Discrete-time k-positive linear systems

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Abstract

Positive systems play an important role in systems and control theory and have found many applications in multi-agent systems, neural networks, systems biology, and more. Positive systems map the nonnegative orthant to itself (and also the nonpositive orthant to itself). In other words, they map the set of vectors with zero sign variations to itself. In this note, discrete-time linear systems that map the set of vectors with up to $k-1$ sign variations to itself are introduced. For the special case $k=1$ these reduce to discrete-time positive linear systems. Properties of these systems are analyzed using tools from the theory of sign-regular matrices. In particular, it is shown that almost every solution of such systems converges to the set of vectors with up to $k-1$ sign variations. Also, the operation of such systems on k-dimensional parallelotopes are studied.

Index Terms

Sign-regular matrices, cones of rank k, exterior products, compound matrices, stability analysis.

1. Introduction

For two vectors $a, b \in \mathbb{R}^n$, we write $b \leq a$ if $b_\ell \leq a_\ell$ for all $\ell \in \{1, \dots, n\}$. Inequalities between matrices are also understood as entry-wise.

Consider the discrete-time (DT) linear time-varying (LTV) system

$$x(i+1) = A(i)x(i), \quad x(0) = x_0 \in \mathbb{R}^n. \tag{1}$$

Let $x(i, x_0)$ denote the solution of (1) at time i. The LTV (1) is called *positive* if $A(i) \geq 0$ for all $i \geq 0$. Then clearly

$$b \leq a \implies x(i,b) \leq x(i,a) \text{ for all } i \geq 0. \tag{2}$$

In particular,

$$0 \leq a \implies 0 \leq x(i,a) \text{ for all } i \geq 0,$$

i.e., a positive system maps the nonnegative orthant

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i\}$$

to itself (and also $\mathbb{R}^n_-$ to itself). The system is called *strongly positive* if it maps $\mathbb{R}^n_+ \setminus \{0\}$ to $\operatorname{int}(\mathbb{R}^n_+)$ (the interior of $\mathbb{R}^n_+$).

Positive systems appear naturally when the state-variables represent quantities that can only take nonnegative values, e.g., probabilities, concentrations of molecules, densities of particles, etc. Positive LTVs play an important role in linear systems and control theory, see, e.g., [7], [24], and via *differential analysis* [11], [19], also in the analysis of nonlinear systems. To explain this, consider the *nonlinear* time-varying system

$$x(i+1) = f(i, x(i)), \tag{3}$$

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and suppose that its trajectories evolve on a convex state-space $\Omega \subseteq \mathbb{R}^n$, and that f is C^1 with respect to x. For $y \in \Omega$, let $x(i,y)$ denote the solution of (3) at time i for $x(0) = y$. Pick $a, b \in \Omega$ and let

$$z(i) := x(i,a) - x(i,b),$$

that is, the difference at time i between the trajectories emanating from a and from b at time zero. Then

$$\begin{aligned} z(i+1) &= f(i, x(i,a)) - f(i, x(i,b)) = J^{ab}(i) z(i) \\ \text{with } J^{ab}(i) &:= \int_0^1 \frac{\partial}{\partial r} f(i, r x(i,a) + (1-r) x(i,b)) \, \mathrm{d}r. \end{aligned} \tag{4}$$

If $J^{ab}(i) \geq 0$ for all $a, b \in \Omega$ and all $i \geq 0$, then the *variational system* (4) is a positive LTV, and this has important consequences for the behavior of (3). Roughly speaking, almost every bounded trajectory of a smooth strongly positive system converges to a periodic trajectory (a cycle) [23]. This is quite different from the behavior in the continuous-time case, where almost every bounded trajectory of the nonlinear system converges to the set of equilibria [29].

The dynamics of a DT positive LTV maps the set of vectors with zero sign variations to itself. A natural question is: what systems map the set of vectors with up to $k-1$ sign variations to itself? We call such a system a *DT k-positive system*. Then a 1-positive system is just a positive system, but for $k > 1$ the system may be k-positive yet not positive.

Continuous-time (CT) k-positive systems have been recently defined and analyzed in [30]. In the CT and time-invariant case, i.e., $\dot{x}(t) = Ax(t)$, the *matrix exponential* of A should satisfy for all time a property called strict sign-regularity of order k, for the definition see the next paragraph, and this can be mapped to simple to check sign conditions on A itself [30]. In the DT case studied here, the matrix A itself must have this property, and verifying this is nontrivial.

A matrix $A \in \mathbb{R}^{n \times m}$ is called *sign-regular of order k* (denoted by SR_k) if all its minors of order k are non-negative or all are non-positive. For example, if all the entries of A are non-negative then it is SR_1. A matrix is called *strictly sign-regular of order k* (denoted by SSR_k) if it is SR_k, and all the minors of order k are non-zero. In other words, all minors of order k are non-zero and have the same sign.[1]

For example, consider the matrix

$$A := \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0.1 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

This matrix is SR_1 (but not SSR_1 as some entries are zero). It has both positive and negative 2-minors (e.g., $\det(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}) = 1$, $\det(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}) = -2$), so it is not SR_2. All its 3-minors are positive, so it is SSR_3, and $\det(A) > 0$, so it is SSR_4.

After the first consideration of SR_k matrices in [16], these matrices have been the subject of only a few studies. In Ref. [1], the authors analyze the spectral properties of nonsingular matrices that are SSR_k for a specific value of k. These results are extended to matrices that are SSR_k for several values of k, for example for all odd k.

To refer to the common sign of the minors of order k, we introduce the signature $\epsilon \in \{-1, 1\}$. A matrix $A \in \mathbb{R}^{n \times m}$ is called *[strictly] sign-regular* ($[S]SR$) if it is $[S]SR_k$ for all $k = 1, \dots, \min\{n, m\}$.

The most important examples of SR [SSR] matrices are the totally nonnegative *(TN)* [totally positive *(TP)*] matrices, that is, matrices with all minors nonnegative [positive]. Such matrices have applications in numerous fields including approximation theory, combinatorics, probability theory, computer aided geometric design, differential and integral equations, and more [6], [14], [16], [22].

[1] We note that the terminology in this field is not uniform and some authors refer to such matrices as sign-consistent of order k.

A very important property of SSR matrices is that multiplying a vector x by such a matrix cannot increase the number of sign variations in x [14]. To explain this *variation diminishing property* (VDP), we introduce some notation. For $y \in \mathbb{R}^n$, let $s^-(y)$ denote the number of sign variations in y after deleting all its zero entries, and let $s^+(y)$ denote the maximal possible number of sign variations in y after each zero entry is replaced by either $+1$ or -1. For example, for $n = 4$ and $y = \begin{bmatrix} 1 & -1 & 0 & -\pi \end{bmatrix}^T$ (where the superscript T denotes transposition), we have $s^-(y) = 1$ and $s^+(y) = 3$. Obviously,

$$0 \leq s^-(y) \leq s^+(y) \leq n-1 \text{ for all } y \in \mathbb{R}^n. \tag{5}$$

The first important results on the VDP of matrices were obtained by Fekete and Pólya [8] and Schoenberg [27]. Later on, Gantmacher and Krein [14, Chapter V] elaborated rather completely the various forms of VDPs and worked out the spectral properties of SR matrices. Two important examples of such VDPs are: if $A \in \mathbb{R}^{n\times m}$ ($m \leq n$) is SR and of rank m then

$$s^-(Ax) \leq s^-(x) \text{ for all } x \in \mathbb{R}^m,$$

whereas if A is SSR then

$$s^+(Ax) \leq s^-(x) \text{ for all } x \in \mathbb{R}^m \setminus \{0\}.$$

Note that combining this with (5) implies that both $s^-(x(i))$ and $s^+(x(i))$ can be used as an integer-valued Lyapunov function for the system $x(i+1) = Ax(i)$.

For $k \in \{1, \dots, n\}$, let

$$\begin{aligned} P_-^k &:= \{z \in \mathbb{R}^n : s^-(z) \leq k-1\}, \\ P_+^k &:= \{z \in \mathbb{R}^n : s^+(z) \leq k-1\}. \end{aligned} \tag{6}$$

Then positive systems map the set P_-^1 to P_-^1, whereas strongly positive systems map P_-^1 to P_+^1. This naturally leads to the question: which linear systems map P_-^k to P_-^k and which map P_-^k to P_+^k? In this this paper, we define and analyze such systems, called DT k-positive linear systems. We show that such systems have interesting dynamical properties that generalize the properties of positive systems.

The remainder of this paper is organized as follows. In the next section, we review notations, definitions, and basic properties that will be used later on. Section 3 defines DT k-positive linear systems and analyzes their properties. The final section concludes. In passing we note that our results are part of a growing body of research on the applications of sign-regularity (and, in particular, total positivity) to dynamical systems [1], [4], [17], [20], [28], [30].

2. Preliminaries

We briefly review several known results that will be used later on.

A. Basic notation and definitions

For an integer $n \geq 1$ and $k \in \{1, \dots, n\}$, let $Q_{k,n}$ denote the set of all strictly increasing sequences of k integers chosen from $\{1, \dots, n\}$. For example, $Q_{2,3} = \{12, 13, 23\}$.

For $A \in \mathbb{R}^{n\times m}$, $\alpha \in Q_{k,n}$, and $\beta \in Q_{j,m}$, we denote the submatrix of A lying in the rows indexed by α and columns indexed by β by $A[\alpha, \beta]$. Thus, $A[\alpha, \beta] \in \mathbb{R}^{k\times j}$. If $k = j$ then we set

$$A(\alpha|\beta) := \det(A[\alpha, \beta]),$$

that is, the minor corresponding to the rows indexed by α and columns indexed by β. We often suppress the brackets associated with an index sequence if we enumerate its entries explicitly.

B. Multiplicative compound

Let $A \in \mathbb{R}^{n\times m}$. For any $k \in \{1,\dots,\min\{n,m\}\}$, the *kth multiplicative compound* of A is the $\binom{n}{k}\times\binom{m}{k}$ matrix that includes all the minors of order k of A organized in lexicographic order. For example, if $A \in \mathbb{R}^{3\times 3}$ then

$$A^{(2)} = \begin{bmatrix} A(12|12) & A(12|13) & A(12|23) \\ A(13|12) & A(13|13) & A(13|23) \\ A(23|12) & A(23|13) & A(23|23) \end{bmatrix}.$$

Note that $A^{(1)} = A$ and that if $m = n$ then $A^{(n)} = \det(A)$. Note also that A is SSR_k [SR_k] if either $A^{(k)} > 0$ or $A^{(k)} < 0$ [either $A^{(k)} \geq 0$ or $A^{(k)} \leq 0$].

The Cauchy-Binet Formula [6, Theorem 1.1.1] provides an expression for the minors of the product of two matrices. Pick $A \in \mathbb{R}^{n\times p}$ and $B \in \mathbb{R}^{p\times m}$. Let $C := AB$. Pick $k \in \{1,\dots,\min\{n,p,m\}\}$, $\alpha \in Q_{k,n}$, and $\beta \in Q_{k,m}$. Then

$$C(\alpha|\beta) = \sum_{\gamma\in Q_{k,p}} A(\alpha|\gamma)B(\gamma|\beta). \tag{7}$$

For $n = p = m$ and $k = n$ this reduces to the familiar formula $\det(AB) = \det(A)\det(B)$. Note that (7) implies that

$$(AB)^{(k)} = A^{(k)}B^{(k)} \tag{8}$$

for all $k \in \{1,\dots,\min\{n,p,m\}\}$. This justifies the term multiplicative compound.

C. Sets of vectors with sign variations

Consider the sets defined in (6). It is straightforward to show using the definitions of s^-, s^+ and (5) that P^k_- is closed and that

$$P^k_+ = \operatorname{int}(P^k_-).$$

It is clear that

$$P^1_- = \mathbb{R}^n_+ \cup \mathbb{R}^n_-, \quad P^1_+ = \operatorname{int}(\mathbb{R}^n_+) \cup \operatorname{int}(\mathbb{R}^n_-). \tag{9}$$

Also, the sets are nested, as

$$\begin{aligned} &P^1_- \subset P^2_- \subset \cdots \subset P^n_- = \mathbb{R}^n, \\ &P^1_+ \subset P^2_+ \subset \cdots \subset P^n_+ = \mathbb{R}^n. \end{aligned} \tag{10}$$

If $x \in P^k_-$ then $\alpha x \in P^k_-$ for all $\alpha \in \mathbb{R}$, and if $x \in P^k_+$ then $\beta x \in P^k_+$ for all $\beta \in \mathbb{R}\setminus\{0\}$, so both P^k_- and $P^k_+ \cup \{0\}$ are cones. Yet, in general P^k_- and P^k_+ are *not* convex sets. For example, for $n = 2$ and the vectors $x := \begin{bmatrix} 2 & 0 \end{bmatrix}^T$, $y := \begin{bmatrix} 0 & -2 \end{bmatrix}^T$, we have $x, y \in P^1_-$ yet $\frac{x}{2} + \frac{y}{2} = \begin{bmatrix} 1 & -1 \end{bmatrix}^T \notin P^1_-$.

Recall that a set $C \subseteq \mathbb{R}^n$ is called *a cone of rank* k [18] if

(i) C is closed;
(ii) $x \in C$ implies that $\alpha x \in C$ for all $\alpha \in \mathbb{R}$;
(iii) C contains a linear subspace of dimension k and no linear subspace of higher dimension.

For example, $\mathbb{R}^2_+ \cup \mathbb{R}^2_-$ (and more generally, $\mathbb{R}^n_+ \cup \mathbb{R}^n_-$ [13]) is a cone of rank 1. A cone C of rank k is called *solid* if its interior is non empty, and k-*solid* if there is a linear subspace W of dimension k such that $W \setminus \{0\} \subseteq \operatorname{int}(C)$; k-solid cones are useful in the analysis of dynamical systems [9], [10], [12], [26]. Roughly speaking, if a trajectory of the system is confined to an invariant set C that is a k-solid cone then the trajectory can be projected onto a k-dimensional subspace contained in C. If this projection is one-to-one then the trajectory is topologically conjugate to a trajectory of a k-dimensional dynamical system.

It was shown in [30] (see also [18]) that for any $k \in \{1, \dots, n\}$, the set P^k_- is a k-solid cone, and that its complement

$$(P^k_-)^c := \operatorname{clos}(\mathbb{R}^n \setminus P^k_-)$$

is an $(n-k)$-solid cone. This implies, in particular, that there exists a k-dimensional subspace W^k such that $W^k \subseteq P^k_-$, and that there is no $(k+1)$-dimensional subspace contained in P^k_-. For example, let $e^i \in \mathbb{R}^n$ denote the vector with all entries zero, except for entry i that is one. Then the k-dimensional subspace $\operatorname{span}\{e^1, \dots, e^k\}$ is contained in P^k_-.

D. *Linear mappings that preserve the number of sign variations in a vector*

Recall that a matrix $A \in \mathbb{R}^{n\times m}$ is termed SR_k if all its minors of order k are either all nonnegative or all nonpositive and it is called SSR_k if all its minors of order k are all positive or all negative, see the example in the Introduction.

Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Pick $k \in \{1, \dots, n\}$. It was shown in [4] that A maps $P^k_- \setminus \{0\}$ to P^k_+ if and only if A is SSR_k, and a continuity argument [30] implies that A maps P^k_- to P^k_- if and only if A is SR_k. For example, the nonsingular matrix

$$A := \begin{bmatrix} 10 & 4 & 1 \\ 1 & 3 & 1 \\ 2 & 4 & 6 \end{bmatrix}$$

is not SR_2, as it has both positive and negative minors of order 2 (e.g., $A(1,2|1,2) = 26$ and $A(2,3|1,2) = -2$), so it does not map P^2_- to itself. Indeed, for $x = \begin{bmatrix} 19 & -6 & -2 \end{bmatrix}^T$, we have $x \in P^2_-$ and $Ax = \begin{bmatrix} 164 & -1 & 2 \end{bmatrix}^T \notin P^2_-$.

We can now introduce and analyze a new class of DT linear systems.

3. Discrete-time k-positive linear systems

Definition 1. *Consider the DT LTV* (1) *with every matrix $A(i)$ nonsingular. The system is called a* k-positive system *if it maps P^k_- to P^k_-, and a* strongly k-positive system *if it maps $P^k_- \setminus \{0\}$ to P^k_+.*

Note that (9) implies that a [strongly] 1-positive system is simply a [strongly] positive system. Note also that since $P^k_+ = \operatorname{int}(P^k_-)$, both P^k_- and P^k_+ are invariant sets of a strongly k-positive system.

The next result follows from [4, Theorem 1] and [30, Theorem 2].

Corollary 1. The system (1) is a [strongly] k-positive system if and only if $A(i)$ is $[S]SR_k$ for all $i \geq 0$.

For the case $k = 1$ this is a generalization of [strongly] positive linear systems. For example, a system is typically defined as strongly positive if all the entries of $A(k)$ are positive, yet it is strongly 1-positive if all its entries are either all positive or all negative.

Form here on we focus on the time-invariant linear system

$$x(j+1) = Ax(j), \quad x(0) = x_0 \in \mathbb{R}^n, \tag{11}$$

where A is nonsingular and SSR_k for some $k \in \{1, \dots, n-1\}$, leaving the time-varying case and nonlinear systems to a sequel paper. Note that even for this LTI case our results are new.

Example 1. *Consider the system* (11) *with $n = 4$ and*

$$A := \begin{bmatrix} 9 & 2 & -2 & 1 \\ 3 & 10 & 1 & -1 \\ -4 & 1.5 & 12 & 4 \\ 1 & -1 & 2 & 15 \end{bmatrix}, \tag{12}$$

for all $j \geq 0$. Note that A is not SSR$_1$ (as it has both positive and negative entries), nor SSR$_2$ (as it has both positive and negative minors of order two, e.g., $A(1,2|1,2) = 84$, $A(3,4|1,3) = -20$). All

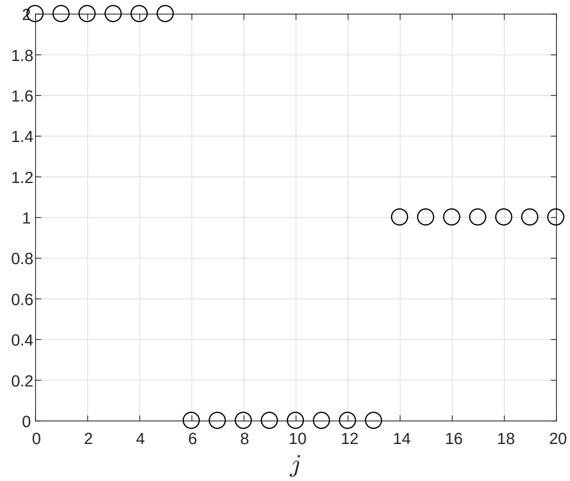

Fig. 1. $s^+(x(j))$ as a function of j for the trajectory in Example 1.

the 16 *minors of order three are positive, and* $\det(A) \neq 0$*, so* A *is SSR*$_3$ *and nonsingular. Figure 1 shows* $s^+(x(j))$ *as a function of* j *for* $x(0) = \begin{bmatrix} 1 & 1 & -1 & 1 \end{bmatrix}^T$. *Note that* $s^-(x(0)) = 2$. *It may be seen that, as expected,* $s^+(x(j)) \leq 2$ *for all* $j \geq 0$. $\square$

A. *k-exponential separation and its implications*

Let $C \subseteq \mathbb{R}^n$ be closed and a *convex cone*, i.e., $x, y \in C$ implies that $\alpha x + \beta y \in C$ for all $\alpha, \beta \geq 0$. Furthermore, let C be *pointed*, i.e., $C \cap (-C) = \{0\}$. Then C induces a (partial) order defined by $a \leq_C b$ if $b - a \in C$. For example, for $C = \mathbb{R}^n_+$ we have $a \leq_C b$ if and only if $b_i \geq a_i$ for all $i \in \{1, \dots, n\}$. Dynamical systems whose flow preserves such an order are called *monotone*, see, e.g., the excellent monograph [29].

Since P^k_- and P^k_+ are not convex sets, k-positive systems are not monotone systems in the usual sense. However, the fact that P^k_- is a k-solid cone has strong implications for the dynamics of such systems.

The first demonstration of this is a k-exponential separation property of (11). This is closely related to the generalization of the Perron Theorem in [13], see also [18], but we give a direct proof based on the spectral properties of a nonsingular SSR_k matrix, see Theorem 1 below. We now review these properties following the presentation in [1].

Fix a nonsingular matrix $A \in \mathbb{R}^{n \times n}$ that is SSR_k for some $k \in \{1, \dots, n-1\}$. Let $\epsilon \in \{-1, 1\}$ denote the common sign of all the minors of order k . Denote the eigenvalues of A by λ_i, $i = 1, \dots, n$, ordered such that

$$|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n| > 0, \tag{13}$$

and let

$$v^1,\ v^2, \dots, v^n \tag{14}$$

denote the corresponding eigenvectors, with complex conjugate eigenvalues appearing in consecutive pairs (we say, with a mild abuse of notation, that $z \in \mathbb{C}^n$ is *complex* if $z \neq \bar{z}$, where $\bar{z}$ denotes the complex conjugate of z). We may assume that every v^i is not purely imaginary. Indeed, otherwise we can replace v^i by $\operatorname{Im}(v^i)$ that is a real eigenvector. Also, the fact that A is real means that if v^i is complex then its real and imaginary parts can be chosen as linearly independent.

Define a set of real vectors $u^1, u^2, \dots, u^n \in \mathbb{R}^n$ by going through the v^i's as follows. If v^1 is real then $u^1 := v^1$ and proceed to examine v^2. If v^1 is complex (and whence $v^2 = \bar{v}^1$) then $u^1 := \operatorname{Re}(v^1)$, $u^2 := \operatorname{Im}(v^1)$ and proceed to examine v^3, and so on.

Suppose that for some i, j the eigenvector v^i is real and v^j is complex. Then is not difficult to show that since A is real and nonsingular, the real vectors $v^i, \operatorname{Re}(v^j), \operatorname{Im}(v^j)$ are linearly independent.

Note that if $v^i, v^{i+1} \in \mathbb{C}^n$ is a complex conjugate pair and $c \in \mathbb{C} \setminus \{0\}$ is complex then

$$cv^i + \bar{c}v^{i+1} = 2(\operatorname{Re}(c)\operatorname{Re}(v^i) - \operatorname{Im}(c)\operatorname{Im}(v^i)) \in \mathbb{R}^n \setminus \{0\},$$

so by choosing an appropriate $c \in \mathbb{C} \setminus \{0\}$ we can get any nonzero real linear combination of the real vectors $\operatorname{Re}(v^i)$ and $\operatorname{Im}(v^i)$.

For $p \leq q$, we say that a set $c_p, \dots, c_q \in \mathbb{C}$ *matches* the set $v^p, \dots, v^q$ of consecutive eigenvectors (14) if the c_i's are not all zero and for every i if the vector v^i is real then c_i is real, and if v^i, v^{i+1} is a complex conjugate pair then $c_{i+1} = \bar{c}_i$. In particular, this implies that $\sum_{i=p}^q c_i v^i \in \mathbb{R}^n$.

It was shown in [1] that if $A \in \mathbb{R}^{n \times n}$ is nonsingular and SSR_k with signature ϵ, then the product $\epsilon\lambda_1\lambda_2 \dots \lambda_k$ is real and positive,

$$|\lambda_k| > |\lambda_{k+1}|, \tag{15}$$

and if $c_1, \dots, c_k \in \mathbb{C}$ $[c_{k+1}, \dots, c_n \in \mathbb{C}]$ match the eigenvectors $v^1, \dots, v^k$ $[v^{k+1}, \dots, v^n]$ of A, then

$$s^+(\sum_{i=1}^k c_i v^i) \leq k - 1, \tag{16}$$

$$s^-(\sum_{i=k+1}^n c_i v^i) \geq k. \tag{17}$$

Furthermore, let $\{u^1, \dots, u^n\}$ be the set of real vectors constructed from $\{v^1, \dots, v^n\}$ as described above. Then $u^1, \dots, u^k$ are linearly independent. In particular, if $v^1, \dots, v^k$ are real then they are linearly independent.

Example 2. *Let*

$$A := \begin{bmatrix} 2 & 6 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 4 & 2 \\ 2 & 0 & 0 & 4 \end{bmatrix}. \tag{18}$$

It is straightforward to verify that this matrix is nonsingular, and that all minors of order 3 *are positive, so* A *is* SSR_3 *with* $\epsilon = 1$*. Its eigenvalues are* [2]

$$\lambda_1 = 3 + s_1, \ \lambda_2 = 3 + \mathrm{i}s_2, \ \lambda_3 = 3 - \mathrm{i}s_2, \ \lambda_4 = 3 - s_1,$$

where $\mathrm{i}^2 = -1$*,* $s_1 := \sqrt{1 + 4\sqrt{3}} \approx 2.8157$*, and* $s_2 := \sqrt{-1 + 4\sqrt{3}} \approx 2.4348$*. Note that* $\lambda_1\lambda_2\lambda_3$ *is real and positive, and that* $|\lambda_3| > |\lambda_4|$*. The matrix of corresponding eigenvectors is*

$$\begin{aligned} V &:= \begin{bmatrix} v^1 & v^2 & v^3 & v^4 \end{bmatrix} \\ &= \begin{bmatrix} \frac{s_1-1}{2} & \frac{\mathrm{i}s_2-1}{2} & \frac{-(\mathrm{i}s_2+1)}{2} & \frac{-(s_1+1)}{2} \\ \frac{s_1^2-1}{12} & \frac{-(1+s_2^2)}{12} & \frac{-(1+s_2^2)}{12} & \frac{s_1^2-1}{12} \\ \frac{2}{s_1-1} & \frac{-2(1+\mathrm{i}s_2)}{1+s_2^2} & \frac{2(-1+\mathrm{i}s_2)}{1+s_2^2} & \frac{-2}{s_1+1} \\ 1 & 1 & 1 & 1 \end{bmatrix}, \end{aligned}$$

[2]All numerical values in this paper are subject to 4-digits accuracy.

and thus

$$\begin{aligned} U &:= \begin{bmatrix} u^1 & u^2 & u^3 & u^4 \end{bmatrix} \\ &= \begin{bmatrix} v^1 & \mathrm{Re}(v^2) & \mathrm{Im}(v^2) & v^4 \end{bmatrix} \\ &= \begin{bmatrix} \frac{s_1-1}{2} & \frac{-1}{2} & \frac{s_2}{2} & \frac{-(s_1+1)}{2} \\ \frac{s_1^2-1}{12} & \frac{-(1+s_2^2)}{12} & 0 & \frac{s_1^2-1}{12} \\ \frac{2}{s_1-1} & \frac{-2}{1+s_2^2} & \frac{-2s_2}{1+s_2^2} & \frac{-2}{s_1+1} \\ 1 & 1 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Note that $s^-(u^i) = s^+(u^i) = i-1$, $i = 1, 2, 4$, *and*

$$1 = s^-(u^3) < s^+(u^3) = 2.$$

□

We now state the main result in this subsection. Let $||\cdot|| : \mathbb{R}^n \to \mathbb{R}_+$ denote some vector norm.

Theorem 1. Suppose that $A \in \mathbb{R}^{n\times n}$ is nonsingular and SSR_k for some $k \in \{1, \dots, n-1\}$. Let $u^1, \dots, u^n$ be the real vectors constructed from the eigenvectors of A as described and let

$$E := \mathrm{span}\{u^1, \dots, u^k\}, \quad E^c := (\mathbb{R}^n \setminus E) \cup \{0\}.$$

Then the following properties hold:

(i) $\dim(E) = k$ and $\dim(E^c) = n-k$;

(ii) both E and E^c are invariant under A;

(iii) $E \subseteq \mathrm{int}(P_-^k) \cup \{0\}$, and $E^c \cap P_-^k = \{0\}$.

(iv) There exist $a > 0$ and $b \in (0, 1)$ such that for any $x(0) \in E$, $\tilde{x}(0) \in E^c$, with $||x(0)|| = ||\tilde{x}(0)|| = 1$, the corresponding solutions of (11) satisfy

$$||\tilde{x}(j)|| \le ab^j ||x(j)||. \tag{19}$$

(v) For any $x(0)$ satisfying

$$x(0) = f + g, \text{ where } f \in E \setminus \{0\} \text{ and } g \in E^c, \tag{20}$$

there exists an $q = q(x(0)) \ge 0$ such that the corresponding solution of (11) satisfies

$$s^+(x(j)) \le k-1 \text{ for all } j \ge q.$$

Condition (v) does not necessarily mean that $x(0)$ is an element of E, as it may also include some non-zero combination of the vectors $u^{k+1}, \dots, u^n$ that are not in E. Note that assertion (v) implies that for almost any initial condition, the corresponding solution of the dynamical system converges to P_+^k in finite time.

Proof. We begin by noting that the eigenvalues of A are ordered as

$$|\lambda_1| \ge \dots \ge |\lambda_k| > |\lambda_{k+1}| \ge \dots \ge |\lambda_n| > 0. \tag{21}$$

Assertion (i) follows immediately from the fact that $u^1, \dots, u^k$ are linearly independent.

Pick $z \in E \setminus \{0\}$. Since $\prod_{\ell=1}^k \lambda_\ell$ is real, either λ_{k-1}, λ_k are both real, or they are a complex conjugate pair. Combining this with the definition of E implies that $z = \sum_{i=1}^k c_i v^i$, for some $c_1, \dots, c_k$ that match $v^1, \dots, v^k$. Hence, $Az = \sum_{i=1}^k c_i \lambda_i v^i$. Clearly, $\{c_1\lambda_1, \dots, c_k\lambda_k\}$ also match $\{v^1, \dots, v^k\}$, so E is invariant under A.

It follows from (16) and the construction of the u^i's that $s^+(z) \le k-1$ for any $z \in E \setminus \{0\}$, that is, $E \setminus \{0\} \subseteq P_+^k$. Since $P_+^k = \mathrm{int}(P_-^k)$, we conclude that $E \subseteq \mathrm{int}(P_-^k) \cup \{0\}$.

In the remainder of the proof we consider without loss of generality the generic case, where $u^1,\dots,u^n$ are linearly independent. Then $E^c=\operatorname{span}\{u^{k+1},\dots,u^n\}$. The proofs of the properties of E^c are then very similar to the proofs for E, and thus we present here only the proofs for E.

To prove (iv), pick $x(0)\in E\setminus\{0\}$ and $\tilde{x}(0)\in E^c\setminus\{0\}$. Then $x(0)=\sum_{i=1}^k c_i v^i$ and $\tilde{x}(0)=\sum_{i=k+1}^n \tilde{c}_i v^i$, where $c_1,\dots,c_k\in\mathbb{C}$ $[\tilde{c}_{k+1},\dots,\tilde{c}_n\in\mathbb{C}]$ match $v^1,\dots,v^k$ $[v^{k+1},\dots,v^n]$. Using (21), a straightforward argument shows that there exists $m>0$ such that

$$\begin{aligned}||x(j)|| &= ||A^j x(0)|| \\ &\geq m|\lambda_k|^j ||x(0)||.\end{aligned}$$

Similarly, there exists $M>0$ such that $||\tilde{x}(j)||\leq M|\lambda_{k+1}|^j||\tilde{x}(0)||$. Thus,

$$\frac{||\tilde{x}(j)||}{||x(j)||}\leq \frac{M}{m}\left|\frac{\lambda_{k+1}}{\lambda_k}\right|^j\frac{||\tilde{x}(0)||}{||x(0)||},$$

and combining this with (15) proves (19).

To prove (v), pick $x(0)$ such that (20) is satisfied. Then $x(0)=\sum_{i=1}^n c_i v^i$, where $c_1,\dots,c_n\in\mathbb{C}$ match $v^1,\dots,v^n$, and $\sum_{i=1}^k c_i v^i\neq 0$. Thus,

$$\frac{x(j)}{||\sum_{i=1}^k c_i\lambda_i^j v^i||}=\frac{\sum_{i=1}^k c_i\lambda_i^j v^i}{||\sum_{i=1}^k c_i\lambda_i^j v^i||}+\frac{\sum_{i=k+1}^n c_i\lambda_i^j v^i}{||\sum_{i=1}^k c_i\lambda_i^j v^i||}.$$

The first term on the right-hand side of this equation is a unit vector in E, and the second term goes to zero as $j\to\infty$. Thus, there exists $r\geq 0$ such that $x(r)\in P_-^k$. Then $x(r+1)\in P_+^k$, and the invariance of P_+^k implies that $x(j)\in P_+^k$ for all $j\geq r+1$. $\square$

B. Dynamics of exterior products

Recall that if $Z\in\mathbb{R}^{n\times k}$, with columns $z^1,\dots,z^k\in\mathbb{R}^n$, then its kth multiplicative compound $Z^{(k)}\in\mathbb{R}^{\binom{n}{k}\times\binom{n}{k}}$ is the exterior product $z^1\wedge\dots\wedge z^k$, represented as a column vector [21]. For example, for $z^1=\begin{bmatrix}r_1 & r_2 & r_3\end{bmatrix}^T$ and $z^2=\begin{bmatrix}w_1 & w_2 & w_3\end{bmatrix}^T$, we have

$$\begin{aligned}Z^{(2)} &= \begin{bmatrix}r_1 & w_1\\ r_2 & w_2\\ r_3 & w_3\end{bmatrix}^{(2)} \\ &= \begin{bmatrix}r_1w_2-r_2w_1 & r_1w_3-r_3w_1 & r_2w_3-r_3w_2\end{bmatrix}^T.\end{aligned}$$

Consider the dynamics (11), where $A\in\mathbb{R}^{n\times n}$ is SSR_k, and pick k initial conditions $w^1,\dots,w^k\in\mathbb{R}^n$. Let

$$X(j):=\begin{bmatrix}x(j,w^1) & \dots & x(j,w^k)\end{bmatrix}\in\mathbb{R}^{n\times k}. \tag{22}$$

Then $X(j+1)=AX(j)$. Taking the kth multiplicative compound on both sides of this equation and using (8) yields

$$\eta(j+1)=A^{(k)}\eta(j), \tag{23}$$

where

$$\eta(j):=x(j,w^1)\wedge\dots\wedge x(j,w^k). \tag{24}$$

The magnitude of this wedge product is the volume of the k-dimensional parallelotope whose edges are the given vectors.

Example 3. *Suppose that* $n = 3$*,* $A := \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$*,* $k = 2$*,* $w^1 = e^p$ *and* $w^2 = e^q$ *for some* $p, q \in \{1, 2, 3\}$*. Then*

$$\begin{aligned} \eta(j) &= x(j, e^p) \wedge x(j, e^q) \\ &= (\lambda_p^j e^p) \wedge (\lambda_q^j e^q) \\ &= \lambda_p^j \lambda_q^j (e^p \wedge e^q) \\ &= (\lambda_p \lambda_q)^j \eta(0). \end{aligned}$$

This implies that under the dynamics (11) *the unsigned area of the parallelogram having* e^p *and* e^q *as two of its sides scales as* $(\lambda_p \lambda_q)^j$*. On the other-hand,* $A^{(2)} = \begin{bmatrix} \lambda_1\lambda_2 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 0 \\ 0 & 0 & \lambda_2\lambda_3 \end{bmatrix}$. □

If A is SSR_k then either every entry of $B := A^{(k)}$ is positive or negative. We assume that $B > 0$ (the case $B < 0$ can be treated similarly). By the Perron Theorem, the spectral radius of B, denoted $\rho(B)$, is a positive eigenvalue and there exist positive vectors v^B, w^B, such that $Bv^B = \rho(B)v^B$ and $B^T w^B = \rho(B)w^B$. By normalization, we may assume that $(v^B)^T w^B = 1$. Then furthermore

$$\lim_{j \to \infty} \left(\frac{B}{\rho(B)} \right)^j = v^B (w^B)^T \tag{25}$$

(see, e.g., [15, Chapter 8]). This yields the following result.

Lemma 1. Suppose that A is SSR_k and that $B := A^{(k)} > 0$. Pick k initial conditions $w^1, \dots, w^k \in \mathbb{R}^n$, and define $X(j)$ and $\eta(j)$ as in (22) and (24). Then

$$\lim_{j \to \infty} \frac{\eta(j)}{(\rho(B))^j} = (w^B)^T \eta(0) v^B. \tag{26}$$

Proof. By (23), $\eta(j) = B^j \eta(0)$, i.e., $\frac{\eta(j)}{(\rho(B))^j} = \left(\frac{B}{\rho(B)}\right)^j \eta(0)$. Taking $j \to \infty$ and using (25) completes the proof. □

Remark 1. *Suppose that the spectral radius* $\rho(A)$ *of* A *satisfies* $\rho(A) < 1$*. Then* $\lim_{j\to\infty} A^j x = 0$ *for all* $x \in \mathbb{R}^n$ *and thus*

$$\eta(j) = (A^j w^1) \wedge \cdots \wedge (A^j w^k),$$

satisfies $\lim_{j\to\infty} \eta(j) = 0$*. Since every eigenvalue of* $A^{(k)}$ *is the product of* k *eigenvalues of* A*, in this case* $\rho(B) < 1$ *so* (26) *also shows that* $\eta(j)$ *goes to zero as* $j \to \infty$*.*

Example 4. *Consider the case* $n = 3, k = 2$*,*

$$A := \begin{bmatrix} 0.79 & 0.2 & 0.01 \\ 0.1 & 0.8 & 0.1 \\ 0.01 & 0.1 & 0.89 \end{bmatrix},$$

$w^1 = e^1$*, and* $w^2 = e^2$*. In other words, we consider the evolution of the unsigned area of the parallelogram with* e^1 *and* e^2 *as two of its sides. A calculation yields*

$$B := A^{(2)} = \begin{bmatrix} 0.612 & 0.078 & 0.012 \\ 0.077 & 0.703 & 0.177 \\ 0.002 & 0.088 & 0.702 \end{bmatrix}$$

(so A is SSR_2), $\rho(B) = 0.8430$,

$$v^B = \begin{bmatrix} 0.2991 & 0.8075 & 0.5084 \end{bmatrix}^T,$$

and

$$w^B = \begin{bmatrix} 0.2203 & 0.6394 & 0.8217 \end{bmatrix}^T,$$

(note that $(w^B)^T v^B = 1$). We compute $\eta(15)$ in two different ways. First,

$$\begin{aligned} \eta(15) &= (A^{15}e^1) \wedge (A^{15}e^2) \\ &= \begin{bmatrix} 0.2397 & 0.2190 & 0.1858 \end{bmatrix}^T \\ &\quad \wedge \begin{bmatrix} 0.4228 & 0.4103 & 0.3859 \end{bmatrix}^T \\ &= 0.0057e^1 + 0.0139e^2 + 0.0083e^3. \end{aligned} \tag{27}$$

Second, it follows from (26) *that*

$$\begin{aligned} \eta(15) &\approx (\rho(B))^{15} (w^B)^T \eta(0) v^B \\ &= (\rho(B))^{15} w_1^B v^B \\ &= \begin{bmatrix} 0.0051 & 0.0137 & 0.0086 \end{bmatrix}^T, \end{aligned}$$

and this is indeed an approximation of (27). □

4. Discussion

Positive systems and their nonlinear counterpart of monotone systems form a class of dynamical systems of fundamental importance in systems biology, neuroscience, and bio-chemical networks, and has recently also found important applications in control engineering for large-scale systems [25].

We introduced a new class of DT linear systems that generalize the important notion of DT positive linear systems. Such systems map the set of vectors with up to $k-1$ sign variations to itself.

An interesting research direction is to study DT nonlinear systems whose variational equation is a k-positive linear system. Since the variational equation (4) includes the integral of a matrix, this raises the following question: when is the integral of a matrix SSR_k?

Lemma 1 describes a convergence to a ray for the exterior product. We believe that this can generalized to the DT time-varying linear system (1), with the matrices $A(i)$ taken from a compact set, using the Birkhoff-Hopf theory [5].

Another interesting research direction may be the extension of k-positive systems to DT control systems as was done for CT monotone systems in [3]. Finally, our results highlight the importance of an efficient algorithm for determining if a given matrix is SSR_k for some k. This issue is currently under study [2].

References

[1] R. Alseidi, M. Margaliot, and J. Garloff, "On the spectral properties of nonsingular matrices that are strictly sign-regular for some order with applications to totally positive discrete-time systems," *J. Math. Anal. Appl.*, vol. 474, pp. 524–543, 2019.

[2] R. Alseidi, M. Margaliot, and J. Garloff, "Recognition of matrices which are strictly sign regular of a given order and a generalization of oscillatory matrices," *submitted*.

[3] D. Angeli and E. D. Sontag, "Monotone control systems," *IEEE Trans. Automat. Control*, vol. 48, pp. 1684–1698, 2003.

[4] T. Ben-Avraham, G. Sharon, Y. Zarai, and M. Margaliot, "Dynamical systems with a cyclic sign variation diminishing property," *IEEE Trans. Automat. Control*, 2019, to appear. [Online]. Available: http://arxiv.org/abs/1807.02779

[5] S. P. Eveson and R. D. Nussbaum, "An elementary proof of the Birkhoff-Hopf theorem," *Math. Proc. Camb. Philos. Soc.*, vol. 117, no. 1, pp. 31–55, 1995.

[6] S. M. Fallat and C. R. Johnson, *Totally Nonnegative Matrices*. Princeton, NJ: Princeton University Press, 2011.

[7] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*. John Wiley, 2000.

[8] M. Fekete and G. Pólya, "Über ein problem von Laguerre," *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, vol. 34, no. 1, pp. 89–120, 1912.

[9] L. Feng, Y. Wang, and J. Wu, "Semiflows "monotone with respect to high-rank cones" on a Banach space," *SIAM J. Math. Anal.*, vol. 49, no. 1, pp. 142–161, 2017.

[10] L. Feng, Y. Wang, and J. Wu, "Generical behavior of flows strongly monotone with respect to high-rank cones." [Online]. Available: https://arxiv.org/abs/1905.06787
[11] F. Forni and R. Sepulchre, "Differentially positive systems," *IEEE Trans. Automat. Control*, vol. 61, no. 2, pp. 346–359, 2016.
[12] F. Forni and R. Sepulchre, "A dissipativity theorem for p-dominant systems," in *Proc. 56th IEEE Conf. on Decision and Control*, Melbourne, Australia, 2017, pp. 3467–3472.
[13] G. Fusco and W. M. Oliva, "A Perron theorem for the existence of invariant subspaces," *Ann. Mat. Pur. Appl.*, vol. 160, pp. 63–76, 1991.
[14] F. R. Gantmacher and M. G. Krein, *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems*. Providence, RI: American Mathematical Society, 2002, translation based on the 1941 Russian original.
[15] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed. Cambridge University Press, 2013.
[16] S. Karlin, *Total Positivity*. Stanford, CA: Stanford University Press, 1968, vol. 1.
[17] R. Katz, M. Margaliot, and E. Fridman, "Entrainment to subharmonic solutions in oscillatory discrete-time systems," *Automatica*, 2019, to appear. [Online]. Available: https://arxiv.org/abs/1904.06547
[18] M. A. Krasnoselskii, E. A. Lifshitz, and A. V. Sobolev, *Positive Linear Systems: The Method of Positive Operators*. Berlin: Heldermann Verlag, 1989.
[19] W. Lohmiller and J.-J. E. Slotine, "On contraction analysis for non-linear systems," *Automatica*, vol. 34, pp. 683–696, 1998.
[20] M. Margaliot and E. D. Sontag, "Revisiting totally positive differential systems: A tutorial and new results," *Automatica*, vol. 101, pp. 1–14, 2019.
[21] J. S. Muldowney, "Compound matrices and ordinary differential equations," *The Rocky Mountain J. Math.*, vol. 20, no. 4, pp. 857–872, 1990.
[22] A. Pinkus, *Totally Positive Matrices*. Cambridge, UK: Cambridge University Press, 2010.
[23] P. Poláčik and I. Tereščák, "Convergence to cycles as a typical asymptotic behavior in smooth strongly monotone discrete-time dynamical systems," *Archive for Rational Mechanics and Analysis*, vol. 116, no. 4, pp. 339–360, 1992.
[24] A. Rantzer and M. E. Valcher, "A tutorial on positive systems and large scale control," in *Proc. 57th IEEE Conf. on Decision and Control*, Miami Beach, FL, USA, 2018, pp. 3686–3697.
[25] A. Rantzer, "Scalable control of positive systems," *Euro. J. Control*, vol. 24, pp. 72–80, 2015.
[26] L. A. Sanchez, "Cones of rank 2 and the Poincaré-Bendixson property for a new class of monotone systems," *J. Diff. Eqns.*, vol. 246, no. 5, pp. 1978–1990, 2009.
[27] I. Schoenberg, "Über variationsvermindernde lineare transformationen," *Mathematische Zeitschrift*, vol. 32, no. 1, pp. 321–328, 1930.
[28] B. Schwarz, "Totally positive differential systems," *Pacific J. Math.*, vol. 32, no. 1, pp. 203–229, 1970.
[29] H. L. Smith, *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, ser. Mathematical Surveys and Monographs. Providence, RI: Amer. Math. Soc., 1995, vol. 41.
[30] E. Weiss and M. Margaliot, "A generalization of linear positive systems with applications to nonlinear systems: Invariant sets and the Poincaré-Bendixon property," 2019, submitted. [Online]. Available: arXiv:1902.01630