Adding integral action for open-loop exponentially stable semigroups and application to boundary control of PDE systems

A. Terrand-Jeanne, V. Andrieu, V. Dos Santos Martins, C.-Z. Xu

***Abstract*—The paper deals with output feedback stabilization of exponentially stable systems by an integral controller. We propose appropriate Lyapunov functionals to prove exponential stability of the closed-loop system. An example of parabolic PDE (partial differential equation) systems and an example of hyperbolic systems are worked out to show how exponentially stabilizing integral controllers are designed. The proof is based on a novel Lyapunov functional construction which employs the forwarding techniques.**

I. Introduction

The use of integral action to achieve output regulation and cancel constant disturbances for infinite dimensional systems has been initiated by S. Pojohlainen in [12]. It has been extended in a series of papers by the same author (see [13] for instance) and some other (see [23]) always considering bounded control operator and following a spectral approach (see also [11]).

In the last two decades, Lyapunov approaches have allowed to consider a large class of boundary control problems (see for instance [2]). In this work our aim is to follow a Lyapunov approach to solve an output regulation problem. The results are separated into two parts.

In a first part, abstract Cauchy problems are considered. It is shown how a Lyapunov functional can be constructed for a linear system in closed loop with an integral controller when some bounds are assumed on the control operator and for an admissible measurement operator. This gives an alternative proof to the results of S. Pojohlainen in [12] (and [23]). It allows also to give explicit value to the integral gain that solves the output regulation problem.

In a second part, following the same Lyapunov functional design procedure, we consider a boundary regulation problem for a class of hyperbolic PDE systems. This result generalizes many others which have been obtained so far in the regulation of PDE hyperbolic systems (see for instance [8], [24], [3], [20], [21], [18]).

The paper is organized as follows. Section II is devoted to the regulation of the measured output for stable abstract Cauchy problems. It is given a general procedure, for an exponentially stable semigroup in open-loop, to construct a Lyapunov functional for the closed loop system obtained with an integral controller. Inspired by this procedure, the case of boundary regulation is considered for a general class of hyperbolic PDE systems in Section III. The proof of the theorem obtained in the context of hyperbolic systems is given in Section IV.

This paper is an extended version of the paper presented in [16]. Compare to this preliminary version all proofs are given and moreover more general classes of hyperbolic systems are considered.

Notation: subscripts t, s, tt, $\dots$ denote the first or second derivative w.r.t. the variable t or s. For an integer n, $\mathrm{I}_{\mathrm{d}n}$ is the identity matrix in $\mathbb{R}^{n\times n}$. Given an operator $\mathcal{A}$ over a Hilbert space, $\mathcal{A}^*$ denotes the adjoint operator. $\mathcal{D}_n$ is the set of diagonal matrix in $\mathbb{R}^{n\times n}$.

II. General Abstract Cauchy problems

A. Problem description

Let $\mathcal{X}$ be a Hilbert space with scalar product $\langle,\rangle_{\mathcal{X}}$ and $\mathcal{A}: D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ be the infinitesimal generator of a C_0-semigroup denoted $t \mapsto e^{\mathcal{A}t}$. Let $\mathcal{B}$ and $\mathcal{C}$ be linear operators, $\mathcal{B}$ from $\mathbb{R}^m$ to $\mathcal{X}$ and $\mathcal{C}$ from $D(\mathcal{C}) \subseteq \mathcal{X}$ to $\mathbb{R}^m$.

In this section, we consider the controlled Cauchy problem with output $\Sigma(\mathcal{A},\mathcal{B},\mathcal{C})$ in Kalman form, as follows

$$\varphi_t = \mathcal{A}\varphi + \mathcal{B}u + w \ , \ y = \mathcal{C}\varphi, \tag{1}$$

where $w \in \mathcal{X}$ is an unknown constant vector and $u : \mathbb{R}_+ \mapsto \mathbb{R}^m$ is the controlled input. We consider the following exponential stability property for the operator $\mathcal{A}$.

Assumption 1 (Exponential Stability): The operator $\mathcal{A}$ generates a C_0-semigroup which is exponentially stable. In other words, there exist ν and k both positive constants such that, $\forall \varphi_0 \in \mathcal{X}$ and $t \in \mathbb{R}_+$

$$\|e^{\mathcal{A}t}\varphi_0\|_{\mathcal{X}} \leqslant k\exp(-\nu t)\|\varphi_0\|_{\mathcal{X}}. \tag{2}$$

We are interested in the regulation problem. More precisely we are concerned with the problem of regulation of the output y via the integral control

$$u = k_i K_i z \ , \ z_t = y - y_{ref}, \tag{3}$$

where $y_{ref} \in \mathbb{R}^m$ is a prescribed reference, $z \in \mathbb{R}^m$, $K_i \in \mathbb{R}^{m\times m}$ is a full rank matrix and k_i a positive real number.

The control law being dynamical, the state space has been extended. Considering the system $\Sigma(\mathcal{A},\mathcal{B},\mathcal{C})$ in closed loop, with integral control law given in (3) the state space is now $\mathcal{X}_e = \mathcal{X} \times \mathbb{R}^m$ which is a Hilbert space with inner product

$$\langle \varphi_{ea}, \varphi_{eb}\rangle_{\mathcal{X}_e} = \langle \varphi_a, \varphi_b\rangle_{\mathcal{X}} + z_a^\top z_b \ ,$$

All authors are with LAGEP-CNRS, Université Claude Bernard Lyon1, Université de Lyon, Domaine Universitaire de la Doua, 43 bd du 11 Novembre 1918, 69622 Villeurbanne Cedex, France.

where $\varphi_{ea} = \begin{bmatrix}\varphi_a \\ z_a\end{bmatrix}$ and $\varphi_{eb} = \begin{bmatrix}\varphi_b \\ z_b\end{bmatrix}$. The associated norm is denoted $\|\cdot\|_{\mathcal{X}_e}$. Let $\mathcal{A}_e : (D(\mathcal{A}) \cap D(\mathcal{C})) \times \mathbb{R}^m \to \mathcal{X}_e$ be the extended operator defined as

$$\mathcal{A}_e = \begin{bmatrix}\mathcal{A} & \mathcal{B}K_i k_i \\ \mathcal{C} & 0\end{bmatrix} \ . \tag{4}$$

The regulation problem to solve can be rephrased as the following.

Regulation problem: We wish to find a positive real number k_i and a full rank matrix K_i such that $\forall (w, y_{ref}) \in \mathcal{X} \times \mathbb{R}^m$:

1) The system (1)-(3) is well-posed. In other words, for all $\varphi_{e0} = (\varphi_0, z_0) \in \mathcal{X}_e$ there exists a unique (weak) solution denoted $\varphi_e(t) = \begin{bmatrix}\varphi(t) \\ z(t)\end{bmatrix} \in C^0(\mathbb{R}_+, \mathcal{X}_e)$ defined $\forall t \geqslant 0$ and initial condition $\varphi_e(0) = \varphi_{e0}$.
2) There exists an equilibrium point denoted $\varphi_{e\infty} = \begin{bmatrix}\varphi_\infty \\ z_\infty\end{bmatrix} \in \mathcal{X}_e$, depending on w and y_{ref}, which is exponentially stable for the system (1)-(3). In other words, there exist positive real numbers ν_e and k_e such that for all $t \geqslant 0$
$$\|\varphi_e(t) - \varphi_{e\infty}\|_{\mathcal{X}_e} \leqslant k_e \exp^{-\nu_e t} \|\varphi_{e0} - \varphi_{e\infty}\|_{\mathcal{X}_e}.$$
3) The output y is regulated toward the reference y_{ref}. More precisely,
$$\forall \varphi_{e0}, \ \lim_{t \to +\infty} |\mathcal{C}\varphi(t) - y_{ref}| = 0. \tag{5}$$

We know with the work of S. Pohjolainen in [13] that when $\mathcal{A}$ generates an exponentially stable analytic semi-group, when $\mathcal{B}$ is bounded and when $\mathcal{C}$ is $\mathcal{A}$-bounded, with a rank condition, the regulation may be achieved. This result has been extended to more general exponentially stable semi-groups in [23].

Theorem 1 ([23]): Assume that $\mathcal{X}$ is separable and that $\mathcal{A}$ satisfies Assumption 1. Assume moreover that :

1) the operator $\mathcal{B}$ is bounded;
2) the operator $\mathcal{C}$ is $\mathcal{A}$-admissible (see [22]), i.e.
   - it is $\mathcal{A}$-bounded :
   $$|\mathcal{C}\varphi| \leqslant c(\|\varphi\|_{\mathcal{X}} + \|\mathcal{A}\varphi\|_{\mathcal{X}}) \ , \ \forall \ \varphi \in D(\mathcal{A}), \tag{6}$$
   for some positive real number c;
   - there exist $T > 0$ and $c_T > 0$ such that
   $$\int_0^T |\mathcal{C}e^{\mathcal{A}t}\varphi|^2 dt \leqslant c_T^2 \|\varphi\|_{\mathcal{X}}^2 \ , \ \forall \ \varphi \in D(\mathcal{A});$$
3) the rank condition holds. In other words operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ satisfy
$$\texttt{rank}\{\mathcal{C}\mathcal{A}^{-1}\mathcal{B}\} = m; \tag{7}$$

then there exists a positive real number k_i^* and a $m \times m$ matrix K_i, such that for all $0 < k_i < k_i^*$ the operator $\mathcal{A}_e$ given in (4) is the generator of an exponentially stable C_0-semigroup in the extended state space $\mathcal{X}_e$. More precisely, the system (1) in closed loop with (3) is well-posed and the equilibrium is exponentially stable. Moreover, for all w and y_{ref}, equation (5) holds (i.e the regulation is achieved).

On another hand, if one wants to address nonlinear abstract Cauchy problems or unbounded operators, we may need to follow a Lyapunov approach. For instance in the context of boundary control, a Lyapunov functional approach has allowed to tackle feedback stabilization of a large class of PDEs (see for instance [2] or [5]).

It is well known (see for instance [9, Theorem 8.1.3]) that exponential stability of the operator $\mathcal{A}$ is equivalent to existence of a bounded positive and self adjoint operator $\mathcal{P}$ in $\mathfrak{L}(X)$ such that

$$\langle \mathcal{A}\varphi, \mathcal{P}\varphi\rangle_{\mathcal{X}} + \langle \mathcal{P}\varphi, \mathcal{A}\varphi\rangle_{\mathcal{X}} \leqslant -\mu\|\varphi\|_{\mathcal{X}} \ , \ \forall \ \varphi \in D(\mathcal{A}), \tag{8}$$

where μ is a positive real number. We assume that this Lyapunov operator $\mathcal{P}$ is given. The first question, we intend to solve is the following: *Knowing the Lyapunov operator $\mathcal{P}$, is it possible to construct a Lyapunov operator $\mathcal{P}_e$ associated to the extended operator $\mathcal{A}_e$?*

To answer this question, we first give a construction based on a well-known technique in the nonlinear finite dimensional control community named *the forwarding* (see for instance [10], [15] or more recently [4], or [1]).

B. A Lyapunov approach for regulation

Inspired by the forwarding techniques, the following result can be obtained.

Theorem 2 (Forwarding Lyapunov functional): Assume that all assumptions of Theorem 1 are satisfied and let $\mathcal{P}$ in $\mathfrak{L}(X)$ be a positive self adjoint operator such that (8) holds. Then there exist a bounded operator $\mathcal{M} : \mathcal{X} \to \mathbb{R}^m$ and positive real numbers p and k_i^*, such that for all $0 < k_i < k_i^*$, there exists $\mu_e > 0$ such that the operator

$$\mathcal{P}_e = \begin{bmatrix}\mathcal{P} + p\mathcal{M}^*\mathcal{M} & -p\mathcal{M}^* \\ -p\mathcal{M} & p\,\mathrm{I_d}\end{bmatrix} \tag{9}$$

is positive and satisfies $\forall \ \varphi_e = (\varphi, z) \in D(\mathcal{A}) \times \mathbb{R}^m$

$$\langle \mathcal{A}_e\varphi_e, \mathcal{P}_e\varphi_e\rangle_{\mathcal{X}_e} + \langle \mathcal{P}_e\varphi_e, \mathcal{A}_e\varphi_e\rangle_{\mathcal{X}_e} \leqslant -\mu_e(\|\varphi\|_{\mathcal{X}}^2 + |z|^2). \tag{10}$$

Proof: The operator $\mathcal{A}$ satisfying Assumption 1, 0 is in its resolvent set and consequently $\mathcal{A}^{-1} : \mathcal{X} \mapsto D(\mathcal{A})$ is well defined and bounded. Let $\mathcal{M} : \mathcal{X} \to \mathbb{R}^m$ be defined by $\mathcal{M} = \mathcal{C}\mathcal{A}^{-1}$ which is well defined due to the fact that $D(\mathcal{A}) \subseteq D(\mathcal{C})$ since $\mathcal{C}$ is $\mathcal{A}$-bounded. Moreover, with (6) $\forall \varphi \in \mathcal{X}$

$$|\mathcal{M}\varphi| = |\mathcal{C}\mathcal{A}^{-1}\varphi| \leqslant c\left(\|\mathcal{A}^{-1}\varphi\|_{\mathcal{X}} + \|\varphi\|_{\mathcal{X}}\right) \leqslant \tilde{c}\|\varphi\|_{\mathcal{X}},$$

where $\tilde{c}$ is a positive real number. Hence, $\mathcal{M}$ is a bounded linear operator. Moreover, $\mathcal{M}$ satisfies the following equation

$$\mathcal{M}\mathcal{A}\varphi = \mathcal{C}\varphi \ , \ \forall \varphi \in D(\mathcal{A}) \ . \tag{11}$$

Let $K_i = (\mathcal{C}\mathcal{A}^{-1}\mathcal{B})^{-1}$ which exists due to the third assumption of Theorem 1. Note that,

$$\langle \varphi_e, \mathcal{P}_e\varphi_e\rangle_{\mathcal{X}_e} = \langle \varphi, \mathcal{P}\varphi\rangle_{\mathcal{X}} + p(z - \mathcal{M}\varphi)^\top(z - \mathcal{M}\varphi), \tag{12}$$

hence $\mathcal{P}_e$ is positive. This candidate Lyapunov functional is similar to the one given in [4, Equation (34)]. It is selected following a forwarding approach.

Moreover, we have

$$\begin{aligned}\langle \mathcal{A}_e\varphi_e, \mathcal{P}_e\varphi_e\rangle_{\mathcal{X}_e} + \langle \mathcal{P}_e\varphi_e, \mathcal{A}_e\varphi_e\rangle_{\mathcal{X}_e} = \\ \langle \mathcal{A}\varphi, \mathcal{P}\varphi\rangle_{\mathcal{X}} + \langle \mathcal{P}\varphi, \mathcal{A}\varphi\rangle_{\mathcal{X}} \\ + 2p(z - \mathcal{M}\varphi)^\top(\mathcal{C}\varphi - \mathcal{M}\mathcal{A}\varphi) + k_i\langle \varphi, \mathcal{P}\mathcal{B}K_i z\rangle_{\mathcal{X}} \\ + k_i\langle \mathcal{P}\mathcal{B}K_i z, \varphi\rangle_{\mathcal{X}} - 2p(z - \mathcal{M}\varphi)^\top \mathcal{M}\mathcal{B}K_i k_i z.\end{aligned}$$

Employing equation (11) and $\mathcal{M}\mathcal{B}K_i = \mathrm{I}_{\mathrm{d}m}$, the former inequality becomes

$$\begin{aligned}\langle \mathcal{A}_e\varphi_e, \mathcal{P}_e\varphi_e\rangle_{\mathcal{X}_e} + \langle \mathcal{P}_e\varphi_e, \mathcal{A}_e\varphi_e\rangle_{\mathcal{X}_e} = \\ \langle \mathcal{A}\varphi, \mathcal{P}\varphi\rangle_{\mathcal{X}} + \langle \mathcal{P}\varphi, \mathcal{A}\varphi\rangle_{\mathcal{X}} + k_i\langle \varphi, \mathcal{P}\mathcal{B}K_i z\rangle_{\mathcal{X}} \\ + k_i\langle \mathcal{P}\mathcal{B}K_i z, \varphi\rangle_{\mathcal{X}} - 2p(z - \mathcal{M}\varphi)^\top k_i z.\end{aligned} \tag{13}$$

Let $\|\mathcal{P}\mathcal{B}K_i\|_{\mathcal{X}}^2 = \alpha$ which is well defined due to the boundedness assumption on $\mathcal{B}$. Given a, b positive constants, the following inequalities hold

$$\langle \varphi, \mathcal{P}\mathcal{B}K_i z\rangle_{\mathcal{X}} \leqslant \frac{1}{2a}\|\varphi\|_{\mathcal{X}}^2 + \frac{a\alpha}{2}|z|^2, \tag{14}$$

$$z^\top \mathcal{M}\varphi \leqslant \frac{1}{2b}\|\varphi\|^2 + \frac{b\|\mathcal{M}\|^2}{2}|z|^2, \tag{15}$$

it yields given (8) that

$$\begin{aligned}\langle \mathcal{A}_e\varphi_e, \mathcal{P}_e\varphi_e\rangle_{\mathcal{X}_e} + \langle \mathcal{P}_e\varphi_e, \mathcal{A}_e\varphi_e\rangle_{\mathcal{X}_e} \\ \leqslant \left[-\mu + \frac{k_i}{a} + \frac{pk_i}{b}\right]\|\varphi\|_{\mathcal{X}}^2 \\ + k_i\left(p(-2 + b\|\mathcal{M}\|^2) + a\alpha\right)|z|^2.\end{aligned} \tag{16}$$

We pick b sufficiently small such that

$$-2 + b\|\mathcal{M}\|^2 < 0. \tag{17}$$

In a second step, we select a sufficiently small and p sufficiently large such that

$$p(-2 + b\|\mathcal{M}\|^2) + a\alpha < 0. \tag{18}$$

Finally, picking k_i^* sufficiently small such that

$$-\mu + \frac{k_i^*}{a} + \frac{pk_i^*}{b} < 0 \tag{19}$$

the result is obtained with
$\mu_e = \min\left\{\mu - \frac{k_i}{a} - \frac{pk_i}{b}, p(2 - b\|\mathcal{M}\|^2) - a\alpha\right\}$. □

C. Discussion on the result

A direct interest of the Lyapunov approach given in Theorem 2, is that it allows to give an explicit value for k_i^* which appears in Theorem 1. We may compute the largest value of k_i^* following this route. First of all, from (19)

$$k_i^* = \sup_{a,b,p,\text{such that }(17)-(18)}\left\{\frac{\mu p}{\frac{p}{a} + \frac{p^2}{b}}\right\}, \tag{20}$$

$$= \mu \sup_{a,b,p,\text{such that }(17)-(18)}\left\{\frac{ab}{pa + b}\right\} \tag{21}$$

On another hand, taking the the value of a and b given by (17) and (18), one can rewritten them with $0 < \beta < 1$ and $0 < \theta < 1$ as

$$b = \frac{2}{\|\mathcal{M}\|^2}\beta, \; a = 2(1-\beta)\theta\frac{p}{\alpha}. \tag{22}$$

Then

$$\frac{ab}{pa + b} = \frac{2(1-\beta)\beta\theta p}{(1-\beta)\theta\|\mathcal{M}\|^2 p^2 + \alpha\beta} \tag{23}$$

which right expression is a function of p taking its maximum value when

$$p = \sqrt{\frac{\alpha\beta}{(1-\beta)\theta\|\mathcal{M}\|^2}}$$

then

$$k_i^* = \sup_{a,b,p,\text{such that }(17)-(18)}\left\{\frac{\mu p}{\frac{p}{a} + \frac{p^2}{b}}\right\}, \tag{24}$$

$$= \mu \sup_{0<\theta<1,\; 0<\beta<1}\left\{\frac{\sqrt{\beta(1-\beta)\theta}}{\sqrt{\alpha}\|\mathcal{M}\|}\right\} \tag{25}$$

It is reached for $\beta = \frac{1}{2}$ and $\theta = 1$ and this yields

$$k_i^* = \frac{\mu}{2\|\mathcal{M}\|\sqrt{\alpha}} = \frac{\mu}{2\|\mathcal{C}\mathcal{A}^{-1}\|\|\mathcal{P}\mathcal{B}(\mathcal{C}\mathcal{A}^{-1}\mathcal{B})^{-1}\|_{\mathcal{X}}}.$$

Of course, this optimal value depends on the considered Lyapunov operator $\mathcal{P}$ solution of (8). Note that a possible solution to this equation with $\mu = 1$ is given for all (φ_1, φ_2) in $\mathcal{X}^2$ by (see [9])

$$\langle \varphi_1, \mathcal{P}\varphi_2\rangle_{\mathcal{X}} = \lim_{t\to+\infty}\int_0^t \langle e^{\mathcal{A}s}\varphi_1, e^{\mathcal{A}s}\varphi_2\rangle_{\mathcal{X}} ds.$$

Due to (2) it is well defined and positive. Note also that we have

$$\|\mathcal{P}\|_{\mathcal{X}} \leqslant \frac{k^2}{2\nu}.$$

This implies, the following corollary.

Corollary 1 (Explicit integral gain): Given a system $\Sigma(\mathcal{A}, \mathcal{B}, \mathcal{C})$ satisfying the assumptions of the Theorem 1, points 1), 2), and 3) of Theorem 1 hold with $K_i = (\mathcal{C}\mathcal{A}^{-1}\mathcal{B})^{-1}$ and

$$k_i^* = \frac{\nu}{\|\mathcal{C}\mathcal{A}^{-1}\|k^2\|\mathcal{B}(\mathcal{C}\mathcal{A}^{-1}\mathcal{B})^{-1}\|}. \tag{26}$$

An interesting question would now to know in which aspect this value may be optimal.

D. Illustration on a parabolic systems

Consider the problem of heating a bar of length $L = 10$ with both endpoints at temperature zero. We control the heat flow in and out around the points $s = 2,\ 5,$ and 7 and measure the temperature at points $3,\ 6,$ and 8. The problem is to find an integral controller such that the measurements at $s = 3,\ 6,$ and 8 are regulated to (for instance) 1, 3, and 2, respectively. Thus the control system is governed by the following PDE

$$\begin{aligned}\phi_t(s,t) = \phi_{ss}(s,t) + \mathbb{1}_{[\frac{3}{2},\frac{5}{2}]}(s)u_1(t) + \mathbb{1}_{[\frac{9}{2},\frac{11}{2}]}(s)u_2(t) \\ + \mathbb{1}_{[\frac{13}{2},\frac{15}{2}]}(s)u_3(t), \quad (s,t) \in (0,\ 10) \times (0,\infty)\end{aligned} \tag{27}$$

where $\phi : [0, +\infty) \times [0, 10] \to \mathbb{R}$ with boundary conditions

$$\phi(0, t) = \phi(10, t) = 0$$
$$\phi(s, 0) = \phi_0(s), \tag{28}$$

where $\mathbb{1}_{[a,b]} : [0, 10] \to \mathbb{R}$ denotes the characteristic function on the interval $[a, b]$, i.e.,

$$\mathbb{1}_{[a,b]}(s) = \begin{cases} 1 & \forall\, s \in [a, b], \\ 0 & \forall\, s \notin [a, b]. \end{cases}$$

The output and the reference are given as

$$y(t) = \begin{bmatrix} \phi(t, 3) \\ \phi(t, 6) \\ \phi(t, 8) \end{bmatrix}, \; y_{ref} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

Let the state space be the Hilbert space $\mathcal{X} = L^2((0, 10), \mathbb{R})$ with usual inner product, and let the input space and the output space be equal to $\mathbb{R}^3$. Clearly, from (28), we get the semigroup generator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{X}$, the input operator $\mathcal{B} : \mathbb{R}^3 \to \mathcal{X}$ and the output operator $\mathcal{C} : D(\mathcal{A}) \to \mathbb{R}^3$ as follows:

$$D(\mathcal{A}) = \{\varphi \in H^2(0, 10) \mid \varphi(0) = \varphi(10) = 0\},$$

and

$$\mathcal{A}\varphi = \varphi_{ss} \;\; \forall\, \varphi \in D(\mathcal{A}),$$

$$\mathcal{B}u = \mathbb{1}_{[\frac{3}{2}, \frac{5}{2}]} u_1 + \mathbb{1}_{[\frac{9}{2}, \frac{11}{2}]} u_2 + \mathbb{1}_{[\frac{13}{2}, \frac{15}{2}]} u_3,$$

and

$$C\varphi = \begin{bmatrix} \varphi(3) \\ \varphi(6) \\ \varphi(8) \end{bmatrix}.$$

Moreover, note that with Sobolev embedding, an integration by part and by completing the square, we have for all φ in $D(\mathcal{A})$

$$\begin{aligned} \sup_{s \in (0,10)} |\varphi(s)| &\le c \int_0^{10} \varphi(s)^2 ds + c \int_0^{10} \varphi_s(s)^2 ds \\ &\le c\|\varphi\|_{\mathcal{X}} + c \int_0^{10} |\varphi(s)\varphi_{ss}(s)| ds \\ &\le \frac{3}{2} c\|\varphi\|_{\mathcal{X}} + \frac{1}{2} c\|\varphi_{ss}\|_{\mathcal{X}}. \end{aligned}$$

Hence $\mathcal{C}$ is $\mathcal{A}$-bounded.

Moreover, by direct computation we find that

$$\mathcal{C}\mathcal{A}^{-1}\mathcal{B} = \frac{-1}{10} \begin{bmatrix} 14 & 15 & 9 \\ 8 & 20 & 18 \\ 4 & 10 & 14 \end{bmatrix}.$$

It is easy to see that the above matrix is regular. Consequently all Assumptions of Theorem 1 hold. With Corollary 1, it is possible to compute explicitly the integral controller gain. By direct computation we have for all φ in $\mathcal{X}$

$$\mathcal{C}\mathcal{A}^{-1}\varphi = \begin{bmatrix} \frac{3}{10} \int_0^{10} (s - 10)\varphi(s) ds + \int_0^3 (3 - s)\varphi(s) ds \\ \frac{3}{5} \int_0^{10} (s - 10)\varphi(s) ds + \int_0^6 (6 - s)\varphi(s) ds \\ \frac{4}{5} \int_0^{10} (s - 10)\varphi(s) ds + \int_0^8 (8 - s)\varphi(s) ds \end{bmatrix},$$

which gives $\|\mathcal{C}\mathcal{A}^{-1}\| \le 6.2466$. We have

$$K_i = \begin{bmatrix} -1.250 & 1.500 & -1.125 \\ 0.500 & -2.000 & 2.250 \\ 0 & 1.000 & -2.000 \end{bmatrix}.$$

For the open-loop system, consider the Lyapunov operator $\mathcal{P} = \mathrm{I_d}$. Then the growth rate may be taken as $\mu = \frac{\pi^2}{50}$. It is easy to see that $\|K_i\| = 4.2433$, and $\|\mathcal{B}\| \le \sqrt{3}$. Putting together the numerical values into the formula (26) allows to estimate the tuning parameter

$$k_i^* = \frac{\omega}{2\|\mathcal{B}K_i\| \, \|\mathcal{C}\mathcal{A}^{-1}\|} \approx 2.1498 * 10^{-3}.$$

With Corollary 1, the integral controller (3) with $0 < k_i < 2.1498 * 10^{-3}$ stabilizes exponentially the equilibrium along solutions of the closed-loop system and drives asymptotically the measured temperatures to the reference values for any initial condition.

III. Case of boundary regulation for hyperbollic PDEs

In the following section we adapt this framework to hyperbolic PDE systems with boundary control.

A. System description

To illustrate the former abstract theory, we consider the case of hyperbolic partial differential equations as studied in [6]. More precisely, the system is given by a one dimensional $n \times n$ hyperbolic system

$$\phi_t(s, t) + \Lambda_0(s)\phi_s(s, t) + \Lambda_1(s)\phi(s, t) = 0 \qquad s \in (0, 1), \; t \in [0, +\infty), \tag{29}$$

where $\phi : [0, +\infty) \times [0, 1] \to \mathbb{R}^n$

$$\begin{aligned} &\Lambda_0(s) = \mathtt{diag}\{\lambda_1(s), \ldots, \lambda_n(s)\} \\ &\lambda_i(s) > 0 \; \forall i \in \{1, \ldots, \ell\} \\ &\lambda_i(s) < 0 \; \forall i \in \{\ell + 1, \ldots, n\}, \end{aligned}$$

where the maps Λ_0 is in $C^1([0, 1]; \mathcal{D}_n)$ and Λ_1 is in $C^1([0, 1]; \mathbb{R}^{n\times n})$ with the initial condition $\phi(0, s) = \phi_0(s)$ for s in $[0, 1]$ where $\phi_0 : [0, 1] \to \mathbb{R}^n$ and with the boundary conditions

$$\begin{bmatrix} \phi_+(t, 0) \\ \phi_-(t, 1) \end{bmatrix} = K \begin{bmatrix} \phi_+(t, 1) \\ \phi_-(t, 0) \end{bmatrix} + Bu(t) + w_b \tag{30}$$

$$= \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \phi_+(t, 1) \\ \phi_-(t, 0) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) + w_b \tag{31}$$

where $\phi = \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix}$ with ϕ^+ in $\mathbb{R}^\ell$, ϕ^- in $\mathbb{R}^{n-\ell}$ and where w_b in $\mathbb{R}^p$ is an unknown disturbance, $u(t)$ is a control input taking values in $\mathbb{R}^m$ and K, B are matrices of appropriate dimensions.

The output to be regulated to a prescribed value denoted by y_{ref}, is given as a disturbed linear combination of the boundary conditions. Namely, the outputs to regulate are in $\mathbb{R}^m$ given as

$$y(t) = L_1 \begin{bmatrix} \phi_+(t, 0) \\ \phi_-(t, 1) \end{bmatrix} + L_2 \begin{bmatrix} \phi_+(t, 1) \\ \phi_-(t, 0) \end{bmatrix} + w_y, \tag{32}$$

where L_1 and L_2 are two matrices in $\mathbb{R}^{m\times n}$ and w_y is an unknown disturbance in $\mathbb{R}^m$. We wish to find a positive real number k_i and a full rank matrix K_i such that

$$u(t) = k_i K_i z(t)\ ,\ z_t(t) = y(t) - y_{ref}\ , z(0) = z_0 \tag{33}$$

where $z(t)$ takes value in $\mathbb{R}^m$ and $z_0 \in \mathbb{R}^m$ solves the regulation problem $\forall y_{ref} \in \mathbb{R}^m$.

The state space denoted by $\mathcal{X}_e$ of the system (29)-(30) in closed loop with the control law (33) is the Hilbert space defined as:

$$\mathcal{X}_e = (L^2(0,1), \mathbb{R}^n) \times \mathbb{R}^m,$$

equipped with the norm defined for $\varphi_e = (\phi, z)$ in $\mathcal{X}_e$ as:

$$\|v\|_{\mathcal{X}_e} = \|\phi\|_{L^2((0,1),\mathbb{R}^m)} + |z|.$$

We introduce also a smoother state space defined as:

$$\mathcal{X}_{e1} = (H^1(0,1), \mathbb{R}^n) \times \mathbb{R}^m.$$

B. Output regulation result

In this section, we give a set of sufficient conditions allowing to solve the regulation problem as described in the introduction. Our approach follows what we have done in the former section. Following [2, Proposition 5.1, p161] we consider the following assumption.

Assumption 2 (Input-to-State Exponential Stability): There exist a C^1 function $P : [0,1] \to \mathcal{D}_n$, a real numbers $\mu > 0$, $\underline{P}$, $\overline{P}$ and a positive definite matrix S in $\mathbb{R}^{n\times n}$ such that

$$(P(s)\Lambda_0(s))_s - P(s)\Lambda_1(s) - \Lambda_1^\top(s)P(s) \leqslant -\mu P(s), \tag{34}$$

$$\underline{P}\,\mathrm{I}_{\mathrm{d}n} \leqslant P(s) \leqslant \overline{P}\,\mathrm{I}_{\mathrm{d}n}\ ,\ \forall s \in [0,1], \tag{35}$$

and

$$-K_+^\top P(1)\Lambda_0(1)K_+ + K_-^\top P(0)\Lambda_0(0)K_- \leqslant -S. \tag{36}$$

where

$$K_+ = \begin{bmatrix} \mathrm{I}_{\mathrm{d}\ell} & 0 \\ K_{21} & K_{22} \end{bmatrix}\ ,\ K_- = \begin{bmatrix} K_{11} & K_{12} \\ 0 & \mathrm{I}_{\mathrm{d}n-\ell} \end{bmatrix} \tag{37}$$

As it will be seen in the following section, this assumption is a sufficient condition for exponential stability of the equilibrium of the open loop system. It can be found in [2] in the case in which S may be semi-definite positive. The positive definiteness of S is fundamental to get an input-to-state stability (ISS) property of the open loop system with respect to the disturbances on the boundary. More general results are given in [14].

The second assumption is related to the rank condition. Let $\Phi : [0,1] \to \mathbb{R}^{n\times n}$ be the matrix function solution to the system

$$\Phi_s(s) = \Lambda_0(s)^{-1}\Lambda_1(s)\Phi(s),$$
$$\Phi(0) = \mathrm{I}_{\mathrm{d}n}\,.$$

We denote $\Phi(s) = \begin{bmatrix} \Phi_{11}(s) & \Phi_{12}(s) \\ \Phi_{21}(s) & \Phi_{22}(s) \end{bmatrix}$ and

$$\Phi_+(1) = \begin{bmatrix} \Phi_{11}(1) & \Phi_{12}(1) \\ 0 & \mathrm{I}_{\mathrm{d}n-\ell} \end{bmatrix}\ ,\ \Phi_-(1) = \begin{bmatrix} \mathrm{I}_{\mathrm{d}\ell} & 0 \\ \Phi_{21}(1) & \Phi_{22}(1) \end{bmatrix}$$

Assumption 3 (Rank condition 1): The matrix in $\mathbb{R}^{n\times n}$ $\Phi_-(1) - K\Phi_+(1)$ is full rank and so is the matrix T defined as

$$T_1 = (L_1\Phi_-(1) + L_2\Phi_+(1))\,(\Phi_-(1) - K\Phi_+(1))^{-1}\,B. \tag{38}$$

Another rank condition has to be introduced. This one is used when solving the forwarding equation. Let $\Psi : [0,1] \mapsto \mathbb{R}^{n\times n}$ be the matrix function solution to the system

$$\begin{aligned} \Psi_s(s) &= \Psi(s)\,(\Lambda_1(s) - \Lambda_{0s}(s))\,\Lambda_0(s)^{-1}, \\ \Psi(0) &= \mathrm{I}_{\mathrm{d}n}\,. \end{aligned} \tag{39}$$

Assumption 4 (Rank condition 2): The matrix in $\mathbb{R}^{n\times n}$

$$\Psi(1)\Lambda_0(1)K_+ - \Lambda_0(0)K_- \tag{40}$$

is full rank and so is the matrix

$$T_2 = -L_1 B + M\left(\Lambda_0(0)\begin{bmatrix} B_1 \\ 0 \end{bmatrix} - \Psi(1)\Lambda_0(1)\begin{bmatrix} 0 \\ B_2 \end{bmatrix}\right)$$

where

$$M = (L_1 K + L_2)\,(\Lambda_0(0)K_- - \Psi(1)\Lambda_0(1)K_+)^{-1}\,. \tag{41}$$

With these assumptions, the following result may be obtained.

Theorem 3 (Regulation for hyperbolic PDE systems): Assume that Assumptions 2, 3 and 4 are satisfied then with $K_i = T_2^{-1}$ there exists $k_i^* > 0$ such that for all $0 < k_i < k_i^*$ the output regulation is obtained. More precisely, for all (w_b, w_y, y_{ref}) in $\mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^m$, the following holds.

1) For all (ϕ_0, z_0) in $\mathcal{X}_e$ (resp. $\mathcal{X}_{1e}$) which satisfies the boundary conditions (30) (resp. the C^1 compatibility condition), there exists a unique weak solution to (29)-(30)-(33) that we denote v and which belongs to $C^0([0,+\infty);\mathcal{X}_e)$ (Respectively, strong solution in:
$$C^0([0,+\infty);\mathcal{X}_{e1}) \cap C^1([0,+\infty);\mathcal{X}_e)\ ). \tag{42}$$
2) There exists an equilibrium state denoted v_∞ in $\mathcal{X}_e$ which is globally exponentially stable in $\mathcal{X}_e$ for system (29)-(30)-(33). More precisely, we have for all $t \geqslant 0$:
$$\|v(t) - v_\infty\|_{\mathcal{X}_e} \leqslant k\exp(-\nu t)\|v_0 - v_\infty\|_{\mathcal{X}_e}. \tag{43}$$
3) Moreover, if v_0 satisfies the C^1-compatibility condition and is in $\mathcal{X}_{1e}$, the regulation is achieved, i.e.
$$\lim_{t\to+\infty} |y(t) - y_{ref}| = 0. \tag{44}$$

The next section is devoted to the proof of this result.

C. About this result

The first assumption needed in Theorem 3 is Assumption 2. When considering only integral control laws, there is no hope to obtain the result without assuming exponential stability of the open loop system. Assumption 2 is slightly more restrictive than exponential stability since it requires an ISS property with respect to the input u. In the case in which this assumption is not satisfied for a given hyperbolic system, a possibility is to modify the boundary condition via a static output feedback (or proportional feedback) following the route of [2] in order to satisfy this assumptions.

One interest of our approach is that, part of the exponential stability of the closed loop system, only Assumptions 3 and 4 which are rank conditions involving the boundary conditions have to be satisfied. In the case in which the two above mentioned assumptions are not satisfied, we may obtain these properties by adding a proportional feedback and consequently changing the value of K in T_1 and T_2 to obtain these rank conditions. These Assumptions 3 and 4 are version of Point 3) in Theorem 1.

In the particular case in which Λ_0 is constant and $\Lambda_1 = 0$, the matrix function $\Phi(s)$ and $\Psi(s)$ are simply equal to identity for all s in $[0, 1]$. In that case, it yields

$$T_1 = (L_1 + L_2)(\mathrm{Id}_n - K)^{-1}B, \tag{45}$$

and,

$$\begin{aligned}
T_2 &= -L_1B + (L_1K + L_2)(K_- - K_+)^{-1}\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix} \\
&= -L_1B + (L_1K + L_2)\begin{bmatrix} K_{11} - \mathrm{Id}_\ell & K_{12} \\ -K_{21} & \mathrm{Id}_{n-\ell} - K_{22} \end{bmatrix}^{-1} \\
&\qquad\qquad \times \begin{bmatrix} \mathrm{Id}_\ell & 0 \\ 0 & -\mathrm{Id}_\ell \end{bmatrix} B \\
&= -L_1B + (L_1K + L_2) \\
&\quad \times \left(\begin{bmatrix} \mathrm{Id}_\ell & 0 \\ 0 & -\mathrm{Id}_\ell \end{bmatrix}\begin{bmatrix} K_{11} - \mathrm{Id}_\ell & K_{12} \\ -K_{21} & \mathrm{Id}_{n-\ell} - K_{22} \end{bmatrix}\right)^{-1} B \\
&= -L_1B - (L_1K + L_2)(\mathrm{Id}_n - K)^{-1}B \\
&= [-L_1(\mathrm{Id}_n - K) - (L_1K + L_2)]\,(\mathrm{Id}_n - K)^{-1}B \\
&= -(L_1 + L_2)(\mathrm{Id}_n - K)^{-1}B.
\end{aligned} \tag{46}$$

Hence when Λ_0 is constant and $\Lambda_1 = 0$, Assumption 3 and Assumption 4 are equivalent.

Also, an interesting aspect of this Lyapunov approach is that explicit values of the supremum value of the gain k_i^* may be given. For instance, as in [20] consider the very particular case of a transport equation. In this case the system is simply

$$\begin{aligned}
&\phi_t(s,t) + \phi_s(s,t) = 0, s \in (0,1), t \in [0,+\infty) \\
&\phi(t,0) = u(t) + w_b \\
&y(t) = \phi(1,t) + w_y
\end{aligned}$$

We can apply Theorem 3 with $n = 1$, $\Lambda_0(s) = -1$, $\Lambda_1(s) = 0$, $K = 0$, $B = 1$, $L_1 = 0$, $L_2 = 1$. This yields $\Psi(s) = 1$, $\Psi(s) = 1$, $T_1 = 1$, $T_2 = -1$. Hence, Assumptions 3 and 4 are satisfied. Assumption 2 is satisfied for all $\mu > 0$ with $P(s) = e^{-\mu s}$, $S = 1$, $\overline{P} = 1$, $\underline{P} = e^{-\mu}$. In that case, employing theorem 3, it yields that there exists $k_i^* > 0$ such that for all $0 < k_i < k_i^*$ with $u(t) = -k_i z$, $\dot{z} = y$, the output regulation is obtained and so the output converges asymptotically to zero. Following the proof of Theorem 3, equation (85) gives

$$k_i^* = \sqrt{\mu e^{-\mu}}.$$

This bound is better then the one obtained in [20] for the linear transport equation (its maximal value is obtained for $\mu = 1$ and is $\frac{1}{\sqrt{e}}$. Note however that similar to the bound of [20], the result obtained with our novel Lyapunov functional is far from the value we get following a frequency approach ($\frac{\pi}{2}$ in this case). Recently in [7], the Lyapunov functional obtained in [20] has been modified to reach this optimal value of the integral gain. A natural question for future research topic is to know if it is possible to modify the Lyapunov functional obtained in Theorem 3 following the methods of [7] to remove the conservatism.

D. Illustration in a 2×2 hyperbolic system

Theorem 3 generalizes many available results on output regulation via integral action for hyperbolic PDEs available in the literature. For instance, the case of 2×2 linear hyperbolic systems has been considered in [19], [8], (see also [2, Section 2.2.4]). The case of cascade of such systems is also considered in [21]. Note also that in [17], this procedure is applied on a Drilling model which is composed of a hyperbolic PDE coupled with a linear ordinary differential equation.

In order to compare the way we improve existing results, the same example as in [8] is considered. In this context, the linearized de Saint-Venant equations can be written in the form of (29)-(30). After normalization, one gets :

$$\Lambda_0(s) = \begin{bmatrix} c & 0 \\ 0 & -d \end{bmatrix} \text{ and } \Lambda_1(s) = 0_{2\times 2}\ , \forall s \tag{47}$$

where $c > 0$ and $d > 0$ and

$$K = \begin{bmatrix} 0 & k_0 \\ k_1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_0 & 0 \\ 0 & b_1 \end{bmatrix}, \tag{48}$$

with $b_0 \neq 0$ and $b_1 \neq 0$. For the system (29)-(30) with these parameters, it is shown in [8] that the output of dimension $m = 2$ defined in (32) with

$$L_1 = \begin{bmatrix} \frac{c}{c+d} & 0 \\ 0 & \frac{-1}{c+d} \end{bmatrix} \text{ and } L_2 = \begin{bmatrix} 0 & \frac{d}{c+d} \\ \frac{1}{c+d} & 0 \end{bmatrix} \tag{49}$$

can be regulated with an integral control law provided

$$|k_0k_1| < 1\ ,\ |k_0| < 1\ ,\ |k_1| < \frac{c}{d}. \tag{50}$$

On another hand, employing ([7]-[8]), Assumptions 2 is satisfied assuming that $|k_0k_1| < 1$. Moreover, with equations (45) and (46), it yields,

$$T_1 = -T_2 = \frac{1}{c+d}\begin{bmatrix} c & d \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -k_0 \\ -k_1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} b_0 & 0 \\ 0 & b_1 \end{bmatrix}.$$

This matrix is well defined and full rank if $|k_0k_1| < 1$ and consequently Assumptions 3 and 4 are always satisfied. Hence, employing Theorem 3, both outputs defined in (49) can be regulated with an integral control law with the only assumption that $|k_0k_1| < 1$.
Then

$$\begin{aligned}
K_i &= T_2^{-1} \\
&= \vartheta\begin{bmatrix} b_1(1-k_0) & b_1(d+ck_0) \\ b_0(1-k_1) & b_0(c+dk_1) \end{bmatrix}
\end{aligned} \tag{51}$$

$$\text{with } \vartheta = \frac{-(c+d)^2(1-k_0k_1)^2 b_0^{-1} b_1^{-1}}{[(1-k_0)(c+dk_1) + (1-k_1)(d+ck_0)]} \tag{52}$$

and

$$k_i^* = \frac{\sqrt{\mu \underline{P}}}{|M|\overline{\Psi}\sqrt{c\left|T_2^{-1}\right|}} \tag{53}$$

μ is given in [8], and $\underline{P}$ the lower bound of the Lyapunov can be deduced easily from the expression of the Lyapunov function involved.
M has been defined above, with T_2. As Ψ is the identity matrix, $\overline{\Psi}$ is 1. Remark that in [8], K_i is diagonal and here is full matrix. Note that some other choices of K_i are possible as long as

$$T_2 K_i + K_i^\top T_2^\top > 0.$$

To conclude, a work is needed to transpose this approach to the global de Saint-Venant equations this is the aim of another paper.

IV. Proof of Theorem 3

The proof of Theorem 3 is divided into three steps. In a first part, it is shown that with Assumption 3, it can be shown that the closed loop system (29)-(30)-(33) admits a steady state. In a second step, it is established that the desired regulation is obtained provided the steady state is exponentially stable. Finally, the construction of an appropriate Lyapunov functional is performed to show the exponential stability of the equilibrium.

A. Stabilization implies regulation

In this first subsection, we explicitly give the equilibrium state of the system (29)-(30)-(33). We show also that if we assume that k_i and K_i are selected such that this equilibrium point is exponentially stable along the closed loop, then the regulation is achieved.

1) Definition of the equilibrium: The first step of the study is to exhibit equilibrium denoted ϕ_∞, z_∞ of the disturbed hyperbolic PDE in closed loop with the boundary integral control (i.e. system (29)-(33)).

We have the following proposition.

Proposition 1: Assumption 3 is a necessary and sufficient condition for the existence of an equilibrium of the system (29)-(30)-(33). Moreover, if Assumption 3 holds then point 1) of Theorem 3 holds.

Proof: First of all, equilibria are such that

$$\phi_{\infty s}(s) = -\Lambda_0(s)^{-1}\Lambda_1(s)\phi_\infty(s),$$

for all s in $[0,1]$. Hence,

$$\phi_\infty(s) = \Phi(s)\phi_\infty(0). \tag{54}$$

Hence,

$$\begin{bmatrix}\phi_{\infty+}(0)\\ \phi_{\infty-}(1)\end{bmatrix} = \Phi_-(1)\phi_\infty(0),$$
$$\begin{bmatrix}\phi_{\infty+}(1)\\ \phi_{\infty-}(0)\end{bmatrix} = \Phi_+(1)\phi_\infty(0)$$

Moreover, with $z_t = 0$, we have

$$L_1\begin{bmatrix}\phi_{\infty+}(0)\\ \phi_{\infty-}(1)\end{bmatrix} + L_2\begin{bmatrix}\phi_{\infty+}(1)\\ \phi_{\infty-}(0)\end{bmatrix} = y_{ref} - w_y,$$

Hence,

$$\left(L_1\Phi_-(1) + L_2\Phi_+(1)\right)\phi_\infty(0) = y_{ref} - w_y, \tag{55}$$

On another side, boundary conditions (30) gives

$$\left(\Phi_-(1) - K\Phi_+(1)\right)\phi_\infty(0) = Bk_iK_iz_\infty + w_b \tag{56}$$

For all w_y and y_{ref} both in $\mathbb{R}^m$, w_b in $\mathbb{R}^p$, by Assumption 3 and since the matrix K_i is full rank the former equation and (55) admit a unique solution $(z_\infty, \phi_\infty(0))$ given as

$$\begin{aligned} z_\infty = {} & \frac{K_i^{-1}}{k_i}T_1^{-1}(y_{ref} - w_y) \\ & - \frac{K_i^{-1}}{k_i}T_1^{-1}\left(L_1\Phi_-(1) + L_2\Phi_+(1)\right) \\ & \times \left(\Phi_-(1) - K\Phi_+(1)\right)^{-1} w_b \end{aligned} \tag{57}$$

and,

$$\begin{aligned} \phi_\infty(0) = {} & \left(\Phi_-(1) - K\Phi_+(1)\right)^{-1} k_iBK_iz_\infty \\ & + \left(\Phi_-(1) - K\Phi_+(1)\right)^{-1} w_b. \end{aligned} \tag{58}$$

Finally, in that case, we can introduce $\tilde{\phi}(s,t) = \phi(s,t) - \phi_\infty(s)$ and $\tilde{z}(t) = z(t) - z_\infty$. It can be checked that $\tilde{\phi}, \tilde{z}$ satisfies the following system:

$$\begin{aligned} \tilde{\phi}_t(s,t) + \Lambda_0(s)\tilde{\phi}_s(s,t) + \Lambda_1(s)\tilde{\phi}(s,t) & \\ = 0\ ,\ s \in (0,1), & \end{aligned} \tag{59}$$

$$\tilde{z}_t = L_1\begin{bmatrix}\tilde{\phi}_+(t,0)\\ \tilde{\phi}_-(t,1)\end{bmatrix} + L_2\begin{bmatrix}\tilde{\phi}_+(t,1)\\ \tilde{\phi}_-(t,0)\end{bmatrix} \tag{60}$$

with the boundary conditions

$$\begin{bmatrix}\tilde{\phi}_+(t,0)\\ \tilde{\phi}_-(t,1)\end{bmatrix} = K\begin{bmatrix}\tilde{\phi}_+(t,1)\\ \tilde{\phi}_-(t,0)\end{bmatrix} + Bu(t), \tag{61}$$

$$u(t) = k_iK_i\tilde{z}(t). \tag{62}$$

As it is shown in [2], for each initial condition $\tilde{v}_0 = (\tilde{\phi}_0, \tilde{z}_0)$ in $\mathcal{X}_e$ which satisfies the boundary conditions (30), there exists a unique weak solution that we denoted $\tilde{v}$ and which belongs to $C^0([0,+\infty);\mathcal{X}_e)$. Moreover, if the initial condition $\tilde{v}_0$ satisfies also the C^1-compatibility condition (see [2] for more details) and lies in $\mathcal{X}_{e1}$ then the solution lies in the set defined in (42). □

2) Sufficient conditions for Regulation: In the following, we show that the regulation problem can be rephrased as a stabilization of the equilibrium state introduced previously.

Proposition 2: Assume Assumption 3 holds and that there exist a functional $V_e : \mathcal{X}_e \to \mathbb{R}_+$, and positive real numbers μ_e and L_e such that:

$$\frac{\|v_\infty - v\|_{\mathcal{X}_e}^2}{L_e} \leqslant V_e(v) \leqslant L_e\|v_\infty - v\|_{\mathcal{X}_e}^2. \tag{63}$$

Assume moreover that for all v_0 in $\mathcal{X}_e$ and all t_0 in $\mathbb{R}_+$ such that the solution v of system (29)-(30)-(33) initiated from v_0 is C^1 at $t=t_0$, we have:

$$\dot{V}_e(t) \leqslant -\mu_e V_e(t), \tag{64}$$

where with a slight abuse of notation $V_e(t) = V_e(v(t))$. Then points 1), 2) and 3) of Theorem 3 hold.

Proof: Point 1) is directly obtained from Proposition 1. The proof of point 2) is by now standard. Let v_0 be in $\mathcal{X}_{e1}$ and satisfies the C^0 and C^1-compatibility conditions. It yields that v is C^1 for all t. Consequently, (64) is satisfied for all $t \geqslant 0$. With Grönwall's lemma, this implies that:

$$V_e(v(t)) \leqslant e^{-\mu_e t} V_e(v_0).$$

Hence with (63), this implies that (43) holds with $k = L_e$ and $\nu = \frac{\mu_e}{2}$ for initial conditions in $\mathcal{X}_{e1}$. $\mathcal{X}_{e1}$ being dense in $\mathcal{X}_e$, the result holds also with initial condition in $\mathcal{X}_e$ and point 2) is satisfied.

On another hand, we have

$$y(t) - y_{ref} = L_1 \begin{bmatrix} \phi_+(t,0) \\ \phi_-(t,1) \end{bmatrix} + L_2 \begin{bmatrix} \phi_+(t,1) \\ \phi_-(t,0) \end{bmatrix} + w_y - y_{ref}, \tag{65}$$

$$= L_1 \begin{bmatrix} \tilde{\phi}_+(t,0) \\ \tilde{\phi}_-(t,1) \end{bmatrix} + L_2 \begin{bmatrix} \tilde{\phi}_+(t,1) \\ \tilde{\phi}_-(t,0) \end{bmatrix}, \tag{66}$$

with $\tilde{\phi}(t,x) = \phi(t,x) - \phi_\infty$. To show that equation (44) holds, we need to show that the right hand side of the former equation tends to zero. This may be obtained provided the initial condition is in $\mathcal{X}_1$. Indeed, let v_0 be in $\mathcal{X}_1$ and satisfies C^1-compatibility conditions. With (42), we know that $v_t \in C([0,\infty); \mathcal{X}_e)$. Moreover, v_t satisfies the dynamics system (29)-(30)-(33) with $w_b = 0$, $w_y = 0$, $y_{ref=0}$ (simply differentiate with time these equations). Hence, $\|v_t(t)\|_{\mathcal{X}_e}$ converges exponentially toward 0 and in particular

$$\|\tilde{\phi}_t(t,\cdot)\|_{(L^2(0,1),\mathbb{R}^n)} \leqslant k e^{-\nu t} \|v_t(0)\|.$$

On another hand, employing (29), it yields:

$$\|\tilde{\phi}_s(t,\cdot)\|_{L^2((0,1),\mathbb{R}^n)} = \|\Lambda_0^{-1}(\tilde{\phi}_t(t,\cdot) + \Lambda_1(\cdot)\phi(t,\cdot))\|_{L^2((0,1),\mathbb{R}^n)}.$$

Hence,

$$\|\tilde{\phi}_s(t,\cdot)\|_{L^2((0,1),\mathbb{R}^n)} \leqslant c\left(\|\tilde{\phi}_t(t,\cdot)\|_{L^2((0,1),\mathbb{R}^n)} + \|\tilde{\phi}(t,\cdot))\|_{L^2((0,1),\mathbb{R}^n)} \right). \tag{67}$$

where c is a positive constant. Consequently $\|\tilde{\phi}_s(t,\cdot)\|_{L^2((0,1),\mathbb{R}^n)}$ converges also to zero and so is $\|\tilde{\phi}(t,\cdot)\|_{H^1((0,1),\mathbb{R}^n)}$. With Sobolev embedding

$$\sup_{x\in[0,1]} |\tilde{\phi}(t,x)| \leqslant C \|\tilde{\phi}(t,\cdot)\|_{H^1((0,1),\mathbb{R}^n)},$$

where C is a positive real number. It implies that:

$$\lim_{t\to+\infty} |\tilde{\phi}(t,1)| + |\tilde{\phi}(t,0)| = 0.$$

Consequently, with (65), it yields that (44) holds and point 3) is satisfied. □

With this proposition in hand, to prove the Theorem 3, it is sufficient to construct a Lyapunov functional V_e which satisfies (63)-(64) along C^1-solutions of (29)-(30)-(33) or equivalently along C^1-solutions of (59)-(61). This is considered in the next section following the route of Section II-B.

B. Lyapunov functional construction

1) Open loop ISS: Inspired by the Lyapunov functional construction introduced in [6] (see also [2]), we know that typical Lyapunov functionals allowing to exhibit stability property for this type of hyperbolic PDE are given as functional $V : L^2((0,1),\mathbb{R}^n) \to \mathbb{R}_+$ defined as

$$V(\varphi) = \int_0^1 \varphi(s)^\top P(s) \varphi(s) ds \ , \tag{68}$$

where $P : [0,1] \to \mathcal{D}_n$ is a C^1 function. Typically in [6], these functions are taken as exponential.

With a slight abuse of notation, we write $V(t) = V(\tilde{\phi}(\cdot,t))$ and we denote by $\dot{V}(t)$ the time derivative of the Lyapunov functional along solutions which are C^1 in time. In our context, with Assumption 2, it yields the following proposition.

Proposition 3: If Assumption 2 holds, there exists a positive real number c such that for every solution ϕ of (29)-(30) initiated from $(\tilde{\phi}_0, \tilde{z}_0)$ in $\mathcal{X}_e$ which satisfies (61)

$$\dot{V}(t) \leqslant -\mu V(t) + c|u(t)|^2 \ . \tag{69}$$

Proof: First of all, with (29),

$$\begin{aligned} \dot{V}(t) = &- \int_0^1 2\phi(t,s)^\top P(s) \Lambda_0(s) \phi_s(t,s) ds \\ &- \int_0^1 \phi(t,s)^\top \left(P(s)\Lambda_1(s) + \Lambda_1(s)^\top P(s) \right) \phi(t,s) ds \end{aligned}$$

With an integration by part, this implies

$$\begin{aligned} \dot{V}(t) = &\int_0^1 \phi(t,s)^\top \left[(P(s)\Lambda_0(s))_s \right] \phi(t,s) ds \\ &- \int_0^1 \phi(t,s)^\top (P(s)\Lambda_1(s) + \Lambda_1(s)^\top P(s)) \phi(t,s) ds \\ &- \phi(t,1)^\top P(1)\Lambda_0(1)\phi(t,1) \\ &+ \phi(t,0)^\top P(0)\Lambda_0(0)\phi(t,0). \end{aligned}$$

With (34), it gives

$$\begin{aligned} \dot{V}(t) \leqslant &-\mu V(t) - \phi(t,1)^\top P(1)\Lambda_0(1)\phi(t,1) \\ &+ \phi(t,0)^\top P(0)\Lambda_0(0)\phi(t,0). \end{aligned}$$

With the boundary condition (31) and (36), this implies

$$\begin{aligned} \dot{V}(t) \leqslant &-\mu V(t) - \begin{bmatrix} \phi_+(1)^\top & \phi_-(0)^\top \end{bmatrix} S \begin{bmatrix} \phi_+(1) \\ \phi_-(0) \end{bmatrix} \\ &+ 2\begin{bmatrix} \phi_+(1)^\top & \phi_-(0)^\top \end{bmatrix} Q u(t) + u(t)^\top R u(t), \end{aligned} \tag{70}$$

where,

$$\begin{aligned} R = &- \begin{bmatrix} 0 & B_2^\top \end{bmatrix} (P(1)\Lambda_0(1) + \Lambda_0(1)P(1)) \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \\ &+ \begin{bmatrix} B_1^\top & 0 \end{bmatrix} (P(0)\Lambda_0(0) + \Lambda_0(0)P(0)) \begin{bmatrix} 0 \\ B_1 \end{bmatrix}, \end{aligned} \tag{71}$$

and,

$$Q = -K_+^\top (P(1)\Lambda_0(1) + \Lambda_0(1)P(1)) \begin{bmatrix} 0 \\ B_2 \end{bmatrix} + K_-^\top (P(0)\Lambda_0(0) + \Lambda_0(0)P(0)) \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

Since S is positive definite, selecting c sufficiently large, it yields

$$\begin{bmatrix} -S & Q \\ Q^\top & R - c\,\mathrm{I}_{\mathrm{d}m} \end{bmatrix} \leqslant 0 \ .$$

Consequently, (70) implies that (69) holds. □

2) Forwarding approach to deal with the integral part: Following the route of Section II-B, a Lyapunov functional is designed from V adding some terms to take into account the state of the integral controller. Let the operator $\mathcal{M}$: $L^1((0,1);\mathbb{R}^n) \to \mathbb{R}^m$ be given as

$$\mathcal{M}\varphi = \int_0^1 M\Psi(s)\varphi(s)ds \tag{72}$$

where Ψ is the matrix function defined in (39), and M is a matrix in $\mathbb{R}^{m\times n}$ defined in (41).

Following the Lyapunov functional construction in Theorem 2, we consider the candidate Lyapunov functional V_e : $L^2((0,1);\mathbb{R}^n) \times \mathbb{R}^m$ given as

$$V_e(\varphi, z) = V(\varphi) + p(z - \mathcal{M}\varphi)^\top (z - \mathcal{M}\varphi). \tag{73}$$

In the following theorem, it is shown that by selecting properly K_i, k_i and p, this function is indeed a Lyapunov functional for the closed loop system. Again, with a slight abuse of notation, we write $V_e(t) = V_e(\tilde{\phi}(\cdot,t), \tilde{z}(t))$ and we denote by $\dot{V}_e(t)$ the time derivative of the Lyapunov functional along solutions which are C^1 in time.

Proposition 4: Assume that Assumptions 2 and 3 hold. Then there exists a matrix K_i in $\mathbb{R}^{m\times m}$ and $k_i^* > 0$ such that for all $0 < k_i < k_i^*$, there exist positive real numbers L_e and μ_e such that for all (φ, z) in $\mathcal{X}_e$

$$\frac{1}{L_e}\left(\|\varphi\|_{\mathcal{X}}^2 + |z|^2\right) \leqslant V_e(\varphi, z) \leqslant L_e\left(\|\varphi\|_{\mathcal{X}}^2 + |z|^2\right), \tag{74}$$

and along C^1 solution of the system (59)-(61)-(62)

$$\dot{V}_e(t) \leqslant -\mu_e V_e(t)\ ,\ \forall t \in \mathbb{R}_+. \tag{75}$$

Proof: With (35), it yields for all φ in $L^2((0,1);\mathbb{R}^n)$,

$$\underline{P}\|\varphi\|^2_{L^2((0,1);\mathbb{R}^n)} \leqslant V(\varphi) \leqslant \overline{P}\|\varphi\|^2_{L^2((0,1);\mathbb{R}^n)}. \tag{76}$$

Let $\overline{\Psi} > 0$ be such that

$$|\Psi(s)| \leqslant \overline{\Psi}\ ,\ \forall s \in [0,1].$$

Note that for all φ in $L^2((0,1);\mathbb{R}^n$, by Cauchy-Schwartz inequality,

$$|\mathcal{M}\varphi| \leqslant |M|\,\overline{\Psi}\,\|\varphi\|_{L^2((0,1);\mathbb{R}^n)}. \tag{77}$$

Hence, for each $p > 0$ equation (74) holds.

Note that along C_1 solution to system (59), we have

$$\begin{aligned}\mathcal{M}\tilde{\phi}_t(t,\cdot) &= \mathcal{M}(-\Lambda_0(\cdot)\tilde{\phi}_s(t,\cdot) - \Lambda_1(\cdot)\tilde{\phi}(t,\cdot)) \\ &= \int_0^1 M\Psi(s)(-\Lambda_0(s)\tilde{\phi}_s(t,s) - \Lambda_1(s)\tilde{\phi}(t,s))ds\end{aligned}$$

With an integration by part this implies

$$\begin{aligned}\mathcal{M}\tilde{\phi}_t(t,\cdot) = &\int_0^1 M(\Psi(s)\Lambda_0(s))_s\tilde{\phi}(t,s)ds \\ &- \int_0^1 M\Psi(s)\Lambda_1(s)\tilde{\phi}(t,s))ds \\ &- M\left(\Psi(1)\Lambda_0(1)\tilde{\phi}(t,1) - \Lambda_0(0)\tilde{\phi}(t,0)\right).\end{aligned}$$

This gives,

$$\begin{aligned}&\mathcal{M}\tilde{\phi}_t(t,\cdot) = \\ &\int_0^1 M\left(\Psi_s(s)\Lambda_0(s) + \Psi(s)(\Lambda_{0s}(s) - \Lambda_1(s))\right)\tilde{\phi}(t,s)ds \\ &\qquad - M\left(\Psi(1)\Lambda_0(1)\tilde{\phi}(t,1) - \Lambda_0(0)\tilde{\phi}(t,0)\right).\end{aligned}$$

With the definition of Ψ, it yields

$$\mathcal{M}\tilde{\phi}_t(t,\cdot) = -M\left(\Psi(1)\Lambda_0(1)\tilde{\phi}(t,1) - \Lambda_0(0)\tilde{\phi}(t,0)\right).$$

With the boundary condition (61), it yields

$$\begin{aligned}\mathcal{M}\tilde{\phi}_t(t,\cdot) = &-M\left(\Psi(1)\Lambda_0(1)K_+ - \Lambda_0(0)K_-\right)\begin{bmatrix}\tilde{\phi}_+(t,1) \\ \tilde{\phi}_-(t,0)\end{bmatrix} \\ &-M\Psi(1)\Lambda_0(1)\begin{bmatrix} 0 \\ B_2\end{bmatrix}u(t) + M\Lambda_0(0)\begin{bmatrix} B_1 \\ 0\end{bmatrix}u(t)\end{aligned}$$

Hence, with the definition of M, it implies

$$\begin{aligned}\mathcal{M}\tilde{\phi}_t(t,\cdot) = &(L_1K + L_2)\begin{bmatrix}\tilde{\phi}_+(t,1) \\ \tilde{\phi}_-(t,0)\end{bmatrix} \\ &+ M\left(\Lambda_0(0)\begin{bmatrix} B_1 \\ 0\end{bmatrix} - \Psi(1)\Lambda_0(1)\begin{bmatrix} 0 \\ B_2\end{bmatrix}\right)u(t)\end{aligned}$$

On another hand,

$$z_t(t) = (L_1K + L_2)\begin{bmatrix}\tilde{\phi}_+(t,1) \\ \tilde{\phi}_-(t,0)\end{bmatrix} + L_1Bu(t).$$

This gives,

$$\begin{aligned}\mathcal{M}\tilde{\phi}_t(t,\cdot) = &z_t(t) - L_1Bu(t) \\ &+ M\left(\Lambda_0(0)\begin{bmatrix} B_1 \\ 0\end{bmatrix} - \Psi(1)\Lambda_0(1)\begin{bmatrix} 0 \\ B_2\end{bmatrix}\right)u(t).\end{aligned}$$

Hence, it yields

$$\mathcal{M}\tilde{\phi}_t(t,\cdot) = z_t(t) + T_2u(t). \tag{78}$$

We recognize here equation (11) when $u = 0$. This gives with (69)

$$\begin{aligned}\dot{V}_e(t) \leqslant &-\mu V(t) + c|u(t)|^2 \\ &- 2p(z(t) - \mathcal{M}\phi(\cdot,t))^\top T_2u(t).\end{aligned} \tag{79}$$

Let now, $K_i = T_2^{-1}$. Hence, this gives with $u = k_i K_i z$,

$$\begin{aligned}\dot{V}_e(t) \leqslant -\mu V(t) + c k_i^2 |K_i z(t)|^2 \\ - 2p|z(t)|^2 k_i + 2p k_i (\mathcal{M}\phi(\cdot,t))^\top z(t), \quad (80)\end{aligned}$$

With (77), and completing the square it yields for all φ in $L^2((0,1);\mathbb{R}^n)$ and z in $\mathbb{R}^m$,

$$\begin{aligned}2(\mathcal{M}\varphi)^\top z &\leqslant |\mathcal{M}\varphi|^2 + |z|^2, &(81)\\ &\leqslant |M|^2 \overline{\Psi}^2 \|\varphi\|^2_{L^2((0,1);\mathbb{R}^n)} + |z|^2. &(82)\end{aligned}$$

Merging the last two inequality yields,

$$\begin{aligned}\dot{V}_e(t) \leqslant -\mu V(t) + p k_i |M|^2 \overline{\Psi}^2 \|\phi(\cdot,t)\|^2_{L^2((0,1);\mathbb{R}^n)} \\ + \left(c k_i^2 |K_i|^2 - p k_i\right) |z(t)|^2. \quad (83)\end{aligned}$$

With (76), this yields

$$\begin{aligned}\dot{V}_e(t) \leqslant \left(-\mu + p k_i \frac{|M|^2 \overline{\Psi}^2}{\underline{P}}\right) V(t) \\ + \left(c k_i^2 |K_i|^2 - p k_i\right) |z(t)|^2. \quad (84)\end{aligned}$$

Note that if

$$p k_i < \frac{\mu \underline{P}}{|M|^2 \overline{\Psi}^2} \ , \ k_i^2 < \frac{p k_i}{c\left|T_2^{-1}\right|},$$

this yields the existence of μ_e such that equation (75) holds. This is obtained for all $k_i < k_i^*$ when

$$k_i^* = \frac{\sqrt{\mu \underline{P}}}{|M|\overline{\Psi}\sqrt{c\left|T_2^{-1}\right|}}, \quad (85)$$

and

$$p < \frac{\mu \underline{P}}{k_i |M|^2 \overline{\Psi}^2}.$$

□

With this proposition, the proof of Theorem 3 is completed.

V. Conclusion

In the last three decades, the regulation problem has been studied for different classes of distributed parameter systems. Most of existing results follow a semigroup approach and the perturbation theory for linear operator. In this paper we have shown that is was also possible to construct Lyapunov functionals to address the regulation problem in the case in which is used an integral action. This framework allows to explicitly give an integral gain. Moreover, it is no more necessary to impose boundedness of control or measurement operators to guarantee the regulation. This is applied to PDE hyperbolic systems and this allows to generalize many available results in this field.

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Alexandre Terrand-Jeanne graduated in electrical engineering from ENS Cachan, France, in 2013. After one year in the robotic laboratory "Centro E.Piaggio" in Pisa, Italy, he is currently a doctoral student in LAGEP, university of Lyon 1. His PhD topic concerns the stability analysis and control laws design for systems involving hyperbolic partial differential equations coupled with nonlinear ordinary differential equations. This work is under the supervision of V. Dos Santos Martins, V. Andrieu and M. Tayakout-Fayolle.

Vincent Andrieu graduated in applied mathematics from INSA de Rouen, France, in 2001. After working in ONERA (French aerospace research company), he obtained a PhD degree from Ecole des Mines de Paris in 2005. In 2006, he had a research appointment at the Control and Power Group, Dept. EEE, Imperial College London. In 2008, he joined the CNRS-LAAS lab in Toulouse, France, as a CNRS-chargé de recherche. Since 2010, he has been working in LAGEP-CNRS, Université de Lyon 1, France. In 2014, he joined the functional analysis group from Bergische Universität Wuppertal in Germany, for two sabbatical years. His main research interests are in the feedback stabilization of controlled dynamical nonlinear systems and state estimation problems. He is also interested in practical application of these theoretical problems, and especially in the field of aeronautics and chemical engineering. Since 2018 he is an associate editor of the IEEE Transactions on Automatic Control, System & Control Letters and IEEE Control Systems Letters.

Cheng-Zhong XU received the Ph.D. degree in automatic control and signal processing from Institut National Polytechnique de Grenoble, Grenoble, France, in 1989, and the Habilitation degree in applied mathematics and automatic control from University of Metz, Metz, France, in 1997. From 1991 to 2002, he was a Chargé de Recherche (Research Officer) in the Institut National de Recherche en Informatique et en Automatique. Since 2002, he has been a Professor of automatic control at the University of Lyon, Lyon, France. His research interests include control of distributed parameter systems and its applications to mechanical and chemical engineering. He was an associated editor of the IEEE Transactions on Automatic Control, from 1995 to 1998. He was an associated editor of the SIAM Journal on Control and Optimization, from 2011 to 2017.

Valérie Dos Santos Martins graduated in Mathematics from the University of Orléans, France in 2001. She received the Ph.D degree in 2004 in Applied Mathematics from the University of Orléans. After one year in the laboratory of Mathematics MAPMO in Orléans as ATER, she was post-doct in the laboratory CESAME/INMA of the University Catholic of Louvain, Belgium. Currently, she is professor assistant in the laboratory LAGEP, University of Lyon 1. Her current research interests include nonlinear control theory, perturbations theory of operators and semigroup, spectral theory and control of nonlinear partial differential equations.