}_i) = (0,0)\ i \leq N\}$ is uniformly globally asymptotically stable. □

Proof. We use cascades-systems theory (in particular, [35, Lemma 2]) and Proposition 2. Let u_i be a given controller for the dynamics equations (2), depending on the leader and follower's states, as well as on the virtual control laws (13). Then, by a suitable change of variable the closed-loop equations take the generic form

$$\dot{\tilde{\eta}}_i = F_{\eta i}(t, \tilde{\eta}_i, e_i), \tag{37}$$

while (21) may be written in the compact form

$$\dot{e}_i = F_{ei}(t, e_i) + G_{ei}(t, e_i)\tilde{\eta}_i. \tag{38}$$

Note that the dependence on t of the functions F_{ei} and G_{ei} comes from the fact that these depend actually on the leader *trajectories* $\eta_{i-1}(t)$ and $e_{i-1}(t)$ which are forward complete. Similarly, in (37) we replace e_i by complete trajectories $e_i(t)$ so the overall closed-loop equations cover a cascaded form

$$\dot{e}_i = F_{ei}(t, e_i) + G_{ei}(t, e_i)\tilde{\eta}_i \tag{39}$$

$$\dot{\tilde{\eta}}_i = \tilde{F}_{\eta i}(t, \tilde{\eta}_i) \tag{40}$$

where $\tilde{F}_{\eta i}(t, \tilde{\eta}_i) := F_{\eta i}(t, \tilde{\eta}_i, e_i(t))$ —*cf.* [36], [37, p. 627].

After [35, Lemma 2] the origin $(e_i, \tilde{\eta}_i) = (0, 0)$ is uniformly globally asymptotically stable if so are the respective origins for the systems (40) and $\dot{e}_i = F_{ei}(t, e_i)$ and if the solutions of (39) are uniformly globally bounded. UGAS for (40) holds by assumption. Then, after Proposition 2, if (10) holds the system (39) is integral-input-to-state stable while, if (11) holds it is small input-to-state stable. On the other hand, either of these conditions implies the so-called 0-UGAS property, that is, uniform global asymptotic stability of the origin without input —this corresponds to the statement of Corollary 1.

Finally, uniform global boundedness follows, under condition (10), from the integral-input-to-state-stability property and the assumption that $\tilde{\eta}_i \in \mathcal{L}_2$. Under condition (11) it follows from the property of small input-to-state stability and forward completeness (see Proposition 2).

This completes the proof of Proposition 3. ■

B. Leader-follower formation control

Consider now a swarm of autonomous vehicles ($N \geq 2$) which are required to follow a reference vehicle that is modeled by (3) and describes a trajectory that either converges, diverges or has both behaviors sequentially. The standing assumption is that the vehicles communicate with each other over a spanning-tree-topology network hence, each vehicle has only one leader but may have several followers.

Proposition 4: Consider the system (1), (2). Let $\eta_r = [v_r \ \omega_r]^\top$ be a given piece-wise continuous function satisfying either (10) or (11) and assume that there exists $\bar{\eta}_r > 0$ such that

$$\max\left\{|\eta_r|_\infty, \ |\dot{\eta}_r|_\infty\right\} \leq \bar{\eta}_r. \tag{41}$$

For each $i \leq N$ consider the expressions of v_i^* and ω_i^* as in (13) (with $v_0 := v_r$ and $\omega_0 := \omega_r$) where:

(i) k_{xi}, k_{yi}, $k_{\theta i}$ are positive constants;
(ii) the functions p_i and $\dot{p}_i$ are bounded and persistently exciting.

Then, for any given control laws u_{i1} and u_{i2} guaranteeing that $\tilde{\eta}_i$ is square integrable and converges to zero, the control objective (7) holds.

Furthermore, define $\tilde{\boldsymbol{\eta}} := [\tilde{\eta}_1 \cdots \tilde{\eta}_N]^\top$, $\boldsymbol{\eta}^* := [\eta_1^* \cdots \eta_N^*]^\top$, and $\boldsymbol{e} := [e_1 \cdots e_N]^\top$. If $\{\tilde{\boldsymbol{\eta}} = 0\}$ for (40) is uniformly globally asymptotically stable (UGAS) and $\tilde{\boldsymbol{\eta}} \in \mathcal{L}_2$ then, for the closed-loop system (39)-(40), $\{(\boldsymbol{e}, \tilde{\boldsymbol{\eta}}) = (0,0)\}$ is also UGAS. Consequently, if $\tilde{\boldsymbol{\eta}} \equiv 0$ then $\{\boldsymbol{e} = 0\}$ for (1) in closed loop with $\boldsymbol{\eta}^*$ is UGAS. □

Proof. The proof consists in applying recursively the statement of Proposition 2 for each $i \leq N$ that is, for each pair of leader-follower vehicles whose closed-loop equations are given by (19) which, all together correspond to

$$\begin{aligned}
\dot{e}_N &= A_{v_{N-1}}(t, e_N)e_N + B_{1N}(t, e_N)\rho_N + B_{2N}(e_N)\tilde{\eta}_N \\
&\vdots \qquad\qquad (42a)\\
\dot{e}_2 &= A_{v_1}(t, e_2)e_2 + B_{12}(t, e_2)\rho_2 + B_{22}(e_2)\tilde{\eta}_2 \qquad (42b)\\
\dot{e}_1 &= A_{v_r}(t, e_1)e_1 + B_{11}(t, e_1)\rho_1 + B_{21}(e_1)\tilde{\eta}_1 \qquad (42c)
\end{aligned}$$

Indeed, Proposition 2 guarantees the asymptotic convergence of the formation errors whether the leader velocities are persistently exciting or converging. Therefore, the properties of $(i-1)$th leader's velocities are propagated to the ith follower and, in turn, to the $(i+1)$th vehicle down to the leaf nodes in the graph.

As before, even though the ith vehicle the dynamics equations depend on e_i and, through $\eta_{i-1} = [v_{i-1} \ \omega_{i-1}]^\top$, on the states of the vehicles above in the graph, up to the reference vehicle, in view of forward completeness (see the proof of Proposition 2), for the purpose of analysis the velocities η_{i-1} may be regarded as exogenous signals. This allows us to consider the system as a multi-*cascaded* time-varying one — see [36]. Then, we may invoke Proposition 2 recursively. Such reasoning, however, relies on distinct analyses for the tracking and robust-stabilization scenarios.

Under the tracking scenario. Let (10) hold. Let $i = 1$ and consider the equation (42c) which corresponds to the dynamics of the error trajectories between the virtual vehicle and the swarm leader (indexed $i = 1$). By Proposition 2 the system (42c) is integral input-to-state stable with respect to the input $\tilde{\eta}_1 := [\tilde{v}_1 \ \tilde{\omega}_1]^\top$. Moreover, since by assumption $\tilde{\eta}_1(t)$ is square-integrable and converges to zero, it follows that $e_1 \to 0$ so, consequently, $v_1^* \to v_r$, $\omega_1^* \to \omega_r$ and, in turn,

$$\lim_{t\to\infty} v_1(t) = v_r(t), \qquad \lim_{t\to\infty} \omega_1(t) = \omega_r(t). \tag{43}$$

Furthermore, there exists $\bar{c}_1 > 0$ such that

$$\max\left\{|v_1|_\infty, \ |\dot{v}_1|_\infty, \ |\omega_1|_\infty, \ |\dot{\omega}_1|_\infty\right\} \leq \bar{c}_1. \tag{44}$$

For $i = 2$ we consider the equation (42b). We see that v_1 and ω_1, regarded as functions of complete solutions, have the same properties as v_r and ω_r. Therefore, A_{v_1} may be considered as a function of time and the state e_2. Consequently, it has similar properties to those of A_{v_r} and, by Proposition 2, we conclude that (42b) is integral input-to-state stable with respect to the input $\tilde{\eta}_2 := [\tilde{v}_2 \ \tilde{\omega}_2]^\top$, provided that η_1 is persistently exciting. The latter indeed follows from (10), (43) and (44) —see [32]. Thus, after Proposition 2, the system (42b) is integral input-to-state stable with respect to $\tilde{\eta}_2$. Next, in view of forward completeness, the assumption that $\tilde{\eta}_2(t) = [\tilde{v}_2(t) \ \tilde{\omega}_2(t)]^\top$ is square integrable and converges, we conclude that

$$\lim_{t\to\infty} |e_2(t)| = 0, \quad \lim_{t\to\infty} v_2(t) = v_1(t), \quad \lim_{t\to\infty} \omega_2(t) = \omega_1(t).$$

and, moreover, there exists $\bar{c}_2 > 0$ such that

$$\max\left\{|v_2|_\infty, \ |\dot{v}_2|_\infty, \ |\omega_2|_\infty, \ |\dot{\omega}_2|_\infty\right\} \leq \bar{c}_2. \tag{45}$$

The previous arguments apply for any $i \geq 2$ so the statement of Proposition 4 under condition (10) follows by induction.

Under the robust-stabilization scenario. By assumption, (11) holds. As in the previous scenario, the proof follows using Proposition 2 recursively. Indeed, for $i = 1$, we conclude that the error dynamics corresponding to the swarm leader and the virtual reference vehicle is small-input-to-state stable with respect to $\tilde{\eta}_1 := [\tilde{v}_1 \ \tilde{\omega}_1]^\top$. Consequently, after forward completeness of trajectories, we have

$$\tilde{\eta}_1 \to 0 \quad \Longrightarrow \quad e_1 \to 0 \quad \Longrightarrow \quad v_1 \to 0, \quad \omega_1 \to 0.$$

The last implication follows from (11). In turn, in view of the convergence of v_1 and ω_1, it follows that for $i = 2$ the closed-loop (42b) is small-input-to-state stable with respect to the input $\tilde{\eta}_2 := [\tilde{v}_2 \ \tilde{\omega}_2]^\top$. Consequently, after forward completeness of trajectories, we have

$$\tilde{\eta}_2 \to 0 \quad \Longrightarrow \quad e_2 \to 0 \quad \Longrightarrow \quad v_2 \to 0, \quad \omega_2 \to 0.$$

The statement that (7) holds, follows by induction.

The proof of UGAS follows by applying the same cascades argument as in the proof of Proposition 3, recursively. ■

Remark 2: It should be clear from the proof of Proposition 3 that it is the construction of strict Lyapunov functions and the statement of Proposition 2 (integral input-to-state stability and small-input-to-state stability) which allow the generalization to the multi-agent case under a spanning tree communication topology. Therefore, such extension is not possible, *e.g.*, for the main results in [14] and [19] which rely on the assumption that the leader velocities are integrable.

Another fundamental statement in the proof is Lemma 1 in the appendix, which is one more original contribution of this note, but its proof is omitted due to page constraints. •

IV. Conclusion

We have established the stability and robustness of a controller for simultaneous tracking and stabilization of swarms of autonomous nonholonomic vehicles via Lyapunov's direct method for multiagent systems. We believe that they may contribute to pave the way to incorporate realistic scenarios, such as output-feedback control, more general interconnection topologies, and time-varying graphs; research in such directions is being carried out.

Other interesting extensions to be explored involve generalizing the setting considered here in terms of the graph topology: connected graphs, time-varying interconnections *etc*.

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Appendix

Lemma 1: Consider the dynamical system

$$\dot{x} = f(x, u(t)), \quad x \in \mathbb{R}^n \tag{46}$$

$$0 = f(0, v), \quad \forall\ v \in \mathcal{D} \subset \mathbb{R}^m, \tag{47}$$

where $u : \mathbb{R}_{\geq 0} \to \mathcal{D} \subset \mathbb{R}^m$ is locally integrable and the function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz in x uniformly in v for all $v \in \mathcal{D}$.

Assume that $\dot{x} = f(x, u)$ is strong iISS with respect to u. Then, if in addition $u(t) \to 0$ as $t \to \infty$ then the origin of (46) is uniformly globally asymptotically stable. □

Proof. See [27]. ■