# Local Optimality of Almost Piecewise-Linear Quantizers for Witsenhausen's Problem 

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#### Abstract

We pose Witsenhausen's problem as a leader-follower game of incomplete information. The follower makes a noisy observation of the leader's action (who moves first) and chooses an action minimizing her expected deviation from the leader's action. Knowing this, leader who observes the realization of the state, chooses an action that minimizes her distance to the state of the world and the ex-ante expected deviation from the follower's action. We study the perfect Bayesian equilibria of the game and identify a class of "near piecewise-linear equilibria" when leader cares much more about being close to the follower than the state, and the state is highly volatile. As a major consequence of this result, we prove the existence of a set of local minima for Witsenhausen's problem in form of slopey quantizers, which are at most a constant factor away from the optimal cost.


## Index Terms

Decentralized control, optimal stochastic control, incomplete information games, perfect Bayesian equilibrium, asymptotic quantization theory.

## I. Introduction

In his seminal work [1], Witsenhausen constructed a simple two-stage Linear-QuadraticGaussian (LQG) decentralized control problem where the optimal controller happens to be nonlinear. This example showed for the first time that linear quadratic Gaussian team problems can have nonlinear solutions. Using this counterexample, [2] produced an example showing that the standard decentralized static output feedback optimal control problem of linear deterministic

[^0]systems could also admit nonlinear solutions. ${ }^{1}$ For nearly half a century, this counterexample has been a subject of intense research across multiple communities ([5]-[10]).

The endogenous information structure of Witsenhausen's counterexample, where the signal observed in the second stage is a noisy version of the control action in the first stage, gives rise to a nonclassical information structure. While the problem looks deceptively simple with quadratic cost, it is actually a very complicated, nonconvex, functional optimization problem. This counterexample has shed light on intricacies of optimal decisions in stochastic team optimization problems with similar information structure. Naturally, this problem has given rise to a large body of literature. For example, [11] provides a variant of Witsenhausen's counterexample with discrete primitive random variables and finite support, where no optimal solution exists. Another interesting variant, with the same information structure but different cost function, is the Gaussian test channel ([6], [12]) where the linear strategies can be shown to be optimal. Interestingly, [13] shows that if the objective function in [1] is changed to a worst case induced norm, the linear controllers dominate nonlinear policies.

Although the optimal strategy and optimal cost for Witsenhausen's counterexample are still unknown, it can be shown that carefully designed nonlinear strategies can largely outperform the linear strategies (see, e.g., the multi-point quantization strategies proposed by [7]). This result, in particular, implies the fragility of the comparative statics and policies solely derived based on the linear strategies in problems with similar setting. A relevant line of research is to provide error bounds on the proximity to optimality for approximate solutions. [14], [15] use information theoretic techniques and vector versions of the original problem to provide such bounds. In [16], authors provide a general result on when one can approximate a continuous team decision problem with a finite one through quantized approximations, using which they show that quantized policies are asymptotically optimal for Witsenhausen's counterexample. There are also several works aiming to approximate the optimal solution. [17]-[20] employ different heuristic approaches, all confirming what one might intuitively call an almost piecewise-linear form for the optimal controller. However, a complete optimality proof for such strategies has been elusive.

[^1]Although Witsenhausen's counterexample has been around for half a century, a little is known about the topological properties of its optimal solution (e.g., whether it is continuous or not, the number of its fixed points, etc.). A breakthrough in this direction is [8], where authors view Witsenhausen's setup in an optimal transport theory framework. This enables them not only to prove the existence of the optimal solution in a much more condense fashion, but also to derive some important characteristics of the optimal solution. In particular, they show that the optimal controller is a strictly increasing function with a real analytic left inverse. ${ }^{2}$ As a consequence, no piecewise-linear strategy can be optimal, though not ruling out the possibility of a "near piecewise-linear" optimal solution.

In this paper, following Witsenhausen's original intuition, we view the problem as a leaderfollower coordination game in which the action of the leader is corrupted by an additive noise, before reaching the follower. The leader aims to coordinate with the follower while staying close to the observed state, recognizing that her action is not observed perfectly. As a result, she needs to signal the follower in a manner that can be decoded efficiently. More than a mere academic counterexample, the above setup could model a scenario where coordination happens across generations and the insights of the leader who is from a different generation is corrupted/lost by the time the message reaches the future generations. If the leader can internalize the fact that her actions will not be observed perfectly, how should she act to make sure coordination occur? When the leader cares far more about coordination with the follower than staying "on the message", the near piecewise-linear equilibrium strategy of the leader coarsens the observation in well-spaced intervals, rather than merely broadcasting a linearly scaled version of the observed state (as the linear strategy would suggest).

To this end, we analyze the perfect Bayesian equilibria of this game and show that strong complementarity ${ }^{3}$ between the leader and the follower combined with a prior with poor enough precision can give rise to nonlinear equilibria, and in particular, equilibria in form of what has been deemed in the literature as slopey quantizers [22]. We subsequently show that these equilibria are indeed local minima of the original Witsenhausen's problem. Using some related results from asymptotic quantization theory ([23]-[25]) together with analytical lower bounds

[^2]on the optimal cost of Witsenhausen's problem derived in [15], we further show that these local minima include near-optimal solutions in the sense that their corresponding cost is at most a constant factor away from the optimal one. Our work thus provides an analytical support for the local optimality of slopey quantization strategies for Witsenhausen's counterexample for a highly volatile state.

The main idea behind the proof is to carefully construct a class of what we informally refer to as near piecewise-linear or slopey quantization strategies for the leader that stays invariant under the best response operator. These strategies can be viewed as small-slope variations of a fixed-rate scalar quantizer minimizing the mean squared quantization error ([23]-[25]). Such an optimal quantizer is characterized by optimality conditions on the threshold levels which determine the boundaries of the quantization cells (or segments) and quantization levels: i) quantization levels must be the centroid of the segments, and ii) thresholds in between two adjacent quantization levels must be equidistant from them. For any fixed number of segments, we consider the strategies whose segments are in a vicinity of the optimal MSE quantizer, have a unique fixed point in each segment close to the quantization level, and are almost linear within each segment with a near-zero derivative. For such strategies, leader's actions remain very close to fixed points of the strategy in each segment. Therefore, well-spaced fixed points (combined with appropriate relative prior of the state in different segments) reveal the leader's actions to the follower with high probability, making the "signal" easily decodable. As a consequence, we can characterize the best response of the follower to leader's strategy. Using this characterization, we show that the best response of the leader to follower's strategy also varies very little, essentially remaining near piecewise-linear over most of the range of the observed signals.

A key challenge in deriving the invariance property for this set of strategies for the leader is to bound and tightly control the displacement in the fixed points and endpoints of the segments of leader's strategy under the best response operator. One major observation here is that the fixed points of the leader's best responses are local minimizers of the expected deviation of the leader's action from the follower (which is known to be a non-convex functional [8]). This insight allows us to show that the fixed points of the leader's best response lie in a tight neighborhood of the fixed points of the follower's strategy. We then show that the fixed points of the follower's strategy in turn lie in a vicinity of a convex combination of the leader's fixed points and the expected value of the state of the world within each segment. Combining the two, we can
derive an approximate dynamics for the displacement in the fixed points and endpoints of the segments in leader's strategy under the best response. Using this approximate dynamics, we then characterize an invariant set of fixed points and interval endpoints for leader's strategy, which we can then use in order to prove the existence of a near piecewise-linear equilibrium strategy for the leader.

## II. Model

We view Witsenhausen's problem ([1]) as a game between a leader $L$ and a follower $F$. Before the agents act, the state of the world $\theta$ is drawn from a normal distribution with zero mean and variance $\sigma^{2}$. The leader can observe the realization of $\theta$ and acts first. The payoff of the leader is given as follows

$$
\begin{equation*}
u_{L}=-r_{L}\left(\theta-a_{L}\right)^{2}-\left(1-r_{L}\right)\left(a_{F}-a_{L}\right)^{2}, \tag{1}
\end{equation*}
$$

where $a_{F}$ is the action of the follower and $0<r_{L}<1$. The follower makes a private, noisy observation of the leader's action, $s=a_{L}+\delta$ where $\delta \sim N(0,1)$. The payoff of the follower is given by

$$
\begin{equation*}
u_{F}=-\left(a_{L}-a_{F}\right)^{2} . \tag{2}
\end{equation*}
$$

We consider the perfect Bayesian equilibria of the game and show that they reduce to the Bayes Nash equilibria due to the Gaussian noise in the observation. ${ }^{4}$ Denote with $a_{L}^{*}(\theta)$ and $a_{F}^{*}(s)$ the equilibrium strategies, and with $\nu^{*}(\cdot \mid s)$ the follower's belief on leader's action given $s$. Due to the normal noise in the observation, $\nu^{*}(\cdot \mid s)$ is fully determined by $a_{L}^{*}(\theta)$ and the prior as there are no off-equilibrium-path information sets. Equilibrium strategies should thus satisfy

$$
\begin{align*}
a_{F}^{*}(s)= & \mathbb{E}_{\nu^{*}}\left[a_{L}^{*} \mid s\right]=\int_{-\infty}^{\infty} a_{L} \nu^{*}\left(a_{L} \mid s\right) d a_{L}, \\
a_{L}^{*}(\theta)= & \underset{a_{L}}{\operatorname{argmax}}-r_{L}\left(\theta-a_{L}\right)^{2} \\
& -\left(1-r_{L}\right) \int_{-\infty}^{\infty}\left(a_{F}^{*}(s)-a_{L}\right)^{2} \phi\left(s-a_{L}\right) d s, \tag{3}
\end{align*}
$$

where $\phi(\cdot)$ denotes the standard normal density function.

[^3]Our model yields the original setup in [1] by choosing $\frac{r_{L}}{1-r_{L}}=k^{2}$. The expected control cost then maps to the (negated) expected payoff of the leader. It is a simple exercise to find the optimal solution to Witsenhausen's problem in the class of linear strategies (see Lemma 11 in [1]), which is also an equilibrium of the game described above. Witsenhausen ([1]) showed that, for sufficiently large $\sigma$, this linear solution is not optimal. In fact, the linear solution can be extremely suboptimal in the sense that the asymptotic ratio of the corresponding cost to the optimal one is infinity ([7]). Our objective in this paper is to characterize a set of local minima for the problem in [1], with a near piecewise-linear strategy for the leader and a cost within a constant factor of the optimal one, given a sufficiently large $\sigma$. To this end, we analyze the equilibria of the game described above in regime $\frac{1}{2} \leq r_{L} \sigma^{2} \leq 1$ and sufficiently large $\sigma .{ }^{5}$

## III. Nonlinear EQuilibria

We first prove the existence of a collection of equilibria with a near piecewise-linear strategy for the leader for sufficiently large values of $\sigma$. Our approach is to identify a set of such strategies for the leader which is invariant under the best response operator. We characterize such a set in the next section.

Given $m \in \mathbb{N}$, consider a partition of the normal distribution $N\left(0, \sigma^{2}\right)$ into $2 m+1$ segments $\cup_{k=-m}^{m} B_{k}^{\mathrm{Q}}$, with $B_{k}^{\mathrm{Q}}=\left[b_{k}^{\mathrm{Q}}, b_{k+1}^{\mathrm{Q}}\right)$ for $k \in \mathbb{N}_{m}, B_{0}^{\mathrm{Q}}=\left(b_{-1}^{\mathrm{Q}}, b_{1}^{\mathrm{Q}}\right)$, and $B_{-k}^{\mathrm{Q}}=\left(b_{-k-1}^{\mathrm{Q}}, b_{-k}^{\mathrm{Q}}\right]$, with $b_{-k}^{\mathrm{Q}}=-b_{k}^{\mathrm{Q}}$ and $b_{m+1}^{\mathrm{Q}}=-b_{-m-1}^{\mathrm{Q}}=+\infty$. Denote with $c_{k}^{\mathrm{Q}}$ the centroid of segment $B_{k}^{\mathrm{Q}}$, that is, $c_{k}^{\mathrm{Q}}=\mathbb{E}_{N\left(0, \sigma^{2}\right)}\left[\theta \mid \theta \in B_{k}^{\mathrm{Q}}\right]$. Clearly, $c_{0}^{\mathrm{Q}}=0$ and $c_{-k}^{\mathrm{Q}}=-c_{k}^{\mathrm{Q}}$ for $k \in \mathbb{N}_{m}$. We now specifically focus on a partition where the interval endpoints $b_{k}^{\mathrm{Q}}$ are equidistant from the centroids adjacent to them, i.e., $b_{k}^{\mathrm{Q}}=\frac{c_{k-1}^{Q}+c_{k}^{Q}}{2}$ for $k \in \mathbb{N}_{m}$. We can show that such a partition exists and is unique. This partition in fact corresponds to the $(2 m+1)$-level fixed-rate scalar quantizer that minimizes the mean-square distortion for a source characterized by $\theta \sim N\left(0, \sigma^{2}\right)$ ([23]-[25]). The properties of this quantizer as $m \rightarrow \infty$ are extensively studied in asymptotic quantization theory, as will be discussed and used in analyzing the asymptotic performance of our proposed local minima in Section IV.

Roughly speaking, the set of strategies we propose for the leader are a class of $(2 m+1)$ segmented strategies with segments being close to $B_{k}^{\mathrm{Q}}(-m \leq k \leq m)$, with a fixed point in each segment in a certain vicinity of $c_{k}^{\mathrm{Q}}(-m \leq k \leq m)$, and almost linear with a slope close to $r_{L}$

[^4]

Fig. 1. Partition of the normal distribution in a $(2 m+1)$-level optimal MSE quantizer, for $m=2$.
over each segment. Before proceeding further, we present some (non-asymptotic) properties of this base configuration, which will facilitate the proof of the invariance property for the proposed set of strategies.

Lemma 1. Given $m \in \mathbb{N}$, consider the partition of the normal distribution $N\left(0, \sigma^{2}\right)$ in a $(2 m+1)$ level optimal MSE quantizer as described above. Define the $k$-th half-step of the quantizer as $x_{k}^{\mathrm{Q}}=\frac{c_{k}^{\mathrm{Q}}-c_{k-1}^{\mathrm{Q}}}{2}$, for $1 \leq k \leq m$. Then,
i) $1-\left(\frac{x_{m}^{\mathrm{Q}}}{\sigma}\right)^{2} \leq \frac{c_{m}^{\mathrm{Q}} x_{m}^{\mathrm{Q}}}{\sigma^{2}} \leq 1$. If $m \geq 2$, then $\frac{3}{4} \leq \frac{c_{m}^{\mathrm{Q}} x_{m}^{\mathrm{Q}}}{\sigma^{2}} \leq 1$.
ii) For $1 \leq k<m$,

$$
\begin{equation*}
\frac{\phi\left(\frac{b_{k}^{\mathrm{Q}}}{\sigma}\right)}{\phi\left(\frac{\left(_{k}^{Q}\right.}{\sigma}\right)} \leq\left(\frac{x_{k+1}^{\mathrm{Q}}}{\left.x_{k}^{\mathrm{Q}}\right)^{2} \leq \frac{\phi\left(\frac{b_{k}^{\mathrm{Q}}}{\sigma}\right)}{\phi\left(\frac{b_{k+1}^{\mathrm{Q}}}{\sigma}\right)} . . . .}\right. \tag{4}
\end{equation*}
$$

As a result, $1 \leq \frac{x_{k+1}^{Q}}{x_{k}^{Q}} \leq e$, for $1 \leq k<m$.
iii) For $0 \leq j \leq k \leq m$, we have $\frac{\operatorname{Prob}\left[\theta \mid \theta \in B_{k}^{Q}\right]}{\operatorname{Prob}\left[\theta \mid \theta \in B_{j}^{Q}\right]} \leq \frac{1+e}{2}$.
iv) For any $m \geq 5$,

$$
\frac{\sqrt{3 \pi}}{e \sqrt{10} m} \leq \frac{x_{1}^{\mathrm{Q}}}{\sigma} \leq \frac{\sqrt{2 \pi e}}{2 m}
$$

$$
\begin{gather*}
1.1 \sqrt{\ln m}-\frac{1}{1.1 \sqrt{\ln m}} \leq \frac{c_{m}^{\mathrm{Q}}}{\sigma} \leq 2 \sqrt{2 \ln m+1.4}+1.45, \\
\frac{1}{2 \sqrt{2 \ln m+6}} \leq \frac{x_{m}^{\mathrm{Q}}}{\sigma} \leq \frac{1}{1.1 \sqrt{\ln m}} . \tag{5}
\end{gather*}
$$

Proof. See the appendix.
Next, we construct a set of $(2 m+1)$-segmented increasing odd functions, denoted by $A_{L}^{m}\left(r_{L}, \sigma\right)$ satisfying the following properties:

Property 1. For every $a_{L}(\theta) \in A_{L}^{m}\left(r_{L}, \sigma\right)$, there exist $2 m+1$ segments $B_{k}=\left[b_{k}, b_{k+1}\right)$, for $k \in \mathbb{N}_{m}, B_{0}=\left(-b_{1}, b_{1}\right)$, and $B_{-k}=\left(b_{-k-1}, b_{-k}\right]$, with $b_{m+1}=-b_{-m-1}=+\infty$ such that:

- $a_{L}(\theta)$ is increasing and odd (i.e., $a_{L}(-\theta)=-a_{L}(\theta)$ ), and is smooth over each interval.
- $a_{L}(\theta)$ has a unique fixed point in each segment. That is, for each interval $B_{k},(-m \leq k \leq$ $m)$, there exists a unique $c_{k} \in B_{k}$ such that $a_{L}\left(c_{k}\right)=c_{k}$, with $c_{0}=0$.

We also impose the constraint that interval endpoints $b_{k}$ remain close to midpoints of $\left[c_{k-1}, c_{k}\right]$ and that fixed points $c_{k}$ remain within certain vicinity of $c_{k}^{\mathrm{Q}}$, .

Property 2. For every $k \in \mathbb{N}_{m},\left|b_{k}-\frac{c_{k-1}+c_{k}}{2}\right| \leq 0.1 r_{L}$. Moreover, $\left|c_{k}-c_{k}^{Q}\right| \leq 2.9$.
From the above property, if we define $\bar{x}_{k}=x_{k}^{\mathrm{Q}}+3$ and $\underline{x}_{k}=x_{k}^{\mathrm{Q}}-3$ for $1 \leq k \leq m$, then $\bar{x}_{k}$ and $\underline{x}_{k}$ represent upper and lower bounds on the lengths of both half-intervals $\left[c_{k-1}, b_{k}\right]$ and $\left[b_{k}, c_{k+1}\right]$.

Finally, we impose a constraint on the slope of $a_{L}(\theta)$ in each interval, keeping the slope very close to $r_{L}$, as well as a linear bound on $a_{L}(\theta)$ in the tail. We impose the following property:

Property 3. For every $-m<k<m$ and $\theta \in B_{k}, \underline{r} \leq \frac{d}{d \theta} a_{L}(\theta) \leq \bar{r}$, where $\underline{r}=r_{L}\left(1-0.5 r_{L}^{2} \sigma^{2}\right)$ and $\bar{r}=r_{L}\left(1+0.5 r_{L}^{2} \sigma^{2}\right)$. For the tail interval $B_{m}, \underline{r} \leq \frac{d}{d \theta} a_{L}(\theta) \leq \bar{r}$ for $b_{m}<\theta<c_{m}+\sqrt{e} \sigma \bar{x}_{m}$. For $\theta>c_{m}+\sqrt{e} \sigma \bar{x}_{m}$ we have $a_{L}(\theta) \leq c_{m}+3 r_{L}\left(\theta-c_{m}\right) .{ }^{6}$

For any $\sigma>0$, define $M(\sigma)=\left\{m \in \mathbb{N} \mid x_{1}^{\mathrm{Q}}>2 \sqrt{2 \ln \sigma}+5, m \geq 25\right\} .{ }^{7}$ We then claim that the set of strategies $A_{L}^{m}\left(r_{L}, \sigma\right)$ for $m \in M(\sigma)$, characterized by Property 1-3, is invariant

[^5]under the best response operator for sufficiently large values of $\sigma$ in the regime $\frac{1}{2} \leq r_{L} \sigma^{2} \leq 1$. We formally state this result here, and provide the proof which is based on the best response analysis carried out in Section V, in the appendix.

Theorem 1. Consider the regime $\frac{1}{2} \leq r_{L} \sigma^{2} \leq 1$ with $\sigma>0$. Then, the set of $(2 m+1)$-segmented strategies $A_{L}^{m}\left(r_{L}, \sigma\right)$ for the leader characterized by Property 1-3, where $m \in M(\sigma)=\{m \in$ $\left.\mathbb{N} \mid x_{1}^{\mathrm{Q}}>2 \sqrt{2 \ln \sigma}+5, m \geq 25\right\}$ and $\sigma \geq 300$, is nonempty and invariant under the best response operator. ${ }^{8}$ Moreover, the game described in Section II has an equilibrium for which:
i) $a_{L}^{*}\left(\theta, r_{L}, \sigma\right) \in A_{L}^{m}\left(r_{L}, \sigma\right)$, and
ii) $\left(a_{L}^{*}\left(\theta, r_{L}, \sigma\right), a_{F}^{*}(s)=\mathbb{E}_{\delta}\left[a_{L}^{*} \mid s\right]\right)$ maximizes the expected payoff of the leader over all pair of strategies $\left(a_{L}\left(\theta, r_{L}, \sigma\right), a_{F}(s)=\mathbb{E}_{\delta}\left[a_{L} \mid s\right]\right)$ where $a_{L}\left(\theta, r_{L}, \sigma\right) \in A_{L}^{m}\left(r_{L}, \sigma\right)$.

Proof. See the appendix.
We expect the above theorem to hold for much smaller values of $\sigma$ and much smaller number of levels $(2 m+1)$, by optimizing/tightening the bounds used in deriving this result. While this is in principle doable, we believe this would sacrifice clarity given the tedious calculations required, and would substantially increase the length of the paper.

## IV. Local Minima and Asymptotic Performance Guarantees

Let

$$
\begin{align*}
& U\left(a_{L}, a_{F}\right)=-\mathbb{E}_{\theta}\left[u_{L}\left(\theta, a_{L}, a_{F}\right)\right]=r_{L} \int_{-\infty}^{\infty}\left(\theta-a_{L}(\theta)\right)^{2} \frac{\phi\left(\frac{\theta}{\sigma}\right)}{\sigma} d \theta \\
& +\left(1-r_{L}\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty}\left(a_{F}(s)-a_{L}(\theta)\right)^{2} \phi\left(s-a_{L}(\theta)\right) \frac{\phi\left(\frac{\theta}{\sigma}\right)}{\sigma} d s d \theta \tag{6}
\end{align*}
$$

for any two measurable functions $a_{L}, a_{F}: \mathbb{R} \rightarrow \mathbb{R}$. As discussed in Section II, $U\left(a_{L}, a_{F}\right)$ defined above maps to the expected cost of the original Witsenhausen's problem in [1]. The aim of this section is to study the performance of the equilibrium strategies characterized by Theorem 1 in view of the above cost function. Being an equilibrium implies that the cost cannot be improved by changing one of the strategies $a_{L}^{*}$ or $a_{F}^{*}$ while keeping the other fixed, although this does not rule out the possibility of obtaining a lower cost by simultaneously changing both strategies. ${ }^{9}$

[^6]With a bit of manipulation, however, we can show that the image of the strategy obtained from an infinitesimal variation in $a_{L}^{*}$ also lies within $A_{L}^{m}$, using which we can show that $\left(a_{L}^{*}, a_{F}^{*}\right)$ is indeed a local minimum of $U$.

Lemma 2. Any pair of equilibrium strategies $\left(a_{L}^{*}, a_{F}^{*}\right)$ characterized by Theorem 1 , where $a_{L}^{*} \in$ $A_{L}^{m}\left(r_{L}, \sigma\right)$ and $a_{F}^{*}(s)=\mathbb{E}_{\delta}\left[a_{L}^{*} \mid s\right]^{10}$, is a local minimum of the cost functional $U$ in (6).

Proof. See the appendix.
We now have the first main result of the paper: a near piecewise-linear strategy for the leader (or first controller) that leads to a local minimum of Witsenhausen's problem. Although the importance of such strategies has been already noticed in the literature ([17]-[20]), no analytical result concerning the optimality of such strategies is reported in the literature. Theorem 1 also presents an important result in the context of two-stage games of incomplete information.

We next aim to evaluate the asymptotic performance of these local minima with respect to the optimal cost. According to Theorem $1,\left(a_{L}^{*}, a_{F}^{*}\right)$ is a minimizer of $U$ over the pair of strategies ( $a_{L}, a_{F}$ ) with $a_{L} \in A_{L}^{m}\left(r_{L}, \sigma\right)$. Therefore, we can use any other pair of strategies with the leader's strategy being in $A_{L}^{m}$ (for which it is easier to evaluate the cost) to find an upper bound for $U\left(a_{L}^{*}, a_{F}^{*}\right)$. For this purpose we use $U\left(a_{L}^{*}, a_{F}^{*}\right) \leq U\left(a_{L}^{\mathrm{Q}}, a_{F}^{\mathrm{Q}}\right)$, where $a_{L}^{\mathrm{Q}}$ is the piecewise-linear strategy with segments $B_{k}^{\mathrm{Q}}$ and fixed points $c_{k}^{\mathrm{Q}}$ specified in the base configuration in Section III and $\frac{d}{d \theta} a_{L}^{\mathrm{Q}}(\theta)=r_{L}$ over each interval, and $a_{F}^{\mathrm{Q}}$ is the optimal $(2 m+1)$-level MSE quantizer (i.e., constant value of $c_{k}^{\mathrm{Q}}$ over segment $B_{k}^{\mathrm{Q}}$ ). It is easy to see that $\left(\theta-a_{L}^{\mathrm{Q}}(\theta)\right)^{2}=\left(1-r_{L}\right)^{2}\left(\theta-a_{F}^{\mathrm{Q}}(\theta)\right)^{2}$. We can thus write $U\left(a_{L}^{\mathrm{Q}}, a_{F}^{\mathrm{Q}}\right)=r_{L}\left(1-r_{L}\right)^{2} D_{L}^{\mathrm{Q}}+\left(1-r_{L}\right) D_{F}^{\mathrm{Q}}$, with

$$
\begin{align*}
D_{L}^{\mathrm{Q}} & =\int_{-\infty}^{\infty}\left(\theta-a_{F}^{\mathrm{Q}}(\theta)\right)^{2} \frac{\phi\left(\frac{\theta}{\sigma}\right)}{\sigma} d \theta \\
D_{F}^{\mathrm{Q}} & =\int_{-\infty}^{\infty} \int_{-\infty}^{\infty}\left(a_{F}^{\mathrm{Q}}(s)-a_{L}^{\mathrm{Q}}(\theta)\right)^{2} \phi\left(s-a_{L}^{\mathrm{Q}}(\theta)\right) \frac{\phi\left(\frac{\theta}{\sigma}\right)}{\sigma} d s d \theta . \tag{7}
\end{align*}
$$

$D_{F}^{\mathrm{Q}}$ can be upper-bounded as $D_{F}^{\mathrm{Q}} \leq 4 \sqrt{\frac{2}{e}} \frac{\left(2-r_{L}\right)^{2}}{\left(1-r_{L}\right)^{2}} \phi\left(\frac{x_{1}^{\mathrm{Q}}}{\sqrt{2}}\right)+r_{L}^{2} D_{L}^{\mathrm{Q}}$ (see the proof of Lemma 3). We can find the exact asymptotic value of $D_{L}^{Q}$ using results from asymptotic quantization theory ([23]-[25]): $D_{L}^{Q}$ is the mean-square error of an optimal $(2 m+1)$-level MSE quantizer for a source $\theta \sim N\left(0, \sigma^{2}\right)$ (see, e.g., [25]). It is known that for large $m, D_{L}^{\mathrm{Q}} \approx \frac{c_{\infty}}{(2 m+1)^{2}}$, where $c_{\infty}$ is

[^7]the Panter-Dite constant of a normal source given by
\[

$$
\begin{equation*}
c_{\infty}=\frac{1}{12}\left(\int_{-\infty}^{\infty}\left(\frac{\phi\left(\frac{\theta}{\sigma}\right)}{\sigma}\right)^{\frac{1}{3}} d \theta\right)^{3}=\frac{\sqrt{3} \pi}{2} \sigma^{2} . \tag{8}
\end{equation*}
$$

\]

Another interesting exact asymptotic equality is $(2 m+1) \frac{x_{1}^{Q}}{\sigma} \approx \frac{\sqrt{6 \pi}}{2}$ using which we can alternatively write $D_{L}^{\mathrm{Q}} \approx \frac{\left(x_{1}^{\mathrm{Q}}\right)^{2}}{\sqrt{3}}$ as $m \rightarrow \infty$. We have the following lemma.

Lemma 3. For the pair of equilibrium strategies $\left(a_{L}^{*}, a_{F}^{*}\right)$ characterized by Theorem 1, where $a_{L}^{*} \in A_{L}^{m}\left(r_{L}, \sigma\right), a_{F}^{*}(s)=\mathbb{E}_{\delta}\left[a_{L}^{*} \mid s\right]$, and $m \in M(\sigma)$ we have

$$
\begin{equation*}
\liminf _{m \rightarrow \infty} \frac{\frac{r_{L}\left(1-r_{L}\right)\left(x_{1}^{\mathrm{Q}}\right)^{2}}{\sqrt{3}}+4 \sqrt{\frac{2}{e}} \frac{\left(2-r_{L}\right)^{2}}{\left(1-r_{L}\right)} \phi\left(\frac{x_{1}^{\mathrm{Q}}}{\sqrt{2}}\right)}{U\left(a_{L}^{*}, a_{F}^{*}\right)} \geq 1 . \tag{9}
\end{equation*}
$$

Proof. See the appendix.
The above asymptotic upper bound on $U\left(a_{L}^{*}, a_{F}^{*}\right)$ is minimized when $x_{1}^{\mathrm{Q}} \approx 2 \sqrt{2 \ln \sigma}$ for large $\sigma$, with number of levels $(2 m+1) \approx \frac{\sqrt{3 \pi} \sigma}{4 \sqrt{\ln \sigma}}$ and yielding a cost $\approx \frac{8 r_{L} \ln \sigma}{\sqrt{3}}$. Recalling that $M(\sigma)=\left\{m \in \mathbb{N} \mid x_{1}^{\mathrm{Q}}>2 \sqrt{2 \ln \sigma}+5, m \geq 25\right\}$, this implies the existence of a local minimum with near piecewise-linear strategy for the leader with a cost asymptotically as low as $\frac{8 r_{L} \ln \sigma}{\sqrt{3}}$. To compare with the optimal solution, we use the lower bounds on the optimal cost of Witsenhausen's problem derived in [15]. The following lemma is an immediate result of Theorem 4 in [15].

Lemma 4. Denote with $U^{*}(\sigma)$ the minimum value of the cost functional $U\left(a_{L}, a_{F}\right)$ given by (6) in the regime $r_{L} \sigma^{2}=1$. Then

$$
\begin{equation*}
\limsup _{\sigma \rightarrow \infty} \frac{\frac{\ln \sigma}{6 \sigma^{2}}}{U^{*}(\sigma)} \leq 1 \tag{10}
\end{equation*}
$$

Proof. See the appendix.
This lower bound is quite loose (as also pointed out by the authors in [15]), but still serves our purpose of showing that our proposed set of local optima include solutions that are only a constant factor away from the optimal cost as $\sigma \rightarrow \infty .{ }^{11}$ We summarize the main findings of this section in the theorem below.

[^8]Theorem 2. Any pair of equilibrium strategies $\left(a_{L}^{*}, a_{F}^{*}\right)$ characterized by Theorem 1, where $a_{L}^{*} \in A_{L}^{m}\left(r_{L}, \sigma\right)$ and $a_{F}^{*}(s)=\mathbb{E}_{\delta}\left[a_{L}^{*} \mid s\right]$, is a local minimum of the cost functional $U$ in (6). Moreover,

$$
\begin{equation*}
\liminf _{\sigma \rightarrow \infty} \frac{\frac{8 r_{L} \ln \sigma}{\sqrt{3}}}{\min _{m \in M(\sigma)} U\left(a_{L}^{*}, a_{F}^{*}\right)} \geq 1 \tag{11}
\end{equation*}
$$

In the regime $r_{L} \sigma^{2}=1$, at least one of these local minima are less than 27.8 times away from the optimal value, as $\sigma \rightarrow \infty$.

Proof. See the appendix.

## V. Best Response Analysis

The objective of this section is to prove Theorem 1 on the invariance of the set of strategies $A_{L}^{m}\left(r_{L}, \sigma\right)$ (for $m \in M(\sigma)$ ), and the existence of an equilibrium with the leader's strategy in this set. The first step in verifying the invariance of $A_{L}^{m}\left(r_{L}, \sigma\right)$ is to characterize the best response of the follower $a_{F}(s)$ to the leader's strategy $a_{L}(\theta) \in A_{L}^{m}\left(r_{L}, \sigma\right)$. We can then use these properties to find the updated best response of the leader to $a_{F}(s)$, denoted by $\tilde{a}_{L}(\theta)$ and enforce its inclusion in $A_{L}^{m}\left(r_{L}, \sigma\right)$.

The follower's best response to the strategy of the leader $a_{L}(\theta)$ is the expected action of the leader given the observation $s=a_{L}+\delta$, that is $a_{F}(s)=\mathbb{E}_{\delta}\left[a_{L} \mid s\right]$. Following a simple application of Bayes rule we can obtain

$$
\begin{equation*}
a_{F}(s)=\frac{\int_{-\infty}^{\infty} a_{L}(\theta) \phi\left(s-a_{L}(\theta)\right) \phi\left(\frac{\theta}{\sigma}\right) d \theta}{\int_{-\infty}^{\infty} \phi\left(s-a_{L}(\theta)\right) \phi\left(\frac{\theta}{\sigma}\right) d \theta} . \tag{12}
\end{equation*}
$$

Using this, we can easily show that $a_{F}(s)$ is analytic and increasing, with $\frac{d}{d s} a_{F}(s)=\operatorname{Var}\left[a_{L} \mid s\right]$ (see [1] for a proof).

In order to characterize $a_{F}(s)$, we start by estimating the expected action of the leader and its variance conditioned on the interval to which $\theta$ belongs. Actions of the leader in interval $B_{k}$ $(k \neq \pm m)$ are well-concentrated around $c_{k}$. In fact $a_{L}(\theta) \in\left[c_{k}-\bar{r} \bar{x}_{k}, c_{k}+\bar{r} \bar{x}_{k+1}\right]$ for $\theta \in B_{k}$, from which the lemma below follows immediately.

Lemma 5. For $0 \leq k<m$, $\left|\mathbb{E}\left[a_{L}(\theta) \mid s, \theta \in B_{k}\right]-c_{k}\right| \leq \bar{r} \bar{x}_{k+1}$ and $\operatorname{Var}\left[a_{L}(\theta) \mid s, \theta \in B_{k}\right] \leq$ $\bar{r}^{2}\left(\frac{\bar{x}_{k}+\bar{x}_{k+1}}{2}\right)^{2}$.

Proof. See the appendix.
The analysis is a bit involved in the tail, since for $\theta>c_{m}$ the leader's actions are not in a bounded vicinity of $c_{m}$ anymore. However, we can derive several useful properties for the tail as well.

Lemma 6. Consider a tail observation by the leader (i.e., $\theta \in B_{m}$ ). Then,

$$
\begin{equation*}
\mathbb{E}\left[a_{L}(\theta) \mid s, \theta \in B_{m}\right]-c_{m} \leq \bar{r} \bar{x}_{m+1}, \tag{13}
\end{equation*}
$$

for $s \leq c_{m}+\bar{x}_{m+1}$, where $\bar{x}_{m+1}=\sqrt{e} \bar{x}_{m}$. For $s>c_{m}+\bar{x}_{m+1}$, we have

$$
\begin{equation*}
\mathbb{E}\left[a_{L}(\theta) \mid s, \theta \in B_{m}\right]-c_{m} \leq 3 r_{L} \sigma\left(s-c_{m}+1\right) \tag{14}
\end{equation*}
$$

Also, $\mathbb{E}\left[a_{L}(\theta) \mid s, \theta \in B_{m}\right]-c_{m} \geq-\bar{r} \bar{x}_{m}$. As for the variance,

$$
\operatorname{Var}\left[a_{L}(\theta) \mid s, \theta \in B_{m}\right] \leq\left\{\begin{array}{l}
\frac{1}{3}, \text { for } s<c_{m-1}  \tag{15}\\
\frac{3}{4} \bar{r}^{2}\left(\frac{\bar{x}_{m}+\bar{x}_{m+1}}{2}\right)^{2}, \text { for } c_{m-1} \leq s \leq c_{m}+\bar{x}_{m+1} \\
2.5 r_{L}^{2} \sigma^{2}\left(s-c_{m}\right)^{2}, \text { for } s>c_{m}+\bar{x}_{m+1}
\end{array}\right.
$$

Proof. See the appendix.
Let the signal observed by the follower be between $c_{k}$ and $c_{k+1}$, i.e., $s=c_{k}+\delta$ with $0 \leq \delta \leq$ $c_{k+1}-c_{k}$. Then, we claim that the follower's posterior on $\theta$ given $s$ has a negligible probability out of the neighboring intervals $B_{k} \cup B_{k+1}$.

Lemma 7. Let the observed signal by the follower be $s=c_{k}+\delta$, where $0 \leq \delta \leq c_{k+1}-c_{k}$, with $k \geq 0$. Then, for any $j \geq 1$,

$$
\begin{equation*}
\frac{\operatorname{Prob}\left[\theta \in B_{k-j} \mid s\right]}{\operatorname{Prob}\left[\theta \in B_{k} \mid s\right]} \leq e^{-\frac{\left(c_{k}-c_{k-j}\right)^{2}}{2}+3 j+1} . \tag{16}
\end{equation*}
$$

Similarly,

$$
\begin{equation*}
\frac{\operatorname{Prob}\left[\theta \in B_{k+j+1} \mid s\right]}{\operatorname{Prob}\left[\theta \in B_{k+1} \mid s\right]} \leq e^{-\frac{\left(c_{k+j+1}-c_{k+1}\right)^{2}}{2}+3 j+1} . \tag{17}
\end{equation*}
$$

Proof. See the appendix.
Using this lemma and the fact that the fixed points $c_{k}$ are well-spaced, we can show that the effect of the intervals other than $B_{k}$ and $B_{k+1}$ on $a_{F}(s)$ are negligible. In order to characterize
the follower's best response $a_{F}(s)$, we then need to focus only on the segments adjacent to the observed signal, and in particular figure out the weight of each of these two neighboring intervals in the follower's posterior on $\theta$. We do this in the following lemma.

## Lemma 8. Define

$$
\begin{equation*}
m_{k+1}=\frac{c_{k}+c_{k+1}}{2}+\frac{1}{\Delta_{k+1}} \ln \left(\frac{\operatorname{Prob}\left[\theta \in B_{k}\right]}{\operatorname{Prob}\left[\theta \in B_{k+1}\right]}\right), \tag{18}
\end{equation*}
$$

where $\Delta_{k+1}=c_{k+1}-c_{k}$. Also, write the signal observed by the follower as $s=m_{k+1}+\delta$. Then, for $0 \leq k<m-1$,

$$
\begin{equation*}
e^{-\Delta_{k+1}\left(\delta+\bar{r} \bar{x}_{k+1}\right)-\frac{\bar{r}^{2} \bar{x}_{k+1}^{2}}{2}} \leq \frac{\operatorname{Prob}\left[\theta \in B_{k} \mid s\right]}{\operatorname{Prob}\left[\theta \in B_{k+1} \mid s\right]} \leq e^{\Delta_{k+1}\left(\bar{r} \bar{x}_{k+2}-\delta\right)+\frac{\bar{r}^{2} \bar{x}_{k+2}^{2}}{2}} \tag{19}
\end{equation*}
$$

For the case involving the tail segment $B_{m}$,

$$
\begin{equation*}
e^{-\Delta_{m}\left(\delta+\bar{r} \bar{x}_{m}\right)-\frac{\bar{r}^{2} \bar{x}_{m}^{2}}{2}} \leq \frac{\operatorname{Prob}\left[\theta \in B_{m-1} \mid s\right]}{\operatorname{Prob}\left[\theta \in B_{m} \mid s\right]} \leq 1.16 e^{-\Delta_{m}\left(\delta-\bar{r} \bar{x}_{m}\right)+\frac{\bar{r}^{2} \bar{x}_{m}^{2}}{2}} . \tag{20}
\end{equation*}
$$

Proof. See the appendix.
It is worth mentioning that $m_{k+1}$ defined in the above lemma is quite close to the midpoint of $c_{k}$ and $c_{k+1}$. In particular, using Lemma 1 we can show that $-\frac{1.1}{\Delta_{k+1}}<m_{k+1}-\frac{c_{k}+c_{k+1}}{2}<\frac{2.3}{\Delta_{k+1}} .{ }^{12}$ We can now characterize the best response of the follower $a_{F}(s)$ to the leader's strategy $a_{L}(\theta) \in$ $A_{L}^{m}\left(r_{L}, \sigma\right)$ up to the first order.

Lemma 9. Let $s=m_{k+1}+\delta$, with $c_{k} \leq s \leq c_{k+1}$. Then

$$
\begin{align*}
& a_{F}(s) \geq c_{k}+\frac{\Delta_{k+1}}{1+1.17 e^{-\Delta_{k+1} \delta}}-1.01 \bar{r} \bar{x}_{k+2} \\
& a_{F}(s) \leq c_{k}+\frac{1.17 \Delta_{k+1}}{1.17+e^{-\Delta_{k+1} \delta}}+1.01 \bar{r} \bar{x}_{k+2} . \tag{21}
\end{align*}
$$

Also,

$$
\begin{align*}
& 0 \leq \frac{d}{d s} a_{F}(s) \leq 1.17 e^{-\Delta_{k+1}|\delta|} \Delta_{k+1}^{2}+1.01 \bar{r}^{2}\left(\frac{\bar{x}_{k}+\bar{x}_{k+1}}{2}\right)^{2} \text { for } \delta \leq-0.5 \\
& 0 \leq \frac{d}{d s} a_{F}(s) \leq 1.17 e^{-\Delta_{k+1}|\delta|} \Delta_{k+1}^{2}+1.01 \bar{r}^{2}\left(\frac{\bar{x}_{k+1}+\bar{x}_{k+2}}{2}\right)^{2} \text { for } \delta>-0.5 . \tag{22}
\end{align*}
$$

[^9]Proof. See the appendix.

Corollary 1. A useful consequence of Lemma 9 is that

$$
\begin{align*}
& a_{F}(s) \geq c_{k+1}-1.17 e^{-\Delta_{k+1} \delta} \Delta_{k+1}-1.01 \bar{r} \bar{x}_{k+2} \\
& a_{F}(s) \leq c_{k}+1.17 e^{\Delta_{k+1} \delta} \Delta_{k+1}+1.01 \bar{r} \bar{x}_{k+2}, \tag{23}
\end{align*}
$$

and

$$
\begin{equation*}
0 \leq \frac{d}{d s} a_{F}(s) \leq 1.17 e^{-\Delta_{k+1}|\delta|} \Delta_{k+1}^{2}+1.01 \bar{r}^{2} \bar{x}_{k+2}^{2}, \tag{24}
\end{equation*}
$$

where $s=m_{k+1}+\delta$, with $c_{k} \leq s \leq c_{k+1}$.

Note that the exponential terms in the above bounds vanish quite fast for large $\Delta_{k+1}$ and $|\delta|$. for small $|\delta|$, another useful upper bound on the derivative of $a_{F}(s)$ is

$$
\begin{equation*}
\frac{d}{d s} a_{F}(s) \leq \frac{1}{4}\left(\Delta_{k+1}+2 \bar{r} \bar{x}_{k+2}\right)^{2}+0.01 \bar{r}^{2} \bar{x}_{k+2}^{2} . \tag{25}
\end{equation*}
$$

Corollary 2. Let $s=m_{k+1}+\delta$, with $c_{k} \leq s \leq c_{k+1}$. Then,

$$
\begin{gather*}
c_{k}-1.1 \bar{r} \bar{x}_{k+2} \leq a_{F}(s) \leq c_{k}+1.1 \bar{r} \bar{x}_{k+2} \text { for } \delta<-\frac{2 \sqrt{2 \ln \sigma}}{5} \\
c_{k+1}-1.1 \bar{r} \bar{x}_{k+2} \leq a_{F}(s) \leq c_{k+1}+1.1 \bar{r} \bar{x}_{k+2} \text { for } \delta>\frac{2 \sqrt{2 \ln \sigma}}{5} . \tag{26}
\end{gather*}
$$

Roughly speaking, the above corollary says that, if the observed signal by the follower is far enough from the midpoint of $c_{k}$ and $c_{k+1}$, then the optimal action of the follower is wellconcentrated around $c_{k}$ or $c_{k+1}$ (whichever that is closer), and changes very slowly according to Lemma $9 .{ }^{13}$ However, $a_{F}(s)$ may have very high variations for $s$ close to $m_{k+1}$ as can be seen from Lemma 9.

The following lemma characterizes $a_{F}(s)$ when follower makes a tail observation.

Lemma 10. Let $s=c_{m}+\delta$, where $\delta>0$. Then,
i) for $\delta \leq \bar{x}_{m+1}, c_{m}-1.01 \bar{r} \bar{x}_{m+1} \leq a_{F}(s) \leq c_{m}+\bar{r} \bar{x}_{m+1}$, and $0 \leq \frac{d}{d s} a_{F}(s) \leq 0.8 \bar{r}^{2}\left(\frac{\bar{x}_{m}+\bar{x}_{m+1}}{2}\right)^{2}$. ii) for $\delta>\bar{x}_{m+1}, c_{m}-1.01 \bar{r} \bar{x}_{m} \leq a_{F}(s) \leq c_{m}+3 r_{L} \sigma(\delta+1)$, and $0 \leq \frac{d}{d s} a_{F}(s) \leq 3 r_{L}^{2} \sigma^{2} \delta^{2}$.

[^10]Proof. See the appendix.
Lemma 9 and 10 provide the first order characteristics of the best response of the follower to a leader's strategy $a_{L}(\theta) \in A_{L}^{m}\left(r_{L}, \sigma\right)$. We are now ready to analyze the leader's best response $\tilde{a}_{L}(\theta)$ to $a_{F}(s)$ and see if we can keep it in $A_{L}^{m}\left(r_{L}, \sigma\right)$. We have $\tilde{a}_{L}(\theta) \in \operatorname{argmax}_{a_{L}} \tilde{u}_{L}\left(\theta, a_{L}\right)$, where

$$
\begin{align*}
\tilde{u}_{L}\left(\theta, a_{L}\right)= & -r_{L}\left(\theta-a_{L}\right)^{2} \\
& -\left(1-r_{L}\right) \int_{-\infty}^{\infty}\left(a_{F}(s)-a_{L}\right)^{2} \phi\left(s-a_{L}\right) d s \tag{27}
\end{align*}
$$

Lemma 11. Consider $\theta \in\left[c_{k}, c_{k+1}\right], 0 \leq k<m$. Then, there exists a unique $\tilde{b}_{k+1} \in\left[c_{k}, c_{k+1}\right]$ such that

$$
\begin{align*}
\left|\tilde{a}_{L}(\theta)-c_{k}\right|<5 \bar{r} \bar{x}_{k+2} & \text { for } \theta<\tilde{b}_{k+1} \\
\left|\tilde{a}_{L}(\theta)-c_{k+1}\right|<5 \bar{r} \bar{x}_{k+2} & \text { for } \theta>\tilde{b}_{k+1} \tag{28}
\end{align*}
$$

Proof. See the appendix.
The points $\tilde{b}_{k+1}$ determine the segments of the best response strategy $\tilde{a}_{L}(\theta)$. The leader's best response strategy clearly has a discontinuity at $\tilde{b}_{k+1}$. However, as we show in the next lemma, $\tilde{a}_{L}(\theta)$ is differentiable at all points $\theta \in\left[c_{k}, c_{k+1}\right] \backslash\left\{\tilde{b}_{k+1}\right\}$. We further bound the derivative of $\tilde{a}_{L}(\theta)$ using the bounds on the follower's strategy and its derivative derived in Lemma 9 , Corollary 1, and Corollary 2.

Lemma 12. Consider $\theta \in\left[c_{k}, c_{k+1}\right], 0 \leq k<m$, with $\theta \neq \tilde{b}_{k+1}$. Then,

$$
\begin{align*}
\frac{d}{d \theta} \tilde{a}_{L}(\theta) & \geq \frac{r_{L}}{r_{L}+\left(1-r_{L}\right)\left(1+0.4 \bar{r}^{2} \sigma^{2}\right)} \\
\frac{d}{d \theta} \tilde{a}_{L}(\theta) & \leq \frac{r_{L}}{r_{L}+\left(1-r_{L}\right)\left(1-0.45 \bar{r}^{2} \sigma^{2}\right)} \tag{29}
\end{align*}
$$

Proof. See the appendix.
Using this lemma and the values $\underline{r}=r_{L}\left(1-0.5 r_{L}^{2} \sigma^{2}\right)$ and $\bar{r}=r_{L}\left(1+0.5 r_{L}^{2} \sigma^{2}\right)$, we can easily verify that $\underline{r} \leq \frac{d}{d \theta} \tilde{a}_{L}(\theta) \leq \bar{r}$. This means that Property 3 is preserved by the best response for $\theta \in\left[-c_{m}, c_{m}\right]$. We study the tail case later in Lemma 14. Next lemma identifies $2 m+1$ local minima of the MSE term in the leader's payoff, each located in a tiny interval around a fixed
point of $a_{L}(\theta)$, establishing that they are indeed the fixed points of the best response strategy $\tilde{a}_{L}(\theta)$.

## Lemma 13. Define

$$
\begin{equation*}
\tilde{J}_{L}\left(a_{L}\right)=\int_{-\infty}^{\infty}\left(a_{F}(s)-a_{L}\right)^{2} \phi\left(s-a_{L}\right) d s \tag{30}
\end{equation*}
$$

Then, $\tilde{J}_{L}\left(a_{L}\right)$ is strongly convex over $\left[c_{k}-5 \bar{r} \bar{x}_{k+2}, c_{k}+5 \bar{r} \bar{x}_{k+2}\right]$ with $\frac{d^{2}}{d a_{L}^{2}} \tilde{J}_{L}\left(a_{L}\right) \geq 2\left(1-0.45 \bar{r}^{2} \sigma^{2}\right)$. Let $\tilde{c}_{k}$ be the unique solution of

$$
\begin{equation*}
\tilde{c}_{k}=\underset{a_{L} \in\left[c_{k}-5 \bar{r} \bar{x}_{k+2}, c_{k}+5 \bar{r} \bar{x}_{k+2}\right] \cap\left[\tilde{b}_{k}, \tilde{b}_{k+1}\right]}{\operatorname{argmin}} \tilde{J}_{L}\left(a_{L}\right) . \tag{31}
\end{equation*}
$$

Then, $\tilde{a}_{L}\left(\tilde{c}_{k}\right)=\tilde{c}_{k}$.
Proof. See the appendix.
The fixed point characterized in the above lemma is the unique fixed point in $\left[\tilde{b}_{k}, \tilde{b}_{k+1}\right]$ from Property 3. Therefore, Property 1 is also preserved under the best response. Next lemma describes the tail properties of $\tilde{a}_{L}(\theta)$.

Lemma 14. If $\tilde{b}_{m}<\theta<\tilde{c}_{m}+\sigma \bar{x}_{m+1}$, then $\underline{r} \leq \frac{d}{d \theta} \tilde{a}_{L}(\theta) \leq \bar{r}$. For $\theta>\tilde{c}_{m}+\sigma \bar{x}_{m+1}$, we have $\tilde{a}_{L}(\theta) \leq \tilde{c}_{m}+3 r_{L}\left(\theta-\tilde{c}_{m}\right)$.

Proof. See the appendix.
Now, in order to verify that the updated strategy $\tilde{a}_{L}(\theta)$ satisfies Property 2 , we need to bound the displacements in the fixed points $\tilde{c}_{k}$ and endpoints $\tilde{b}_{k}$.

Lemma 15. For the endpoints of the intervals corresponding to $\tilde{a}_{L}(\theta)$, we have $\left|\tilde{b}_{k+1}-\frac{\tilde{c}_{k}+\tilde{c}_{k+1}}{2}\right| \leq$ $0.1 r_{L}$.

Proof. See the appendix.
Bounding the displacement in $\tilde{c}_{k}$ can be done in multiple steps: first we need to relate the fixed point of the leader's best response $\tilde{a}_{L}(\theta)$ in interval $\tilde{B}_{k}$ to the fixed point of $a_{F}(s)$ in $B_{k}$ (i.e., $s_{k}$ ), followed by estimating $s_{k}$ in terms of $c_{k}$ and $e_{k}$ where $e_{k}=\mathbb{E}_{N\left(0, \sigma^{2}\right)}\left[\theta \mid \theta \in B_{k}\right]$, that is, the expected value of $\theta$ over $B_{k}$. Finally we bound the displacement in $e_{k}$ with the displacement of the interval endpoints using properties of truncated normal distribution.

Lemma 16. Let $s_{k}$ be the fixed point of $a_{F}(s)$ in the interval $\left[c_{k}-5 \bar{r} \bar{x}_{m+1}, c_{k}+5 \bar{r} \bar{x}_{m+1}\right]$, i.e., $a_{F}\left(s_{k}\right)=s_{k}$. Then,

$$
\begin{equation*}
\left|\tilde{c}_{k}-s_{k}\right| \leq 0.42 r_{L}^{2}\left(\frac{\bar{x}_{k}+\bar{x}_{k+1}}{2}\right)^{2}+0.08 r_{L}^{2} \underline{x}_{1} \tag{32}
\end{equation*}
$$

Proof. See the appendix.

Lemma 17. $s_{k}$ can be located based on $c_{k}$ and $e_{k}$ as

$$
\begin{equation*}
\left|s_{k}-\left(1-r_{L}\right) c_{k}-r_{L} e_{k}\right| \leq 1.9 r_{L}^{2} \bar{x}_{k+1} \tag{33}
\end{equation*}
$$

Proof. See the appendix.
Using Lemma 15-17, we can reach at

$$
\begin{equation*}
\left|\tilde{c}_{k}-\left(1-r_{L}\right) c_{k}-r_{L} \hat{e}_{k}\right| \leq 0.42 r_{L}^{2}\left(\frac{\bar{x}_{k}+\bar{x}_{k+1}}{2}\right)^{2}+2 r_{L}^{2} \bar{x}_{k+1} \tag{34}
\end{equation*}
$$

where $\hat{e}_{k}=\mathbb{E}_{N\left(0, \sigma^{2}\right)}\left[\theta \mid \theta \in \hat{B}_{k}\right]$, with $\hat{B}_{k}=\left[\hat{b}_{k}, \hat{b}_{k+1}\right], \hat{b}_{k}=\frac{c_{k-1}+c_{k}}{2}$ and $\hat{b}_{k+1}=\frac{c_{k}+c_{k+1}}{2}$. We can now use (34) and Lemma 15 to verify that Property 2 is also preserved by the best response, completing the proof of the invariance of $A_{L}^{m}\left(r_{L}, \sigma\right)$ for $m \in M(\sigma)$ and $\sigma \geq 300$ in the regime $\frac{1}{2} \leq r_{L} \sigma^{2} \leq 1$. This is carried out in the proof of the following theorem.

## VI. Conclusions

We studied Witsenhausen's counterexample in a leader-follower game setup where the follower makes noisy observations from the leader's action and aims to choose her action as close as possible to that of the leader. Leader who moves first and can see the realization of the state of the world chooses her action to minimize her ex-ante distance from the follower's action as well as the state of the world. We showed the existence of nonlinear perfect Bayesian equilibria in the regime $\frac{1}{2} \leq r_{L} \sigma^{2} \leq 1$, where the leader's strategy is a perturbed near-piecewise-linear version of an optimal MSE quantizer. We then proved that these equilibria are indeed local minima of the original Witsenhausen's problem. Incorporating some relevant results from asymptotic quantization theory and lower bounds on the optimal cost of Witsenhausen's problem from the literature, we showed that the proposed local minima include solutions that are at most a constant factor away from the optimal one.

## Appendix

Proof of Lemma 1. To avoid lengthy expressions, we introduce and work with normalized variables $\hat{x}_{k}=\frac{x_{k}^{\mathrm{Q}}}{\sigma}, \hat{b}_{k}=\frac{b_{k}^{\mathrm{Q}}}{\sigma}, \hat{c}_{k}=\frac{c_{k}^{\mathrm{Q}}}{\sigma}$, and $\hat{B}_{k}=\frac{B_{k}^{\mathrm{Q}}}{\sigma}$. All the expectations are then taken using $N(0,1)$ as the probability measure.
i) Let $\rho(h)=\frac{\phi(h)}{1-\Phi(h)}$ denote the Mill's ratio. Then, $0 \leq 1-\rho(\rho-h) \leq(\rho-h)^{2}$ (see, e.g. [27] for a proof). This implies that $0 \leq 1-\hat{c}_{m} \hat{x}_{m} \leq \hat{x}_{m}^{2}$. Moreover, $\hat{x}_{m}$ is decreasing with $m$ and it follows from direct calculation that $\hat{x}_{2}=0.48<\frac{1}{2}$, hence completing the proof.
ii) Since $\hat{c}_{k}$ is the centroid of segment $\hat{B}_{k}$, we should have

$$
\begin{equation*}
\int_{\hat{b}_{k}}^{\hat{c}_{k}}\left(\hat{c}_{k}-\theta\right) \phi(\theta) d \theta=\int_{\hat{c}_{k}}^{\hat{b}_{k+1}}\left(\theta-\hat{c}_{k}\right) \phi(\theta) d \theta . \tag{35}
\end{equation*}
$$

This, together with the fact that $\int_{\hat{b}_{k}}^{\hat{c}_{k}}\left(\hat{c}_{k}-\theta\right) \phi(\theta) d \theta \leq \phi\left(\hat{b}_{k}\right) \int_{\hat{b}_{k}}^{\hat{c}_{k}}\left(\hat{c}_{k}-\theta\right) d \theta$ and $\int_{\hat{c}_{k}}^{\hat{b}_{k+1}}(\theta-$ $\left.\hat{c}_{k}\right) \phi(\theta) d \theta \geq \phi\left(\hat{b}_{k+1}\right) \int_{\hat{c}_{k}}^{\hat{b}_{k+1}}\left(\theta-\hat{c}_{k}\right) d \theta$ proves the RHS inequality in (4). To derive the LHS, let $p_{k}^{1}=\operatorname{Prob}\left[\theta \mid \hat{b}_{k} \leq \theta \leq \hat{c}_{k}\right]$ and $p_{k}^{2}=\operatorname{Prob}\left[\theta \mid \hat{c}_{k} \leq \theta \leq \hat{b}_{k+1}\right]$. Noting that $\phi(\theta)$ (for $\theta \geq 0$ ) and $\hat{c}_{k}-\theta$ are decreasing with $\theta$, we apply algebraic Chebyshev inequality to (35) to obtain $p_{k}^{1} \hat{x}_{k} \leq p_{k}^{2} \hat{x}_{k+1}$. On the other hand,

$$
\begin{equation*}
\frac{p_{k}^{2}}{p_{k}^{1}} \leq \frac{\hat{x}_{k+1}}{\hat{x}_{k}} \times \frac{\phi\left(\hat{c}_{k}\right)}{\phi\left(\hat{b}_{k}\right)} \tag{36}
\end{equation*}
$$

Combining the two, we can derive the LHS in (4). It then immediately follows from the LHS inequality that $\hat{x}_{k} \leq \hat{x}_{k+1}$ for $1 \leq k$. Applying the result of part (i) to the RHS inequality we can easily show that $\frac{\hat{x}_{k+1}}{\hat{x}_{k}} \leq e$.
iii) A useful property here is that $g(x)=\frac{1}{x} \int_{a}^{a+x} \phi(t) d t$ is decreasing in $x$ for $x, a>0$. Using this, we can obtain

$$
\begin{align*}
& \frac{\operatorname{Prob}\left[\theta \mid \theta \in \hat{B}_{k}\right]}{\operatorname{Prob}\left[\theta \mid \theta \in \hat{B}_{j}\right]} \leq \frac{\hat{x}_{k}+\hat{x}_{k+1}}{\hat{x}_{j}+\hat{x}_{j+1}} \times \frac{\phi\left(\hat{b}_{k}\right)}{\phi\left(\hat{b}_{j}\right)} \\
& \leq \frac{\hat{x}_{k}+\hat{x}_{k+1}}{\hat{x}_{j}+\hat{x}_{j+1}} \times \frac{\hat{x}_{j}^{2}}{\hat{x}_{k}^{2}} \leq \frac{1+e}{2} \frac{\hat{x}_{j}}{\hat{x}_{k}} \leq \frac{1+e}{2} \tag{37}
\end{align*}
$$

where the last two lines follow from part (ii). We have to modify the proof for $k=$ $m$. To extend the proof to the case $k=m$, it suffices to show that $\operatorname{Prob}\left[\theta \mid \theta \in \hat{B}_{m}\right] \leq$
$\operatorname{Prob}\left[\theta \mid \theta \in \hat{B}_{m-1}\right]$. We first note that

$$
\begin{equation*}
\operatorname{Prob}\left[\theta \mid \theta \in \hat{B}_{m}\right]=\frac{\phi\left(\hat{b}_{m}\right)}{\hat{c}_{m}} \leq \frac{4}{3} \phi\left(\hat{b}_{m}\right) \hat{x}_{m}, \tag{38}
\end{equation*}
$$

where we have used part (i) in the last inequality. The proof now follows from the fact that $\operatorname{Prob}\left[\theta \mid \theta \in \hat{B}_{m-1}\right]>\left(\hat{x}_{m-1}+\hat{x}_{m}\right) \phi\left(\hat{b}_{m}\right) \geq\left(1+\frac{1}{e}\right) \hat{x}_{m} \phi\left(\hat{b}_{m}\right)$.
iv) We start by showing that

$$
\begin{equation*}
e^{\frac{1}{2}\left(\hat{x}_{1}+\ldots+\hat{x}_{k}\right)^{2}} \leq \frac{\hat{x}_{k+1}}{\hat{x}_{1}} \leq e^{\frac{5}{6}\left(\hat{x}_{1}+\ldots+\hat{x}_{k+1}\right)^{2}} . \tag{39}
\end{equation*}
$$

The LHS easily follows from part (ii), while the RHS requires a more involved analysis as we elaborate below. The idea here is to find an appropriate lower bound for the RHS of (35). Using Jensen's inequality for the function $e^{-x}$, we can obtain

$$
\begin{equation*}
\int_{\hat{c}_{k}}^{\hat{b}_{k+1}}\left(\theta-\hat{c}_{k}\right) \phi(\theta) d \theta \geq \frac{\hat{x}_{k+1}^{2}}{2 \sqrt{2 \pi}} e^{-\frac{1}{2} \int_{\hat{c}_{k}}^{\hat{b}_{k+1}} \frac{2 \theta^{2}\left(\theta-\hat{c}_{k}\right)}{\hat{x}_{k+1}^{2}} d \theta} \tag{40}
\end{equation*}
$$

Combining this with the same upper bound of $\frac{\hat{x}_{k}^{2}}{2} \phi\left(\hat{b}_{k}\right)$ as in part (ii) for the LHS of (35) and after some simplification we can reach at

$$
\begin{equation*}
\frac{\hat{x}_{k+1}^{2}}{\hat{x}_{k}^{2}} \leq e^{\hat{c}_{k} \hat{x}_{k}+\frac{2}{3} \hat{c}_{k} \hat{x}_{k+1}-\frac{\hat{x}_{k}^{2}}{2}+\frac{\hat{x}_{k+1}^{2}}{4}} . \tag{41}
\end{equation*}
$$

Substituting $k$ with $1, \ldots, k-1$ and multiplying all these $k$ inequalities we can prove the RHS inequality in (39).
Incorporating the simple inequality $\frac{\phi\left(\hat{x}_{1}+\ldots+\hat{x}_{k}\right)}{\phi\left(\hat{x}_{1}+\ldots+\hat{x}_{k+1}\right)} \leq e^{\frac{\hat{c}_{m} \hat{x}_{m}}{2}} \leq \sqrt{e}$ into (39), we can find that $\hat{x}_{1} \leq \sqrt{2 \pi e} \hat{x}_{k+1} \phi\left(\hat{x}_{1}+\ldots+\hat{x}_{k+1}\right)$ for $k=1, \ldots, m-1$. Adding up all these inequalities and $\hat{x}_{1} \leq \sqrt{2 \pi e} \hat{x}_{1} \phi\left(\hat{x}_{1}\right)$ yields

$$
\begin{equation*}
\frac{m \hat{x}_{1}}{\sqrt{2 \pi e}} \leq \sum_{k=1}^{m} \hat{x}_{k} \phi\left(\hat{x}_{1}+\ldots+\hat{x}_{k}\right) \leq \int_{0}^{\infty} \phi(\theta) d \theta=\frac{1}{2} \tag{42}
\end{equation*}
$$

proving $\hat{x}_{1} \leq \frac{\sqrt{2 \pi e}}{2 m}$.
Based on the RHS of (39) and following a similar approach we can show that

$$
\begin{equation*}
\sum_{k=1}^{m-1} \hat{x}_{k+1} e^{-\frac{5}{6}\left(\hat{x}_{1}+\ldots+\hat{x}_{k}\right)^{2}} \leq e(m-1) \hat{x}_{1} \tag{43}
\end{equation*}
$$

On the other hand,

$$
\begin{align*}
\int_{\hat{x}_{1}+\ldots+\hat{x}_{m}}^{\infty} e^{-\frac{5}{6} \theta^{2}} d \theta & \leq \frac{6 e^{-\frac{5}{6}\left(\hat{x}_{1}+\ldots+\hat{x}_{m}\right)^{2}}}{10\left(\hat{x}_{1}+\ldots+\hat{x}_{m}\right)} \\
& \leq \frac{6 e^{-\frac{5}{6}\left(\hat{x}_{1}+\ldots+\hat{x}_{m}\right)^{2}} \hat{x}_{m}}{10\left(\hat{x}_{1}+\ldots+\hat{x}_{m}\right) \hat{x}_{m}} \leq 1.6 \hat{x}_{1} \tag{44}
\end{align*}
$$

where the last inequality follows from (39) and part (i). Putting (43) and (44) together we can show that $e m \hat{x}_{1} \geq \int_{0}^{\infty} e^{-\frac{5}{6} \theta^{2}} d \theta=\sqrt{\frac{3 \pi}{10}}$, which yields $\hat{x}_{1} \geq \frac{\sqrt{3 \pi}}{\sqrt{10} e m}$. Using (39) for $k=m-1$, together with $\hat{x}_{m} \hat{c}_{m} \leq 1$ and $\hat{x}_{1} \leq \frac{\sqrt{2 \pi e}}{2 m}$, we can find

$$
\begin{equation*}
\frac{1}{2 \sqrt{1.2}} \geq \hat{x}_{m} \sqrt{\ln \frac{2 m \hat{x}_{m}}{\sqrt{2 \pi e}}} \tag{45}
\end{equation*}
$$

using which we can show that $\hat{x}_{m} \leq \frac{1}{1.1 \sqrt{\ln m}}$, for $m \geq 5$. This in turn implies that

$$
\begin{equation*}
\hat{c}_{m} \geq \frac{1}{\hat{x}_{m}}-\hat{x}_{m} \geq 1.1 \sqrt{\ln m}-\frac{1}{1.1 \sqrt{\ln m}} \tag{46}
\end{equation*}
$$

Finally, using the LHS of (39) for $k=m-1$, together with $\hat{x}_{m} \leq \frac{1}{1.1 \sqrt{\ln m}}$ and $\hat{x}_{1} \geq \frac{\sqrt{3 \pi}}{\sqrt{10 e m}}$, and some manipulation we can obtain

$$
\begin{equation*}
\hat{c}_{m} \leq 2 \sqrt{2 \ln m+1.4}+\frac{2}{1.1 \sqrt{\ln m}} \leq 2 \sqrt{2 \ln m+1.4}+1.45 \tag{47}
\end{equation*}
$$

for $m \geq 5$. Along with $\hat{c}_{m} \hat{x}_{m} \geq 1-\hat{x}_{m}^{2}$, this leads to

$$
\begin{equation*}
\hat{x}_{m} \geq \frac{1}{\frac{\hat{c}_{m}}{2}+\sqrt{\frac{\hat{c}_{m}^{2}}{4}+1}} \geq \frac{1}{2 \sqrt{2 \ln m+6}} . \tag{48}
\end{equation*}
$$

Remark 1. Particular consequences of assuming $m \geq 25$ and $\sigma \geq 300$ and that $m$ is such that $x_{1}^{\mathrm{Q}}>2 \sqrt{2 \ln \sigma}+5$, are frequently used in the proofs concerning best response analysis. We summarize these properties here to avoid confusion in case they are not explicitly mentioned when used in the proofs.

1) $x_{m}^{\mathrm{Q}}>6 x_{1}^{\mathrm{Q}}$ and $c_{m}^{\mathrm{Q}}>13.5 x_{m}^{\mathrm{Q}}$. This follows from direct calculation of the optimal $(2 m+1)$ level MSE quantizer for $m=25$ and that $\frac{x_{m}^{Q}}{x_{1}^{Q}}$ and $\frac{c_{m}^{Q}}{x_{m}^{Q}}$ are both increasing with $m$.
2) $x_{m}^{\mathrm{Q}}<0.262 \sigma$. This follows from direct calculation of the optimal $(2 m+1)$-level MSE
quantizer for $m=25$ and that $\frac{x_{m}^{\mathrm{Q}}}{\sigma}$ is decreasing with $m$. As a result, $\bar{x}_{m}=x_{m}^{\mathrm{Q}}+3<0.272 \sigma$ since $\sigma \geq 300$.
3) $x_{1}^{\mathrm{Q}}>11.5$. This follows from $x_{1}^{\mathrm{Q}}>2 \sqrt{2 \ln \sigma}+5$ and $\sigma \geq 300$. As a result, $\underline{x}_{1}=x_{1}^{\mathrm{Q}}-3>$ 8.5.
4) $m<\frac{\sigma}{2}$. This follows from part iv) of Lemma 1: $\frac{2 m}{\sigma} \leq \frac{\sqrt{2 \pi e}}{x_{1}^{Q}}<\frac{\sqrt{2 \pi e}}{2 \sqrt{2 \ln \sigma}+5}<1$.

Proof of Lemma 5. This is an immediate result of Property 2.
Proof of Lemma 6. We start with the case where $a_{L}\left(c_{m}+\sigma \bar{x}_{m+1}\right) \leq s \leq c_{m}+\bar{x}_{m+1}$. Let $\theta_{c}=c_{m}+\sigma \bar{x}_{m+1}$ and $\delta_{c}=s-a_{L}\left(\theta_{c}\right)$. With some manipulation, we can show that for every $b_{m} \leq \theta, \theta^{\prime} \leq \theta_{c}$,

$$
\begin{equation*}
\frac{\operatorname{Prob}\left[\theta^{\prime} \mid s\right]}{\operatorname{Prob}[\theta \mid s]}=\frac{\phi\left(s-a_{L}\left(\theta^{\prime}\right)\right) \phi\left(\frac{\theta^{\prime}}{\sigma}\right)}{\phi\left(s-a_{L}(\theta)\right) \phi\left(\frac{\theta}{\sigma}\right)} \geq \frac{\phi\left(\frac{\theta^{\prime}}{\sigma}\right) \phi\left(\delta_{c}+\bar{r}\left(\theta_{c}-\theta^{\prime}\right)\right)}{\phi\left(\frac{\theta}{\sigma}\right) \phi\left(\delta_{c}+\underline{r}\left(\theta_{c}-\theta\right)\right)} . \tag{49}
\end{equation*}
$$

Integrating with respect to $\theta^{\prime}$ and after some simplification, we arrive at

$$
\begin{align*}
\operatorname{Prob}\left[\theta \mid s, b_{m} \leq \theta \leq \theta_{c}\right] & \leq \frac{\phi\left(\delta_{c}+\underline{r}\left(\theta_{c}-\theta\right)\right)}{\phi\left(\delta_{c}+\bar{r}\left(\theta_{c}-\theta\right)\right)} \frac{\bar{\phi}(\theta)}{\bar{\Phi}\left(\theta_{c}\right)-\bar{\Phi}\left(b_{m}\right)} \\
& \leq \xi \frac{\bar{\phi}(\theta)}{\bar{\Phi}\left(\theta_{c}\right)-\bar{\Phi}\left(b_{m}\right)} \tag{50}
\end{align*}
$$

where $\bar{\phi} \sim N\left(\bar{\mu}, \bar{\nu}^{2}\right)$, with $\bar{\mu}=\frac{\bar{r} \sigma^{2}\left(\delta_{c}+\bar{r} \theta_{c}\right)}{1+\bar{r}^{2} \sigma^{2}}$ and $\bar{\nu}^{2}=\frac{\sigma^{2}}{1+\bar{r}^{2} \sigma^{2}}<\sigma^{2}$, and

$$
\begin{equation*}
\xi=\frac{\phi\left(\delta_{c}+\underline{r}\left(\theta_{c}-b_{m}\right)\right)}{\phi\left(\delta_{c}+\bar{r}\left(\theta_{c}-b_{m}\right)\right)} \tag{51}
\end{equation*}
$$

Using $\delta_{c} \leq \bar{x}_{m+1}-\underline{r} \sigma \bar{x}_{m+1}$, it is easy to verify that $\delta_{c}+r_{L}\left(\theta_{c}-b_{m}\right)<\bar{x}_{m+1}+1$. It then follows that

$$
\begin{align*}
\ln \xi & =(\bar{r}-\underline{r})\left(\theta_{c}-b_{m}\right)\left(\delta_{c}+r_{L}\left(\theta_{c}-b_{m}\right)\right) \\
& \leq r_{L}^{3} \sigma^{2}(\sqrt{e} \sigma+1) \bar{x}_{m}\left(\sqrt{e} \bar{x}_{m}+1\right)<0.001, \tag{52}
\end{align*}
$$

for $\sigma \geq 300$ and $m \geq 25$ (for which $x_{m}^{\mathrm{Q}}<0.262 \sigma$ ), implying that $\xi<1.01$.
A useful formula is

$$
\begin{align*}
\frac{1}{\sigma^{2}} \operatorname{Var}_{N\left(\mu, \sigma^{2}\right)}[\theta \mid c \leq \theta] & =1-\frac{\left(\mathbb{E}_{N\left(\mu, \sigma^{2}\right)}[\theta \mid c \leq \theta]-c\right)\left(\mathbb{E}_{N\left(\mu, \sigma^{2}\right)}[\theta \mid c \leq \theta]-\mu\right)}{\sigma^{2}} \\
& \leq\left(\frac{\mathbb{E}_{N\left(\mu, \sigma^{2}\right)}[\theta \mid c \leq \theta]-c}{\sigma}\right)^{2} \tag{53}
\end{align*}
$$

for $\mu \leq c$. Moreover, the above variance is increasing in $\mu$ for $\mu \leq c$ (see, e.g. [27]).
Let $e_{\bar{\mu}}=\mathbb{E}_{N\left(\bar{\mu}, \sigma^{2}\right)}\left[\theta \mid b_{m} \leq \theta\right]-b_{m}$ and $e_{0}=\mathbb{E}_{N\left(0, \sigma^{2}\right)}\left[\theta \mid b_{m} \leq \theta\right]-b_{m}$. Then, from the increasing property mentioned above, we find

$$
\begin{align*}
& \left(b_{m}+e_{\bar{\mu}}-\bar{\mu}\right) e_{\bar{\mu}} \leq\left(b_{m}+e_{0}\right) e_{0} \Rightarrow \\
& e_{\bar{\mu}} \leq \frac{-\left(b_{m}-\bar{\mu}\right)+\sqrt{\left(b_{m}-\bar{\mu}\right)^{2}+4 e_{0}\left(b_{m}+e_{0}\right)}}{2} \\
& \leq \frac{-\left(b_{m}-\bar{\mu}\right)+\sqrt{\left(b_{m}-\bar{\mu}\right)^{2}+4 \bar{x}_{m}\left(b_{m}+\bar{x}_{m}\right)}}{2} \tag{54}
\end{align*}
$$

and hence,

$$
\begin{equation*}
\frac{e_{\bar{\mu}}}{\bar{x}_{m}} \leq \frac{2\left(b_{m}+\bar{x}_{m}\right)}{\left(b_{m}-\bar{\mu}\right)+\sqrt{\left(b_{m}-\bar{\mu}\right)^{2}+4 \bar{x}_{m}\left(b_{m}+\bar{x}_{m}\right)}} \tag{55}
\end{equation*}
$$

Based on this inequality and that $x_{1}^{\mathrm{Q}}>11.5$ (from the definition of $M(\sigma)$ and that $\sigma \geq 300$ ), and that $x_{m}^{\mathrm{Q}}>6 x_{1}^{\mathrm{Q}}$ and $b_{m}^{\mathrm{Q}} \geq 12.5 x_{m}^{\mathrm{Q}}$ for $m \geq 25$, we can show that $e_{\bar{\mu}} \leq 1.134 \bar{x}_{m}$. Therefore,

$$
\begin{align*}
& \mathbb{E}\left[a_{L}(\theta) \mid s, b_{m} \leq \theta \leq \theta_{c}\right]-c_{m} \leq \xi \bar{r} \mathbb{E}_{\bar{\phi}}\left[\theta-b_{m} \mid b_{m} \leq \theta \leq \theta_{c}\right] \\
& \quad<\xi \bar{r} \mathbb{E}_{N\left(\bar{\mu}, \sigma^{2}\right)}\left[\theta-b_{m} \mid b_{m} \leq \theta\right]<1.01 \times 1.134 \bar{r} \bar{x}_{m}<0.75 \bar{r} \bar{x}_{m+1} . \tag{56}
\end{align*}
$$

As for the variance, we start with

$$
\begin{equation*}
\operatorname{Var}\left[a_{L}(\theta) \mid s, b_{m} \leq \theta \leq \theta_{c}\right] \leq \xi \bar{r}^{2} \operatorname{Var}_{\bar{\phi}}\left[\theta \mid b_{m} \leq \theta \leq \theta_{c}\right] \leq \xi \bar{r}^{2} \operatorname{Var}_{N\left(\bar{\mu}, \sigma^{2}\right)}\left[\theta \mid b_{m} \leq \theta\right] \tag{57}
\end{equation*}
$$

Combining this with the bound in (53), we get

$$
\begin{equation*}
\operatorname{Var}\left[a_{L}(\theta) \mid s, b_{m} \leq \theta \leq \theta_{c}\right] \leq 1.01 \times 1.134^{2} \bar{r}^{2} \bar{x}_{m}^{2}<1.3 \bar{r}^{2} \bar{x}_{m}^{2} \tag{58}
\end{equation*}
$$

Similar results to the above can be derived for the case where $a_{L}\left(b_{m}\right) \leq s<a_{L}\left(\theta_{c}\right)$, using $\theta_{s}$ instead of $\theta_{c}$, where $s=a_{L}\left(\theta_{s}\right)$ with $b_{m} \leq \theta_{s}<\theta_{c}$. The same for the case $s<a_{L}\left(b_{m}\right)$, following a similar argument with $\bar{\phi}_{b} \sim N\left(\bar{\mu}_{b}, \bar{\nu}^{2}\right)$, where $\bar{\mu}_{b}=\frac{\bar{r} \sigma^{2}\left(\bar{r} b_{m}-\delta_{b}\right)}{1+\bar{r}^{2} \sigma^{2}}, \bar{\nu}^{2}=\frac{\sigma^{2}}{1+\bar{r}^{2} \sigma^{2}}$, and $\delta_{b}=a_{L}\left(b_{m}\right)-s>0$.

Now we bring into play the tail effect. For every $\theta \geq \theta_{c}$, we use

$$
\begin{equation*}
\frac{\operatorname{Prob}[\theta \mid s]}{\operatorname{Prob}\left[c_{m} \leq \theta^{\prime} \leq \theta_{c} \mid s\right]}=\frac{\phi\left(s-a_{L}(\theta)\right) \phi\left(\frac{\theta}{\sigma}\right)}{\int_{c_{m}}^{\theta_{c}} \phi\left(s-a_{L}\left(\theta^{\prime}\right)\right) \phi\left(\frac{\theta^{\prime}}{\sigma}\right) d \theta^{\prime}}, \tag{59}
\end{equation*}
$$

using which for $s=c_{m}+\delta$ with $0 \leq \delta \leq \bar{x}_{m+1}$, we get

$$
\begin{equation*}
\frac{\operatorname{Prob}[\theta \mid s]}{\operatorname{Prob}\left[c_{m} \leq \theta^{\prime} \leq \theta_{c} \mid s\right]} \leq \frac{e^{\frac{\delta^{2}}{2}} \phi\left(\frac{\theta}{\sigma}\right)}{\sigma\left(\Phi\left(\frac{\theta_{c}}{\sigma}\right)-\Phi\left(\frac{c_{m}}{\sigma}\right)\right)} . \tag{60}
\end{equation*}
$$

Therefore, for $\theta \geq \theta_{c}$

$$
\begin{equation*}
\operatorname{Prob}\left[\theta \mid s, \theta \geq c_{m}\right] \leq \frac{e^{\frac{\delta^{2}}{2}}\left(1-\Phi\left(\frac{\theta_{c}}{\sigma}\right)\right)}{\Phi\left(\frac{\theta_{c}}{\sigma}\right)-\Phi\left(\frac{c_{m}}{\sigma}\right)} \times \frac{\phi\left(\frac{\theta}{\sigma}\right)}{\sigma\left(1-\Phi\left(\frac{\theta_{c}}{\sigma}\right)\right)} . \tag{61}
\end{equation*}
$$

Using the inequality $h \leq \rho(h) \leq \frac{h^{2}+1}{h}$ for $h>0$, we can show that $\frac{1-\Phi\left(\frac{\theta_{c}}{\sigma}\right)}{\Phi\left(\frac{\theta_{c}}{\sigma}\right)-\Phi\left(\frac{c_{m}}{\sigma}\right)} \leq \frac{\phi\left(\frac{\theta_{c}}{\sigma}\right)}{\phi\left(\frac{\sigma_{m}}{\sigma}\right)}$. This, along with (61) and $0 \leq \delta \leq \bar{x}_{m+1}$ and $\theta_{c}=c_{m}+\sigma \bar{x}_{m+1}$ yields

$$
\begin{align*}
& \operatorname{Prob}\left[\theta \mid s, \theta \in B_{m}\right] \leq \operatorname{Prob}\left[\theta \mid s, \theta \geq c_{m}\right] \\
& \leq \frac{e^{-\frac{c_{m} \bar{x}_{m+1}}{\sigma}} \phi\left(\frac{\theta}{\sigma}\right)}{\sigma\left(1-\Phi\left(\frac{\theta_{c}}{\sigma}\right)\right)} \leq \frac{e^{-\sigma} \phi\left(\frac{\theta}{\sigma}\right)}{\sigma\left(1-\Phi\left(\frac{\theta_{c}}{\sigma}\right)\right)}, \tag{62}
\end{align*}
$$

for $\theta \geq \theta_{c}$, where we have used $c_{m} \bar{x}_{m+1} \geq \sigma^{2}$ (which easily follows from Lemma 1 ). Using this along with (56), we can obtain

$$
\begin{align*}
& \mathbb{E}\left[a_{L}(\theta) \mid s, \theta \in B_{m}\right]-c_{m} \\
& \leq 0.75 \bar{r} \bar{x}_{m+1}+e^{-\sigma}\left(3 r_{L}\left(\theta_{c}-c_{m}+\sigma\left(\rho\left(\frac{\theta_{c}}{\sigma}\right)-\frac{\theta_{c}}{\sigma}\right)\right)\right) \\
& \leq 0.75 \bar{r} \bar{x}_{m+1}+3 r_{L} \sigma e^{-\sigma}\left(\bar{x}_{m+1}+\frac{\sigma}{c_{m}+\sigma \bar{x}_{m+1}}\right) \leq \bar{r} \bar{x}_{m+1} \tag{63}
\end{align*}
$$

Also, $\mathbb{E}\left[a_{L}(\theta) \mid s, \theta \in B_{m}\right] \geq a_{L}\left(b_{m}\right) \geq c_{m}-\bar{r} \bar{x}_{m}$. To bound the variance, let $\kappa=\mathbb{E}\left[a_{L}(\theta) \mid s, b_{m} \leq\right.$ $\left.\theta \leq \theta_{c}\right]\left(\kappa>a_{L}\left(b_{m}\right)\right)$. Then,

$$
\begin{align*}
& \operatorname{Var}\left[a_{L}(\theta) \mid s, \theta \in B_{m}\right] \leq \mathbb{E}\left[\left(a_{L}(\theta)-\kappa\right)^{2} \mid s, \theta \in B_{m}\right] \\
& \leq 1.3 \bar{r}^{2} \bar{x}_{m}^{2}+e^{-\sigma} \mathbb{E}_{N\left(0, \sigma^{2}\right)}\left[\left(a_{L}(\theta)-a_{L}\left(b_{m}\right)\right)^{2} \mid \theta \geq \theta_{c}\right] \\
& \leq 1.3 \bar{r}^{2} \bar{x}_{m}^{2}+e^{-\sigma} \mathbb{E}_{N\left(0, \sigma^{2}\right)}\left[\left(3 r_{L}\left(\theta-c_{m}\right)+\bar{r} \bar{x}_{m}\right)^{2} \mid \theta \geq \theta_{c}\right] \\
& \leq 1.3 \bar{r}^{2} \bar{x}_{m}^{2}+9 \bar{r}^{2} e^{-\sigma}\left(\left(\sigma \bar{x}_{m+1}+1\right)^{2}+\sigma^{2}\right) \\
& \leq 0.75 \bar{r}^{2}\left(\frac{\bar{x}_{m}+\bar{x}_{m+1}}{2}\right)^{2}, \tag{64}
\end{align*}
$$

on noting $\sigma \geq 300$.

For $s<c_{m-1}$, we use the fact that $\operatorname{Var}\left[\theta \mid s, b_{m} \leq \theta \leq \theta_{c}\right] \leq\left(\frac{\theta_{c}-b_{m}}{2}\right)^{2}$ to get

$$
\begin{equation*}
\operatorname{Var}\left[a_{L}(\theta) \mid s, b_{m} \leq \theta \leq \theta_{c}\right] \leq \bar{r}^{2}\left(\frac{\theta_{c}-b_{m}}{2}\right)^{2}<0.3 \tag{65}
\end{equation*}
$$

As for the effect of $\theta>\theta_{c}$, we can easily see that for $s<c_{m}$ (62) becomes

$$
\begin{equation*}
\operatorname{Prob}\left[\theta \mid s, \theta \in B_{m}\right] \leq \frac{e^{-\sigma-\frac{\bar{x}_{m+1}^{2}}{2}} \phi\left(\frac{\theta}{\sigma}\right)}{\sigma\left(1-\Phi\left(\frac{\theta_{c}}{\sigma}\right)\right)} \tag{66}
\end{equation*}
$$

We can use this to bound the variance similar to (64):

$$
\operatorname{Var}\left[a_{L}(\theta) \mid s, \theta \in B_{m}\right] \leq \begin{cases}0.75 \bar{r}^{2}\left(\frac{\bar{x}_{m}+\bar{x}_{m+1}}{2}\right)^{2}, & c_{m-1} \leq s \leq c_{m}  \tag{67}\\ \frac{1}{3}, & s<c_{m-1}\end{cases}
$$

For the case where $s>c_{m}+\bar{x}_{m+1}$ (i.e., $\delta>\bar{x}_{m+1}$ ), let $\theta_{s}=c_{m}+\sigma \delta$. Then, similar to (62) we can obtain

$$
\begin{equation*}
\operatorname{Prob}\left[\theta \mid s, \theta \in B_{m}\right] \leq \frac{e^{-\sigma} \phi\left(\frac{\theta}{\sigma}\right)}{\sigma\left(1-\Phi\left(\frac{\theta_{s}}{\sigma}\right)\right)} \tag{68}
\end{equation*}
$$

for $\theta \geq \theta_{s}$. Using this and similar to (63), we can reach at

$$
\begin{equation*}
\mathbb{E}\left[a_{L}(\theta) \mid s, \theta \in B_{m}\right]<c_{m}+3 r_{L} \sigma(\delta+1) \tag{69}
\end{equation*}
$$

To bound the variance, similar to (64) we can show

$$
\begin{equation*}
\operatorname{Var}\left[a_{L}(\theta) \mid s, \theta \in B_{m}\right]<2.5 r_{L}^{2} \sigma^{2} \delta^{2} \tag{70}
\end{equation*}
$$

which completes the proof.
Proof of Lemma 7. First, we use the properties of the base configuration listed in Lemma 1 to show that

$$
\begin{equation*}
\frac{\operatorname{Prob}\left[\theta \in B_{k-j}\right]}{\operatorname{Prob}\left[\theta \in B_{k}\right]} \leq e^{2 j+1} \tag{71}
\end{equation*}
$$

 Then, it is straightforward to verify that

$$
\begin{equation*}
\frac{\operatorname{Prob}\left[\theta \in B_{k-j}\right]}{\operatorname{Prob}\left[\theta \in B_{k}\right]} \leq \frac{\operatorname{Prob}\left[\theta \in \tilde{B}_{k-j}\right]}{\operatorname{Prob}\left[\theta \in \tilde{B}_{k}\right]} \tag{72}
\end{equation*}
$$

Similar to part iii) of Lemma 1, we can write

$$
\begin{equation*}
\frac{\operatorname{Prob}\left[\theta \in \tilde{B}_{k-j}\right]}{\operatorname{Prob}\left[\theta \in \tilde{B}_{k}\right]} \leq \frac{\phi\left(\frac{\tilde{b}_{k-j+1}}{\sigma}\right)}{\phi\left(\frac{\tilde{\tilde{b}}_{k+1}+\delta}{\sigma}\right)} \times \frac{\tilde{b}_{k+1}-\tilde{b}_{k}+\delta}{\tilde{b}_{k+1}-\tilde{b}_{k}} \tag{73}
\end{equation*}
$$

where $\delta=\max \left\{0,\left(\tilde{b}_{k-j+1}-\tilde{b}_{k-j}\right)-\left(\tilde{b}_{k+1}-\tilde{b}_{k}\right)\right\} \leq 6$. As a result,

$$
\begin{align*}
\frac{\operatorname{Prob}\left[\theta \in B_{k-j}\right]}{\operatorname{Prob}\left[\theta \in B_{k}\right]} & \leq \frac{\phi\left(\frac{c_{k-j}^{\mathrm{Q}}+c_{k-j+1}^{Q}}{2 \sigma}\right)}{\phi\left(\frac{c_{k}^{\mathrm{Q}}+c_{k+1}^{Q}+12}{2 \sigma}\right)} \times \frac{x_{k+1}^{Q}+x_{k}^{Q}-3+\delta}{x_{k+1}^{Q}+x_{k}^{Q}-3} \\
& \leq 1.3 e^{\frac{1}{\sigma^{2}}\left(c_{m}^{Q}+3\right)\left(c_{m}^{\mathrm{Q}}-c_{m-j}^{Q}+6\right)} \tag{74}
\end{align*}
$$

where we have used the fact that $x_{1}^{Q}>11.5$. Based on the above inequality, together with $c_{m}^{\mathrm{Q}}-c_{m-j}^{\mathrm{Q}} \leq 2 j x_{m}^{Q}$, and $\frac{c_{m}^{\mathrm{Q}} x_{m}^{\mathrm{Q}}}{\sigma^{2}} \leq 1$, and that $\frac{c_{m}^{\mathrm{Q}}}{\sigma} \leq 2 \sqrt{2 \ln \sigma+1.5}+1.5$ (which follows from Lemma 1 and that $m<\sigma$ as stated in Remark 1), and a bit of manipulation we can show that $\frac{\operatorname{Prob}\left[\theta \in B_{k-j}\right]}{\operatorname{Prob}\left[\theta \in B_{k}\right]} \leq e^{2 j+1}$.

Now, to prove the lemma for $k<m$, we write

$$
\begin{align*}
& \frac{\operatorname{Prob}\left[\theta \in B_{k-j} \mid s\right]}{\operatorname{Prob}\left[\theta \in B_{k} \mid s\right]} \leq \frac{\operatorname{Prob}\left[\theta \in B_{k-j}\right] \phi\left(\delta+\left(c_{k}-c_{k-j}\right)-\bar{r} \bar{x}_{m}\right)}{\operatorname{Prob}\left[\theta \in B_{k}\right] \phi\left(\delta+\bar{r} \bar{x}_{m}\right)} \\
& \leq \frac{\operatorname{Prob}\left[\theta \in B_{k-j}\right]}{\operatorname{Prob}\left[\theta \in B_{k}\right]} e^{-\frac{\left(c_{k}-c_{k-j}\right)^{2}}{2}+\bar{r} \bar{x}_{m}\left(c_{k}-c_{k-j}\right)-\delta\left(c_{k}-c_{k-j}-2 \bar{r} \bar{x}_{m}\right)} \\
& \leq e^{-\frac{\left(c_{k}-c_{k-j}\right)^{2}}{2}+2 j+1+2 j \bar{r} \bar{x}_{m}^{2}} \leq e^{-\frac{\left(c_{k}-c_{k-j}\right)^{2}}{2}+3 j+1}, \tag{75}
\end{align*}
$$

using $\bar{r} \bar{x}_{m}^{2}<\frac{1}{2}$ (which follows from $x_{m}^{Q}<0.262 \sigma$ for $m \geq 25$ ). The case $k=m$ needs separate treatment. Define $\hat{B}_{m}=\left[b_{m}, c_{m}+\bar{x}_{m}\right]$. Then, it is easy to verify that (71) still holds if we replace $B_{m}$ with $\hat{B}_{m}$. Therefore, the proof in this case follows from an argument similar to above on noting that $\operatorname{Prob}\left[\theta \in \hat{B}_{m} \mid s\right] \leq \operatorname{Prob}\left[\theta \in B_{m} \mid s\right]$.

Proof of Lemma 8. If $s \geq c_{k}+\bar{r} \bar{x}_{k+2}$, then

$$
\begin{align*}
& \frac{\operatorname{Prob}\left[\theta \in B_{k} \mid s\right]}{\operatorname{Prob}\left[\theta \in B_{k+1} \mid s\right]} \leq \frac{\operatorname{Prob}\left[\theta \in B_{k}\right] \phi\left(m_{k+1}+\delta-c_{k}-\bar{r} \bar{x}_{k+2}\right)}{\operatorname{Prob}\left[\theta \in B_{k+1}\right] \phi\left(m_{k+1}+\delta-c_{k+1}-\bar{r} \bar{x}_{k+2}\right)} \\
& \leq \frac{\operatorname{Prob}\left[\theta \in B_{k}\right]}{\operatorname{Prob}\left[\theta \in B_{k+1}\right]} e^{\Delta_{k+1}\left(\frac{c_{k}+c_{k+1}}{2}+\bar{r} \bar{x}_{k+2}-\delta\right)} \leq e^{\Delta_{k+1}\left(\bar{r} \bar{x}_{k+2}-\delta\right)} \tag{76}
\end{align*}
$$

where the last inequality follows from the definition of $m_{k+1}$. However, for the case where $s<c_{k}+\bar{r} \bar{x}_{k+2}$, the upper bound on the likelihood $\operatorname{Prob}\left[s \mid \theta \in B_{k}\right]$ in the first inequality may be
less and hence is replaced by 1 , which will thereby lead to

$$
\begin{equation*}
\frac{\operatorname{Prob}\left[\theta \in B_{k} \mid s\right]}{\operatorname{Prob}\left[\theta \in B_{k+1} \mid s\right]} \leq e^{\Delta_{k+1}\left(\bar{r} \bar{x}_{k+2}-\delta\right)+\frac{\bar{r}^{2} \bar{x}_{k+2}^{2}}{2}} \tag{77}
\end{equation*}
$$

The other side of the inequality can be proved similarly.
For the case $k=m-1$, the lower bound $\phi\left(m_{k+1}+\delta-c_{k+1}-\bar{r} \bar{x}_{k+2}\right)$ for the likelihood $\operatorname{Prob}\left[s \mid \theta \in B_{k+1}\right]$ is not valid anymore. To fix this, as in Lemma 7, we use $\hat{B}_{m}=\left[b_{m}, c_{m}+\bar{x}_{m}\right]$ instead of $B_{m}$ to obtain,

$$
\begin{align*}
\frac{\operatorname{Prob}\left[\theta \in B_{m-1} \mid s\right]}{\operatorname{Prob}\left[\theta \in B_{m} \mid s\right]} & \leq \frac{\operatorname{Prob}\left[\theta \in B_{m-1} \mid s\right]}{\operatorname{Prob}\left[\theta \in \hat{B}_{m} \mid s\right]} \\
& \leq \frac{\operatorname{Prob}\left[\theta \in B_{m}\right]}{\operatorname{Prob}\left[\theta \in \hat{B}_{m}\right]} e^{\Delta_{m}\left(\bar{r} \bar{x}_{m}-\delta\right)+\frac{\bar{r}^{2} \bar{x}_{m}^{2}}{2}} \tag{78}
\end{align*}
$$

On the other hand, we can show that $\frac{\operatorname{Prob}\left[\theta \in B_{m}\right]}{\operatorname{Prob}\left[\theta \in \hat{B}_{m}\right]}<1.16$, which completes the proof. It is easy to see that the inequality in LHS stays as before for $k=m-1$.

Proof of Lemma 9. As the first step we bound the effect of intervals other than $B_{k} \cup B_{k+1}$. Let $\eta=\mathbb{E}\left[a_{L}(\theta) \mid s, \theta \in B_{k} \cup B_{k+1}\right]$ and $\eta_{j}=\mathbb{E}\left[a_{L}(\theta) \mid s, \theta \in B_{j}\right]$ for $-m \leq j \leq m$. Using Lemma 7, we can write

$$
\begin{align*}
& \sum_{j=1}^{k+m} \frac{\operatorname{Prob}\left[\theta \in B_{k-j} \mid s\right]}{\operatorname{Prob}\left[\theta \in B_{k} \mid s\right]}\left(\eta-\eta_{k-j}\right) \\
& \leq \sum_{j=1}^{k+m}\left(c_{k}-c_{k-j}+2 \bar{x}_{m}+2 \bar{r} \bar{x}_{m}\right) e^{-\frac{\left(c_{k}-c_{k-j}\right)^{2}}{2}+3 j+1} \\
& \leq \sum_{j=1}^{k+m}\left(2 j \underline{x}_{1}+2 \bar{x}_{m}+2 \bar{r} \bar{x}_{m}\right) e^{-2 j^{2} \underline{x}_{1}^{2}+3 j+1} \\
& \leq 4 e^{-2 \underline{x}_{1}^{2}+4} \bar{x}_{m} \sum_{j=1}^{k+m} j e^{-2\left(j^{2}-1\right) \underline{x}_{1}^{2}+3(j-1)} \\
& \leq 4 e^{-2 \underline{x}_{1}^{2}+4} \bar{x}_{m} \sum_{j=1}^{\infty} e^{-(j-1)^{2}} \leq \frac{5.6 e^{4}}{\sigma^{15}}<10^{-10} \bar{r}^{2} \underline{x}_{1}^{2} \tag{79}
\end{align*}
$$

where we have used the identity $\sum_{j=0}^{\infty} e^{-j^{2}} \approx 1.386$. Similarly, we can bound the effect of
non-neighboring intervals on the variance:

$$
\begin{equation*}
\sum_{j=1}^{k+m} \frac{\operatorname{Prob}\left[\theta \in B_{k-j} \mid s\right]}{\operatorname{Prob}\left[\theta \in B_{k} \mid s\right]}\left(\eta-\eta_{k-j}\right)^{2} \leq \frac{5.6 e^{4}}{\sigma^{14}} . \tag{80}
\end{equation*}
$$

On the other hand,

$$
\begin{align*}
& \sum_{j=1}^{k+m} \frac{\operatorname{Prob}\left[\theta \in B_{k-j} \mid s\right]}{\operatorname{Prob}\left[\theta \in B_{k} \mid s\right]} \operatorname{Var}\left[a_{L}(\theta) \mid s, \theta \in B_{k-j}\right] \\
& \leq \sum_{j=1}^{k+m} \frac{1}{3} e^{-\frac{\left(c_{k}-c_{k-j}\right)^{2}}{2}+3 j+1} \leq \frac{0.5 e^{4}}{\sigma^{16}} \tag{81}
\end{align*}
$$

Combining the two, we obtain

$$
\begin{equation*}
\sum_{\substack{j=-m \\ j \notin k, k+1\}}}^{m} \operatorname{Prob}\left[\theta \in B_{j} \mid s\right]\left(\left(\eta-\eta_{j}\right)^{2}+\operatorname{Var}\left[a_{L}(\theta) \mid s, \theta \in B_{j}\right]\right) \leq 10^{-10} \bar{r}^{2} \underline{x}_{1}^{2} \tag{82}
\end{equation*}
$$

Therefore, the effect of intervals other than $B_{k}$ and $B_{k+1}$ on $a_{F}(s)$ (and its derivative given by $\operatorname{Var}\left[a_{L} \mid s\right]$ ) is quite negligible. Now, focusing on these two intervals (i.e., $B_{k}$ and $B_{k+1}$ ), we have

$$
\begin{align*}
& \mathbb{E}\left[a_{L}(\theta) \mid s, \theta \in B_{k} \cup B_{k+1}\right] \\
& =p \mathbb{E}\left[a_{L}(\theta) \mid s, \theta \in B_{k}\right]+(1-p) \mathbb{E}\left[a_{L}(\theta) \mid s, \theta \in B_{k+1}\right] \\
& \leq p\left(c_{k}+\bar{r} \bar{x}_{k+1}\right)+(1-p)\left(c_{k}+\Delta_{k+1}+\bar{r} \bar{x}_{k+2}\right) \\
& \leq c_{k}+(1-p) \Delta_{k+1}+\bar{r} \bar{x}_{k+2}, \tag{83}
\end{align*}
$$

where $p=\frac{\operatorname{Prob}\left[\theta \in B_{k} \mid s\right]}{\operatorname{Prob}\left[\theta \in B_{k} \cup B_{k+1} \mid s\right]}$. The proof for the upper bound on $a_{F}(s)$ now follows from Lemma 8. The proof for the lower bound on $a_{F}(s)$ is similar. Now, as for the derivative, we first note that $\frac{d}{d s} a_{F}(s)=\operatorname{Var}\left[a_{L} \mid s\right]$. Again, focusing on $B_{k} \cup B_{k+1}$, we can write

$$
\begin{align*}
& \operatorname{Var}\left[a_{L} \mid s, \theta \in B_{k} \cup B_{k+1}\right] \\
& \leq p \operatorname{Var}\left[a_{L} \mid s, \theta \in B_{k}\right]+(1-p) \operatorname{Var}\left[a_{L} \mid s, \theta \in B_{k+1}\right] \\
& +p(1-p)\left(\mathbb{E}\left[a_{L} \mid s, \theta \in B_{k}\right]-\mathbb{E}\left[a_{L} \mid s, \theta \in B_{k+1}\right]\right)^{2} \tag{84}
\end{align*}
$$

The rest easily follows from Lemma 5 and Lemma 8.
Proof of Lemma 10. Exploiting the term $e^{-\delta\left(c_{m}-c_{m-r}-2 \bar{r} \sigma\right)}$ in (75) (for $k=m$ ), it is easy to
observe that the same upper bounds given by (79) and (82) hold for the effect of intervals other than $B_{m}$ on $a_{F}(s)$ provided

$$
\begin{equation*}
e^{-\delta\left(c_{m}-c_{m-r}-2 \bar{r} \bar{x}_{m}\right)}\left(\mathbb{E}\left[a_{L}(\theta) \mid s, \theta \in B_{m}\right]-c_{m}\right)<2 \bar{x}_{m} \tag{85}
\end{equation*}
$$

Verifying the above inequality is quite straightforward using Lemma 6, and specially noting that $\mathbb{E}\left[a_{L}(\theta) \mid s, \theta \in B_{m}\right]-c_{m}<3 r_{L} \sigma(\delta+1)$ for $\delta>\bar{x}_{m+1}$. The proof of the lemma is now an immediate consequence of Lemma 6.

Proof of Lemma 11. We start by showing that

$$
\begin{equation*}
\tilde{J}_{L}\left(c_{k}\right)=\int_{-\infty}^{\infty}\left(a_{F}(s)-c_{k}\right)^{2} \phi\left(s-c_{k}\right) d s \leq 1.1 \bar{r}^{2} \bar{x}_{k+2}^{2} . \tag{86}
\end{equation*}
$$

Using the upper bound on $a_{F}(s)$ given in Corollary 1, we can write

$$
\begin{align*}
& \int_{c_{k}}^{m_{k+1}}\left(a_{F}(s)-c_{k}\right)^{2} \phi\left(s-c_{k}\right) d s \\
& \leq \int_{0}^{m_{k+1}-c_{k}}\left(1.01 \bar{r} \bar{x}_{k+2}+1.17 \Delta_{k+1} e^{-\delta \Delta_{k+1}}\right)^{2} \phi\left(m_{k+1}-c_{k}-\delta\right) d \delta \tag{87}
\end{align*}
$$

A useful inequality here is

$$
\begin{equation*}
\int_{0}^{\Lambda} e^{-\delta \Delta} \phi(\Lambda-\delta) d \delta \leq \phi(\Lambda) \int_{0}^{\Lambda} e^{-\delta(\Delta-\Lambda)} d \delta \leq \frac{\phi(\Lambda)}{\Delta-\Lambda} \tag{88}
\end{equation*}
$$

Another useful property is that

$$
\begin{equation*}
\frac{\Delta_{k+1}}{2}+\frac{2.3}{\Delta_{k+1}} \geq m_{k+1}-c_{k} \geq \frac{\Delta_{k+1}}{2}-\frac{1.1}{\Delta_{k+1}} \geq \underline{x}_{k} \tag{89}
\end{equation*}
$$

The proof of the RHS is similar to part iii) of Lemma 1 :

$$
\begin{align*}
& \frac{\operatorname{Prob}\left[\theta \mid \theta \in B_{k+1}\right]}{\operatorname{Prob}\left[\theta \mid \theta \in B_{k}\right]} \leq \frac{b_{k+2}-b_{k+1}}{b_{k+1}-b_{k}} \leq \frac{0.5\left(c_{k+2}-c_{k}\right)+0.2 r_{L}}{0.5\left(c_{k+1}-c_{k-1}\right)-0.2 r_{L}} \\
& \leq \frac{x_{k+2}^{\mathrm{Q}}+x_{k+1}^{\mathrm{Q}}+3}{x_{k+1}^{\mathrm{Q}}+x_{k}^{\mathrm{Q}}-3} \leq \frac{e(1+e) x_{k}^{\mathrm{Q}}+3}{(1+e) x_{k}^{\mathrm{Q}}-3} . \tag{90}
\end{align*}
$$

As a result,

$$
\begin{equation*}
m_{k+1}-c_{k} \geq \frac{\Delta_{k+1}}{2}-\ln \left(\frac{e(1+e) x_{k}^{\mathrm{Q}}+3}{(1+e) x_{k}^{\mathrm{Q}}-3}\right) \frac{1}{\Delta_{k+1}} \geq \frac{\Delta_{k+1}}{2}-\frac{1.1}{\Delta_{k+1}} \geq \underline{x}_{k} \tag{91}
\end{equation*}
$$

on noting that $x_{k}^{\mathrm{Q}} \geq 11.5$ and $\Delta_{k+1} \geq 17$. This also implies that $\Delta_{k+1} \leq 2 \Delta_{k+1}^{m}+0.14$, where
we define $\Delta_{k+1}^{m}=m_{k+1}-c_{k}$. The LHS of (89) follows from an analysis similar to (74) for the adjacent intervals of $B_{k}$ and $B_{k+1}$ :

$$
\begin{align*}
\frac{\operatorname{Prob}\left[\theta \in B_{k}\right]}{\operatorname{Prob}\left[\theta \in B_{k+1}\right]} & \leq 1.3 \times \frac{\phi\left(\frac{c_{k-1}^{\mathrm{Q}}+c_{k}^{\mathrm{Q}}+6}{2 \sigma}\right)}{\phi\left(\frac{c_{k}^{\mathrm{Q}}+c_{k+1}^{\mathrm{Q}}}{2 \sigma}\right)} \\
& \leq 1.3 e^{\frac{1}{2 \sigma^{2}}\left(2 x_{m}^{\mathrm{Q}}-3\right)\left(2 c_{m}^{\mathrm{Q}}+3\right)} \leq 1.3 e^{2} \tag{92}
\end{align*}
$$

from which the proof follows from the definition of $m_{k+1}$.
Incorporating (88) and (89) into (87) and after some manipulation, we can find

$$
\begin{align*}
& \int_{c_{k}}^{m_{k+1}}\left(a_{F}(s)-c_{k}\right)^{2} \phi\left(s-c_{k}\right) d s \leq \frac{1.01^{2} \bar{r}^{2} \bar{x}_{k+2}^{2}}{2} \\
& +\left(\frac{1.37\left(2 \Delta_{k+1}^{m}+0.14\right)^{2}}{2\left(2 \Delta_{k+1}^{m}+0.14\right)-\Delta_{k+1}^{m}}+\frac{2.37 \bar{r} \bar{x}_{k+2}\left(2 \Delta_{k+1}^{m}+0.14\right)}{\left(2 \Delta_{k+1}^{m}+0.14\right)-\Delta_{k+1}^{m}}\right) \phi\left(\Delta_{k+1}^{m}\right) \\
& \leq \frac{1.01^{2} \bar{r}^{2} \bar{x}_{k+2}^{2}}{2}+\left(\frac{1.37\left(2 \Delta_{k+1}^{m}+0.14\right)}{1.5}+\frac{2.37 \bar{r} \bar{x}_{k+2}}{0.5}\right) \phi\left(\Delta_{k+1}^{m}\right) \\
& \leq \frac{1.01^{2} \bar{r}^{2} \bar{x}_{k+2}^{2}}{2}+\left(\frac{1.37\left(2 \underline{x}_{1}+0.14\right)}{1.5}+4.74 \bar{r} \bar{x}_{k+2}\right) \phi\left(\underline{x}_{1}\right) \\
& \leq \frac{1.01^{2} \bar{r}^{2} \bar{x}_{k+2}^{2}}{2}+\left(1.83 \underline{x}_{1}+0.1+4.74 \bar{r} \bar{x}_{k+2}\right) \phi\left(\underline{x}_{1}\right) \\
& \leq \frac{1.01^{2} \bar{r}^{2} \bar{x}_{k+2}^{2}}{2}+\frac{2 \underline{x}_{1} e^{-2\left(\underline{x}_{1}-2\right)}}{\sqrt{2 \pi e^{4}} \sigma^{4}} \tag{93}
\end{align*}
$$

where we have also used $\underline{x}_{k} \geq \underline{x}_{1} \geq 2 \sqrt{2 \ln \sigma}+2$.
For $s \in\left[m_{k+1}, c_{k+1}\right]$, we have $a_{F}(s) \leq c_{k+1}+1.1 \bar{r} \bar{x}_{k+2}$ according to Corollary 2. Therefore,

$$
\begin{align*}
& \int_{m_{k+1}}^{c_{k+1}}\left(a_{F}(s)-c_{k}\right)^{2} \phi\left(s-c_{k}\right) d s \leq\left(\Delta_{k+1}+1.1 \bar{r} \bar{x}_{k+2}\right)^{2} \Phi\left(c_{k}-m_{k+1}\right) \\
& \leq\left(2 \Delta_{k+1}^{m}+0.14+1.1 \bar{r} \bar{x}_{k+2}\right)^{2} \frac{\phi\left(\Delta_{k+1}^{m}\right)}{\Delta_{k+1}^{m}} \\
& \leq\left(2 \underline{x}_{1}+0.14+1.1 \bar{r} \bar{x}_{k+2}\right)^{2} \frac{\phi\left(\underline{x}_{1}\right)}{\underline{x}_{1}} \leq \frac{5 \underline{x}_{1} e^{-2\left(x_{1}-2\right)}}{\sqrt{2 \pi e^{4}} \sigma^{4}} \tag{94}
\end{align*}
$$

Using $a_{F}(s) \leq s+1.1 \bar{x}_{m}$, we can show

$$
\begin{align*}
& \int_{c_{k+1}}^{\infty}\left(a_{F}(s)-c_{k}\right)^{2} \phi\left(s-c_{k}\right) d s \leq \frac{\sqrt{2 \pi}}{2} \phi\left(2 \underline{x}_{1}\right)\left(1.1 \bar{x}_{m}+2 \underline{x}_{1}+1\right)^{2} \\
& \leq \frac{\sqrt{2 \pi}}{2} \phi(4 \sqrt{2 \ln \sigma})\left(1.1 \bar{x}_{m}+\underline{x}_{1}+1\right)^{2} \leq \frac{\left(1.1 \bar{x}_{m}+\underline{x}_{1}+1\right)^{2}}{2 \sigma^{16}}<10^{-4} \bar{r}^{2} \bar{x}_{k+2}^{2} \tag{95}
\end{align*}
$$

Combining (93), (94), and (95), we can arrive at

$$
\begin{align*}
\int_{-\infty}^{\infty}\left(a_{F}(s)-c_{k}\right)^{2} \phi\left(s-c_{k}\right) d s & \leq 1.01^{2} \bar{r}^{2} \bar{x}_{k+2}^{2}+2 \times 10^{-4} \bar{r}^{2} \bar{x}_{k+2}^{2}+\frac{14 \underline{x}_{1} e^{-2\left(\underline{x}_{1}-2\right)}}{\sqrt{2 \pi e^{4}} \sigma^{4}} \\
& \leq 1.1 \bar{r}^{2} \bar{x}_{k+2}^{2} \tag{96}
\end{align*}
$$

For the case of $k=m, \int_{c_{m}}^{\infty}\left(a_{F}(s)-c_{m}\right)^{2} \phi\left(s-c_{m}\right) d s$ needs a different treatment. First we note that

$$
\begin{align*}
& \int_{c_{m}+\bar{x}_{m+1}}^{\infty}\left(a_{F}(s)-c_{m}\right)^{2} \phi\left(s-c_{m}\right) d s \leq \int_{\bar{x}_{m+1}}^{\infty} 9 r_{L}^{2} \sigma^{2}(\delta+1)^{2} \phi(\delta) d \delta \\
& \leq 9 r_{L}^{2} \sigma^{2} \phi\left(\bar{x}_{m+1}\right) \frac{\left(\bar{x}_{m+1}+2\right)^{2}}{2}<10^{-4} \bar{r}^{2} \bar{x}_{m+1}^{2} . \tag{97}
\end{align*}
$$

Also,

$$
\begin{equation*}
\int_{c_{m}}^{c_{m}+\bar{x}_{m+1}}\left(a_{F}(s)-c_{m}\right)^{2} \phi\left(s-c_{m}\right) d s<0.5 \bar{r}^{2} \bar{x}_{m+1}^{2} \tag{98}
\end{equation*}
$$

using which it is straightforward to verify that (86) holds for $k=m$ as well (define $\bar{x}_{m+2}=\bar{x}_{m+1}$ for consistency).

Let $\theta=c_{k}+\epsilon$, with $0 \leq \epsilon \leq \frac{\Delta_{k+1}}{2}$. We first show that $\tilde{a}_{L}(\theta)$ lies in a $5 \bar{r} \bar{x}_{k+2}$-vicinity of either $c_{k}$ or $c_{k+1}$. We begin with the case where $\tilde{a}_{L}(\theta) \in\left[c_{k}, c_{k+1}\right]$. Let $\tilde{a}_{L}(\theta)=c_{k}+\epsilon^{\prime}$, with $5 \bar{r} \bar{x}_{k+2} \leq \epsilon^{\prime} \leq \Delta_{k+1}-5 \bar{r} \bar{x}_{k+2}$. We can use Corollary 2 to obtain a lower bound for $\tilde{J}_{L}\left(\tilde{a}_{L}\right)$ :

$$
\begin{equation*}
\tilde{J}\left(\tilde{a}_{L}\right) \geq\left(\epsilon^{\prime}-1.1 \bar{r} \bar{x}_{k+2}\right)^{2}\left(1-\Phi\left(\frac{4 \sqrt{2 \ln \sigma}}{5}\right)\right) \geq \frac{5\left(\epsilon^{\prime}-1.1 \bar{r} \bar{x}_{k+2}\right)^{2}}{16 \sqrt{\pi \ln \sigma} \sigma^{\frac{16}{25}}} \tag{99}
\end{equation*}
$$

where the last inequality follows from the property that $1-\Phi(x) \geq \frac{x \phi(x)}{1+x^{2}}$. Putting this together with $\tilde{u}_{L}\left(\theta, \tilde{a}_{L}\right) \geq \tilde{u}_{L}\left(\theta, c_{k}\right) \geq-r_{L}\left(\frac{\Delta_{k+1}}{2}\right)^{2}-1.1\left(1-r_{L}\right) \bar{r}^{2} \bar{x}_{k+2}^{2}$, it is easy to show that $\epsilon^{\prime}-$ 1.1 $\bar{r} \bar{x}_{k+2}<\frac{\Delta_{k+1}}{\sqrt{\sigma}}$ for $\sigma \geq 300$. A second use of Corollary 2 now yields $\tilde{J}_{L}\left(\tilde{a}_{L}\right) \geq\left(\epsilon^{\prime}-\right.$ $\left.1.1 \bar{r} \bar{x}_{k+2}\right)^{2} \Phi\left(\frac{\Delta_{k+1}}{4}\right) \geq 0.99\left(\epsilon^{\prime}-1.1 \bar{r} \bar{x}_{k+2}\right)^{2}$ noting $\Delta_{k+1} \geq 2 \underline{x}_{1}>17$. Therefore,

$$
\begin{equation*}
\tilde{u}_{L}\left(\theta, \tilde{a}_{L}\right) \leq-r_{L}\left(\epsilon^{\prime}-\epsilon\right)^{2}-0.99\left(\epsilon^{\prime}-1.1 \bar{r} \bar{x}_{k+2}\right)^{2} . \tag{100}
\end{equation*}
$$

On the other hand, using (86), we get $\tilde{u}_{L}\left(\theta, c_{k}\right) \geq-r_{L} \epsilon^{2}-0.99\left(1-r_{L}\right)\left(1.1 \bar{r} \bar{x}_{k+2}\right)^{2}$. The RHS in (100) is maximized for $\epsilon^{*}=\frac{r_{L} \epsilon+0.99\left(1-r_{L}\right) \times 1.1 \bar{r} \bar{x}_{k+2}}{r_{L}+0.99\left(1-r_{L}\right)}$. Having $\tilde{u}_{L}\left(\theta, \tilde{a}_{L}\right) \geq \tilde{u}_{L}\left(\theta, c_{k}\right)$ requires that $\left|\epsilon^{\prime}-\epsilon^{*}\right|<\left|0-\epsilon^{*}\right|$, that is, $\epsilon^{\prime} \leq 2 \epsilon^{*}$ (recall the assumption by contradiction that $5 \bar{r} \bar{x}_{k+2} \leq \epsilon^{\prime}$ ). This then requires $\epsilon^{\prime}<5 \bar{r} \bar{x}_{k+2}$.

For the case $\tilde{a}_{L}(\theta) \notin\left[c_{k}, c_{k+1}\right]$, suppose $\tilde{a}_{L}(\theta)<c_{k}$ (the other case is similar), and let $\tilde{a}_{L}(\theta)=$ $c_{k}-\epsilon^{\prime}$. Then, it follows from $\tilde{u}_{L}\left(\theta, \tilde{a}_{L}\right) \geq \tilde{u}_{L}\left(\theta, c_{k}\right)$ that $-r_{L} \epsilon^{2}-1.1\left(1-r_{L}\right) \bar{r}^{2} \bar{x}_{k+2}^{2} \leq-r_{L}\left(\epsilon+\epsilon^{\prime}\right)^{2}$, from which it easily follows that $\epsilon^{\prime}<1.1$. Now an argument similar to the case $\tilde{a}_{L}(\theta) \in\left[c_{k}, c_{k+1}\right]$ shows that $\tilde{J}_{L}\left(\tilde{a}_{L}\right) \geq 0.99\left(\epsilon^{\prime}-1.1 \bar{r} \bar{x}_{k+2}\right)^{2}$. Combining this with $\left|\theta-c_{k}\right|<\left|\theta-\tilde{a}_{L}\right|$ and $\tilde{u}_{L}\left(\theta, \tilde{a}_{L}\right) \geq$ $\tilde{u}_{L}\left(\theta, c_{k}\right)$, we get $1.1 \bar{r}^{2} \bar{x}_{k+2}^{2}>0.99\left(\epsilon^{\prime}-1.1 \bar{r} \bar{x}_{k+2}\right)^{2}$ resulting in $\epsilon^{\prime}<2.5 \bar{r} \bar{x}_{k+2}$.

Similar to Lemma 7 in [1], we can show that $\tilde{a}_{L}(\theta)$ is increasing. The fact that $\tilde{a}_{L}(\theta)$ is increasing implies that it cannot swing between the two neighborhoods. Therefore, there exists a unique $\tilde{b}_{k+1}$ separating the two regimes of $\left[c_{k}-5 \bar{r} \bar{x}_{k+2}, c_{k}+5 \bar{r} \bar{x}_{k+2}\right]$ and $\left[c_{k+1}-5 \bar{r} \bar{x}_{k+2}, c_{k+1}+\right.$ $\left.5 \bar{r} \bar{x}_{k+2}\right]$, thus completing the proof.

Proof of Lemma 12. We start by

$$
\begin{align*}
& \frac{d}{d a_{L}} \int_{-\infty}^{\infty}\left(a_{F}(s)-a_{L}\right)^{2} \phi\left(s-a_{L}\right) d s \\
& =\frac{d}{d a_{L}} \int_{-\infty}^{\infty}\left(a_{F}\left(s+a_{L}\right)-a_{L}\right)^{2} \phi(s) d s \\
& =2 \int_{-\infty}^{\infty}\left(a_{L}-a_{F}\left(s+a_{L}\right)\right)\left(1-\frac{d}{d s} a_{F}\left(s+a_{L}\right)\right) \phi(s) d s \\
& =2 \int_{-\infty}^{\infty}\left(a_{L}-a_{F}(s)\right)\left(1-\frac{d}{d s} a_{F}(s)\right) \phi\left(s-a_{L}\right) d s \tag{101}
\end{align*}
$$

Changing the order of differentiation and integration requires the map $a_{L} \mapsto\left(a_{F}\left(s+a_{L}\right)-a_{L}\right)^{2}$ to be (i) continuously differentiable, and (ii) its partial derivative be bounded by an integrable function in an open interval around $a_{L} .{ }^{14}$ These are easy to verify, noting that the follower's strategy $a_{F}(s)$ (i) is analytic, from which the continuity of partial derivatives is immediate, and (ii) $\left|a_{F}(s)\right|<|s|+\sigma$ and $\left|\frac{d}{d s} a_{F}(s)\right|<s^{2}+c_{m}^{2}$, from which integrability of the partial derivatives follows from the finiteness of the moments of normal distribution. We can similarly, verify the following identity derived using the integration by part for any given $a_{L} \in \mathbb{R}$ :

$$
\begin{align*}
& \int_{-\infty}^{\infty}\left(a_{L}-a_{F}(s)\right)\left(s-a_{L}\right) \phi\left(s-a_{L}\right) d s=-\int_{-\infty}^{\infty}\left(a_{L}-a_{F}(s)\right) d \phi\left(s-a_{L}\right) \\
& =\int_{-\infty}^{\infty} \phi\left(s-a_{L}\right) d\left(a_{L}-a_{F}(s)\right)=-\int_{-\infty}^{\infty} \frac{d}{d s} a_{F}(s) \phi\left(s-a_{L}\right) d s \tag{102}
\end{align*}
$$

[^11]Using (30), the first order condition for the optimal leader's response $\tilde{a}_{L}$ gives

$$
\begin{equation*}
r_{L}\left(\tilde{a}_{L}-\theta\right)+\left(1-r_{L}\right) \int_{-\infty}^{\infty}\left(\tilde{a}_{L}-a_{F}(s)\right)\left(1-\frac{d}{d s} a_{F}(s)\right) \phi\left(s-\tilde{a}_{L}\right) d s=0 . \tag{103}
\end{equation*}
$$

Define the real analytic function $\Theta: \mathbb{R} \rightarrow \mathbb{R}$ as

$$
\begin{equation*}
\Theta(x)=x+\frac{\left(1-r_{L}\right)}{r_{L}} \int_{-\infty}^{\infty}\left(x-a_{F}(s)\right)\left(1-\frac{d}{d s} a_{F}(s)\right) \phi(s-x) d s \tag{104}
\end{equation*}
$$

From (103) it then follows that $\Theta(\cdot)$ is the left inverse of the leader's best response strategy $\tilde{a}_{L}(\cdot)$, that is, $\Theta\left(\tilde{a}_{L}(\theta)\right)=\theta$ for all $\theta \in \mathbb{R}$.

Differentiating $\Theta(x)$ we get

$$
\begin{align*}
& r_{L} \frac{d}{d x} \Theta(x)=r_{L}+\left(1-r_{L}\right) \int_{-\infty}^{\infty}\left(1-\frac{d}{d s} a_{F}(s)\right) \phi(s-x) d s \\
& +\left(1-r_{L}\right) \int_{-\infty}^{\infty}\left(x-a_{F}(s)\right)\left(1-\frac{d}{d s} a_{F}(s)\right)(s-x) \phi(s-x) d s \\
& =r_{L}+\left(1-r_{L}\right)\left(1-\int_{-\infty}^{\infty}\left(2+\left(x-a_{F}(s)\right)(s-x)\right) \frac{d}{d s} a_{F}(s) \phi(s-x) d s\right), \tag{105}
\end{align*}
$$

where the last equality is obtained using (102). Next, we use (105) to bound $\frac{d}{d x} \Theta(x)$ for $x=$ $\tilde{a}_{L} \in\left[c_{k}-5 \bar{r} \bar{x}_{k+2}, c_{k}+5 \bar{r} \bar{x}_{k+2}\right]$.

Using (24) to bound $\frac{d}{d s} a_{F}(s)$ and then applying (88), we can obtain

$$
\begin{align*}
& \int_{\tilde{a}_{L}}^{m_{k+1}}\left(\frac{d}{d s} a_{F}(s)-1.01 \bar{r}^{2} \bar{x}_{k+2}^{2}\right) \phi\left(s-\tilde{a}_{L}\right) d s \\
& \leq 1.17 \Delta_{k+1}^{2} \int_{0}^{m_{k+1}-\tilde{a}_{L}} e^{-\Delta_{k+1} \delta} \phi\left(m_{k+1}-\tilde{a}_{L}-\delta\right) d \delta \\
& \leq \frac{1.17 \Delta_{k+1}^{2} \phi\left(m_{k+1}-\tilde{a}_{L}\right)}{\Delta_{k+1}-m_{k+1}+\tilde{a}_{L}} . \tag{106}
\end{align*}
$$

Another useful inequality similar to (88) is

$$
\begin{equation*}
\int_{0}^{\Lambda} e^{-\delta \Delta} \phi(\Lambda+\delta) d \delta \leq \frac{\phi(\Lambda)}{\Delta+\Lambda} \tag{107}
\end{equation*}
$$

which together with the bound on $\frac{d}{d s} a_{F}(s)$ given in (24) yields

$$
\int_{m_{k+1}}^{c_{k+1}}\left(\frac{d}{d s} a_{F}(s)-1.01 \bar{r}^{2} \bar{x}_{k+2}^{2}\right) \phi\left(s-\tilde{a}_{L}\right) d s
$$

$$
\begin{align*}
& \leq 1.17 \Delta_{k+1}^{2} \int_{0}^{c_{k+1}-m_{k+1}} e^{-\Delta_{k+1} \delta} \phi\left(m_{k+1}-\tilde{a}_{L}+\delta\right) d \delta \\
& \leq \frac{1.17 \Delta_{k+1}^{2} \phi\left(m_{k+1}-\tilde{a}_{L}\right)}{\Delta_{k+1}+m_{k+1}-\tilde{a}_{L}} . \tag{108}
\end{align*}
$$

Let $\Delta_{k+1}^{L}=m_{k+1}-\tilde{a}_{L}$. Then, using (91), we can show that for $\tilde{a}_{L} \in\left[c_{k}-5 \bar{r} \bar{x}_{k+2}, c_{k}+5 \bar{r} \bar{x}_{k+2}\right]$, $\Delta_{k+1}^{L} \geq \underline{x}_{k}$. Also, $\Delta_{k+1} \leq 2 \Delta_{k+1}^{L}+0.14$. This, together with (106) and (108), yields

$$
\begin{align*}
& \int_{\tilde{a}_{L}}^{c_{k+1}}\left(\frac{d}{d s} a_{F}(s)-1.01 \bar{r}^{2} \bar{x}_{k+2}^{2}\right) \phi\left(s-\tilde{a}_{L}\right) d s \\
& \leq \frac{2.34 \Delta_{k+1}^{3} \phi\left(m_{k+1}-\tilde{a}_{L}\right)}{\Delta_{k+1}^{2}-\left(m_{k+1}-\tilde{a}_{L}\right)^{2}} \leq \frac{2.34\left(2 \Delta_{k+1}^{L}+0.14\right)^{3} \phi\left(\Delta_{k+1}^{L}\right)}{\left(2 \Delta_{k+1}^{L}+0.14\right)^{2}-\left(\Delta_{k+1}^{L}\right)^{2}} \\
& \leq \frac{2.34\left(2 \underline{x}_{1}+0.14\right)^{3} \phi\left(\underline{x}_{1}\right)}{\left(2 \underline{x}_{1}+0.14\right)^{2}-\underline{x}_{1}^{2}} \leq\left(6.24 \underline{x}_{1}+1\right) \phi\left(\underline{x}_{1}\right) \leq \frac{\left(6.24 \underline{x}_{1}+1\right) e^{-2\left(\underline{x}_{1}-2\right)}}{\sqrt{2 \pi e^{4}} \sigma^{4}} \tag{109}
\end{align*}
$$

where we have again used $\underline{x}_{1} \geq 2 \sqrt{2 \ln \sigma}+2$. On the other hand,

$$
\begin{align*}
& \int_{c_{k+1}}^{c_{m}+\bar{x}_{m+1}}\left(\frac{d}{d s} a_{F}(s)-1.01 \bar{r}^{2} \bar{x}_{k+2}^{2}\right) \phi\left(s-\tilde{a}_{L}\right) d s \\
& \leq\left(\frac{1}{4}\left(\Delta_{m+1}+2 \bar{r} \bar{x}_{m+1}\right)^{2}+0.01 \bar{r}^{2} \bar{x}_{m+1}^{2}-1.01 \bar{r}^{2} \bar{x}_{k+2}^{2}\right) \frac{\phi\left(c_{k+1}-\tilde{a}_{L}\right)}{c_{k+1}-\tilde{a}_{L}} \\
& \leq \bar{x}_{m+1}^{2}\left((1+\bar{r})^{2}+0.01 \bar{r}^{2}\right) \frac{\phi\left(2 \underline{x}_{1}\right)}{2 \underline{x}_{1}} \\
& \leq \frac{\bar{x}_{m+1}^{2}\left((1+\bar{r})^{2}+0.01 \bar{r}^{2}\right)}{2 \sqrt{2 \pi} \sigma^{16} \underline{x}_{1}} . \tag{110}
\end{align*}
$$

Combining the two, we get

$$
\begin{align*}
& \int_{c_{k}}^{c_{m}+\bar{x}_{m+1}}\left(\frac{d}{d s} a_{F}(s)-1.01 \bar{r}^{2} \bar{x}_{k+2}^{2}\right) \phi\left(s-\tilde{a}_{L}\right) d s \\
& <\frac{\left(6.24 \underline{x}_{1}+1\right) e^{-2\left(\underline{x}_{1}-2\right)}}{\sqrt{2 \pi e^{4}} \sigma^{4}}+\frac{\bar{x}_{m+1}^{2}\left((1+\bar{r})^{2}+0.01 \bar{r}^{2}\right)}{2 \sqrt{2 \pi} \sigma^{16} \underline{x}_{1}}<0.01 \bar{r}^{2} \bar{x}_{k+2}^{2} \tag{111}
\end{align*}
$$

noting that $\sigma \geq 300$.
For $s>c_{m}+\bar{x}_{m+1}$, using Lemma 10 and the fact that $\tilde{a}_{L}(\theta) \leq \tilde{a}_{L}\left(c_{m}\right)<c_{m}+5 r_{L} \sigma$, and similar machinery to the above, we get

$$
\begin{aligned}
& \int_{c_{m}+\bar{x}_{m+1}}^{\infty} \frac{d}{d s} a_{F}(s) \phi\left(s-\tilde{a}_{L}\right) d s \\
& \leq 3 r_{L}^{2} \sigma^{2} \int_{0}^{\infty}\left(\delta+\bar{x}_{m+1}\right)^{2} \phi\left(\delta+\bar{x}_{m+1}-5 r_{L} \bar{x}_{m+1}\right) d \delta
\end{aligned}
$$

$$
\begin{equation*}
\leq \frac{3 \sqrt{2 \pi}}{2} r_{L}\left(\bar{x}_{m+1}+1\right)^{2} \phi\left(\bar{x}_{m+1}-5 r_{L} \bar{x}_{m+1}\right)<10^{-4} \bar{r}^{2} \bar{x}_{k+2}^{2} \tag{112}
\end{equation*}
$$

This, along with (111) yields

$$
\begin{equation*}
\int_{-\infty}^{\infty} \frac{d}{d s} a_{F}(s) \phi\left(s-\tilde{a}_{L}\right) d s<1.05 \bar{r}^{2} \bar{x}_{k+2}^{2} \tag{113}
\end{equation*}
$$

The case $k=m$ is even easier on noting that $\frac{d}{d s} a_{F}(s) \leq 0.8 \bar{r}^{2} \bar{x}_{m+1}^{2}$ over the whole interval $s \in\left[c_{m}, c_{m}+\bar{x}_{m+1}\right]$.

Now, to bound the other term assume $a_{F}\left(\tilde{a}_{L}\right) \leq \tilde{a}_{L}$ (the other case is similar). From (9), we can see

$$
\begin{equation*}
a_{F}\left(m_{k+1}\right) \geq c_{k}+\frac{\Delta_{k+1}}{2.17}-1.01 \bar{r} \bar{x}_{k+2}>c_{k}+5 \bar{r} \bar{x}_{k+2}>\tilde{a}_{L} . \tag{114}
\end{equation*}
$$

Using this, together with $\tilde{a}_{L}-a_{F}\left(\tilde{a}_{L}\right) \leq 5 \bar{r} \bar{x}_{k+2}+1.1 \bar{r} \bar{x}_{k+2}<6.1 \bar{r} \bar{x}_{k+2}$, we get

$$
\begin{align*}
& \int_{-\infty}^{\infty} \frac{d}{d s} a_{F}(s)\left(\tilde{a}_{L}-a_{F}(s)\right)\left(s-\tilde{a}_{L}\right) \phi\left(s-\tilde{a}_{L}\right) d s \\
& \leq\left(\tilde{a}_{L}-a_{F}\left(\tilde{a}_{L}\right)\right) \int_{\tilde{a}_{L}}^{m_{k+1}} \frac{d}{d s} a_{F}(s)\left(s-\tilde{a}_{L}\right) \phi\left(s-\tilde{a}_{L}\right) d s \\
& \leq 6.1 \bar{r} \bar{x}_{k+2}\left(\frac{1.01 \bar{r}^{2} \bar{x}_{k+2}^{2}}{\sqrt{2 \pi}}+\left(m_{k+1}-\tilde{a}_{L}\right) \int_{\tilde{a}_{L}}^{m_{k+1}}\left(\frac{d}{d s} a_{F}(s)-1.01 \bar{r}^{2} \bar{x}_{k+2}^{2}\right) \phi\left(s-\tilde{a}_{L}\right) d s\right) \\
& \leq 6.1 \bar{r} \bar{x}_{k+2}\left(\frac{1.01 \bar{r}^{2} \bar{x}_{k+2}^{2}}{\sqrt{2 \pi}}+\frac{1.17 \Delta_{k+1}^{2}\left(m_{k+1}-\tilde{a}_{L}\right) \phi\left(m_{k+1}-\tilde{a}_{L}\right)}{\Delta_{k+1}-m_{k+1}+\tilde{a}_{L}}\right) \\
& \leq 6.1 \bar{r} \bar{x}_{k+2}\left(\frac{1.01 \bar{r}^{2} \bar{x}_{k+2}^{2}}{\sqrt{2 \pi}}+1.17\left(2 \underline{x}_{1}+0.14\right)^{2} \phi\left(\underline{x}_{1}\right)\right) \\
& <2.46 \bar{r}^{3} \bar{x}_{k+2}^{3}+\frac{7.15 \bar{r} \bar{x}_{k+2}\left(2 \underline{x}_{1}+0.14\right)^{2} e^{-2\left(\underline{x}_{1}-2\right)}}{\sqrt{2 \pi e^{4}} \sigma^{4}}<0.1 \bar{r}^{2} \bar{x}_{k+2}^{2}, \tag{115}
\end{align*}
$$

for $\sigma \geq 300$. For the case $k=m$ and $a_{F}\left(\tilde{a}_{L}\right) \leq \tilde{a}_{L}$,

$$
\begin{align*}
& \int_{-\infty}^{\infty} \frac{d}{d s} a_{F}(s)\left(\tilde{a}_{L}-a_{F}(s)\right)\left(s-\tilde{a}_{L}\right) \phi\left(s-\tilde{a}_{L}\right) d s \\
& \leq\left(\tilde{a}_{L}-a_{F}\left(\tilde{a}_{L}\right)\right) \int_{\tilde{a}_{L}}^{\infty} \frac{d}{d s} a_{F}(s)\left(s-\tilde{a}_{L}\right) \phi\left(s-\tilde{a}_{L}\right) d s \tag{116}
\end{align*}
$$

We break this into two parts. First, using an approach similar to (112), we can obtain

$$
\begin{equation*}
\int_{c_{m}+\bar{x}_{m+1}}^{\infty} \frac{d}{d s} a_{F}(s)\left(s-\tilde{a}_{L}\right) \phi\left(s-\tilde{a}_{L}\right) d s<10^{-4} \bar{r}^{2} \bar{x}_{m+1}^{2} \tag{117}
\end{equation*}
$$

The other part is $\int_{\tilde{a}_{L}}^{c_{m}+\bar{x}_{m+1}} \frac{d}{d s} a_{F}(s)\left(s-\tilde{a}_{L}\right) \phi\left(s-\tilde{a}_{L}\right) d s$. If $\tilde{a}_{L} \geq c_{m}$, then

$$
\begin{equation*}
\int_{\tilde{a}_{L}}^{c_{m}+\bar{x}_{m+1}} \frac{d}{d s} a_{F}(s)\left(s-\tilde{a}_{L}\right) \phi\left(s-\tilde{a}_{L}\right) d s \leq \frac{0.8 \bar{r}^{2} \bar{x}_{m+1}^{2}}{\sqrt{2 \pi}} \tag{118}
\end{equation*}
$$

where we have used that $\frac{d}{d s} a_{F}(s) \leq 0.8 \bar{r}^{2} \bar{x}_{m+1}^{2}$ for $s \in\left[c_{m}, c_{m}+\bar{x}_{m+1}\right]$ according to Lemma 10 . Hence we get

$$
\begin{align*}
& \int_{-\infty}^{\infty} \frac{d}{d s} a_{F}(s)\left(\tilde{a}_{L}-a_{F}(s)\right)\left(s-\tilde{a}_{L}\right) \phi\left(s-\tilde{a}_{L}\right) d s \\
& \leq 6.1 \bar{r} \bar{x}_{m+1}\left(0.32 \bar{r}^{2} \bar{x}_{m+1}^{2}+10^{-4} \bar{r}^{2} \bar{x}_{m+1}^{2}\right)<0.1 \bar{r}^{2} \bar{x}_{m+1}^{2} \tag{119}
\end{align*}
$$

We need to revise (118) when $\tilde{a}_{L}<c_{m}$. On one hand $\tilde{a}_{L}-a_{F}\left(\tilde{a}_{L}\right)<c_{m}-a_{F}\left(\tilde{a}_{L}\right)<1.1 \bar{r} \bar{x}_{m+1}$ (instead of $6.1 \bar{r} \bar{x}_{m+1}$ ). We also use a loose bound such as ${ }^{15}$

$$
\begin{equation*}
\int_{\tilde{a}_{L}}^{c_{m}+\bar{x}_{m+1}} \frac{d}{d s} a_{F}(s)\left(s-\tilde{a}_{L}\right) \phi\left(s-\tilde{a}_{L}\right) d s \leq \frac{1.1 \bar{r}^{2} \bar{x}_{m+1}^{2}}{\sqrt{2 \pi}} \tag{120}
\end{equation*}
$$

in order to get

$$
\begin{align*}
& \int_{-\infty}^{\infty} \frac{d}{d s} a_{F}(s)\left(\tilde{a}_{L}-a_{F}(s)\right)\left(s-\tilde{a}_{L}\right) \phi\left(s-\tilde{a}_{L}\right) d s \\
& \leq 1.1 \bar{r} \bar{x}_{m+1}\left(\frac{1.1 \bar{r}^{2} \bar{x}_{m+1}^{2}}{\sqrt{2 \pi}}+10^{-4} \bar{r}^{2} \bar{x}_{m+1}^{2}\right)<0.1 \bar{r}^{2} \bar{x}_{m+1}^{2} \tag{121}
\end{align*}
$$

Finally, putting all together we get

$$
\begin{align*}
& \int_{-\infty}^{\infty} \frac{d}{d s} a_{F}(s)\left(2+\left(\tilde{a}_{L}-a_{F}(s)\right)\left(s-\tilde{a}_{L}\right)\right) \phi\left(s-\tilde{a}_{L}\right) d s \\
& <(2 \times 1.05+0.1) \bar{r}^{2} \bar{x}_{m+1}^{2}<2.2 e \bar{r}^{2} \bar{x}_{m}^{2}<0.45 \bar{r}^{2} \sigma^{2} \tag{122}
\end{align*}
$$

where we use that $\bar{x}_{m}<0.272 \sigma$ for $m \geq 25$ and $\sigma \geq 300$.
Now for the other side, on noting that $a_{F}(s) \leq s+1.1 \bar{x}_{m+1}$ and that for $s \in\left[c_{k+1}, c_{m}+\bar{x}_{m+1}\right]$ we have $\frac{d}{d s} a_{F}(s) \leq\left((1+\bar{r})^{2}+1.01 \bar{r}^{2}\right) \bar{x}_{m+1}^{2}$, we can write

$$
\begin{aligned}
& \int_{c_{k+1}}^{c_{m}+\bar{x}_{m+1}} \frac{d}{d s} a_{F}(s)\left(\left(a_{F}(s)-\tilde{a}_{L}\right)\left(s-\tilde{a}_{L}\right)-2\right) \phi\left(s-\tilde{a}_{L}\right) d s \\
& <\left((1+\bar{r})^{2}+1.01 \bar{r}^{2}\right) \bar{x}_{m+1}^{2} \int_{c_{k+1}}^{\infty}\left(s+1.1 \bar{x}_{m+1}-\tilde{a}_{L}\right)\left(s-\tilde{a}_{L}\right) \phi\left(s-\tilde{a}_{L}\right) d s
\end{aligned}
$$

[^12]\[

$$
\begin{align*}
& <\sqrt{2 \pi}\left((1+\bar{r})^{2}+1.01 \bar{r}^{2}\right) \bar{x}_{m+1}^{2} \phi\left(c_{k+1}-\tilde{a}_{L}\right) \int_{0}^{\infty}\left(\delta+c_{k+1}-\tilde{a}_{L}+1.1 \bar{x}_{m+1}\right)\left(\delta+c_{k+1}-\tilde{a}_{L}\right) \phi(\delta) d \delta \\
& <\sqrt{2 \pi}\left((1+\bar{r})^{2}+1.01 \bar{r}^{2}\right) \bar{x}_{m+1}^{2} \phi\left(c_{k+1}-\tilde{a}_{L}\right)\left(\frac{1+\left(c_{k+1}-\tilde{a}_{L}\right)\left(c_{k+1}-\tilde{a}_{L}+1.1 \bar{x}_{m+1}\right)}{2}+\frac{2\left(c_{k+1}-\tilde{a}_{L}\right)+1.1 \bar{x}_{m+1}}{\sqrt{2 \pi}}\right) \\
& <\sqrt{2 \pi}\left((1+\bar{r})^{2}+1.01 \bar{r}^{2}\right) \bar{x}_{m+1}^{2} \phi\left(2 \underline{x}_{1}\right)\left(\frac{1+2 \underline{x}_{1}\left(2 \underline{x}_{1}+1.1 \bar{x}_{m+1}\right)}{2}+\frac{4 \underline{x}_{1}+1.1 \bar{x}_{m+1}}{\sqrt{2 \pi}}\right) \\
& <10 \bar{x}_{m+1}^{4} \phi(4 \sqrt{2 \ln \sigma})<10^{-4} \bar{r}^{2} \bar{x}_{m+1}^{2} . \tag{123}
\end{align*}
$$
\]

For $s \geq c_{m}+\bar{x}_{m+1}, a_{F}(s) \leq c_{m}+3 r_{L} \sigma\left(s-c_{m}+1\right)<s$, and $\frac{d}{d s} a_{F}(s) \leq 3 r_{L}\left(s-c_{m}\right)^{2}$. An approach similar to (112) leads to

$$
\begin{equation*}
\int_{c_{m}+\bar{x}_{m+1}}^{\infty} \frac{d}{d s} a_{F}(s)\left(\left(a_{F}(s)-\tilde{a}_{L}\right)\left(s-\tilde{a}_{L}\right)-2\right) \phi\left(s-\tilde{a}_{L}\right) d s<10^{-4} \bar{r}^{2} \bar{x}_{m+1}^{2} \tag{124}
\end{equation*}
$$

If $c_{k} \leq s \leq m_{k+1}-\frac{2 \ln \sigma}{\Delta_{k+1}}$, then, according to Corollary 1 ,

$$
\begin{equation*}
a_{F}(s)<c_{k}+1.17 \frac{\Delta_{k+1}}{\sigma^{2}}+1.01 \bar{r} \bar{x}_{k+2}<c_{k}+2.34 r_{L} \Delta_{k+1}+1.01 \bar{r} \bar{x}_{k+2}<c_{k}+5.8 \bar{r} \bar{x}_{k+2} . \tag{125}
\end{equation*}
$$

Therefore, $\left(a_{F}(s)-\tilde{a}_{L}\right)\left(s-\tilde{a}_{L}\right)<11 \bar{r} \bar{x}_{k+2}^{2}<2$. On the other hand, it is easy to find

$$
\begin{align*}
& \int_{m_{k+1}-\frac{2 \ln \sigma}{\Delta_{k+1}}}^{c_{k+1}} \frac{d}{d s} a_{F}(s)\left(\left(a_{F}(s)-\tilde{a}_{L}\right)\left(s-\tilde{a}_{L}\right)-2\right) \phi\left(s-\tilde{a}_{L}\right) d s \\
& \leq\left(\left(\frac{\Delta_{k+1}}{2}+\bar{r} \bar{x}_{k+2}\right)^{2}+0.01 \bar{r}^{2} \bar{x}_{k+2}^{2}\right) \Delta_{k+1} \phi\left(m_{k+1}-\tilde{a}_{L}-\frac{2 \ln \sigma}{\Delta_{k+1}}\right) . \tag{126}
\end{align*}
$$

Let $\Delta_{k+1}^{L}=m_{k+1}-\tilde{a}_{L}$. Then, from the above inequality (and similar to (109)) we can get

$$
\begin{align*}
& \int_{m_{k+1}-\frac{2 \ln \sigma}{\Delta_{k+1}}}^{c_{k+1}} \frac{d}{d s} a_{F}(s)\left(\left(a_{F}(s)-\tilde{a}_{L}\right)\left(s-\tilde{a}_{L}\right)-2\right) \phi\left(s-\tilde{a}_{L}\right) d s \\
& \leq\left(\left(\Delta_{k+1}^{L}+0.07+\bar{r} \bar{x}_{k+2}\right)^{2}+0.01 \bar{r}^{2} \bar{x}_{k+2}^{2}\right)\left(2 \Delta_{k+1}^{L}+0.14\right) \phi\left(\Delta_{k+1}^{L}-\frac{2 \ln \sigma}{\Delta_{k+1}}\right) \\
& \leq\left(\left(\underline{x}_{1}+0.07+\bar{r} \bar{x}_{k+2}\right)^{2}+0.01 \bar{r}^{2} \bar{x}_{k+2}^{2}\right)\left(2 \underline{x}_{1}+0.14\right) \phi\left(\underline{x}_{1}-\frac{2 \ln \sigma}{4 \sqrt{2 \ln \sigma}}\right) \\
& \leq 2\left(\underline{x}_{1}+0.1\right)^{3} \phi\left(\underline{x}_{1}-\frac{2 \sqrt{2 \ln \sigma}}{8}\right) \leq 2(2 \sqrt{2 \ln \sigma}+2.1)^{3} \phi\left(2 \sqrt{2 \ln \sigma}+2-\frac{2 \sqrt{2 \ln \sigma}}{8}\right) \\
& \leq \frac{54 \ln \sigma \sqrt{\ln \sigma} e^{-\frac{14}{8}\left(\underline{x}_{1}-2\right)}}{\sqrt{2 \pi e^{4}} \sigma^{3}}<0.15 \bar{r}^{2} \sigma^{2}, \tag{127}
\end{align*}
$$

for $\sigma \geq 300$. Overall, we arrive at

$$
\begin{equation*}
\int_{-\infty}^{+\infty} \frac{d}{d s} a_{F}(s)\left(\left(a_{F}(s)-\tilde{a}_{L}\right)\left(s-\tilde{a}_{L}\right)-2\right) \phi\left(s-\tilde{a}_{L}\right) d s<0.4 \bar{r}^{2} \sigma^{2} \tag{128}
\end{equation*}
$$

Similar argument holds for $k=m$.
It follows from the above analysis that

$$
\begin{equation*}
r_{L}+\left(1-r_{L}\right)\left(1-0.45 \bar{r}^{2} \sigma^{2}\right) \leq r_{L} \frac{d}{d x} \Theta(x) \leq r_{L}+\left(1-r_{L}\right)\left(1+0.4 \bar{r}^{2} \sigma^{2}\right) \tag{129}
\end{equation*}
$$

for $x \in\left[c_{k}-5 \bar{r} \bar{x}_{k+2}, c_{k}+5 \bar{r} \bar{x}_{k+2}\right]$, where $\Theta(x)$ defined in (104) is the real analytic left-inverse of the leader's best response strategy $\tilde{a}_{L}(\theta)$, that is, $\Theta\left(\tilde{a}_{L}(\theta)\right)=\theta$ for all $\theta \in \mathbb{R}$. Moreover, $\tilde{a}_{L}(\theta)$ is an increasing function, as already discussed in the proof of the previous lemma.

We next claim that $\tilde{a}_{L}(\theta)$ is continuous over both $\left(c_{k}, \tilde{b}_{k+1}\right)$ and ( $\left.\tilde{b}_{k+1}, c_{k+1}\right)$. Consider $\theta \in$ $\left(c_{k}, \tilde{b}_{k+1}\right)$. From Lemma 11, we know that $\tilde{a}_{L}(\theta) \in\left[c_{k}-5 \bar{r} \bar{x}_{k+2}, c_{k}+5 \bar{r} \bar{x}_{k+2}\right]$. This is a tiny interval around a fixed point of the leader's original strategy $a_{L}(\theta)$ (according to which the follower's best response $a_{F}(s)$ is derived). We next show that leader's utility $\tilde{u}_{L}\left(\theta, a_{L}\right)$ given in (27) is strongly concave (in $a_{L}$ ) over this interval. The good news is that we have already bounded $\frac{\partial^{2}}{\partial a_{L}^{2}} \tilde{u}_{L}\left(\theta, a_{L}\right)$ for $a_{L} \in\left[c_{k}-5 \bar{r} \bar{x}_{k+2}, c_{k}+5 \bar{r} \bar{x}_{k+2}\right]$ while bounding the derivative of $\Theta(x)$ over this interval. In fact, from the definition of both functions given in (27) and (104), it is easy to verify that

$$
\begin{equation*}
\frac{\partial^{2}}{\partial a_{L}^{2}} \tilde{u}_{L}\left(\theta, a_{L}\right)=-2 r_{L} \frac{d}{d x} \Theta(x)_{\mid x=a_{L}} \tag{130}
\end{equation*}
$$

Strong concavity of $\tilde{u}_{L}\left(\theta, a_{L}\right)$ for $a_{L} \in\left[c_{k}-5 \bar{r} \bar{x}_{k+2}, c_{k}+5 \bar{r} \bar{x}_{k+2}\right]$ then follows from (129). Recall that for $\theta \in\left(c_{k}, \tilde{b}_{k+1}\right)$, leader's best response $\tilde{a}_{L}(\theta) \in\left[c_{k}-5 \bar{r} \bar{x}_{k+2}, c_{k}+5 \bar{r} \bar{x}_{k+2}\right]$, meaning that $\tilde{a}_{L}(\theta)$ is the unique maximizer of $\tilde{u}_{L}\left(\theta, a_{L}\right)$ over $\left[c_{k}-5 \bar{r} \bar{x}_{k+2}, c_{k}+5 \bar{r} \bar{x}_{k+2}\right]$. This uniqueness property implies the continuity of $\tilde{a}_{L}(\theta)$ over $\theta \in\left(c_{k}, \tilde{b}_{k+1}\right)$, noting that both left and right limits of $\tilde{a}_{L}(\theta)$ are maximizers of leader's utility over $\left[c_{k}-5 \bar{r} \bar{x}_{k+2}, c_{k}+5 \bar{r} \bar{x}_{k+2}\right]$, and hence have to be identical.

Therefore, leader's best response $\tilde{a}_{L}(\theta)$ is continuous over both $\left(c_{k}, \tilde{b}_{k+1}\right)$ and $\left(\tilde{b}_{k+1}, c_{k+1}\right)$, with its graph coinciding its analytic left inverse $\Theta(x)$ over both intervals. We can thus use (129) to bound its derivative over each of these intervals, hence completing the proof.
Proof of Lemma 13. We have already calculated the second derivative of $\tilde{J}_{L}$ when deriving the
partial derivative in (105). The same argument implies $\frac{d^{2}}{d a_{L}^{2}} \tilde{J}_{L}\left(a_{L}\right) \geq 2\left(1-0.45 \bar{r}^{2} \sigma^{2}\right)$ for $a_{L} \in$ $\left[c_{k}-5 \bar{r} \bar{x}_{k+2}, c_{k}+5 \bar{r} \bar{x}_{k+2}\right]$. This, implies that $\tilde{J}_{L}$ is strongly convex over $\left[c_{k}-5 \bar{r} \bar{x}_{k+2}, c_{k}+5 \bar{r} \bar{x}_{k+2}\right]$. It's unique minimizer $\tilde{c}_{k}$ minimizes both losses in the leader's payoff, hence it is the fixed point of $\tilde{a}_{L}(\theta)$, that is $\tilde{a}_{L}\left(\tilde{c}_{k}\right)=\tilde{c}_{k}$.
Proof of Lemma 14. The case $\tilde{b}_{m}<\theta \leq c_{m}$ follows from Lemma 12, so we only need to consider $\theta>c_{m}$. As the first step, we derive some useful lower bounds on $\tilde{J}_{L}\left(a_{L}\right)$ for $a_{L}=c_{m}+\epsilon$ with $\epsilon \geq 0$. In particular, we claim that for $\epsilon>\bar{r} \bar{x}_{m+1}$ and $\sigma \geq 300$, we have

$$
\begin{equation*}
\tilde{J}_{L}\left(a_{L}\right) \geq 0.99\left(\epsilon-\bar{r} \bar{x}_{m+1}\right)^{2} . \tag{131}
\end{equation*}
$$

We consider two cases: If $\epsilon \leq \frac{3}{4} \bar{x}_{m+1}$, then $a_{F}(s)-c_{m} \leq \bar{r} \bar{x}_{m+1}$ for $s \leq a_{L}+\frac{1}{4} \bar{x}_{m+1}$, and hence $\tilde{J}_{L}\left(a_{L}\right) \geq\left(\epsilon-\bar{r} \bar{x}_{m+1}\right)^{2} \Phi\left(\frac{\bar{x}_{m+1}}{4}\right)$. If $\epsilon>\frac{3}{4} \bar{x}_{m+1}$, then $a_{L}-a_{F}\left(a_{L}+\frac{\bar{x}_{m+1}}{4}-1\right) \geq\left(1-4 r_{L} \sigma\right) \epsilon$, and hence $\tilde{J}_{L}\left(a_{L}\right) \geq\left(1-4 r_{L} \sigma\right)^{2} \Phi\left(\frac{\bar{x}_{m+1}}{4}-1\right) \epsilon^{2}$. These two observations result in the lower bound in (131) noting $\sigma \geq 300$.

Consider now $\theta=c_{m}+\delta, 0 \leq \delta$. We claim that $\tilde{a}_{L}(\theta)<c_{m}+2.2 r_{L}\left(\delta+\bar{x}_{m+1}\right)$. Let $\tilde{a}_{L}=c_{m}+\epsilon$. Then,

$$
\begin{equation*}
\tilde{u}_{L}\left(\theta, \tilde{a}_{L}\right) \leq-r_{L}(\epsilon-\delta)^{2}-0.99\left(1-r_{L}\right)\left(\epsilon-\bar{r} \bar{x}_{m+1}\right)^{2} . \tag{132}
\end{equation*}
$$

Using an approach similar to Lemma 11, we can show that $\epsilon<2 \epsilon^{*}$ where

$$
\begin{equation*}
\epsilon^{*}=\frac{r_{L} \delta+0.99\left(1-r_{L}\right) \bar{r} \bar{x}_{m+1}}{r_{L}+0.99\left(1-r_{L}\right)} \tag{133}
\end{equation*}
$$

from which we easily get $\epsilon<2.02 r_{L}\left(\delta+\bar{x}_{m+1}\right)$. Using this, the proof of the second part of the lemma is quite straightforward.

As for $\frac{d}{d \theta} \tilde{a}_{L}(\theta)$, the case $\tilde{a}_{L}(\theta) \leq c_{m}+5 \bar{r} \bar{x}_{m+1}$ is covered in Lemma 12 . Hence, we study the case $\tilde{a}_{L}(\theta)>c_{m}+5 \bar{r} \bar{x}_{m+1}$. Noting that

$$
\begin{equation*}
\tilde{a}_{L}(\theta)-c_{m} \leq 2.2 r_{L}\left(\sigma \bar{x}_{m+1}+5 \bar{r} \bar{x}_{m+1}+\bar{x}_{m+1}\right)<2.4, \tag{134}
\end{equation*}
$$

and similar to (112), we can obtain

$$
\begin{equation*}
\int_{c_{m}+\bar{x}_{m+1}}^{\infty} \frac{d}{d s} a_{F}(s) \phi\left(s-\tilde{a}_{L}\right) d s<10^{-4} \bar{r}^{2} \bar{x}_{m+1}^{2} \tag{135}
\end{equation*}
$$

Also,

$$
\begin{equation*}
\int_{c_{m}}^{c_{m}+\bar{x}_{m+1}} \frac{d}{d s} a_{F}(s) \phi\left(s-\tilde{a}_{L}\right) d s \leq 0.8 \bar{r}^{2} \bar{x}_{m+1}^{2} \Phi\left(\tilde{a}_{L}-c_{m}\right) . \tag{136}
\end{equation*}
$$

On the other hand, similar to (111), we have

$$
\begin{equation*}
\int_{-\infty}^{c_{m}} \frac{d}{d s} a_{F}(s) \phi\left(s-\tilde{a}_{L}\right) d s<10^{-4} \bar{r}^{2} \bar{x}_{m+1}^{2}+1.01 \bar{r}^{2} \bar{x}_{m+1}^{2} \Phi\left(c_{m}-\tilde{a}_{L}\right) . \tag{137}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\int_{-\infty}^{\infty} \frac{d}{d s} a_{F}(s) \phi\left(s-\tilde{a}_{L}\right) d s<0.91 \bar{r}^{2} \bar{x}_{m+1}^{2}<2.48 \bar{r}^{2} \bar{x}_{m}^{2} \tag{138}
\end{equation*}
$$

since $\bar{x}_{m+1}=\sqrt{e} \bar{x}_{m}$.
To bound the other term, using $\tilde{a}_{L}-a_{F}(s)<2.2 r_{L}\left(\sigma \bar{x}_{m+1}+\bar{x}_{m+1}+5 r_{L} \sigma\right)+1.5 \bar{r} \bar{x}_{m+1}<2.5$ for $s \geq c_{m}+\bar{x}_{m+1}$ and similar to (112), we can show that

$$
\begin{equation*}
\int_{c_{m}+\bar{x}_{m+1}}^{\infty} \frac{d}{d s} a_{F}(s)\left(\tilde{a}_{L}-a_{F}(s)\right)\left(s-\tilde{a}_{L}\right) \phi\left(s-\tilde{a}_{L}\right) d s<10^{-4} \bar{r}^{2} \bar{x}_{m+1}^{2} \tag{139}
\end{equation*}
$$

For $s \leq c_{m}+\bar{x}_{m+1}$, we have $a_{F}(s)<\tilde{a}_{L}$. Therefore,

$$
\begin{align*}
& \int_{-\infty}^{c_{m}+\bar{x}_{m+1}} \frac{d}{d s} a_{F}(s)\left(\tilde{a}_{L}-a_{F}(s)\right)\left(s-\tilde{a}_{L}\right) \phi\left(s-\tilde{a}_{L}\right) d s \\
& \leq \int_{\tilde{a}_{L}}^{c_{m}+\bar{x}_{m+1}} \frac{d}{d s} a_{F}(s)\left(\tilde{a}_{L}-a_{F}(s)\right)\left(s-\tilde{a}_{L}\right) \phi\left(s-\tilde{a}_{L}\right) d s \\
& \leq \frac{1}{\sqrt{2 \pi}}\left(\bar{r} \sigma \bar{x}_{m+1}+1.01 \bar{r} \bar{x}_{m+1}\right) \times 0.8 \bar{r}^{2}\left(\frac{\bar{x}_{m}+\bar{x}_{m+1}}{2}\right)^{2}<0.26 \bar{r}^{2} \bar{x}_{m}^{2} \tag{140}
\end{align*}
$$

Putting all together, we get

$$
\begin{equation*}
\int_{-\infty}^{\infty} \frac{d}{d s} a_{F}(s)\left(2+\left(\tilde{a}_{L}-a_{F}(s)\right)\left(s-\tilde{a}_{L}\right)\right) \phi\left(s-\tilde{a}_{L}\right) d s<5.22 \bar{r}^{2} \bar{x}_{m}^{2}<0.4 \bar{r}^{2} \sigma^{2} \tag{141}
\end{equation*}
$$

since $\bar{x}_{m}<0.272 \sigma$ for $m \geq 25$ and $\sigma \geq 300$.
For the other side, similar to (123), (124), and (126), we have

$$
\begin{align*}
& \int_{-\infty}^{c_{m-1}} \frac{d}{d s} a_{F}(s)\left(\left(a_{F}(s)-\tilde{a}_{L}\right)\left(s-\tilde{a}_{L}\right)-2\right) \phi\left(s-\tilde{a}_{L}\right) d s<10^{-4} \bar{r}^{2} \bar{x}_{m+1}^{2} \\
& \int_{c_{m}+\bar{x}_{m+1}}^{\infty} \frac{d}{d s} a_{F}(s)\left(\left(a_{F}(s)-\tilde{a}_{L}\right)\left(s-\tilde{a}_{L}\right)-2\right) \phi\left(s-\tilde{a}_{L}\right) d s<10^{-4} \bar{r}^{2} \bar{x}_{m+1}^{2}, \\
& \int_{c_{m-1}}^{m_{m}+\sqrt{\ln \sigma}} \frac{d}{d s} a_{F}(s)\left(\left(a_{F}(s)-\tilde{a}_{L}\right)\left(s-\tilde{a}_{L}\right)-2\right) \phi\left(s-\tilde{a}_{L}\right) d s<0.1 \bar{r}^{2} \sigma^{2} . \tag{142}
\end{align*}
$$

Also, same as (123),

$$
\begin{equation*}
\int_{-\infty}^{c_{m-1}} \frac{d}{d s} a_{F}(s)\left(\left(a_{F}(s)-\tilde{a}_{L}\right)\left(s-\tilde{a}_{L}\right)-2\right) \phi\left(s-\tilde{a}_{L}\right) d s<10^{-4} \bar{r}^{2} \bar{x}_{m+1}^{2} \tag{143}
\end{equation*}
$$

Consider now $s \in\left[m_{m}+\sqrt{\ln \sigma}, c_{m}+\bar{x}_{m+1}\right]$. It is easy to verify that $\left(a_{F}(s)-\tilde{a}_{L}\right)\left(s-\tilde{a}_{L}\right)<$ 2 for $s \in\left[\tilde{a}_{L}, c_{m}+\bar{x}_{m+1}\right]$. On the other hand, using Lemma 9 and 10, we can verify that $\frac{d}{d s} a_{F}(s) \leq 1.1 \bar{r}^{2} \bar{x}_{m+1}^{2}$ for $s \in\left[m_{m}+\sqrt{\ln \sigma}, \tilde{a}_{L}\right]$. Moreover, it follows from Corollary 2 that $a_{F}(s) \geq c_{m}-1.1 \bar{r} \bar{x}_{m+1}$ for $s \in\left[m_{m}+\sqrt{\ln \sigma}, \tilde{a}_{L}\right]$, implying that $\tilde{a}_{L}-a_{F}(s)<2.4+1.1 \bar{r} \bar{x}_{m+1}<$ 2.5 (see (134)). This yields

$$
\begin{align*}
& \int_{m_{m}+\sqrt{\ln \sigma}}^{\tilde{a}_{L}} \frac{d}{d s} a_{F}(s)\left(\left(a_{F}(s)-\tilde{a}_{L}\right)\left(s-\tilde{a}_{L}\right)-2\right) \phi\left(s-\tilde{a}_{L}\right) d s \\
& <2.5 \times 1.1 \bar{r}^{2} \bar{x}_{m+1}^{2} \int_{m_{m}+\sqrt{\ln \sigma}}^{\tilde{a}_{L}}\left(\tilde{a}_{L}-s\right) \phi\left(s-\tilde{a}_{L}\right) d s \\
& <\frac{2.5 \times 1.1 \bar{r}^{2} \bar{x}_{m+1}^{2}}{\sqrt{2 \pi}}<1.1 \bar{r}^{2} \bar{x}_{m+1}^{2} . \tag{144}
\end{align*}
$$

From the above analysis it follows that

$$
\begin{align*}
& \int_{-\infty}^{\infty} \frac{d}{d s} a_{F}(s)\left(\left(a_{F}(s)-a_{L}\right)\left(s-a_{L}\right)-2\right) \phi\left(s-a_{L}\right) d s \\
& <0.1 \bar{r}^{2} \sigma^{2}+1.2 \bar{r}^{2} \bar{x}_{m+1}^{2}<\left(0.1+1.2 \times e \times 0.272^{2}\right) \bar{r}^{2} \sigma^{2}<0.4 \bar{r}^{2} \sigma^{2} \tag{145}
\end{align*}
$$

Using this and (141), and following exact same steps as in the proof of Lemma 12, we can show that $\underline{r} \leq \frac{d}{d \theta} \tilde{a}_{L}(\theta) \leq \bar{r}$.
Proof of Lemma 15. We start by finding an upper bound for $\tilde{J}_{L}\left(\tilde{c}_{k}\right)$. We use $\tilde{J}_{L}\left(\tilde{c}_{k}\right) \leq \tilde{J}_{L}\left(s_{k}\right)$, where we recall that $a_{F}\left(s_{k}\right)=s_{k}$. Noting that the upper bound on the derivative of $a_{F}(s)$ in (24) is increasing for $s_{k} \leq s \leq m_{k+1}$, we can write

$$
\begin{align*}
& \int_{s_{k}}^{m_{k+1}}\left(a_{F}(s)-s_{k}\right)^{2} \phi\left(s-s_{k}\right) d s=\int_{s_{k}}^{m_{k+1}}\left(a_{F}(s)-a_{F}\left(s_{k}\right)\right)^{2} \phi\left(s-s_{k}\right) d s \leq \\
& \int_{s_{k}}^{m_{k+1}}\left(s-s_{k}\right)^{2}\left(1.17 e^{-\Delta_{k+1}\left(m_{k+1}-s\right)} \Delta_{k+1}^{2}+1.01 \bar{r}^{2} \bar{x}_{k+2}^{2}\right)^{2} \phi\left(s-s_{k}\right) d s \tag{146}
\end{align*}
$$

Following a similar machinery as the one already used in, e.g., (109), we can arrive at

$$
\int_{s_{k}}^{m_{k+1}}\left(a_{F}(s)-a_{F}\left(s_{k}\right)\right)^{2} \phi\left(s-s_{k}\right) d s \leq \frac{1.01^{2} \bar{r}^{4} \bar{x}_{k+2}^{4}}{2}
$$

$$
\begin{align*}
& +2.4 \bar{r}^{2} \bar{x}_{k+2}^{2} \Delta_{k+1}^{2} \frac{\left(m_{k+1}-s_{k}\right)^{2} \phi\left(m_{k+1}-s_{k}\right)}{\Delta_{k+1}-m_{k+1}+s_{k}} \\
& +1.37 \Delta_{k+1}^{4} \frac{\left(m_{k+1}-s_{k}\right)^{2} \phi\left(m_{k+1}-s_{k}\right)}{2 \Delta_{k+1}-m_{k+1}+s_{k}} \\
& \leq \frac{1.01^{2} \bar{r}^{4} \bar{x}_{k+2}^{4}}{2}+\left(9.6 \bar{r}^{2} \bar{x}_{k+2}^{2} \underline{x}_{1}^{2}\left(\underline{x}_{1}+0.07\right)+7.31 \underline{x}_{1}^{2}\left(\underline{x}_{1}+0.07\right)^{3}\right) \phi\left(\underline{x}_{1}\right) \\
& <\frac{7.5 \underline{x}_{1}^{5} e^{-2\left(\underline{x}_{1}-2\right)}}{\sqrt{2 \pi e^{4}} \sigma^{4}} \tag{147}
\end{align*}
$$

where the last inequality follows from $\underline{x}_{1} \geq 2 \sqrt{2 \ln \sigma}+2$. Similarly, using (25) and Lemma 10 we can show that

$$
\begin{equation*}
\int_{m_{k+1}}^{\infty}\left(a_{F}(s)-s_{k}\right)^{2} \phi\left(s-s_{k}\right) d s<\frac{2 \underline{x}_{1}^{5} e^{-2\left(\underline{x}_{1}-2\right)}}{\sqrt{2 \pi e^{4}} \sigma^{4}} . \tag{148}
\end{equation*}
$$

These two yield $\tilde{J}_{L}\left(s_{k}\right)<\frac{9.5 \underline{x}_{1}^{5} e^{-2\left(\underline{x}_{1}-2\right)}}{\sqrt{2 \pi e^{4} \sigma^{4}}}<0.2 r_{L}^{2}$ using $\underline{x}_{1}>8.75$ for $\sigma \geq 300$. It is easy to verify that the same hold when $k=m$. The analysis in this case is even simpler on noting that $\frac{d}{d s} a_{F}(s) \leq 0.8 \bar{r}^{2} \bar{x}_{m+1}^{2}$ over the whole interval $s \in\left[c_{m}, c_{m}+\bar{x}_{m+1}\right]$.

Applying the Envelope's theorem to (27), we get

$$
\begin{equation*}
\frac{d}{d \theta} \tilde{u}_{L}\left(\theta, \tilde{a}_{L}(\theta)\right)=2 r_{L}\left(\theta-\tilde{a}_{L}(\theta)\right) \tag{149}
\end{equation*}
$$

Integrating this, along the inequality below

$$
\begin{equation*}
\underline{r} \theta+(1-\underline{r}) \tilde{c}_{k} \leq \tilde{a}_{L}(\theta) \leq \bar{r} \theta+(1-\bar{r}) \tilde{c}_{k} \tag{150}
\end{equation*}
$$

we get

$$
\begin{equation*}
r_{L}(1-\bar{r})\left(\theta-\tilde{c}_{k}\right)^{2} \leq \tilde{u}_{L}(\theta, \cdot)-\tilde{u}_{L}\left(\tilde{c}_{k}, \cdot\right) \leq r_{L}(1-\underline{r})\left(\theta-\tilde{c}_{k}\right)^{2}, \tag{151}
\end{equation*}
$$

where we use $\tilde{u}_{L}(\theta, \cdot)$ as a short-note for $\tilde{u}_{L}\left(\theta, \tilde{a}_{L}(\theta)\right)$. Now, we note that at the endpoint $\theta=\tilde{b}_{k+1}$, the above should also hold for $\tilde{c}_{k+1}$. that is,

$$
\begin{gather*}
r_{L}(1-\bar{r})\left(\tilde{b}_{k+1}-\tilde{c}_{k}\right)^{2} \leq \tilde{u}_{L}\left(\tilde{b}_{k+1}, \cdot\right)-\tilde{u}_{L}\left(\tilde{c}_{k}, \cdot\right) \leq r_{L}(1-\underline{r})\left(\tilde{b}_{k+1}-\tilde{c}_{k}\right)^{2} \\
r_{L}(1-\bar{r})\left(\tilde{b}_{k+1}-\tilde{c}_{k+1}\right)^{2} \leq \tilde{u}_{L}\left(\tilde{b}_{k+1}, \cdot\right)-\tilde{u}_{L}\left(\tilde{c}_{k+1}, \cdot\right) \leq r_{L}(1-\underline{r})\left(\tilde{b}_{k+1}-\tilde{c}_{k+1}\right)^{2} \tag{152}
\end{gather*}
$$

Using this and noting $0<\tilde{u}_{L}\left(\tilde{c}_{k}, \cdot\right)<0.2 r_{L}^{2}$ and $0<\tilde{u}_{L}\left(\tilde{c}_{k+1}, \cdot\right)<0.2 r_{L}^{2}$, we can arrive at

$$
\begin{equation*}
\left(1-r_{L}\right)\left|\left(\tilde{b}_{k+1}-\tilde{c}_{k+1}\right)^{2}-\left(\tilde{b}_{k+1}-\tilde{c}_{k}\right)^{2}\right|<0.2 r_{L}+\frac{r_{L}^{2}}{2}\left(\tilde{c}_{k+1}-\tilde{c}_{k}\right)^{2} \tag{153}
\end{equation*}
$$

using which the rest of the proof is straightforward.
Proof of Lemma 16. Using Corollary 2, it is straightforward to show that $\left|s_{k}-c_{k}\right|<1.1 \bar{r} \bar{x}_{m+1}$. Evaluating the derivative of $\tilde{J}_{L}\left(a_{L}\right)$ at $a_{L}=s_{k}$, we get

$$
\begin{equation*}
\frac{d}{d a_{L}} \tilde{J}_{L}\left(s_{k}\right)=2 \int_{-\infty}^{\infty}\left(a_{F}\left(s_{k}\right)-a_{F}(s)\right)\left(1-\frac{d}{d s} a_{F}(s)\right) \phi\left(s-s_{k}\right) d s \tag{154}
\end{equation*}
$$

where we have also used $s_{k}=a_{F}\left(s_{k}\right)$. We consider the case $s_{k} \leq \tilde{c}_{k}$ yielding $\frac{d}{d a_{L}} \tilde{J}_{L}\left(s_{k}\right) \leq 0$ (the other case is quite similar). Noting that for $s<s_{k}, \frac{d}{d s} a_{F}(s)>1$ requires $s<m_{k}+0.5$, we can obtain

$$
\begin{align*}
& -\frac{d}{d a_{L}} \tilde{J}_{L}\left(s_{k}\right) \leq 2 \int_{s_{k}}^{\infty}\left(a_{F}(s)-a_{F}\left(s_{k}\right)\right) \phi\left(s-s_{k}\right) d s \\
& \quad+2 \int_{-\infty}^{m_{k}+0.5}\left(a_{F}\left(s_{k}\right)-a_{F}(s)\right)\left(1-\frac{d}{d s} a_{F}(s)\right) \phi\left(s-s_{k}\right) d s \tag{155}
\end{align*}
$$

With a bit of manipulation similar to the ones used in Lemma 11-15, we can obtain

$$
\begin{align*}
& \int_{s_{k}}^{\infty}\left(a_{F}(s)-a_{F}\left(s_{k}\right)\right) \phi\left(s-s_{k}\right) d s \\
& \leq \frac{1.01 \bar{r}^{2}\left(\bar{x}_{k}+\bar{x}_{k+1}\right)^{2}}{4 \sqrt{2 \pi}}+\frac{4.68 x_{1}^{2} e^{-2\left(\underline{x}_{1}-2\right)}}{\sqrt{2 \pi e^{4}} \sigma^{4}} \tag{156}
\end{align*}
$$

As for the second term, similarly

$$
\begin{align*}
& \int_{-\infty}^{m_{k}+0.5}\left(a_{F}\left(s_{k}\right)-a_{F}(s)\right)\left(1-\frac{d}{d s} a_{F}(s)\right) \phi\left(s-s_{k}\right) d s \\
& <(1+\bar{r})^{2} \bar{x}_{m+1}^{2} \phi\left(2 \underline{x}_{1}-1.1 \bar{r} \bar{x}_{m+1}\right)+(1+\bar{r})^{2} \underline{x}_{1}^{2} \phi\left(\underline{x}_{1}-0.5-1.1 \bar{r} \bar{x}_{m+1}\right) . \tag{157}
\end{align*}
$$

By putting the above inequalities together and noting $\underline{x}_{1} \geq 2 \sqrt{2 \ln \sigma}+2$ and $\sigma \geq 300$, it is a matter of some machinery to verify that

$$
\begin{equation*}
0 \leq-\frac{d}{d a_{L}} \tilde{J}_{L}\left(s_{k}\right)<\frac{1.01 \bar{r}^{2}\left(\bar{x}_{k}+\bar{x}_{k+1}\right)^{2}}{2 \sqrt{2 \pi}}+0.1 \bar{r}^{2} \underline{x}_{1} \tag{158}
\end{equation*}
$$

On the other hand, as we showed before $\frac{d^{2}}{d a_{L}^{2}} \tilde{J}_{L}\left(s_{k}\right) \geq 2\left(1-0.45 \bar{r}^{2} \sigma^{2}\right)$, meaning that $s_{k}$ is at most $\frac{1.01 \bar{r}^{2}\left(\bar{x}_{k}+\bar{x}_{k+1}\right)^{2}}{4 \sqrt{2 \pi}\left(1-0.45 \bar{r}^{2} \sigma^{2}\right)}+\frac{0.1 \bar{r}^{2} \underline{x}_{1}}{2\left(1-0.45 \bar{r}^{2} \sigma^{2}\right)}<0.42 r_{L}^{2}\left(\frac{\bar{x}_{k}+\bar{x}_{k+1}}{2}\right)^{2}+0.08 r_{L}^{2} \underline{x}_{1}$ away from the minimizer at which the first derivative is zero. This completes the proof.
Proof of Lemma 17. We start by finding the fixed point of $\mathbb{E}\left[a_{L}(\theta) \mid s, \theta \in B_{k}\right]$, that is $\mathbb{E}\left[a_{L}(\theta) \mid \hat{s}_{k}, \theta \in\right.$
$\left.B_{k}\right]=\hat{s}_{k} . \hat{s}_{k}$ is the solution of the following equation

$$
\begin{equation*}
\int_{b_{k}}^{b_{k+1}}\left(a_{L}(\theta)-\hat{s}_{k}\right) \phi\left(a_{L}(\theta)-\hat{s}_{k}\right) \phi\left(\frac{\theta}{\sigma}\right) d \theta=0 . \tag{159}
\end{equation*}
$$

Noting that $\hat{s}_{k}$ is close to $c_{k}$ and in particular $\left|a_{L}(\theta)-\hat{s}_{k}\right|<1$ for $\theta \in B_{k}$, and that $x \phi(x)$ is increasing for $|x|<1$, together with the fact that $a_{L}(\theta) \leq a_{L}\left(b_{k}\right)+\bar{r}\left(\theta-b_{k}\right)$, we obtain

$$
\begin{equation*}
\int_{b_{k}}^{b_{k+1}}\left(\bar{r}\left(\theta-b_{k}\right)-\left(\hat{s}_{k}-a_{L}\left(b_{k}\right)\right)\right) \phi\left(\bar{r}\left(\theta-b_{k}\right)-\left(\hat{s}_{k}-a_{L}\left(b_{k}\right)\right)\right) \phi\left(\frac{\theta}{\sigma}\right) d \theta \geq 0 \tag{160}
\end{equation*}
$$

Therefore by finding the solution of

$$
\begin{equation*}
\int_{b_{k}}^{b_{k+1}} \bar{r}(\theta-y) \phi(\bar{r}(\theta-y)) \phi\left(\frac{\theta}{\sigma}\right) d \theta=0, \tag{161}
\end{equation*}
$$

we can upper-bound the fixed point $\hat{s}_{k}$ as $\hat{s}_{k} \leq a_{L}\left(b_{k}\right)+\bar{r}\left(y-b_{k}\right)$. Simplifying (161) yields

$$
\begin{equation*}
y=\mathbb{E}_{\bar{\psi}_{y}}\left[\theta \mid \theta \in B_{k}\right] \tag{162}
\end{equation*}
$$

where $\bar{\psi}_{y} \sim N\left(\frac{\bar{r}^{2} \sigma^{2}}{1+\bar{r}^{2} \sigma^{2}} y, \frac{\sigma^{2}}{1+\bar{r}^{2} \sigma^{2}}\right)$. A quick bound for $y$ can be obtained from $\mathbb{E}_{\bar{\psi}_{y}}\left[\theta \mid \theta \in B_{k}\right] \leq$ $e_{k}+\frac{\bar{r}^{2} \sigma^{2}}{1+\bar{r}^{2} \sigma^{2}} y$, which yields $y-e_{k} \leq \bar{r}^{2} \sigma^{2} e_{k}<0.1 \bar{x}_{k}$. An alternative representation of (162) is

$$
\begin{equation*}
\frac{y}{\sqrt{1+\bar{r}^{2} \sigma^{2}}}=\mathbb{E}_{N\left(0, \sigma^{2}\right)}\left[\theta \left\lvert\, b_{k}-\Delta b_{k}(y) \leq \theta \leq b_{k+1}-\Delta b_{k}(y)+\frac{\bar{r}^{2} \sigma^{2}\left(b_{k+1}-b_{k}\right)}{1+\sqrt{1+\bar{r}^{2} \sigma^{2}}}\right.\right] \tag{163}
\end{equation*}
$$

where $\Delta b_{k}(y)=\frac{\bar{r}^{2} \sigma^{2}}{1+\bar{r}^{2} \sigma^{2}+\sqrt{1+\bar{r}^{2} \sigma^{2}}} y+\frac{\bar{r}^{2} \sigma^{2}}{1+\sqrt{1+\bar{r}^{2} \sigma^{2}}}\left(y-b_{k}\right)>0$. A useful property of normal distribution is that $\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial b}\right) \mathbb{E}_{N\left(0, \sigma^{2}\right)}[\theta \mid a \leq \theta \leq b]=1-\operatorname{Var}_{N\left(0, \sigma^{2}\right)}[\theta \mid a \leq \theta \leq b] \geq 1-\frac{(b-a)^{2}}{(1+\sqrt{3})^{2} \sigma^{2}}$. Also, $\frac{\partial}{\partial b} \mathbb{E}_{N\left(0, \sigma^{2}\right)}[\theta \mid a \leq \theta \leq b] \leq \frac{1}{2}$. Applying these to the above equation we can obtain

$$
\begin{equation*}
0 \leq \frac{y}{\sqrt{1+\bar{r}^{2} \sigma^{2}}}+\Delta b_{k}(y)-e_{k} \leq \frac{\bar{r}^{2} \sigma^{2}\left(b_{k+1}-b_{k}\right)}{2\left(1+\sqrt{1+\bar{r}^{2} \sigma^{2}}\right)}+\frac{\left(b_{k+1}-b_{k}\right)^{2}}{(1+\sqrt{3})^{2} \sigma^{2}} \Delta b_{k}(y) \tag{164}
\end{equation*}
$$

where $e_{k}=\mathbb{E}_{N\left(0, \sigma^{2}\right)}\left[\theta \mid \theta \in B_{k}\right]$. Simplifying this, along with $y\left(b_{k+1}-b_{k}\right)<2.2 \sigma^{2}$ (which follows from $c_{m}^{\mathrm{Q}}\left(c_{m}^{\mathrm{Q}}-b_{m}^{\mathrm{Q}}\right) \leq 1$ ), we can arrive at

$$
\begin{equation*}
\left|y-e_{k}\right| \leq 0.4 \bar{r}^{2} \sigma^{2}\left(b_{k+1}-b_{k}\right) \tag{165}
\end{equation*}
$$

Recalling $\hat{s}_{k} \leq a_{L}\left(b_{k}\right)+\bar{r}\left(y-b_{k}\right)$, and that $a_{L}\left(b_{k}\right) \leq c_{k}-\underline{r}\left(c_{k}-b_{k}\right) \leq c_{k}-r_{L}\left(c_{k}-b_{k}\right)+\left(r_{L}-\underline{r}\right) \bar{x}_{k}$,
we can reach at

$$
\begin{equation*}
\hat{s}_{k} \leq\left(1-r_{L}\right) c_{k}+r_{L} e_{k}+r_{L}^{2} \bar{x}_{k}+0.4 \bar{r}^{3} \sigma^{2}\left(\bar{x}_{k}+\bar{x}_{k+1}\right) \tag{166}
\end{equation*}
$$

Following a similar argument to lower-bound $\hat{s}_{k}$, we can show that $\left|\hat{s}_{k}-\left(1-r_{L}\right) c_{k}-r_{L} e_{k}\right| \leq$ $r_{L}^{2} \bar{x}_{k}+0.4 \bar{r}^{3} \sigma^{2}\left(\bar{x}_{k}+\bar{x}_{k+1}\right)$.

To find the fixed point of $a_{F}(s)$ in $B_{k}$ (that is $a_{F}\left(s_{k}\right)=s_{k}$ ), we first note that $\hat{s}_{k}$ lies within the interval $\left[c_{k}-2 \bar{r} \bar{x}_{m+1}, c_{k}+2 \bar{r} \bar{x}_{m+1}\right]$. Moreover, for $s \in\left[c_{k}-2 \bar{r} \bar{x}_{m+1}, c_{k}+2 \bar{r} \bar{x}_{m+1}\right]$, using Lemma 9 we can obtain $\frac{d}{d s} a_{F}(s)<0.01$. This along with $\left|a_{F}\left(s_{k}\right)-s_{k}\right|<1.5 \bar{r} \bar{x}_{m+1}$ implies that for $s \in\left[c_{k}-2 \bar{r} \bar{x}_{m+1}, c_{k}+2 \bar{r} \bar{x}_{m+1}\right]$, we have $a_{F}(s) \in\left[c_{k}-2 \bar{r} \bar{x}_{m+1}, c_{k}+2 \bar{r} \bar{x}_{m+1}\right]$. Therefore,

$$
\begin{equation*}
\left|s_{k}-\hat{s}_{k}\right|<\frac{\left|\hat{s}_{k}-a_{F}\left(\hat{s}_{k}\right)\right|}{1-0.01}=\frac{\left|\mathbb{E}\left[a_{L}(\theta) \mid \hat{s}_{k}, \theta \in B_{k}\right]-\mathbb{E}\left[a_{L}(\theta) \mid \hat{s}_{k}\right]\right|}{1-0.01} \tag{167}
\end{equation*}
$$

Assume $\hat{s}_{k} \geq c_{k}$ (the other case is similar). We have already shown as part of Lemma 9 that while observing $s \in\left[c_{k}, c_{k+1}\right]$, the effect of intervals other than $B_{k} \cup B_{k+1}$ on $a_{F}(s)$ is negligible (as given by (79)). Similarly, and by using Lemma 8, we can show that

$$
\begin{equation*}
\operatorname{Prob}\left[\theta \in B_{k+1} \mid \hat{s}_{k}\right]\left(\mathbb{E}\left[a_{L}(\theta) \mid \hat{s}_{k}, \theta \in B_{k+1}\right]-\mathbb{E}\left[a_{L}(\theta) \mid \hat{s}_{k}, \theta \in B_{k}\right]\right) \leq 1.17 e^{-\Delta_{k+1} \delta}\left(\Delta_{k+1}+2 \bar{r} \bar{x}_{m+1}\right), \tag{168}
\end{equation*}
$$

where $\delta=m_{k+1}-\hat{s}_{k}$. Combining this and (79), we can arrive at

$$
\begin{equation*}
\left|\mathbb{E}\left[a_{L}(\theta) \mid \hat{s}_{k}, \theta \in B_{k}\right]-\mathbb{E}\left[a_{L}(\theta) \mid \hat{s}_{k}\right]\right|<10^{-4} r_{L}^{2} \bar{x}_{1} \tag{169}
\end{equation*}
$$

After all, we have

$$
\begin{equation*}
\left|s_{k}-\left(1-r_{L}\right) c_{k}-r_{L} e_{k}\right|<1.02 r_{L}^{2} \bar{x}_{k}+0.41 \bar{r}^{3} \sigma^{2}\left(\bar{x}_{k}+\bar{x}_{k+1}\right)<1.9 r_{L}^{2} \bar{x}_{k+1} . \tag{170}
\end{equation*}
$$

The proof has to be modified for the tail case $k=m$, since in the tail $\left|a_{L}(\theta)-\hat{s}_{m}\right|<1$ does not hold for all $\theta \in B_{m}$. To handle this, we define $\mathbb{E}\left[a_{L}(\theta) \mid \hat{s}_{m}, \theta \in \hat{B}_{m}\right]=\hat{s}_{m}$, where $\hat{B}_{m}=\left[b_{m}, b_{m}+\sigma^{2}-\sigma\right]$. It is easy to verify that $\bar{r}\left(\sigma^{2}-\sigma\right)<1$ and hence $\left|a_{L}(\theta)-\hat{s}_{m}\right|<1$ holds in this interval. We also need to find an alternative to the variance inequality $\operatorname{Var}_{N\left(0, \sigma^{2}\right)}\left[\theta \mid \theta_{1} B_{k}\right] \leq$ $\frac{\left(b_{k+1}-b_{k}\right)^{2}}{(1+\sqrt{3})^{2} \sigma^{2}}$. Here we use $\operatorname{Var}_{N\left(0, \sigma^{2}\right)}\left[\theta \mid \theta \in B_{m}\right] \leq\left(e_{m}-b_{m}\right)^{2}$. Also, we can show $\Delta b_{m}(y)<\bar{r} e_{m}$, from which and a bit manipulation we can reach at $\operatorname{Var}_{N\left(0, \sigma^{2}\right)}\left[\theta \mid \theta \geq b_{m}-\Delta b_{m}(y)\right] \leq \frac{\left(e_{m}-b_{m}\right)^{2}}{(1-\bar{r})^{2}}$.

We can then show that the results derived above also holds for $\hat{B}_{m}$. More precisely,

$$
\begin{equation*}
\left|\hat{s}_{m}-\left(1-r_{L}\right) c_{m}-r_{L} \mathbb{E}_{N\left(0, \sigma^{2}\right)}\left[\theta \mid \theta \in \hat{B}_{m}\right]\right|<r_{L}^{2} \bar{x}_{m}+0.4 \bar{r}^{3} \sigma^{2}\left(\bar{x}_{m}+\bar{x}_{m+1}\right) \tag{171}
\end{equation*}
$$

Let $\theta_{b}=b_{m}+\sigma^{2}-\sigma$. We can easily observe that

$$
\begin{align*}
e_{m}-\mathbb{E}_{N\left(0, \sigma^{2}\right)}\left[\theta \mid \theta \in \hat{B}_{m}\right] & \leq \operatorname{Prob}\left[\theta \geq \theta_{b} \mid \theta \in B_{m}\right]\left(\mathbb{E}_{N\left(0, \sigma^{2}\right)}\left[\theta \mid \theta \geq \theta_{b}\right]-\mathbb{E}_{N\left(0, \sigma^{2}\right)}\left[\theta \mid \theta \in \hat{B}_{m}\right]\right) \\
& \leq \frac{\phi\left(\frac{\theta_{b}}{\sigma}\right)}{\phi\left(\frac{b_{m}}{\sigma}\right)}\left(\theta_{b}+\frac{\sigma^{2}}{\theta_{b}}-b_{m}\right)<10^{-4} r_{L}^{3} \sigma \tag{172}
\end{align*}
$$

We also need to bound $\left|\mathbb{E}\left[a_{L}(\theta) \mid \hat{s}_{m}, \theta \in \hat{B}_{m}\right]-\mathbb{E}\left[a_{L}(\theta) \mid \hat{s}_{m}\right]\right|$. We start with $\mid \mathbb{E}\left[a_{L}(\theta) \mid \hat{s}_{m}, \theta \in\right.$ $\left.\hat{B}_{m}\right]-\mathbb{E}\left[a_{L}(\theta) \mid \hat{s}_{m}, \theta \in B_{m}\right] \mid$. First, note that

$$
\begin{equation*}
a_{L}\left(\theta_{b}\right) \geq c_{m}+\underline{r}\left(\sigma^{2}-2 \sigma\right)>c_{m}+3 \bar{r} \sigma \tag{173}
\end{equation*}
$$

which along $\left|\hat{s}_{m}-c_{m}\right|<\bar{r} \sigma$ implies that $\left|a_{L}\left(\theta_{b}\right)-\hat{s}_{m}\right|>2 \bar{r} \sigma$. Consequently, $\left|\hat{s}_{m}-a_{L}\left(\theta^{\prime}\right)\right|<$ $\left|a_{L}\left(\theta_{b}\right)-\hat{s}_{m}\right|$ for all $\theta^{\prime} \in \hat{B}_{m}$. Similar to (62), we can hence derive

$$
\begin{equation*}
\operatorname{Prob}\left[\theta \mid \hat{s}_{m}, \theta \in B_{m}\right] \leq \frac{e^{-\frac{(\sigma-1)^{2}}{2} \phi\left(\frac{\theta}{\sigma}\right)}}{\sigma\left(1-\Phi\left(\frac{\theta_{b}}{\sigma}\right)\right)}, \tag{174}
\end{equation*}
$$

for $\theta \geq \theta_{b}$. Using this along with $\mathbb{E}\left[a_{L}(\theta) \mid \hat{s}_{m}, \theta \in \hat{B}_{m}\right] \geq c_{m}-\bar{r} \sigma$, we can then obtain

$$
\begin{equation*}
\mathbb{E}\left[a_{L}(\theta) \mid \hat{s}_{m}, \theta \in B_{m}\right]-\mathbb{E}\left[a_{L}(\theta) \mid \hat{s}_{m}, \theta \in \hat{B}_{m}\right]<10^{-4} r_{L}^{3} \sigma \tag{175}
\end{equation*}
$$

Therefore, the same steps as in the non-tail case can be followed in this case as well, resulting in (170) to also hold for $k=m$.

Proof of Theorem 1. We can find by direct calculation of the optimal quantizer for $m=25$ that $\frac{x_{1}^{\mathrm{Q}}}{\sigma}>0.041$. On the other hand, $\frac{2 \sqrt{2 \ln \sigma}+5}{\sigma}<0.04$ for $\sigma \geq 300$. This implies that $25 \in M(\sigma)$ for $\sigma \geq 300$ and hence is nonempty.

We next use (34) to verify that Property 2 is also preserved by the best response, completing the proof of the invariance of $A_{L}^{m}$. It suffices to show that

$$
\begin{equation*}
\left|\hat{e}_{k}-c_{k}^{\mathrm{Q}}\right|+0.42 r_{L}\left(\frac{\bar{x}_{k}+\bar{x}_{k+1}}{2}\right)^{2}+2 r_{L} \bar{x}_{k+1} \leq 2.9 \tag{176}
\end{equation*}
$$

To bound $\left|\hat{e}_{k}-c_{k}^{\mathrm{Q}}\right|$, we first note that $\frac{\partial}{\partial \hat{b}_{k}} \hat{e}_{k}+\frac{\partial}{\partial \hat{b}_{k+1}} \hat{e}_{k}=1-\frac{\operatorname{Var}_{N\left(0, \sigma^{2}\right)}\left[\theta \mid \theta \in \hat{B}_{k}\right]}{\sigma^{2}}$. Using the inequality
$\operatorname{Var}\left[\theta \mid \theta \in \hat{B}_{k}\right]>\left(\hat{e}_{k}-\mathbb{E}\left[\theta \mid \hat{b}_{k} \leq \theta \leq \hat{e}_{k}\right]\right)\left(\mathbb{E}\left[\theta \mid \hat{e}_{k} \leq \theta \leq \hat{b}_{k+1}\right]-\hat{e}_{k}\right)$ and Lemma 1 we can show that

$$
\begin{equation*}
\operatorname{Var}_{N\left(0, \sigma^{2}\right)}\left[\theta \mid \theta \in \hat{B}_{k}\right]>\frac{\underline{x}_{k} \underline{x}_{k+1}}{2(1+\sqrt{e})} \tag{177}
\end{equation*}
$$

yielding $\left|\hat{e}_{k}-c_{k}^{\mathrm{Q}}\right|<2.9\left(1-\frac{\underline{x}_{k} \underline{x}_{k+1}}{2(1+\sqrt{e}) \sigma^{2}}\right)$. Therefore to have (176), it suffices to have

$$
\begin{equation*}
\frac{0.42\left(\bar{x}_{k}+\bar{x}_{k+1}\right)^{2}}{4 \underline{x}_{k} \underline{x}_{k+1}}+\frac{2 \bar{x}_{k+1}}{\underline{x}_{k} \underline{x}_{k+1}} \leq \frac{2.9}{2(1+\sqrt{e})}, \tag{178}
\end{equation*}
$$

which is satisfied for sufficiently large values of $\sigma$ (e.g., $\sigma \geq 300$ ) on noting that $\frac{0.42(e+1)^{2}}{4 e} \leq$ $\frac{2.9}{2(1+\sqrt{e})}$. This completes the proof of the invariance of $A_{L}^{m}$.

The existence of an equilibrium with $a_{L}^{*}(\theta) \in A_{L}^{m}$ and $a_{F}^{*}(s)=\mathbb{E}_{\delta}\left[a_{L}^{*} \mid s\right]$ follows from an argument similar to the one used in [1] for the existence of an optimal solution. Let $\left(a_{L}^{n}(\theta), a_{F}^{n}(s)=\right.$ $\left.\mathbb{E}_{\delta}\left[a_{L}^{n} \mid s\right]\right)$ be a maximizing sequence for the ex-ante expected payoff of the leader, that is,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbb{E}_{\theta}\left[u_{L}\left(\theta, a_{L}^{n}(\theta), a_{F}^{n}(s)\right)\right]=\sup \left\{\mathbb{E}_{\theta}\left[u_{L}\left(\theta, a_{L}(\theta), a_{F}(s)\right)\right] \mid a_{L}(\theta) \in A_{L}^{m}, a_{F}(s)=\mathbb{E}_{\delta}\left[a_{L} \mid s\right]\right\} \tag{179}
\end{equation*}
$$

The first step is to show that this supremum is attained for a pair of strategies $\left(a_{L}^{*}(\theta), a_{F}^{*}(s)=\right.$ $\left.\mathbb{E}_{\delta}\left[a_{L}^{*} \mid s\right]\right)$ with $a_{L}^{*}(\theta) \in A_{L}^{m}$. Strategies $a_{L}(\theta) \in A_{L}^{m}$ are increasing and bounded $\left(\left|a_{L}(\theta)\right|<\right.$ $|\theta|+1$ ). Using Lemma 8 in [1] (which is a variation of the Helly's selection principle based on the diagonalisation argument), there exists a subsequence $a_{L}^{n_{k}}(\theta)$ converging pointwise to a limit strategy $a_{L}(\theta)$, and so do the interval endpoints and fixed points, that is, $\left\{b_{j}^{n_{k}}\right\} \rightarrow\left\{b_{j}\right\}$ and $\left\{c_{j}^{n_{k}}\right\} \rightarrow\left\{c_{j}\right\}$. Relabel $\left(a_{L}^{n_{k}}(\theta), a_{F}^{n_{k}}(s)\right)$ as $\left(a_{L}^{n}(\theta), a_{F}^{n}(s)\right)$. From $a_{L}^{n}(\theta) \rightarrow a_{L}(\theta)$, it is easy to see that $a_{L}(\theta)$ satisfies Property 1-2. Property 3, however, concerns the derivative of $a_{L}$ and thus cannot be deduced directly from pointwise convergence. We now turn into the follower's best response sequence $a_{F}^{n}(s)$, claiming that $a_{F}^{n}(s) \rightarrow a_{F}(s)$. The proof is again similar to the approach used in Theorem 1 in [1]: First note that

$$
\begin{equation*}
a_{F}^{n}(s)=\mathbb{E}_{\delta}\left[a_{L}^{n} \mid s\right]=\frac{\int_{-\infty}^{\infty} a_{L}^{n}(\theta) \phi\left(s-a_{L}^{n}(\theta)\right) \phi\left(\frac{\theta}{\sigma}\right) d \theta}{\int_{-\infty}^{\infty} \phi\left(s-a_{L}^{n}(\theta)\right) \phi\left(\frac{\theta}{\sigma}\right) d \theta} \tag{180}
\end{equation*}
$$

For every $s \in \mathbb{R}$, functions $\phi(s-x)$ and $x \phi(s-x)$ are continuous and bounded functions of $x$. Therefore, $\phi\left(s-a_{L}^{n}(\theta)\right) \rightarrow \phi\left(s-a_{L}(\theta)\right)$ and $a_{L}^{n}(\theta) \phi\left(s-a_{L}^{n}(\theta)\right) \rightarrow a_{L}(\theta) \phi\left(s-a_{L}(\theta)\right)$ pointwise in $\theta$, for all $s$. This proves $a_{F}^{n}(s) \rightarrow a_{F}(s)$ using bounded convergence theorem. We can
similarly show that $u_{L}\left(\theta, a_{L}^{n}(\theta), a_{F}^{n}(s)\right) \rightarrow u_{L}\left(\theta, a_{L}(\theta), a_{F}(s)\right)$ pointwise in $\theta$ and subsequently $\mathbb{E}_{\theta}\left[u_{L}\left(\theta, a_{L}^{n}(\theta), a_{F}^{n}(s)\right)\right] \rightarrow \mathbb{E}_{\theta}\left[u_{L}\left(\theta, a_{L}(\theta), a_{F}(s)\right)\right]$, that is, the supremum in (179) is attained by the pair of the strategies $\left(a_{L}(\theta), a_{F}(s)=\mathbb{E}_{\delta}\left[a_{L} \mid s\right]\right)$, though not ruling out the possibility of $a_{L}(\theta) \notin A_{L}^{m}$.

Exploiting the fact that $a_{F}$ is analytic and that $\frac{d}{d s} a_{F}(s)=\operatorname{Var}\left[a_{L} \mid s\right]$ and following an argument similar to above, we can then show that $\frac{d}{d s} a_{F}^{n}(s) \rightarrow \frac{d}{d s} a_{F}(s)$ pointwise in $s$. Therefore, the best response characteristics of the follower to a strategy in $A_{L}^{m}$ (as given in Lemma 6-7 and Corollary 1-2) which also involves bounds on the derivative, hold for $a_{F}(s)$. These conditions force the best response of the leader to $a_{F}(s)$, denoted by $\tilde{a}_{L}(\theta)$, to lie in $A_{L}^{m}$. On the other hand,

$$
\begin{equation*}
\mathbb{E}_{\theta}\left[u_{L}\left(\theta, \tilde{a}_{L}(\theta), \tilde{a}_{F}(s)\right)\right] \geq \mathbb{E}_{\theta}\left[u_{L}\left(\theta, \tilde{a}_{L}(\theta), a_{F}(s)\right)\right] \geq \mathbb{E}_{\theta}\left[u_{L}\left(\theta, a_{L}(\theta), a_{F}(s)\right)\right], \tag{181}
\end{equation*}
$$

where $\tilde{a}_{F}(s)$ is the follower's best response to $\tilde{a}_{L}(\theta) \in A_{L}^{m}$. Therefore, the pair of the strategies $\left(\tilde{a}_{L}(\theta), \tilde{a}_{F}(s)=\mathbb{E}_{\delta}\left[\tilde{a}_{L} \mid s\right]\right)$ where $\tilde{a}_{L}(\theta) \in A_{L}^{m}$ attains the supremum in (180), that is, it is a maximizer for the expected payoff of the leader over $A_{L}^{m}$.

Finally, from (181) we should have $\mathbb{E}_{\theta}\left[u_{L}\left(\theta, \tilde{a}_{L}(\theta), \tilde{a}_{F}(s)\right)\right]=\mathbb{E}_{\theta}\left[u_{L}\left(\theta, \tilde{a}_{L}(\theta), a_{F}(s)\right)\right]$. This implies that $\tilde{a}_{F}(s)=a_{F}(s)$ almost surely, otherwise replacing $a_{F}(s)$ with leader's best response $\tilde{a}_{F}(s)$ would result in a higher expected payoff for the leader. Noting that both $\tilde{a}_{F}(s)$ and $a_{F}(s)$ are analytic, almost surely equivalence implies being identical. This implies that the pair of the strategies $\left(\tilde{a}_{L}(\theta), a_{F}(s)\right)$ are best responses to each other, and hence correspond to an equilibrium of the game.
Proof of Lemma 2. We need to prove that there does not exist an infinitesimal variation of $\left(a_{L}^{*}, a_{F}^{*}\right)$, namely $\left(a_{L}^{\delta}, a_{F}^{\delta}\right)$, for which $U\left(a_{L}^{\delta}, a_{F}^{\delta}\right)<U\left(a_{L}^{*}, a_{F}^{*}\right)$. Noting that $U\left(a_{L}^{\delta}, \mathbb{E}\left[a_{L}^{\delta} \mid s\right]\right) \leq$ $U\left(a_{L}^{\delta}, a_{F}^{\delta}\right)$, we only need to consider the strategies in which the follower's action is the expected action of the leader given the observation $s$ (i.e., $\mathbb{E}\left[a_{L}^{\delta} \mid s\right]$ ). The idea is to show that for a sufficiently small $\delta_{L}>0$ and any strategy $a_{L}^{\delta}$ with $\left\|a_{L}^{\delta}-a_{L}^{*}\right\|_{\infty}<\delta_{L}$, the best response image obtained from $a_{L}^{\delta} \rightarrow a_{F}^{\delta} \rightarrow \tilde{a}_{L}^{\delta}$ lies in $A_{L}^{m}$. The proof then follows from the fact that $U\left(\tilde{a}_{L}^{\delta}, \mathbb{E}\left[\tilde{a}_{L}^{\delta} \mid s\right]\right) \leq U\left(a_{L}^{\delta}, \mathbb{E}\left[a_{L}^{\delta} \mid s\right]\right)$, and that $\left(a_{L}^{*}, a_{F}^{*}\right)$ is the minimizer of $U$ over all pair of strategies $\left(a_{L}, a_{F}\right)$ with $a_{L} \in A_{L}^{m}$ (see the proof of Theorem 1).

To prove the inclusion of $\tilde{a}_{L}^{\delta}$ in $A_{L}^{m}$, it suffices to show that all the properties for the follower's best response to a strategy in $A_{L}^{m}$ given specifically by Lemma 9-10 and Lemma 17 also hold
for $a_{F}^{\delta}$, noting that these are all we need to deduce Property 1-3 for the leader's best response (which define the set $A_{L}^{m}$ ). What is left is then to show that the properties for $a_{F}^{*}$ given by Lemma 9-10 and Lemma 17 also hold for $a_{F}^{\delta}(s)=\mathbb{E}\left[a_{L}^{\delta} \mid s\right]$ for sufficiently small $\delta$. The proof easily follows from a couple of simple observations. First, it is straightforward to verify that all the bounds given for $a_{F}^{*}$ in the aforementioned lemmas are indeed strict. Therefore, by recasting the corresponding inequalities as continuous functions of $\delta_{L}$ we can ensure that all of them will still hold for sufficiently small $\delta_{L}$. We elaborate on this in more details in what follows.

We start by verifying that Lemma 5-6 also hold for $a_{L}^{\delta}$ for small enough $\delta_{L}$. In Lemma 5,

$$
\begin{align*}
a_{L}^{\delta}(\theta)-c_{k}^{*} & \leq \delta_{L}+\bar{r}\left(b_{k+1}^{*}-c_{k}^{*}\right) \leq \delta_{L}+0.1 \bar{r} r_{L}+\bar{r} \frac{c_{k+1}^{*}-c_{k}^{*}}{2} \\
& \leq \delta_{L}+0.1 \bar{r} r_{L}+\bar{r}\left(x_{k+1}^{Q}+2.9\right) \leq \bar{r} \bar{x}_{k+1} \tag{182}
\end{align*}
$$

for small enough $\delta_{L}$, where we recall that $\bar{x}_{k+1}=x_{k+1}^{\mathrm{Q}}+3$. Similarly we can show that $a_{L}^{\delta}(\theta)-$ $c_{k}^{*} \geq-\bar{r} \bar{x}_{k}$, hence Lemma 5 also holds for $a_{L}^{\delta}$. Next, we study the effect of $\delta_{L}$ in Lemma 6. As for (49), using

$$
\begin{equation*}
\frac{\phi\left(s-a_{L}^{\delta}\left(\theta^{\prime}\right)\right)}{\phi\left(s-a_{L}^{\delta}(\theta)\right)} \geq \frac{\phi\left(s-a_{L}^{*}\left(\theta^{\prime}\right)\right)}{\phi\left(s-a_{L}^{*}(\theta)\right)} e^{-\delta_{L}^{2}-2 \delta_{L} \bar{x}_{m+1}} \tag{183}
\end{equation*}
$$

the RHS of the inequality will be multiplied by $e^{-\delta_{L}^{2}-2 \delta_{L} \bar{x}_{m+1}}$. As a result, the value of $\xi$ in (52) will be multiplied by $e^{\delta_{L}^{2}+2 \delta_{L} \bar{x}_{m+1}}$. (56) will then become

$$
\begin{equation*}
\mathbb{E}\left[a_{L}^{\delta}(\theta) \mid s, b_{m}^{*} \leq \theta \leq \theta_{c}^{*}\right]-c_{m}^{*} \leq \delta_{L}+1.128 \times 1.025 e^{\delta_{L}^{2}+2 \delta_{L} \bar{x}_{m+1}} \bar{r}_{m}<0.75 \bar{r} \bar{x}_{m+1}, \tag{184}
\end{equation*}
$$

for sufficiently small $\delta_{L}$. For the bound on variance in (57)-(58), let $\mu^{*}=\mathbb{E}\left[a_{L}^{*}(\theta) \mid s, b_{m}^{*} \leq \theta \leq\right.$ $\left.\theta_{c}^{*}\right]$. Then,

$$
\begin{align*}
& \operatorname{Var}\left[a_{L}^{\delta}(\theta) \mid s, b_{m}^{*} \leq \theta \leq \theta_{c}^{*}\right] \leq \operatorname{Var}\left[a_{L}^{*}(\theta) \mid s, b_{m}^{*} \leq \theta \leq \theta_{c}^{*}\right] \\
& \quad+\delta_{L}^{2}+2 \delta_{L} \sqrt{\operatorname{Var}\left[a_{L}^{*}(\theta) \mid s, b_{m}^{*} \leq \theta \leq \theta_{c}^{*}\right]} \tag{185}
\end{align*}
$$

Hence, (58) becomes

$$
\begin{gather*}
\operatorname{Var}\left[a_{L}^{\delta}(\theta) \mid s, b_{m}^{*} \leq \theta \leq \theta_{c}^{*}\right] \leq 1.025 e^{\delta_{L}^{2}+2 \delta_{L} \bar{x}_{m+1}} \times 1.128^{2} \bar{r}^{2}\left(\bar{x}_{m}+\frac{4 \bar{x}_{m}^{2} \bar{\mu}}{\sigma^{2}}\right)^{2} \\
+\delta_{L}^{2}+2.2 \delta_{L} \bar{r} \bar{x}_{m}<1.2 \bar{r}^{2} \bar{x}_{m}^{2} \tag{186}
\end{gather*}
$$

for sufficiently small $\delta_{L}$. As for the modification required in the tail effect, $e^{\frac{\delta^{2}}{2}}$ in (60) has to be replaced with $e^{\frac{\left(\delta+\delta_{L}\right)^{2}}{2}}$, using which we can verify that (62) still holds for small enough $\delta_{L}$. The rest of the changes are similar.

Lemma 7-8 are based on Lemma 5-6, and Lemma 9-10 are derived using Lemma 5-8, hence also hold for $a_{F}^{\delta}$. Finally, in Lemma 17 which is about the fixed points of the follower's strategy, noting $a_{L}^{\delta}\left(b_{k}^{*}\right) \leq a_{L}^{*}\left(b_{k}^{*}\right)+\delta_{L}$, we need to add $\delta_{L}$ to the RHS of (166). Using this, we can easily verify that this lemma also holds for $a_{F}^{\delta}$. Therefore, all the properties required for the follower's strategy to deduce Property 1-3 for the leader's best response are satisfied for $a_{F}^{\delta}$ for sufficiently small $\delta_{L}$, indicating that $\tilde{a}_{L}^{\delta}$ lies in $A_{L}^{m}$. This completes the proof.
Proof of Lemma 3. Using

$$
\begin{equation*}
\liminf _{m \rightarrow \infty} \frac{\frac{\left(x_{1}^{\mathrm{Q}}\right)^{2}}{\sqrt{3}}}{D_{L}^{Q}}=1, \tag{187}
\end{equation*}
$$

it suffices to show that $D_{F}^{\mathrm{Q}} \leq 4 \sqrt{\frac{2}{e}} \frac{\left(2-r_{L}\right)^{2}}{\left(1-r_{L}\right)^{2}} \phi\left(\frac{x_{1}^{\mathrm{Q}}}{\sqrt{2}}\right)+r_{L}^{2} D_{L}^{\mathrm{Q}}$. Consider an interval $B_{k}^{\mathrm{Q}}$ and some $\theta \in B_{k}^{Q}$. For any $j>k$ (similarly for $j<k$ ), we have

$$
\begin{equation*}
\frac{\left(c_{j}^{\mathrm{Q}}-a_{L}^{\mathrm{Q}}(\theta)\right)^{2}}{\left(b_{j}^{\mathrm{Q}}-a_{L}^{\mathrm{Q}}(\theta)\right)^{2}} \leq \frac{\left(2 x_{j}^{\mathrm{Q}}-r_{L} x_{j}^{\mathrm{Q}}\right)^{2}}{\left(x_{j}^{\mathrm{Q}}-r_{L} x_{j}^{\mathrm{Q}}\right)^{2}}=\frac{\left(2-r_{L}\right)^{2}}{\left(1-r_{L}\right)^{2}} . \tag{188}
\end{equation*}
$$

Using this, we can obtain

$$
\begin{align*}
& \int_{s \notin B_{k}^{Q}}\left(a_{F}^{\mathrm{Q}}(s)-a_{L}^{\mathrm{Q}}(\theta)\right)^{2} \phi\left(s-a_{L}(\theta)\right) d s \\
& \leq \frac{\left(2-r_{L}\right)^{2}}{\left(1-r_{L}\right)^{2}} \int_{s \notin B_{k}^{\mathrm{Q}}}\left(s-a_{L}^{\mathrm{Q}}(\theta)\right)^{2} \phi\left(s-a_{L}^{\mathrm{Q}}(\theta)\right) d s . \tag{189}
\end{align*}
$$

Combining this with the inequality $\int_{a}^{\infty} x^{2} \phi(x) d x \leq 2 \max \left(x e^{-\frac{x^{2}}{4}}\right) \phi\left(\frac{a}{\sqrt{2}}\right)$, we arrive at

$$
\begin{equation*}
\int_{s \notin B_{k}^{\mathrm{Q}}}\left(a_{F}^{\mathrm{Q}}(s)-a_{L}^{\mathrm{Q}}(\theta)\right)^{2} \phi\left(s-a_{L}^{\mathrm{Q}}(\theta)\right) d s \leq 4 \sqrt{\frac{2}{e}} \phi\left(\frac{x_{1}^{\mathrm{Q}}}{\sqrt{2}}\right) . \tag{190}
\end{equation*}
$$

On the other hand,

$$
\begin{align*}
& \int_{s \in B_{k}^{\mathrm{Q}}}\left(a_{F}^{\mathrm{Q}}(s)-a_{L}^{\mathrm{Q}}(\theta)\right)^{2} \phi\left(s-a_{L}(\theta)\right) d s \\
& \leq\left(c_{k}^{\mathrm{Q}}-a_{L}^{\mathrm{Q}}(\theta)\right)^{2}=r_{L}^{2}\left(a_{F}^{\mathrm{Q}}(\theta)-\theta\right)^{2} \tag{191}
\end{align*}
$$

implying that

$$
\begin{align*}
& \int_{-\infty}^{\infty} \int_{s \in B_{k}^{\mathrm{Q}}}\left(a_{F}^{\mathrm{Q}}(s)-a_{L}^{\mathrm{Q}}(\theta)\right)^{2} \phi\left(s-a_{L}^{\mathrm{Q}}(\theta)\right) \frac{\phi\left(\frac{\theta}{\sigma}\right)}{\sigma} d s d \theta \leq \\
& r_{L}^{2} \int_{-\infty}^{\infty}\left(\theta-a_{F}^{\mathrm{Q}}(\theta)\right)^{2} \frac{\phi\left(\frac{\theta}{\sigma}\right)}{\sigma} d \theta=r_{L}^{2} D_{L}^{\mathrm{Q}}, \tag{192}
\end{align*}
$$

which completes the proof.
Proof of Lemma 4. First we note that the minimum value of the cost functional $U\left(a_{L}, a_{F}\right)$ with $r_{L} \sigma^{2}=1$ is asymptotically the same as the optimal cost of Witsenhausen's problem for $k^{2} \sigma^{2}=1$. Using the inequality given by (17) in the proof of Theorem 4 in [15], we can obtain

$$
\begin{equation*}
U^{*}(\sigma)>\min _{P^{*}>0.5}\left\{k^{2} P^{*}+\frac{1}{15} e^{-12 P^{*}}\right\}, \tag{193}
\end{equation*}
$$

noting that in the scalar version of Witsenhausen's problem we have $m=1$. Minimizing the RHS above we can find $U^{*}(\sigma)>\frac{\ln \sigma}{6 \sigma^{2}}+\frac{1-\ln 1.25}{12 \sigma^{2}}$, which completes the proof.
Proof of Theorem 2. The first part of the theorem follows directly from Lemma 2. Using $M(\sigma)=$ $\left\{m \in \mathbb{N} \mid x_{1}^{\mathrm{Q}}>2 \sqrt{2 \ln \sigma}+4, m \geq 25\right\}$ and that $m \frac{x_{1}^{\mathrm{Q}}}{\sigma} \approx \frac{\sqrt{6 \pi}}{4}$ for large $m$, we get

$$
\begin{equation*}
M(\sigma) \approx\left\{m \in \mathbb{N} \left\lvert\, 25<m<\frac{\sqrt{6 \pi} \sigma}{8 \sqrt{2 \ln \sigma}+20}\right.\right\} \tag{194}
\end{equation*}
$$

Denote with $x_{1}^{*}$ the minimizer of the asymptotic upper bound on $U\left(a_{L}^{*}, a_{F}^{*}\right)$ given by Lemma 3. Then, it is easy to verify that $\lim _{\sigma \rightarrow \infty} \frac{x_{1}^{*}}{2 \sqrt{2 \ln \sigma}}=1$, which clearly intersects $M(\sigma)$ for large $\sigma$. The corresponding asymptotic cost is $\approx \frac{8 r_{L} \ln \sigma}{\sqrt{3}}$, hence proving (11). We can use Lemma 3 and Lemma 4 to show that the equilibrium corresponding to $\operatorname{argmin}_{m \in M(\sigma)} U\left(a_{L}^{*}, a_{F}^{*}\right)$ is at most $\frac{\frac{8 r_{L} \ln \sigma}{\sqrt{3}}}{\frac{\ln \sigma}{6 \sigma^{2}}}=16 \sqrt{3}<27.8$ away from the optimal cost as $\sigma \rightarrow \infty$.

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[^1]:    ${ }^{1}$ Another relevant setting in which nonlinear equilibria may emerge is the seminal work of Crawford and Sobel ([3]) on signaling games where misaligned objectives of a sender and a receiver can result in quantized equilibrium strategies. Recently, authors in [4] consider an extension of this model to a noisy channel setup and show that for a Gaussian source and scalar signals the equilibrium encoder is linear.

[^2]:    ${ }^{2}$ Note that this does not imply the continuity of the optimal solution.
    ${ }^{3}$ Games of strategic complementarities are those in which the best response of each player is increasing in actions of others [21].

[^3]:    ${ }^{4}$ See, e.g., [26] for a definition of perfect Bayesian equilibrium and Bayes Nash equilibrium.

[^4]:    ${ }^{5}$ This clearly covers the case $k^{2} \sigma^{2}=1$.

[^5]:    ${ }^{6} \mathrm{We}$ state the properties (and in many cases the analysis) only for $\theta \geq 0$. The counterpart for $\theta \leq 0$ is immediate since the function is odd.
    ${ }^{7}$ As we will see in Section IV, this ensures inclusion of local minima with an expected cost within a constant factor of the optimal cost.

[^6]:    ${ }^{8}$ Recall that $x_{1}^{\mathrm{Q}}$ is the first half-step in a $(2 m+1)$-level optimal MSE quantizer.
    ${ }^{9}$ We thank Anant Sahai for bringing this point into the authors' attention.

[^7]:    ${ }^{10}$ Recall that $\delta \sim N(0,1)$ is the noise in the follower's observation.

[^8]:    ${ }^{11}$ The ratio between the upper and lower bounds in [15] is almost 100 . For the well-known case of $\sigma=5$ and $k^{2} \sigma^{2}=1$, the lowest known cost $\approx 0.167$ is only 12.5 times the value obtained from the lower bound.

[^9]:    ${ }^{12}$ See the proof of Lemma 11 for details.

[^10]:    ${ }^{13}$ Note that $\frac{2 \sqrt{2 \ln \sigma}}{5}<\frac{x_{1}}{5}<\frac{\Delta_{k+1}}{10}$

[^11]:    ${ }^{14}$ See, e.g., Proposition 14.2.2 in [28].

[^12]:    ${ }^{15}$ Note that $\tilde{a}_{L}>a_{F}\left(\tilde{a}_{L}\right)>c_{m}-1.01 \bar{r} \bar{x}_{m+1}$.

