

INVARIANCE FEEDBACK ENTROPY OF UNCERTAIN CONTROL SYSTEMS

MAHENDRA SINGH TOMAR, MATTHIAS RUNGGER, AND MAJID ZAMANI

ABSTRACT. We introduce a novel notion of invariance feedback entropy to quantify the state information that is required by any controller that enforces a given subset of the state space to be invariant. We establish a number of elementary properties, e.g. we provide conditions that ensure that the invariance feedback entropy is finite and show for the deterministic case that we recover the well-known notion of entropy for deterministic control systems. We prove the data rate theorem, which shows that the invariance entropy is a tight lower bound of the data rate of any coder-controller that achieves invariance in the closed loop. We analyze uncertain linear control systems and derive a universal lower bound of the invariance feedback entropy. The lower bound depends on the absolute value of the determinant of the system matrix and a ratio involving the volume of the invariant set and the set of uncertainties. Furthermore, we derive a lower bound of the data rate of any static, memoryless coder-controller. Both lower bounds are intimately related and for certain cases it is possible to bound the performance loss due to the restriction to static coder-controllers by 1 bit/time unit. We provide various examples throughout the paper to illustrate and discuss different definitions and results.

1. INTRODUCTION

In this work we study the classical feedback control loop, in which a controller that is feedback connected with a given system is used to enforce a prespecified control task in the closed loop. Unlike in the classical setting, we do not assume that the sensor (or coder) is able to transmit an infinite amount of information to the controller, but is restricted to use a digital noiseless channel with a bounded data rate to communicate with the controller. The closed loop of such a feedback is illustrated in Fig. 1. In this context, we are interested in characterizing the minimal data rate of the digital channel between coder and controller that enables the controller to achieve the given control task. Or equivalently, we are interested in quantifying the information required by the controller to achieve a given control goal.

Data rate constrained feedback is a mature research topic and has been extensively studied for linear control systems and asymptotic stabilizability, see e.g. [1] and references therein. Remarkably, for this class of synthesis problems, the critical data rate has been characterized in terms of the unstable eigenvalues of the system matrix independent of the particular disturbance model [2–4].

We are interested in minimal data rates necessary for a coder-controller scheme to render a given nonempty subset of the state space invariant. Invariance specifications are one of the most fundamental system requirements and are ubiquitous in the analysis and control of dynamical systems [5, 6]. In [7], Nair et. al extended the well-known notion of topological entropy of dynamical systems [8–10] to discrete-time deterministic

Key words and phrases. Networked control systems, communication channel, entropy, invariance, data rate constrained feedback.

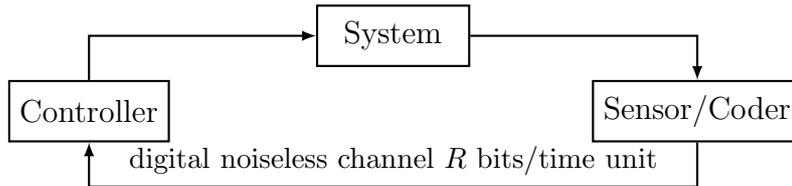


FIGURE 1. Coder-controller feedback loop.

control systems and showed that the topological feedback entropy characterizes the data rate necessary to achieve invariance. Later Colonius and Kawan [11] introduced a notion of invariance entropy for continuous-time deterministic control systems. While the definition in [7] clearly resembles the definition of entropy for dynamical systems in [8] based on open covers, the invariance entropy introduced in [11] is close to the notion of entropy in [9, 10] based on spanning sets. Both notions coincide for discrete-time control systems provided that a strong invariance condition holds [12, 13].

In this paper, we continue this line of research and introduce a notion of invariance feedback entropy for uncertain control systems to characterize the necessary state information required by any controller to enforce the invariance condition in the closed loop. After we introduce the notation used in this paper in Section 2, we motivate the need of the novel notion of invariance feedback entropy in Section 3. We define invariance feedback entropy and establish various elementary properties in Section 4. We show that the entropy is nonincreasing across two systems that are related via a feedback refinement relation [14]. This result generalizes the fact that the invariance entropy of deterministic control systems cannot increase under semiconjugation [11, Thm 3.5], [13, Prp. 2.13]. We provide conditions that ensure that the invariance feedback entropy is finite and show that we recover the notion of invariance feedback entropy known for deterministic control systems, in the deterministic case. We establish the data rate theorem in Section 5. It shows that the invariance entropy provides a tight lower bound on the data rate of any coder-controller that enforces the invariance specification in the closed loop. To this end, we introduce a history-dependent notion of data rate. We discuss possible alternative data rate definitions and motivate our particular choice by two examples. We continue with the analysis of uncertain linear control systems in Section 6. We derive a lower bound on the invariance feedback entropy. The lower bound depends on the absolute value of the determinant of the system matrix and a ratio involving the volume of the invariant set and the set of uncertainties. The lower bound is invariant under state space transformations and recovers the well-known minimal data rate [1] in the absence of uncertainties. Similar to [1, Section II], in the derivation we make use of the Brunn-Minkowsky inequality for compact, measurable sets. Additionally, we derive a lower bound of the data rate of any *static, memoryless* coder-controller. Both lower bounds are intimately related and for certain cases it is possible to bound the performance loss due to the restriction to static coder-controllers by $\log_2(1 + 1/2^{h_{\text{inv}}(Q)})$, where $h_{\text{inv}}(Q)$ is the invariance feedback entropy of the uncertain linear systems, i.e., the best possible (dynamically) achievable data rate, and Q is the set of interest. We show that the lower bounds are tight for certain classes of systems.

A preliminary version of the results presented in Sections 3-5 appeared in [15] and the results on uncertain linear systems (Section 6) appeared in [16]. This paper provides a

detailed and extended elaboration of the results proposed in [15, 16], including the new results presented in Theorem 1 and Theorem 5.

2. NOTATION

We denote by \mathbb{N} , \mathbb{Z} and \mathbb{R} the set of natural, integer and real numbers, respectively. We annotate those symbols with subscripts to restrict the sets in the obvious way, e.g. $\mathbb{R}_{>0}$ denotes the positive real numbers. We denote the closed, open and half-open intervals in \mathbb{R} with endpoints a and b by $[a, b]$, $]a, b[$, $[a, b[$, and $]a, b]$, respectively. The corresponding intervals in \mathbb{Z} are denoted by $[a; b]$, $]a; b[$, $[a; b[$, and $]a; b]$, i.e., $[a; b] = [a, b] \cap \mathbb{Z}$ and $]a; a[= \emptyset$.

For a set A , we use $\#A \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ to denote the number of elements of A , i.e., if A is finite we have $\#A \in \mathbb{Z}_{>0}$ and $\#A = \infty$ otherwise. Given two sets A and B , we say that A is smaller (larger) than B if $\#A \leq \#B$ ($\#A \geq \#B$) holds. A set J of subsets of A is said to *cover* B , where $B \subseteq A$, if B is a subset of the union of elements of J . A *cover* of set B , is a set of subsets of B that covers B .

We use $\exists_{a \in A} x = a$ to refer to: there exists a in A such that $x = a$. In a similar way, $\forall_{a \in A} x = a$ is used. Given two sets $A, B \subseteq \mathbb{R}^n$, we define the set addition by $A + B := \{x \in \mathbb{R}^n \mid \exists_{a \in A}, \exists_{b \in B} x = a + b\}$ and denote by $\dim(A)$ the dimension of set A . For $A = \{a\}$, we slightly abuse notation and use $a + B = \{a\} + B$. The symbols $\text{cl} A$, $\text{int} A$ and $\wp(A)$ denote the closure, the interior and the power set of a set A , respectively. We call a set $A \subseteq \mathbb{R}^n$ measurable if it is Lebesgue measurable and use $\mu(A)$ to denote its measure [17]. We use id to denote an identity map.

We follow [18] and use $f: A \rightrightarrows B$ to denote a *set-valued map* from A into B , whereas $f: A \rightarrow B$ denotes an ordinary map. If f is set-valued, then f is *strict* if for every $a \in A$ we have $f(a) \neq \emptyset$. The restriction of f to a subset $M \subseteq A$ is denoted by $f|_M$. By convention we set $f|_{\emptyset} := \emptyset$. The composition of $f: A \rightrightarrows B$ and $g: C \rightrightarrows A$, $(f \circ g)(x) = f(g(x))$ is denoted by $f \circ g$. We use B^A to denote the set of all functions $f: A \rightarrow B$.

For a relation $R \subseteq A \times B$ and $D \subseteq A$, we define $R(D) := \cup_{d \in D} R(d)$.

The concatenation of two functions $x: [0; a[\rightarrow X$ and $y: [0; b[\rightarrow X$ with $a \in \mathbb{N}$ and $b \in \mathbb{N} \cup \{\infty\}$ is denoted by xy which we define by $xy(t) := x(t)$ for $t \in [0; a[$ and $xy(t) := y(t - a)$ for $t \in [a, a + b[$.

We use $\inf \emptyset = \infty$, $\log_2 \infty = \infty$ and $0 \cdot \infty = 0$.

3. MOTIVATION

We study data rate constrained feedback for discrete-time *uncertain control systems* described by difference inclusions of the form

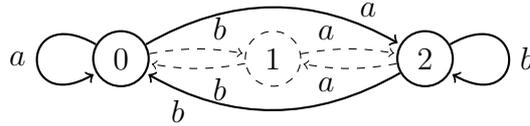
$$\xi(t+1) \in F(\xi(t), \nu(t)) \quad (1)$$

where $\xi(t) \in X$ is the *state signal* and $\nu(t) \in U$ is the *input signal*. The sets X and U are referred to as *state alphabet* and *input alphabet*, respectively. The map $F: X \times U \rightrightarrows X$ is called the *transition function*.

We are interested in coder-controllers that force the system (1) to evolve inside a nonempty set Q of the state alphabet X , i.e., every state signal ξ of the closed loop illustrated in Fig. 1 with $\xi(0) \in Q$ satisfies $\xi(t) \in Q$ for all $t \in \mathbb{Z}_{\geq 0}$. Specifically, we are interested in the average data rate of such coder-controllers.

Notably, our system description is rather general and, depending on the structure of alphabets X and U , we can represent a variety of commonly used system models. If we assume X and U to be discrete, we can use (1) to represent discrete event systems¹ [19] and digital/embedded systems [20]. Let us consider the following simple example.

Example 1. Consider a system with state alphabet and input alphabet given by $X := \{0, 1, 2\}$ and $U := \{a, b\}$, respectively. The transition function is illustrated by



The set of interest is defined to $Q := \{0, 2\}$. The transitions and states that lead, respectively, are outside Q are indicated by dashed lines. When the system is in state 0 the only valid input is given by a . Similarly, if the system is in state 2 the only valid input is given by b . If the input a is applied at 0 at time t , the system can either be in 0 or 2 at time $t + 1$. Note that the valid control inputs for the states 0 and 2 differ and the controller is required to have exact state information at every point in time. Due to the nondeterministic transition function, it is not possible to determine the current state of the system based on the knowledge of the past states, the past control inputs and the transition function. Therefore, the controller can obtain the state information only through measurement, which implies that at least one bit needs to be transmitted at every time step. \square

Current theories [7, 11, 13, 21] are unable to explain the minimal data rate of one bit per time step observed in Example 1.

If we allow X and U to be (subsets of) Euclidean spaces, we are able to recover one of the most fundamental system models in control theory, i.e., the class of nonlinear control systems with bounded uncertainties [6, 22]. If the system description is given in continuous-time, we can use (1) to represent the sampled-data system [23] with sampling time $\tau \in \mathbb{R}_{>0}$ as illustrated in Fig. 2. The disturbance signal ω is assumed to

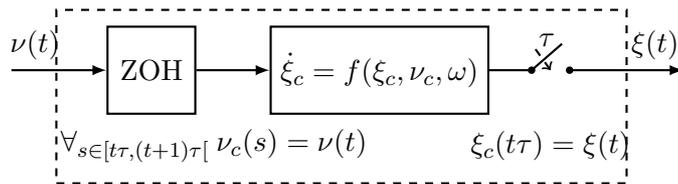


FIGURE 2. Sampled-data discrete-time system.

be bounded $\omega(s) \in W \subseteq \mathbb{R}^p$ for all times $s \in \mathbb{R}_{\geq 0}$. The transition function $F(x, u)$ is defined as the set of states that are reachable by the continuous-time system at time τ from initial state x under constant input signal $\nu_c(s) = u$ and a bounded disturbance signal ω . If the continuous-time dynamic is linear, the sampled-data system results in a discrete-time system of the form

$$\xi(t + 1) \in A\xi(t) + B\nu(t) + W \quad (2)$$

¹If (1) represents a discrete event system, the data rate unit is given in bits/event.

where A and B are matrices of appropriate dimension and W is a nonempty set representing the uncertainties.

Example 2. Consider an instance of (2) with $X := \mathbb{R}$, $U := [-1, 1]$ and

$$F(x, u) := \frac{1}{2}x + u + [-3, 3]$$

with the set of constraints given by $Q := [-4, 4]$. \square

For Example 2, we establish in Section 6, that the smallest possible data rate of a coder-controller that enforces Q to be invariant is one bit per time step. The example demonstrates that in contrast to linear systems without disturbances, where the data rate depends only on the unstable eigenvalues, see e.g. [11, Thm. 5.1] or [4], for systems of the form (2) the data rate depends among other things also on the stable eigenvalues.

4. INVARIANCE FEEDBACK ENTROPY

We introduce the notion of invariance feedback entropy and establish some elementary properties.

4.1. The entropy. Formally, we define a *system* as triple

$$\Sigma := (X, U, F) \tag{3}$$

where X and U are nonempty sets and $F : X \times U \rightrightarrows X$ is assumed to be strict. A *trajectory* of (3) on $[0; \tau[$ with $\tau \in \mathbb{N} \cup \{\infty\}$ is a pair of sequences (ξ, ν) , consisting of a state signal $\xi : [0; \tau + 1[\rightarrow X$ and an input signal $\nu : [0; \tau[\rightarrow U$, that satisfies (1) for all $t \in [0; \tau[$. We denote the set of all trajectories on $[0; \infty[$ by $\mathcal{B}(\Sigma)$.

Throughout the paper, we call a system (X, U, F) *finite* if X and U are finite.

We follow [7] and [11, Sec. 6] and define the invariance feedback entropy with the help of covers of Q . Consider the system $\Sigma = (X, U, F)$ and a nonempty set $Q \subseteq X$. A cover \mathcal{A} of Q and a function $G : \mathcal{A} \rightarrow U$ is called an *invariant cover* (\mathcal{A}, G) of Σ and Q if \mathcal{A} is finite and for all $A \in \mathcal{A}$ we have $F(A, G(A)) \subseteq Q$.

Consider an invariant cover (\mathcal{A}, G) of Σ and Q , fix $\tau \in \mathbb{N}$ and let $\mathcal{S} \subseteq \mathcal{A}^{[0; \tau[}$ be a set of sequences in \mathcal{A} . For $\alpha \in \mathcal{S}$ and $t \in [0; \tau - 1[$ we define

$$P(\alpha|_{[0; t]}) := \{A \in \mathcal{A} \mid \exists \hat{\alpha} \in \mathcal{S} \hat{\alpha}|_{[0; t]} = \alpha|_{[0; t]} \wedge A = \hat{\alpha}(t + 1)\}.$$

The set $P(\alpha|_{[0; t]})$ contains the cover elements A so that the sequence $\alpha|_{[0; t]}A$ can be extended to a sequence in \mathcal{S} . For $t = \tau - 1$ we have $\alpha|_{[0; \tau - 1]} = \alpha$ and we define for notational convenience the set

$$P(\alpha) := \{A \in \mathcal{A} \mid \exists \hat{\alpha} \in \mathcal{S} A = \hat{\alpha}(0)\}$$

which is actually independent of $\alpha \in \mathcal{S}$ and corresponds to the “initial” cover elements A in \mathcal{S} , i.e., there exists $\alpha \in \mathcal{S}$ with $A = \alpha(0)$. A set $\mathcal{S} \subseteq \mathcal{A}^{[0; \tau[}$ is called (τ, Q) -*spanning* in (\mathcal{A}, G) if the set $P(\alpha)$ with $\alpha \in \mathcal{S}$ covers Q and we have

$$\forall \alpha \in \mathcal{S} \forall t \in [0; \tau - 1[F(\alpha(t), G(\alpha(t))) \subseteq \bigcup_{A' \in P(\alpha|_{[0; t]})} A'. \tag{4}$$

We associate with every (τ, Q) -spanning set \mathcal{S} the *expansion number* $N(\mathcal{S})$, which we define by

$$N(\mathcal{S}) := \max_{\alpha \in \mathcal{S}} \prod_{t=0}^{\tau-1} \#P(\alpha|_{[0; t]}).$$

A tight lower bound on the expansion number of any (τ, Q) -spanning set \mathcal{S} in (\mathcal{A}, G) is given by

$$r_{\text{inv}}(\tau, Q) := \min \{N(\mathcal{S}) \mid \mathcal{S} \text{ is } (\tau, Q)\text{-spanning in } (\mathcal{A}, G)\}.$$

We define the *entropy* of an invariant cover (\mathcal{A}, G) by

$$h(\mathcal{A}, G) := \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log_2 r_{\text{inv}}(\tau, Q). \quad (5)$$

As shown in Lemma 1 (stated below), the limit of the sequence in (5) exists so that the entropy of an invariant cover (\mathcal{A}, G) is well-defined.

The *invariance feedback entropy* of Σ and Q follows by

$$h_{\text{inv}}(Q) := \inf_{(\mathcal{A}, G)} h(\mathcal{A}, G) \quad (6)$$

where we take the infimum over all (\mathcal{A}, G) invariant covers of Σ and Q . Let us revisit the examples from the previous section to illustrate the various definitions.

Example 1 (Continued). First, we determine an invariant cover (\mathcal{A}, G) of the system in Example 1 and Q . Since the system is finite, we can set $\mathcal{A} := \{\{x\} \mid x \in Q\}$. Recall that $Q = \{0, 2\}$ and a suitable function G is given by $G(\{0\}) := a$ and $G(\{2\}) := b$. Suppose that $\mathcal{S} \subseteq \mathcal{A}^{[0; \tau[}$ is (τ, Q) -spanning with $\tau \in \mathbb{N}$. Let us look at condition (4) for $t \in [0; \tau - 1[$ and $\alpha \in \mathcal{S}$. If $\alpha(t) = \{0\}$, we have $P(\alpha|_{[0; t]}) = \{\{0\}, \{2\}\}$ since $F(\{0\}, G(\{0\})) = F(0, a) = \{0, 2\}$. If $\alpha(t) = \{2\}$ the same reasoning leads to $P(\alpha|_{[0; t]}) = \{\{0\}, \{2\}\}$. Also for $\alpha \in \mathcal{S}$ we have $P(\alpha) = \{\{0\}, \{2\}\}$ since $P(\alpha)$ is required to be a cover of Q . It follows that $\mathcal{S} = \mathcal{A}^{[0; \tau[}$ and the expansion number $N(\mathcal{S}) = r_{\text{inv}}(\mathcal{A}, G) = 2^\tau$ so that the entropy of the (\mathcal{A}, G) follows to $h(\mathcal{A}, G) = 1$. Since (\mathcal{A}, G) is the only invariant cover we obtain $h_{\text{inv}}(Q) = 1$. \square

Example 2 (Continued). Let us recall the linear system in Example 2. An invariant cover (\mathcal{A}, G) is given by $\mathcal{A} := \{a_0, a_1\}$ with $a_0 := [-4, 0]$, $a_1 := [0, 4]$ and $G(a_0) := 1$, $G(a_1) := -1$. Let \mathcal{S} be any (τ, Q) -spanning set in (\mathcal{A}, G) . As $P(\alpha) \subseteq \mathcal{A}$ is required to cover Q , so $P(\alpha) = \mathcal{A}$. For $a_i \in \mathcal{A}$, $i \in \{0, 1\}$ we have $F(a_i, G(a_i)) = [-4; 4]$ which makes $P(a_i) = \mathcal{A}$. Thus $\mathcal{S} = \mathcal{A}^{[0; \tau[}$. Since $\#\mathcal{A} = 2$, we obtain that $h(\mathcal{A}, G) = 1$. \square

We continue with showing the subadditivity of $\log_2 r_{\text{inv}}(\cdot, Q)$.

Lemma 1. *Consider the system $\Sigma = (X, U, F)$ and a nonempty set $Q \subseteq X$. Let (\mathcal{A}, G) be an invariant cover of Σ and Q , then the function $\tau \mapsto \log_2 r_{\text{inv}}(\tau, Q)$, $\mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ is subadditive, i.e., for all $\tau_1, \tau_2 \in \mathbb{N}$ the inequality*

$$\log_2 r_{\text{inv}}(\tau_1 + \tau_2, Q) \leq \log_2 r_{\text{inv}}(\tau_1, Q) + \log_2 r_{\text{inv}}(\tau_2, Q)$$

holds and we have

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log_2 r_{\text{inv}}(\tau, Q) = \inf_{\tau \in \mathbb{N}} \frac{1}{\tau} \log_2 r_{\text{inv}}(\tau, Q). \quad (7)$$

The following lemma might be of independent interest. We use it in the proof of Theorem 4.

Lemma 2. *Consider an invariant cover (\mathcal{A}, G) of (3) and some nonempty set $Q \subseteq X$. Let \mathcal{S} be a (τ, Q) -spanning set, then we have $\#\mathcal{S} \leq N(\mathcal{S})$.*

The proofs of both lemmas are given in the appendix.

4.2. Entropy across related systems. One of the most important properties of entropy of classical dynamical systems is its invariance under any change of coordinates [8, Thm. 1]. In [11] this property has been shown for deterministic control systems in the context of semiconjugation [11, Thm. 3.5]. In the following, we present a result in the context of feedback refinement relations [14], which contains the result on semiconjugation as a special case.

Definition 1. Let Σ_1 and Σ_2 be two systems of the form

$$\Sigma_i = (X_i, U_i, F_i) \text{ with } i \in \{1, 2\}. \quad (8)$$

A strict relation $R \subseteq X_1 \times X_2$ is a feedback refinement relation from Σ_1 to Σ_2 if there exists a map $r : U_2 \rightarrow U_1$ so that the following inclusion holds for all $(x_1, x_2) \in R$ and $u \in U_2$

$$R(F_1(x_1, r(u))) \subseteq F_2(x_2, u). \quad (9)$$

Theorem 1. Consider two systems Σ_i , $i \in \{1, 2\}$ of the form (8). Let Q_1 and Q_2 be two nonempty subsets of X_1 and X_2 , respectively. Suppose that R is a feedback refinement relation from Σ_1 to Σ_2 , and $Q_1 = R^{-1}(Q_2)$. Then

$$h_{1,\text{inv}}(Q_1) \leq h_{2,\text{inv}}(Q_2) \quad (10)$$

holds, where $h_{i,\text{inv}}(Q_i)$ is the invariance feedback entropy of Σ_i and Q_i .

Proof. If $h_{2,\text{inv}}(Q_2) = \infty$, the inequality holds and subsequently we consider the case $h_{2,\text{inv}}(Q_2) < \infty$. We will make use of Lemma 9 in the Appendix to show (10). Let us pick an invariant cover (\mathcal{A}_2, G_2) of Σ_2 and Q_2 so that $h(\mathcal{A}_2, G_2) < \infty$. Next we define the set $\mathcal{A}_1 := \{A_1 \subseteq Q_1 \mid \exists A_2 \in \mathcal{A}_2 \ R^{-1}(A_2) = A_1\}$.

Now let $M = R^{-1}$ and $r : U_2 \rightarrow U_1$ in Lemma 9, where R and r are, respectively, the relation and map associated with the feedback refinement relation in Def. 1. We observe that all the conditions 1) - 4) in Lemma 9 hold.

Thus there exists a map $G_1^* : \mathcal{A}_1 \rightarrow U_1$ such that (\mathcal{A}_1, G_1^*) is an invariant cover of Σ_1 and Q_1 , and

$$h(\mathcal{A}_1, G_1^*) \leq h(\mathcal{A}_2, G_2).$$

Therefore, inequality (10) holds. \square

4.3. Conditions for finiteness. We analyze two particular instances of systems – finite systems and systems with a topological state alphabet – and provide conditions ensuring that the invariance entropy is finite. The results are based on the following lemma.

Lemma 3. Consider a system $\Sigma = (X, U, F)$ and a nonempty set $Q \subseteq X$. There exists an invariant cover (\mathcal{A}, G) of Σ and Q iff $h_{\text{inv}}(Q) < \infty$.

Proof. It follows immediately from (6) that $h_{\text{inv}}(Q) < \infty$ implies the existence of an invariant cover of Σ and Q . For the reverse direction, we assume that (\mathcal{A}, G) is an invariant cover of Σ and Q . We fix $\tau \in \mathbb{N}$ and define $\mathcal{S} := \{\alpha \in \mathcal{A}^{[0;\tau]} \mid \forall t \in [0;\tau-1] \ \alpha(t+1) \cap F(\alpha(t), G(\alpha(t))) \neq \emptyset\}$. It is easy to verify that \mathcal{S} is (τ, Q) -spanning and $N(\mathcal{S}) \leq (\#\mathcal{A})^\tau$. An upper bound on $h_{\text{inv}}(Q)$ follows by $\log_2 \#\mathcal{A}$. \square

If Σ is finite, it is rather straightforward to show that the controlled invariance of Q w.r.t. Σ is necessary and sufficient for $h_{\text{inv}}(Q)$ to be finite. Let us recall the notion of controlled invariance [6].

We call $Q \subseteq X$ *controlled invariant* with respect to a system $\Sigma = (X, U, F)$, if for all $x \in Q$ there exists $u \in U$ so that $F(x, u) \subseteq Q$. We refer the interested readers to [24] for a discussion on computation of controlled invariant set for controllable linear discrete-time systems.

Theorem 2. *Consider a finite system $\Sigma = (X, U, F)$ and a nonempty set $Q \subseteq X$. Then $h_{\text{inv}}(Q) < \infty$ if and only if Q is controlled invariant.*

Proof. Let $h_{\text{inv}}(Q)$ be finite. Then there exists an invariant cover (\mathcal{A}, G) so that $h(\mathcal{A}, G) < \infty$. Hence, for every $x \in Q$ we can pick an $A \in \mathcal{A}$ with $x \in A$, so that $F(x, G(A)) \subseteq F(A, G(A)) \subseteq Q$. Hence Q is controlled invariant w.r.t. Σ .

Assume Q is controlled invariant w.r.t. Σ . For $x \in Q$, let $u_x \in U$ be such that $F(x, u_x) \subseteq Q$. It is easy to check that (\mathcal{A}, G) with $\mathcal{A} := \{\{x\} \mid x \in Q\}$ and $G(\{x\}) := u_x$ is an invariant cover of Σ and Q , so that the assertion follows from Lemma 3. \square

In general controlled invariance of Q is not sufficient to guarantee finiteness of the invariance feedback entropy as shown in the next example.

Example 3. Consider $\Sigma = (\mathbb{R}, [-1, 1], F)$ with the dynamics given by $F(x, u) := x + u + [-1, 1]$. Let $Q := [-1, 1]$, then for every $x \in Q$ we can pick $u = -x$ so that $F(x, u) = [-1, 1] \subseteq Q$, which shows that Q is controlled invariant. Now suppose that $h_{\text{inv}}(Q)$ is finite. Then according to Lemma 3 there exists an invariant cover (\mathcal{A}, G) of Σ and Q . Since \mathcal{A} is required to be finite, there exists $A \in \mathcal{A}$ with an infinite number of elements and therefore we can pick two different states in A , i.e., $x, x' \in A$ with $x \neq x'$. However, there does not exist a single $u \in U$ so that $F(x, u) \subseteq Q$ and $F(x', u) \subseteq Q$. Hence, (\mathcal{A}, G) cannot be an invariant cover, which implies $h_{\text{inv}}(Q) = \infty$. \square

In the subsequent theorem we present some conditions for systems with a topological state alphabet, which imply the finiteness of the invariance entropy. The conditions may be difficult to verify for a particular problem instance. Nevertheless with these conditions, we follow closely the assumptions based on continuity and strong invariance employed in [1, 12] to ensure finiteness of the invariance entropy for deterministic systems. We use the following notion of continuity of set-valued maps [25] to show the next result.

Let A and B be topological spaces and $f : A \rightrightarrows B$. We say that f is *upper semicontinuous*, if for every $a \in A$ and every open set $V \subseteq B$ containing $f(a)$ there exists an open set $U \subseteq A$ with $a \in U$ so that $f(U) \subseteq V$.

Theorem 3. *Consider a system $\Sigma = (X, U, F)$ and a nonempty compact subset Q of X . Let X be a topological space. If $F(\cdot, u)$ is upper semicontinuous for every $u \in U$ and Q is strongly controlled invariant, i.e., for all $x \in Q$ there exists $u \in U$ so that $F(x, u) \subseteq \text{int } Q$, then $h_{\text{inv}}(Q) < \infty$.*

Proof. For each $x \in Q$, we pick an input $u_x \in U$ so that $F(x, u_x) \subseteq \text{int } Q$. Since $F(\cdot, u_x)$ is upper semicontinuous and $\text{int } Q$ is open, there exists an open subset A_x of X , so that $x \in A_x$ and $F(A_x, u_x) \subseteq \text{int } Q$. Hence, the set $\{A_x \mid x \in Q\}$ of open subsets of X covers Q . Since Q is a compact subset of X , there exists a finite set $\{A_{x_1}, \dots, A_{x_m}\}$ so that $Q \subseteq \cup_{i \in [1; m]} A_{x_i}$ [26, Ch. 2.6]. Let $\mathcal{A} := \{A_{x_1} \cap Q, \dots, A_{x_m} \cap Q\}$ and define for every $i \in [1; m]$ the function $G(A_{x_i}) := u_{x_i}$. Then (\mathcal{A}, G) is an invariant cover of Σ and Q , and the assertion follows from Lemma 3. \square

Example 3 (Continued). Let $\varepsilon > 0$, consider Σ from Example 3 with the modified input set $U_\varepsilon := [-1 - \varepsilon, 1 + \varepsilon]$. Let $Q_\varepsilon := [-1 - \varepsilon, 1 + \varepsilon]$ then we see that Q_ε is strongly controlled invariant. We construct an invariant cover for Σ and Q_ε as follows. We define n as the smallest integer larger than $\frac{1}{2\varepsilon}$ and introduce $\{x_{-n}, \dots, x_0, \dots, x_n\}$ with $x_i := 2i\varepsilon$ and set $A_i := (x_i + [-\varepsilon, \varepsilon]) \cap Q_\varepsilon$. For each $i \in [-n; n]$ we define $G(A_i) := -x_i$ so that $F(A_i, G(A_i)) \subseteq Q_\varepsilon$. By definition of n we have $x_{-n} \leq -1$ and $1 \leq x_n$ and we see that (\mathcal{A}, G) with $\mathcal{A} := \{A_i \mid i \in [-n; n]\}$ is an invariant cover of Σ and Q_ε . Hence, it follows from Lemma 3 that $h_{\text{inv}}(Q_\varepsilon)$ is finite. \square

4.4. Deterministic systems. For deterministic systems we recover the notion of invariance feedback entropy in [7, 12].

Let us consider the map $f : X \times U \rightarrow X$ representing a deterministic system

$$\xi(t+1) = f(\xi(t), \nu(t)). \quad (11)$$

We can interpret (11) as special instance of (3), where F is given by $F(x, u) := \{f(x, u)\}$ for all $x \in X$ and $u \in U$ and the notions of a trajectory of (3) extend to (11) in the obvious way. Given an input $u \in U$, we introduce $f_u : X \rightarrow X$ by $f_u(x) := f(x, u)$ and extend this notation to sequences $\nu \in U^{[0;t]}$, $t \in \mathbb{N}$ by

$$f_\nu(x) := f_{\nu(t)} \circ \dots \circ f_{\nu(0)}(x).$$

We follow [12] to define the entropy of (11). Consider a nonempty set $Q \subseteq X$ and fix $\tau \in \mathbb{N}$. A set $\mathcal{S} \subseteq U^{[0;\tau]}$ is called (τ, Q) -spanning for f and Q , if for every $x \in Q$ there exists $\nu \in \mathcal{S}$ so that the associated trajectory (ξ, ν) on $[0; \tau]$ of (11) with $\xi(0) = x$ satisfies $\xi([0; \tau]) \subseteq Q$. We use $r_{\text{det}}(\tau, Q)$ to denote the number of elements of the smallest (τ, Q) -spanning set

$$r_{\text{det}}(\tau, Q) := \inf\{\#\mathcal{S} \mid \mathcal{S} \text{ is } (\tau, Q)\text{-spanning}\}. \quad (12)$$

The *(deterministic) invariance entropy* of (X, U, f) and Q is defined by

$$h_{\text{det}}(Q) := \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log_2 r_{\text{det}}(\tau, Q). \quad (13)$$

Again the function $\tau \mapsto \frac{1}{\tau} \log_2 r_{\text{det}}(\tau, Q)$ is subadditive [12, Prop. 2.2] which ensures that the limit in (13) exists.

Now, we have the following theorem.

Theorem 4. *Consider the system $\Sigma = (X, U, F)$ and a nonempty set $Q \subseteq X$. Suppose F satisfy $F(x, u) = \{f(x, u)\}$ for all $x \in X$, $u \in U$ for some $f : X \times U \rightarrow X$. Then the invariance feedback entropy of Σ and Q equals the deterministic invariance entropy of (X, U, f) and Q , i.e.,*

$$h_{\text{inv}}(Q) = h_{\text{det}}(Q). \quad (14)$$

Proof. We begin with the inequality $h_{\text{det}}(Q) \geq h_{\text{inv}}(Q)$. If $h_{\text{det}}(Q) = \infty$ the inequality trivially holds and subsequently we assume that $h_{\text{det}}(Q)$ is finite. We fix $\varepsilon > 0$ and pick $\tau \in \mathbb{N}$ so that $\frac{1}{\tau} \log_2 r_{\text{det}}(\tau, Q) \leq h_{\text{det}}(Q) + \varepsilon$. We chose a (τ, Q) -spanning set \mathcal{S}_{det} for f and Q with $\#\mathcal{S}_{\text{det}} = r_{\text{det}}(\tau, Q)$. For every $\nu \in \mathcal{S}_{\text{det}}$ we define the sets

$$A_0(\nu) := Q \cap \bigcap_{t=0}^{\tau-1} f_{\nu|_{[0;t]}}^{-1}(Q)$$

and for $t \in [0; \tau - 1[$ the sets $A_{t+1}(\nu) := f(A_t(\nu), \nu(t))$. The minimality of \mathcal{S}_{det} implies that $A_0(\nu) \neq \emptyset$ and $A_0(\nu) \neq A_0(\nu')$ for all $\nu, \nu' \in \mathcal{S}_{\text{det}}$. Let \mathcal{A} be the set of all sets $A_t(\nu)$

with $t \in [0; \tau[$ and $\nu \in \mathcal{S}_{\det}$. With each $A \in \mathcal{A}$ we associate a single pair (ν, t) , where $\nu \in \mathcal{S}_{\det}$ and $t \in [0; \tau[$, such that it satisfies $A = A_t(\nu)$ and the following condition: $\nu' \in \mathcal{S}_{\det}$ and $t' \in [0; \tau[$ with $A = A_{t'}(\nu')$ implies $t \leq t'$. Then we define the map $G : \mathcal{A} \rightarrow U$ by $G(A) = \nu(t)$ where (ν, t) is associated with A . By the definition of $A_t(\nu)$, it is easy to see that $f(A_t(\nu), G(A_t(\nu))) \subseteq Q$ for all $t \in [0; \tau[$ and $\nu \in \mathcal{S}_{\det}$. Moreover, since \mathcal{S}_{\det} is (τ, Q) -spanning, for every $x \in Q$ there is $\nu \in \mathcal{S}_{\det}$ so that for all $t \in [0; \tau[$ we have $f_{\nu|_{[0;t]}}(x) \in Q$ which implies $x \in A_0(\nu)$ and we see that $\{A_0(\nu) \mid \nu \in \mathcal{S}_{\det}\}$ covers Q . It follows that (\mathcal{A}, G) is an invariant cover of (X, U, F) and Q . Let \mathcal{S}_{inv} be the set of sequences $\alpha : [0; \tau[\rightarrow \mathcal{A}$ defined iteratively as $\alpha(0) \in \{A_0(\nu) \mid \nu \in \mathcal{S}_{\det}\}$ and $\alpha(t+1) = f(\alpha(t), G(\alpha(t)))$. Then $P(\alpha)$ covers Q since $\{A_0(\nu) \mid \nu \in \mathcal{S}_{\det}\}$ covers Q as discussed above. For any distinct $\alpha, \alpha' \in \mathcal{S}_{\text{inv}}$ we have $\alpha(0) \neq \alpha'(0)$ so for every $t \in [0; \tau - 1[$ we have $\#P(\alpha|_{[0;t]}) = 1$, $f(\alpha(t), G(\alpha(t))) = P(\alpha|_{[0;t]})$ and thus \mathcal{S}_{inv} satisfies (4). Therefore, \mathcal{S}_{inv} is (τ, Q) -spanning in (\mathcal{A}, G) . Moreover, as $\nu \neq \nu'$ implies $A_0(\nu) \neq A_0(\nu')$, we have $\#P(\alpha) = \#\mathcal{S}_{\det}$, so that $r_{\text{inv}}(\tau, Q) \leq N(\mathcal{S}_{\text{inv}}) = \#\mathcal{S}_{\det} = r_{\det}(\tau, Q)$ follows. Due to Lemma 1, we have $\log_2 r_{\text{inv}}(n\tau, Q) \leq n \log_2 r_{\text{inv}}(\tau, Q)$ and we see that $\frac{1}{\tau} \log_2 r_{\text{inv}}(\tau, Q)$ (and therefore $\frac{1}{\tau} \log_2 r_{\det}(\tau, Q)$) provides an upper bound for $h(\mathcal{A}, G)$ so that we obtain $h_{\text{inv}}(Q) \leq h(\mathcal{A}, G) \leq h_{\det}(Q) + \varepsilon$. Since this holds for any $\varepsilon > 0$ we obtain the desired inequality.

We continue with the inequality $h_{\det}(Q) \leq h_{\text{inv}}(Q)$. If $h_{\text{inv}}(Q) = \infty$ the inequality trivially holds and subsequently we assume $h_{\text{inv}}(Q) < \infty$. We fix $\varepsilon > 0$ and pick an invariant cover (\mathcal{A}, G) of Σ and Q so that $h(\mathcal{A}, G) \leq h_{\text{inv}}(Q) + \varepsilon$. We fix $\tau \in \mathbb{N}$ and pick a (τ, Q) -spanning set \mathcal{S}_{inv} in (\mathcal{A}, G) so that $N(\mathcal{S}_{\text{inv}}) = r_{\text{inv}}(\tau, Q)$. We define for every $\alpha \in \mathcal{S}_{\text{inv}}$ the input sequence $\nu_\alpha : [0; \tau[\rightarrow U$ by $\nu_\alpha(t) := G(\alpha(t))$ and introduce the set $\mathcal{S}_{\det} := \{\nu_\alpha \mid \alpha \in \mathcal{S}_{\text{inv}}\}$. For $x \in Q$ we iteratively construct $\alpha \in \mathcal{A}^{[0;\tau[}$ and $\nu \in U^{[0;\tau[}$ as follows: for $t = 0$ we pick $\alpha_0 \in \mathcal{S}_{\text{inv}}$ so that $x \in \alpha_0(0)$ and set $\nu(0) := G(\alpha_0(0))$. For $t \in [0; \tau - 1[$ we pick $\alpha_{t+1} \in \mathcal{S}_{\text{inv}}$ so that $\alpha_{t+1}|_{[0;t]} = \alpha_t$ and $f_{\nu|_{[0;t]}}(x) \in \alpha_{t+1}(t+1)$ and set $\nu(t+1) := G(\alpha_{t+1}(t+1))$. Since (\mathcal{A}, G) is an invariant cover of (X, U, F) and Q , it is easy to show that $f_{\nu|_{[0;t]}}(x) \in Q$ holds for all $t \in [0; \tau[$, which implies that \mathcal{S}_{\det} is (τ, Q) -spanning for f and Q . Thus, we obtain $r_{\det}(\tau, Q) \leq \#\mathcal{S}_{\det} \leq \#\mathcal{S}_{\text{inv}} \leq N(\mathcal{S}_{\text{inv}}) = r_{\text{inv}}(\tau, Q)$, where the inequality $\#\mathcal{S}_{\text{inv}} \leq N(\mathcal{S}_{\text{inv}})$ follows from Lemma 2. Since this holds for any $\tau \in \mathbb{N}$, we obtain the inequality $\varepsilon + h_{\text{inv}}(Q) \geq h(\mathcal{A}, G) \geq h_{\det}(Q)$ for arbitrary $\varepsilon > 0$ which shows $h_{\text{inv}}(Q) \geq h_{\det}(Q)$. \square

4.5. Invariant covers with closed elements. We conclude this section with a result on the topological structure of the cover elements for systems with topological state alphabet and lower semicontinuous transition functions and closed sets Q . The result is used in Theorem 7 but might be of interest on its own.

Let A and B be topological spaces and $f : A \rightrightarrows B$. We say that f is *lower semicontinuous* if $f^{-1}(V)$ is open whenever $V \subseteq B$ is open.

Theorem 5. *Consider a system $\Sigma = (X, U, F)$ with topological state alphabet and a nonempty closed set $Q \subseteq X$. Assume that $F(\cdot, u)$ is lower semicontinuous for every $u \in U$. Let (\mathcal{A}, G) be an invariant cover of Σ and Q and let $\mathcal{C} := \{\text{cl } A \subseteq \text{cl } X \mid A \in \mathcal{A}\}$. Then there exists a map $H^* : \mathcal{C} \rightarrow U$ such that (\mathcal{C}, H^*) is an invariant cover of Σ and Q and*

$$h(\mathcal{C}, H^*) \leq h(\mathcal{A}, G). \quad (15)$$

In the proof of the theorem, we use the following lemma.

Lemma 4. *Let X be a topological space and $f : X \rightrightarrows X$. If f is lower semicontinuous then $f(\text{cl } \Omega) \subseteq \text{cl } f(\Omega)$ holds for every nonempty subset $\Omega \subseteq X$.*

Proof. For the sake of contradiction, suppose there exists $x \in \text{cl } \Omega$, $y \in f(x)$ and $y \notin \text{cl } f(\Omega)$. Then the open set $V := X \setminus \text{cl } f(\Omega)$ contains y . Let us define $U := f^{-1}(V) = \{x' \in X \mid f(x') \cap V \neq \emptyset\}$ and since f is lower semicontinuous and V is open so U is open. As $V \cap f(x) \ni y$, thus nonempty, so $x \in U$. By definition, $V \cap f(\Omega) = \emptyset$ so $U \cap \Omega = \emptyset$ and since U is open so $U \cap \text{cl } \Omega = \emptyset$ which is in contradiction with $x \in U$ and $x \in \text{cl } \Omega$. \square

Proof of Theorem 5. In Lemma 9 in the Appendix, let $M = \text{cl}$, $\Sigma_1 = \Sigma_2 = \Sigma$, $Q_2 = Q_1 = Q$, $\mathcal{A}_2 = \mathcal{A}$, $G_2 = G$, $\mathcal{A}_1 = \mathcal{C}$ and $r = \text{id}$, then one can easily verify that conditions 1) - 3) hold, while Lemma 4 implies that 4) is satisfied. Thus there exists a map $H^* : \mathcal{C} \rightarrow U$ such that (\mathcal{C}, H^*) is an invariant cover of Σ and Q , and $h(\mathcal{C}, H^*) \leq h(\mathcal{A}, G)$. \square

5. DATA-RATE-LIMITED FEEDBACK

We present the data rate theorem associated with the invariance feedback entropy of uncertain control systems. It shows that the invariance feedback entropy is a tight lower bound of the data rate of any coder-controller scheme that renders the set of interest invariant.

We introduce a history-dependent definition of data rates of coder-controllers with which we extend previously used time-invariant [1] and time-varying [7, 11] notions. We interpret the history-dependent definition of data rate as a nonstochastic variant of the notion of data rate used e.g. in [27, Def. 4.1] for noisy linear systems, defined as the average of the expected length of the transmitted symbols in the closed loop. We motivate the particular notion of data rate by two examples; one which illustrates that the time-varying definition [7] results in too large data rates and one which shows that the notion of data rate based on the framework of nonstochastic information theory, used in [28, 29] for estimation [29] and control [28] of linear systems, leads to too small data rates.

5.1. The coder-controller. We assume that a coder for the system (3) is located at the sensor side (see Fig. 1), which at every time step, encodes the current state of the system using the finite *coding alphabet* S . It transmits a symbol $s_t \in S$ via the discrete noiseless channel to the controller. The transmitted symbol $s_t \in S$ might depend on all past states and is determined by the *coder function*

$$\gamma : \bigcup_{t \in \mathbb{Z}_{\geq 0}} X^{[0:t]} \rightarrow S.$$

At time $t \in \mathbb{Z}_{\geq 0}$, the controller received $t + 1$ symbols $s_0 \dots s_t$, which are used to determine the control input given by the *controller function*

$$\delta : \bigcup_{t \in \mathbb{Z}_{\geq 0}} S^{[0:t]} \rightarrow U.$$

A *coder-controller* for (3) is a triple $H := (S, \gamma, \delta)$, where S is a coding alphabet and γ and δ are a compatible coder function and controller function, respectively.

Given a coder-controller (S, γ, δ) for (3) and $\xi \in X^{[0:t]}$ with $t \in \mathbb{Z}_{\geq 0}$, let us use the mapping

$$\Gamma_t : X^{[0:t]} \rightarrow S^{[0:t]}$$

to denote the sequence $\zeta = \Gamma_t(\xi)$ of coder symbols generated by ξ , i.e., $\zeta(t') = \gamma(\xi|_{[0;t']})$ holds for all $t' \in [0; t]$. Subsequently, for $\zeta \in S^{[0;t]}$ with $t \in \mathbb{N}$, we use

$$Z(\zeta) := \{s \in S \mid \exists_{(\xi, \nu) \in \mathcal{B}(\Sigma)} \zeta s = \Gamma(\xi|_{[0;t]}) \wedge \forall_{t' \in [0;t]} \nu(t') = \delta(\zeta|_{[0;t']})\} \quad (16)$$

to denote the possible successor coder symbols s of the symbol sequence ζ in the closed loop illustrated in Fig. 1. For notational convenience, let us use the convention $Z(\emptyset) := S$, so that $Z(\zeta|_{[0;0]}) = S$ for any sequence ζ in S . For $\tau \in \mathbb{N} \cup \{\infty\}$, we introduce the set

$$\mathcal{Z}_\tau := \{\zeta \in S^{[0;\tau]} \mid \zeta(0) \in \gamma(X) \wedge \forall_{t \in [0;\tau]} \zeta(t) \in Z(\zeta|_{[0;t]})\}$$

and define the *transmission data rate* of a coder-controller H by

$$R(H) := \limsup_{\tau \rightarrow \infty} \max_{\zeta \in \mathcal{Z}_\tau} \frac{1}{\tau} \sum_{t=0}^{\tau-1} \log_2 \#Z(\zeta|_{[0;t]}) \quad (17)$$

as the asymptotic average numbers of symbols in $Z(\zeta)$ considering the worst-case of possible symbol sequences $\zeta \in \mathcal{Z}_\tau$.

A coder-controller $H = (S, \gamma, \delta)$ for (3) is called *Q-admissible* where Q is a nonempty subset of X , if for every trajectory (ξ, ν) on $[0; \infty[$ of (3) that satisfies

$$\xi(0) \in Q \text{ and } \forall_{t \in \mathbb{Z}_{\geq 0}} \nu(t) = \delta(\Gamma_t(\xi|_{[0;t]})) \quad (18)$$

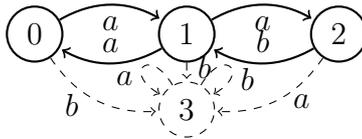
we have $\xi(\mathbb{Z}_{\geq 0}) \subseteq Q$. Let us use $\mathcal{B}_Q(H)$ to denote the set of all trajectories (ξ, ν) on $[0; \infty[$ of (3) that satisfy (18).

5.1.1. Time-varying data rate definition. We follow [7] and introduce a time-varying notion of data rate for a coder-controller $H = (S, \gamma, \delta)$ for (3). Let $(S_t)_{t \geq 0}$ be the sequence in the power set of S that for each $t \in \mathbb{Z}_{\geq 0}$ contains the smallest number of symbols so that $\gamma(\xi) \in S_t$ holds for all $\xi \in X^{[0;t]}$. Then the time-varying data rate of H follows by

$$R_{\text{tv}}(H) := \liminf_{\tau \rightarrow \infty} \frac{1}{\tau} \sum_{t=0}^{\tau-1} \log_2 \#S_t.$$

In the following we use an example to show that there exists a Q -admissible coder-controller H , which satisfies $R(H) < R_{\text{tv}}(\bar{H})$ for any Q -admissible coder-controller \bar{H} . Note that this inequality is purely a nondeterministic phenomenon: if the control system is deterministic, it follows from the deterministic and the nondeterministic data rate theorem ([7, Thm. 1] and Theorem 6 below) and the equivalence $h_{\text{det}}(Q) = h_{\text{inv}}(Q)$ (Theorem 4) that the different notions of data rates coincide in the sense that $\inf_H R(H) = \inf_H R_{\text{tv}}(H)$ (at least if the strong invariance condition in [7, Thm. 1] holds).

Example 4. Consider an instance of (3) with $U := \{a, b\}$, $X := \{0, 1, 2, 3\}$ and F is illustrated by



Let $Q := \{0, 1, 2\}$. The transitions that lead outside Q and the states that are outside Q are marked by dashed lines. Consider the coder-controller $H = (S, \gamma, \delta)$ with $S := X$ and γ and δ are given for $\xi \in X^{[0;t]}$, $t \in \mathbb{Z}_{\geq 0}$, by $\gamma(\xi) := \xi(t)$ and $\delta(\xi) := a$ if $\xi(t) \in \{0, 1, 3\}$

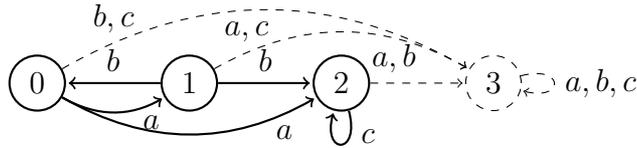
and $\delta(\xi) := b$ if $\xi(t) = 2$. We compute the number of possible successor symbols $Z(\xi)$ for $\xi \in X^{[0;t]}$, $t \in \mathbb{Z}_{\geq 0}$, by $\#Z(\xi) = 1$ if $\xi(t) \in \{0, 2, 3\}$ and $\#Z(\xi) = 2$ if $\xi(t) = 1$. It is easy to verify that H is Q -admissible. Since the state $\xi(t) = 1$ occurs only every other time step for any element (ξ, ν) of the closed loop, we compute the data rate to $R(H) = 1/2$. Consider a time-varying Q -admissible coder-controller $\bar{H} = (\bar{S}, \bar{\gamma}, \bar{\delta})$. Initially, the states $\{0, 1\}$ and $\{2\}$ need to be distinguishable at the controller side in order to confine the system to Q so that $\#\bar{S}_0 \geq 2$ follows. At time $t = 1$, the system is possibly again in any of the states $\{0, 1, 2\}$ (depending on the initial condition) and we have $\#\bar{S}_1 \geq 2$. By continuing this argument we see that $\#\bar{S}_t \geq 2$ for all $t \in \mathbb{Z}_{\geq 0}$ and $R_{\text{tv}}(\bar{H}) \geq 1$ follows. \square

5.1.2. *Zero-error capacity of uncertain channels.* Alternatively to the definition of the data rate of a coder-controller in (17) we could follow [28, 29] and define the data rate of a coder-controller as the zero-error capacity C_0 of an *ideal stationary memoryless uncertain channel* (SMUC) in the nonstochastic information theory framework presented in [29, Def. 4.1]. The input alphabet of the SMUC equals the output alphabet and is given by S . The channel is ideal and does not introduce any error in the transmission. Hence, the transition function is the identity, i.e., $T(s) = s$ holds for all $s \in S$. The input function space $\mathcal{Z}_\infty \subseteq S^{[0;\infty[}$ is the set of all possible symbol sequences that are generated by the closed loop, which represents the total amount of information that needs to be transmitted by the channel. For the ideal SMUC, the zero-error capacity [29, Eq. (25)], for a coder-controller H results in

$$C_0(H) := \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log_2 \#\mathcal{Z}_\tau.$$

We use the following example to demonstrate that the zero-error capacity is too low, i.e., $C_0(H) = 0$ while $R(H) \geq 1$.

Example 5. Consider an instance of (3) with $U := \{a, b\}$, $X := \{0, 1, 2, 3\}$ and F is illustrated by



The transitions and states that lead, respectively, are outside the set of interest $Q := \{0, 1, 2\}$ are dashed. Consider the Q -admissible coder-controller $H = (S, \gamma, \delta)$ with $S := X$ and γ and δ are given for $\xi \in X^{[0;t]}$, $t \in \mathbb{Z}_{\geq 0}$ by $\gamma(\xi) := \xi(t)$ and

$$\delta(\xi) := \begin{cases} a & \text{if } \xi(t) \in \{0, 3\} \\ b & \text{if } \xi(t) = 1 \\ c & \text{if } \xi(t) = 2. \end{cases}$$

We pick the trajectory $(\xi, \nu) \in \mathcal{B}_Q(H)$ given for $t \in \mathbb{Z}_{\geq 0}$ by $\xi(2t) = 0$ and $\xi(2t + 1) = 1$. We obtain $Z(\xi|_{[0;t]}) = \{1, 2\}$ if $\xi(t) = 0$ and $Z(\xi|_{[0;t]}) = \{0, 2\}$ if $\xi(t) = 1$. Since $\#F(x, u) \leq 2$ for all $x \in X$ and $u \in U$, it is straightforward to see that $\sum_{t=0}^{\tau-1} \log_2 \#Z(\xi|_{[0;t]}) = \max_{\zeta \in \mathcal{Z}_\tau} \sum_{t=0}^{\tau-1} \log_2 \#Z(\zeta|_{[0;t]})$ holds for all $\tau \in \mathbb{N}$. Hence, we obtain $R(H) = 1$.

We are going to derive $C_0(H)$. Consider the set $\mathcal{Z}_\tau \subseteq X^{[0;\tau[}$ and the hypothesis for

$\tau \in \mathbb{N}$: there exists at most one $\xi \in \mathcal{Z}_\tau$ with $\xi(\tau - 1) = 1$ and there exists at most one $\xi \in \mathcal{Z}_\tau$ with $\xi(\tau - 1) = 0$. For $\tau = 1$ we have $\mathcal{Z}_1 = X$ and the hypothesis holds. Suppose the hypothesis holds for $\tau \in \mathbb{N}$ and let $\xi \in \mathcal{Z}_\tau$. We have $Z(\xi) = \{0, 2\}$ if $\xi(t) = 1$, $Z(\xi) = \{1, 2\}$ if $\xi(t) = 0$, $Z(\xi) = \{2\}$ if $\xi(t) = 2$ and $Z(\xi) = \{3\}$ if $\xi(t) = 3$, so that the hypothesis holds for $\tau + 1$, which shows that the hypothesis holds for every $\tau \in \mathbb{N}$. Therefore, we obtain a bound of the number of elements in \mathcal{Z}_τ by $4 + 2(\tau - 1)$ and the zero-error capacity of H follows by $C_0(H) = 0$. \square

Example 5 shows that even though, the asymptotic average of the total amount of information that needs to be transmitted (= symbol sequences generated by the closed loop) via the channel is zero, the necessary (and sufficient) data rate to confine the system Σ within Q is one. The discrepancy results from the causality constraints that are imposed on the coder-controller structure by the invariance condition, i.e., at each instant in time the controller needs to be able to produce a control input so that all successor states are inside Q see e.g. [27]. Contrary to this observation, the zero-error capacity is an adequate measure for data rate constraints for deterministic linear systems (without disturbances) [28, 29].

5.1.3. Periodic coder-controllers. In the proof of the data rate theorem, we work with periodic coder-controllers. Given $\tau \in \mathbb{N}$ and a coder-controller $H = (S, \gamma, \delta)$, we say that H is τ -periodic if for all $t \in \mathbb{Z}_{\geq 0}$, $\zeta \in S^{[0;t]}$ and $\xi \in X^{[0;t]}$ we have

$$\begin{aligned} \gamma(\xi) &= \gamma(\xi|_{[\tau \lfloor t/\tau \rfloor; t]}), \\ \delta(\zeta) &= \delta(\zeta|_{[\tau \lfloor t/\tau \rfloor; t]}). \end{aligned} \tag{19}$$

Lemma 5. *The transmission data rate of a τ -periodic coder-controller $H = (S, \gamma, \delta)$ for (3) is given by*

$$R(H) = \max_{\zeta \in \mathcal{Z}_\tau} \frac{1}{\tau} \sum_{t=0}^{\tau-1} \log_2 \#Z(\zeta|_{[0;t]}). \tag{20}$$

Proof. Let L denote the right-hand-side of (20). Consider $T \in \mathbb{N}$, $\zeta \in \mathcal{Z}_T$ and set $a := \lfloor T/\tau \rfloor$ and $\bar{\tau} := T - \tau a$. We define $\zeta_i := \zeta|_{[i\tau; (i+1)\tau]}$ for $i \in [0; a[$ and $\zeta_a := \zeta|_{[a\tau; T]}$. Since γ is τ -periodic, we see that each ζ_i with $i \in [0; a[$ is an element of \mathcal{Z}_τ , and we obtain for $N_i := \sum_{t=0}^{\tau-1} \log_2 \#Z(\zeta_i|_{[0;t]})$ the bound $N_i \leq L\tau$ for all $i \in [0; a[$. We define $N_a := \sum_{t=0}^{\bar{\tau}-1} \log_2 \#Z(\zeta_a|_{[0;t]})$ which is bounded by $N_a \leq \tau \log_2 \#S$. Note that $a\tau + \bar{\tau} = T$, so that for $C := \tau \log_2 \#S$ we have

$$\frac{1}{T} \sum_{t=0}^{T-1} \log_2 \#Z(\zeta|_{[0;t]}) = \frac{1}{T} (\sum_{i=0}^{a-1} N_i + N_a) \leq \frac{1}{T} (aL\tau + L\bar{\tau} + C) = L + \frac{C}{T}.$$

Since C is independent of T , the assertion follows. \square

Lemma 6. *For every coder-controller $H = (S, \delta, \gamma)$ for (3) and $\varepsilon > 0$, there exists a τ -periodic coder-controller $\hat{H} = (S, \hat{\delta}, \hat{\gamma})$ that satisfies*

$$R(\hat{H}) \leq R(H) + \varepsilon.$$

Proof. For $\varepsilon > 0$, we pick $\tau \in \mathbb{N}$ so that $\log_2 \#\mathcal{Z}_0/\tau \leq \varepsilon/2$ and

$$\max_{\zeta \in \mathcal{Z}_\tau} \frac{1}{\tau} \sum_{t=0}^{\tau-2} \log_2 \#Z(\zeta|_{[0;t]}) \leq R(H) + \varepsilon/2.$$

We define $\hat{\gamma}$ and $\hat{\delta}$ for all $\xi \in X^{[0;t]}$, $\zeta \in S^{[0;t]}$ with $t \in \mathbb{Z}_{\geq 0}$ by

$$\hat{\gamma}(\xi) := \gamma(\xi|_{[\tau\lfloor t/\tau\rfloor;t]}) \quad \text{and} \quad \hat{\delta}(\zeta) := \delta(\zeta|_{[\tau\lfloor t/\tau\rfloor;t]}).$$

Let \hat{Z} be defined in (16) w.r.t. $\hat{\gamma}$. Then we have for all $\zeta \in S^{[0;t]}$ with $t \in [0; \tau - 1[$ the equality $Z(\zeta) = \hat{Z}(\zeta)$ and for every $\zeta \in S^{[0;\tau[}$ we have $\hat{Z}(\zeta) = \mathcal{Z}_0$ which follows from the fact that $\hat{\gamma}$ is τ -periodic. The transmission data rate of \hat{H} follows by (20) which is bounded by

$$\max_{\zeta \in \hat{\mathcal{Z}}_\tau} \frac{1}{\tau} \left(\sum_{t=0}^{\tau-2} \log_2 \# \hat{Z}(\zeta|_{[0;t]}) + \log_2 \# \mathcal{Z}_0 \right) \leq R(H) + \varepsilon. \quad \square$$

5.2. The data rate theorem. The next result establishes the main data rate theorem in this work.

Theorem 6. *Consider the system $\Sigma = (X, U, F)$ and a nonempty set $Q \subseteq X$. The invariance feedback entropy of Σ and Q satisfies*

$$h_{\text{inv}}(Q) = \inf_{H \in \mathcal{H}} R(H) \quad (21)$$

where \mathcal{H} is the set of all Q -admissible coder-controllers for Σ .

We use the following two technical lemmas to show the theorem.

Lemma 7. *Let $H = (S, \gamma, \delta)$ be a Q -admissible τ -periodic coder-controller for $\Sigma = (X, U, F)$. Then there exists an invariant cover (\mathcal{A}, G) of Σ and Q and a (τ, Q) -spanning set \mathcal{S} in (\mathcal{A}, G) so that*

$$\frac{1}{\tau} \log_2 N(\mathcal{S}) \leq R(H).$$

Proof. For every $t \in [0; \tau[$ and every $\zeta \in \mathcal{Z}_{t+1}$ we define $A(\zeta) := \{x \in Q \mid \exists (\xi, \nu) \in \mathcal{B}_Q(H) \zeta = \Gamma_t(\xi|_{[0;t]}) \wedge \xi(t) = x\}$, $G(A(\zeta)) := \delta(\zeta)$ and $\mathcal{A} := \{A(\zeta) \mid \zeta \in \mathcal{Z}_{t+1} \wedge t \in [0; \tau[\}$. We show that (\mathcal{A}, G) is an invariant cover of Σ and Q . Clearly, \mathcal{A} is finite and every element of \mathcal{A} is a subset of Q . Since H is Q -admissible, for every $x \in Q$ there exists $(\xi, \nu) \in \mathcal{B}_Q(H)$ so that $\xi(0) = x$. Hence, $\{A(s) \mid s \in \mathcal{Z}_1\}$ covers Q and we see that \mathcal{A} covers Q . Let $A \in \mathcal{A}$ and suppose that there exists $x \in A$ so that $F(x, G(A)) \not\subseteq Q$. Since $A \in \mathcal{A}$, there exists $t \in [0; \tau[$, $\zeta \in \mathcal{Z}_{t+1}$ and $(\xi, \nu) \in \mathcal{B}_Q(H)$ so that $A = A(\zeta)$, $\zeta = \Gamma_t(\xi|_{[0;t]})$ and $x = \xi(t)$. Note that ν satisfies (18) so that $\nu(t) = G(A(\zeta))$ holds. We fix $x' \in F(x, G(A)) \setminus Q$ and pick a trajectory (ξ', ν') of Σ on $[0; \infty[$ such that $\xi'(0) = x'$ and $\nu'(t') = \delta(\Gamma_t((\xi|_{[0;t]}\xi')|_{[t;t'+1]}))$ holds for all $t' \in \mathbb{Z}_{\geq 0}$. We define $(\bar{\xi}, \bar{\nu})$ by $\bar{\xi} := \xi|_{[0;t]}\xi'$ and $\bar{\nu} := \nu|_{[0;t]}\nu'$, which by construction is a trajectory of Σ on $[0; \infty[$ which satisfies (18) but $\bar{\xi}([0; \infty[) \not\subseteq Q$. This contradicts the Q -admissibility of H and we can deduce that $F(A, G(A)) \subseteq Q$ for all $A \in \mathcal{A}$, which shows that (\mathcal{A}, G) is an invariant cover of Σ and Q .

We are going to construct a (τ, Q) -spanning set $\mathcal{S} \subseteq \mathcal{A}^{[0;\tau[}$ with the help of \mathcal{Z}_τ . For each $\zeta \in \mathcal{Z}_\tau$ we define a sequence $\alpha_\zeta : [0; \tau[\rightarrow \mathcal{A}$ by $\alpha_\zeta(t) := A(\zeta|_{[0;t]})$ for all $t \in [0; \tau[$ and use \mathcal{S} to denote the set of all such sequences $\{\alpha_\zeta \mid \zeta \in \mathcal{Z}_\tau\}$. Note that $P(\alpha_\zeta) = \{A(s) \mid s \in \mathcal{Z}_1\}$ holds for all $\alpha_\zeta \in \mathcal{S}$, and we see that $P(\alpha_\zeta)$ covers Q . Let us show (4). Let $\alpha_\zeta \in \mathcal{S}$, $t \in [0; \tau - 1[$ so that $\alpha_\zeta(t) = A(\zeta|_{[0;t]})$. We define $\zeta_t := \zeta|_{[0;t]}$ and fix $x_0 \in A(\zeta_t)$ and $x_1 \in F(x_0, G(A(\zeta_t)))$. Since $x_0 \in A(\zeta_t)$ there exists $(\xi, \nu) \in \mathcal{B}_Q(H)$ so that $\zeta_t = \Gamma_t(\xi|_{[0;t]})$ with $\xi(t) = x_0$ and we use (18) to see that $G(A(\zeta_t)) = \delta(\zeta_t) = \nu(t)$. Therefore, $(\xi, \nu)|_{[0;t]}$ can be extended to a trajectory

in $(\bar{\xi}, \bar{\nu}) \in \mathcal{B}_Q(H)$ with $\bar{\xi}(t+1) = x_1$. Let $s = \gamma(\bar{\xi}|_{[0;t+1]})$, then we have $s \in Z(\zeta_t)$ and $\zeta_{t+1} := \zeta_t s \in \mathcal{Z}_{t+2}$ holds. Moreover, $\zeta_{t+1} = \Gamma_{t+1}(\bar{\xi}|_{[0;t+1]})$ and we conclude that $x_1 \in A(\zeta_{t+1})$. We repeat this process for $x_i \in F(A(\zeta_{t+i}), G(A(\zeta_{t+i})))$, $i \in [0; k]$ until $t+k = \tau-1$ at which point we arrive at $\zeta_{t+k} \in \mathcal{Z}_\tau$ and we see that the associated sequence $\alpha_{\zeta_{t+k}}$ is an element of \mathcal{S} that satisfies $x_1 \in \alpha_{\zeta_{t+k}}(t+1)$ and $\alpha_{\zeta_{t+k}}|_{[0;t]} = \alpha_\zeta|_{[0;t]}$. Since such a sequence can be constructed for every $x_1 \in F(x_0, G(A(\zeta_t)))$ and $x_0 \in A(\zeta_t)$, we see that (4) holds and it follows that \mathcal{S} is (τ, Q) -spanning in (\mathcal{A}, G) .

We claim that $\#P(\alpha_\zeta|_{[0;t]}) \leq \#Z(\zeta|_{[0;t]})$ for every $\alpha_\zeta \in \mathcal{S}$ and $t \in [0; \tau-1[$. Let $A \in P(\alpha_\zeta|_{[0;t]})$, then there exists $\alpha_{\zeta'} \in \mathcal{S}$ such that $A = \alpha_{\zeta'}(t+1)$ and $\zeta'|_{[0;t]} = \zeta|_{[0;t]}$. Hence $\zeta'(t+1) \in Z(\zeta|_{[0;t]})$. Moreover, for $A, \bar{A} \in P(\alpha_\zeta|_{[0;t]})$ with $A \neq \bar{A}$ there exists $\alpha_{\zeta'}, \alpha_{\bar{\zeta}'} \in \mathcal{S}$ such that $A = \alpha_{\zeta'}(t+1)$ and $\bar{A} = \alpha_{\bar{\zeta}'}(t+1)$, which shows that $\zeta'(t+1) \neq \bar{\zeta}'(t+1)$ and $\zeta'(t+1), \bar{\zeta}'(t+1) \in Z(\zeta|_{[0;t]})$ and we obtain $\#P(\alpha_\zeta|_{[0;t]}) \leq \#Z(\zeta|_{[0;t]})$ for all $t \in [0; \tau-1[$ and $\zeta \in \mathcal{Z}_\tau$. For $t = \tau-1$ we have $P(\zeta) = \{A(s) \mid s \in \mathcal{Z}_1\}$. For $Z(\zeta)$ we have $Z(\zeta) = \gamma(X)$, since H is τ -periodic and we obtain $\#P(\alpha_\zeta) \leq \#Z(\zeta)$ for every $\zeta \in \mathcal{Z}_\tau$. Hence, $N(\mathcal{S}) \leq \max_{\zeta \in \mathcal{Z}_\tau} \prod_{t=0}^{\tau-1} \#Z(\zeta|_{[0;t]})$ follows and we obtain $\frac{1}{\tau} \log_2 N(\mathcal{S}) \leq R(H)$. \square

In the proof of the following lemma, we use an *enumeration* of a finite set A , which is a function $e : [1; \#A] \rightarrow A$ such that $e([1; \#A]) = A$.

Lemma 8. *Consider an invariant cover (\mathcal{A}, G) of $\Sigma = (X, U, F)$ and some nonempty set $Q \subseteq X$. Let \mathcal{S} be a (τ, Q) -spanning set in (\mathcal{A}, G) . Then there exists a Q -admissible τ -periodic coder-controller $H = (S, \gamma, \delta)$ for Σ so that*

$$\frac{1}{\tau} \log_2 N(\mathcal{S}) \geq R(H).$$

Proof. We define $\mathcal{S}_t := \{\alpha \in \mathcal{A}^{[0;t]} \mid \exists \hat{\alpha} \in \mathcal{S} \hat{\alpha}|_{[0;t]} = \alpha\}$ for $t \in [0; \tau[$ and observe that $\mathcal{S}_{\tau-1} = \mathcal{S}$ and for every $\alpha \in \mathcal{S}$ we have $P(\alpha) = \mathcal{S}_0$. For $\alpha \in \mathcal{S}_t$ with $t \in [0; \tau-1[$ let $e(\alpha)$ be an enumeration of $P(\alpha)$. We slightly abuse the notation, and use $e(\emptyset)$ to denote an enumeration of \mathcal{S}_0 so that $e(\alpha|_{[0;0]}) = e(\emptyset)$ for all $\alpha \in \mathcal{S}$. Let $m \in \mathbb{N}$ be the smallest number so that every co-domain of $e(\alpha)$ is a subset of $[1; m]$. We use this interval to define the set of symbols $S := [1; m]$. We are going to define $\gamma(\xi)$ and $\delta(\zeta)$ for all sequences $\xi \in X^{[0;t]}$, respectively, $\zeta \in S^{[0;t]}$ with $t \in [0; \tau[$, which determines γ and δ for all elements in their domain, since γ and δ are τ -periodic. We begin with γ , which we define iteratively. For $t = 0$ and $x \in X$ we set $\gamma(x) := e(\emptyset)(A)$ if there exists $A \in \mathcal{S}_0$ with $x \in A$. If there are several $A \in \mathcal{S}_0$ that contain x we simply pick one. If there does not exist any $A \in \mathcal{S}_0$ with $x \in A$ we set $\gamma(x) := 1$. For $t \in]0; \tau[$ and $\xi \in X^{[0;t]}$ we define $\gamma(\xi) := e(\alpha|_{[0;t]})(\alpha(t))$ for $\alpha \in \mathcal{S}_t$ that satisfies i) $\xi(t) \in \alpha(t)$ and ii) $\gamma(\xi|_{[0;t']}) = e(\alpha|_{[0;t']})(\alpha(t'))$ holds for all $t' \in [0; t[$. Again, if there are several such $\alpha \in \mathcal{S}_t$ we simply pick one. If there does not exist any α in \mathcal{S}_t that satisfies i) and ii), we set $\gamma(\xi) := 1$. We define δ for $t \in [0; \tau[$ and $\zeta \in S^{[0;t]}$ as follows: if there exists $\alpha \in \mathcal{S}_t$ that satisfies $e(\alpha|_{[0;t']})(\alpha(t')) = \zeta(t')$ for all $t' \in [0; t[$, we set $\delta(\zeta) := G(\alpha(t))$, otherwise we set $\delta(\zeta) := u$ for some $u \in U$. Let us show that the coder-controller is Q -admissible. We fix $(\xi, \nu) \in \mathcal{B}_Q(H)$ and proceed by induction with the hypothesis parameterized by $t \in [0; \tau[$: there exists $\alpha \in \mathcal{S}_t$ so that $\xi(t) \in \alpha(t)$, $\gamma(\xi|_{[0;t']}) = e(\alpha|_{[0;t']})(\alpha(t'))$ and $\nu(t') = G(\alpha(t'))$ hold for all $t' \in [0; t[$. For $t = 0$, we know that \mathcal{S}_0 covers Q so that for $\xi(0) \in Q$ there exists $A \in \mathcal{S}_0$ with $x \in A$ and it follows from the definition of γ and δ that $\gamma(\xi(0)) = e(\emptyset)(\bar{A})$ for some $\bar{A} \in \mathcal{S}_0$ with

$\xi(0) \in \bar{A}$ and $\nu(0) = \delta(\gamma(\bar{A})) = G(\bar{A})$. Now suppose that the induction hypothesis holds for $t \in]0; \tau - 1[$. Since $\xi(t) \in \alpha(t)$ and $\nu(t) = G(\alpha(t))$ for some $\alpha \in \mathcal{S}_t$, we use (4) to see that there exists $\bar{\alpha} \in \mathcal{S}$ so that $\bar{\alpha}|_{[0;t]} = \alpha$ and $\xi(t+1) \in \bar{\alpha}(t+1)$, so that $\bar{\alpha}$ satisfies i) and ii) in the definition of γ and we have $\gamma(\xi|_{[0;t+1]}) = e(\alpha)(\hat{\alpha}(t+1))$ for some $\hat{\alpha} \in \mathcal{S}_{t+1}$ with $\xi(t+1) \in \hat{\alpha}(t+1)$ and $\hat{\alpha}|_{[0;t]} = \alpha$. Since $\hat{\alpha}$ is uniquely determined by the symbol sequence $\zeta \in S^{[0;t+1]}$ given by $\zeta(t') = e(\hat{\alpha}|_{[0;t']})(\hat{\alpha}(t'))$ for all $t' \in [0; t+1]$, we have $\nu(t+1) = \delta(\zeta) = G(\hat{\alpha}(t+1))$, which completes the induction. Note that the induction hypothesis implies that $F(\xi(t), \nu(t)) \subseteq Q$ for all $t \in [0; \tau]$, since $\xi(t) \in \alpha(t)$ and $\nu(t) = G(\alpha(t))$. We obtain $\xi([0; \infty[) \subseteq Q$ from the τ -periodicity of H and the Q -admissibility follows.

We derive a bound for $R(H)$. Since H is τ -periodic, we have for any $\zeta \in \mathcal{Z}_\tau$ the equality $Z(\zeta) = e(\emptyset)(\mathcal{S}_0)$ and we see that $\#Z(\zeta) = \#e(\emptyset)(\mathcal{S}_0) = \#P(\alpha)$ for any $\alpha \in \mathcal{S}$. We fix $\zeta \in \mathcal{Z}_\tau$ and pick $\alpha \in \mathcal{S}$ so that $\alpha(t) = e^{-1}(\alpha|_{[0;t]})(\zeta(t))$ holds for all $t \in [0; \tau]$. By definition, the set $Z(\zeta|_{[0;t]})$ is the co-domain of an enumeration of $P(\alpha|_{[0;t]})$, which shows $\#Z(\zeta|_{[0;t]}) = \#P(\alpha|_{[0;t]})$. Therefore, we have $\max_{\zeta \in \mathcal{Z}_\tau} \prod_{t=0}^{\tau-1} \#Z(\zeta|_{[0;t]}) \leq \max_{\alpha \in \mathcal{S}} \prod_{t=0}^{\tau-1} \#P(\alpha|_{[0;t]})$ and the assertion follows by (20). \square

We continue with the proof of Theorem 6.

Proof of Theorem 6. Let us first prove the inequality $h_{\text{inv}}(Q) \leq \inf_{H \in \mathcal{H}} R(H)$. If the right-hand-side of (21) equals infinity the inequality trivially holds and subsequently we assume the right-hand-side of (21) is finite. We fix $\varepsilon > 0$ and pick a coder-controller $\bar{H} = (S, \bar{\gamma}, \bar{\delta})$ so that $R(\bar{H}) \leq \inf_{H \in \mathcal{H}} R(H) + \varepsilon$. According to Lemma 6 there exists a τ -periodic coder-controller $H = (S, \gamma, \delta)$ so that $R(H) \leq R(\bar{H}) + \varepsilon$. It is straightforward to see that for every $(\xi, \nu) \in \mathcal{B}_Q(H)$ and $\xi_i := \xi|_{[i\tau; (i+1)\tau[}$, $i \in \mathbb{Z}_{\geq 0}$, there exists $(\bar{\xi}, \bar{\nu}) \in \mathcal{B}_Q(\bar{H})$, so that $\xi_i = \bar{\xi}|_{[0;\tau[}$, which shows that H is Q -admissible. From Lemma 7 it follows that there exists an (\mathcal{A}, G) of Σ and Q and a (τ, Q) -spanning set in (\mathcal{A}, G) so that $\frac{1}{\tau} \log_2 N(\mathcal{S}) \leq R(H)$. We use Lemma 1 to see that $r_{\text{inv}}(n\tau, Q) \leq nr_{\text{inv}}(\tau, Q)$ so that $h(\mathcal{A}, G) = \lim_{n \rightarrow \infty} \frac{1}{n\tau} \log_2 r_{\text{inv}}(n\tau, Q) \leq \frac{1}{\tau} \log_2 r_{\text{inv}}(\tau, Q) \leq \frac{1}{\tau} \log_2 N(\mathcal{S})$. By the choice of H we obtain $2\varepsilon + \inf_{H \in \mathcal{H}} R(H) \geq R(H) \geq h_{\text{inv}}(Q)$. Since this holds for arbitrary $\varepsilon > 0$ we arrive at the desired inequality.

We continue with the inequality $h_{\text{inv}}(Q) \geq \inf_{H \in \mathcal{H}} R(H)$. If $h_{\text{inv}}(Q) = \infty$ the inequality trivially holds and subsequently we consider $h_{\text{inv}}(Q) < \infty$. We fix $\varepsilon > 0$ and pick an invariant cover (\mathcal{A}, G) of Σ and Q so that $h(\mathcal{A}, G) < h_{\text{inv}}(Q) + \varepsilon$. We pick $\tau \in \mathbb{N}$ so that $\frac{1}{\tau} \log_2 r_{\text{inv}}(\tau, Q) < h(\mathcal{A}, G) + \varepsilon$. Let \mathcal{S} be (τ, Q) -spanning set that satisfies $r_{\text{inv}}(\tau, Q) = N(\mathcal{S})$. It follows from Lemma 8 that there exists a Q -admissible coder-controller H so that $\frac{1}{\tau} \log_2 N(\mathcal{S}) \geq R(H)$ holds, and hence, we obtain $2\varepsilon + h_{\text{inv}}(Q) \geq R(H)$. This inequality holds for any $\varepsilon > 0$, which implies that $h_{\text{inv}}(Q) \geq \inf_{H \in \mathcal{H}} R(H)$. \square

6. UNCERTAIN LINEAR CONTROL SYSTEMS

We derive a lower bound of the invariance feedback entropy of uncertain linear control systems (2) and compact sets Q . In this setting, we also derive a lower bound of the data rate of any *static* or *memoryless* coder-controller. Similar to [1, Section II] we employ the Brunn-Minkowsky inequality to obtain a lower bound on the growth of the size of the uncertainty set of the state at the controller side in one time step. For the general case, we use this inequality to derive a lower bound on the expansion number, which in

turn leads to the entropy. For static coder-controllers the derivation of the lower bound is substantially simpler, see the proof of [1, Thm 1] and the proof of Theorem 8.

6.1. Universal lower bound.

Theorem 7. *Consider the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and two nonempty sets $W, Q \subseteq \mathbb{R}^n$ with $W \subseteq Q$ and suppose that W is measurable and Q is compact. Let Σ be given by $X = \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$ with $U \neq \emptyset$ and F according to*

$$\forall x \in X \forall u \in U \quad F(x, u) = Ax + Bu + W. \quad (22)$$

Let $\mathbb{R}^n = \mathbb{E}_1 \oplus \mathbb{E}_2$, where \mathbb{E}_1 is an A invariant subspace of \mathbb{R}^n with $\mathbb{E}_1 \neq \{0\}$, and \oplus stands for the direct sum. Let $\pi_1 : \mathbb{R}^n \rightarrow \mathbb{E}_1$ be the projection onto \mathbb{E}_1 along \mathbb{E}_2 , and² $\mu_1(\pi_1 W) < \mu_1(\pi_1 Q)$, also let $n_1 = \dim(\mathbb{E}_1)$ and μ_1 denote the n_1 -dimensional Lebesgue measure. Then, the invariance feedback entropy of Σ and Q satisfies

$$\log_2 \left(|\det A|_{\mathbb{E}_1} \frac{\mu_1(\pi_1 Q)}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^{n_1}} \right) \leq h_{\text{inv}}(Q). \quad (23)$$

Proof. Let us first point out that every compact set has finite Lebesgue measure.

If $|\det A|_{\mathbb{E}_1} = 0$ the left-hand-side is $-\infty$ and (23) holds. In the remainder we consider the case $|\det A|_{\mathbb{E}_1} > 0$. If $h_{\text{inv}}(Q) = \infty$ the inequality (23) holds independent of the left-hand-side and subsequently we assume that $h_{\text{inv}}(Q) < \infty$. We pick $\varepsilon \in \mathbb{R}_{>0}$ and an invariant cover (\mathcal{C}, H) of Σ and Q , so that $h(\mathcal{C}, H) \leq h_{\text{inv}}(Q) + \varepsilon$. Given Theorem 5, we can assume that the cover elements of \mathcal{C} are closed, which yields by the compactness of Q that the cover elements are compact and therefore Lebesgue measurable.

We fix $\tau \in \mathbb{N}$ and pick a (τ, Q) -spanning set \mathcal{S} so that $r_{\text{inv}}(\tau, Q) = N(\mathcal{S})$, which exists, since for fixed τ , the number of (τ, Q) -spanning set is finite.

We are going to show that there exists $\alpha \in \mathcal{S}$ that satisfies

$$\left(|\det A|_{\mathbb{E}_1} \frac{\mu_1(\pi_1 Q)}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^{n_1}} \right)^\tau \leq \prod_{t=0}^{\tau-1} \#P(\alpha|_{[0;t]}). \quad (24)$$

We construct $\alpha \in \mathcal{S}$ iteratively over $t \in [0; \tau[$. For $t = 0$ we introduce $S_0 := \{\alpha(0) \mid \alpha \in \mathcal{S}\}$ and define

$$m_0 := \max\{\mu_1(\pi_1 \alpha(0))^{1/n_1} \mid \alpha \in \mathcal{S}\}.$$

We pick $\Omega_0 \in S_0$ so that $m_0 = \mu_1(\pi_1 \Omega_0)^{1/n_1}$. For $t \in [1; \tau - 1[$ we set $\alpha_{t'} := \Omega_0 \cdots \Omega_{t'}$ for $t' \in [0; t]$ and assume that $\Omega_{t'} \in P(\alpha|_{[0;t']})$ and $\mu_1(\pi_1 \Omega_{t'})^{1/n_1} = m_{t'}$ holds for all $t' \in [1; t]$ where

$$m_{t'} := \max\{\mu_1(\pi_1 \Omega)^{1/n_1} \mid \Omega \in P(\alpha|_{[0;t']})\}.$$

Then we set $m_{t+1} := \max\{\mu_1(\pi_1 \Omega)^{1/n_1} \mid \Omega \in P(\alpha|_{[0;t+1]})\}$ and pick $\Omega_{t+1} \in P(\alpha|_{[0;t+1]})$ so that $m_{t+1} = \mu_1(\pi_1 \Omega_{t+1})^{1/n_1}$. For $t = \tau - 1$ we obtain a sequence $\alpha := \Omega_0 \cdots \Omega_{\tau-1}$ that is an element of \mathcal{S} . Hence, it follows from (4) that α satisfies for all $t \in [0; \tau[$ the inclusion

$$\pi_1(A\alpha(t) + BH(\alpha(t)) + W) \subseteq \pi_1 \left(\bigcup_{\Omega \in P(\alpha|_{[0;t]})} \Omega \right). \quad (25)$$

²Since map π_1 is linear, we use notation $\pi_1 A$ instead of $\pi_1(A)$, $\forall A \subseteq \mathbb{R}^n$, for the sake of simpler presentation.

For $t \in [0; \tau - 1[$, we use the Brunn-Minkowsky inequality for compact, measurable sets [30]

$$\mu_1(\pi_1 A\alpha(t))^{1/n_1} + \mu_1(\pi_1 W)^{1/n_1} \leq \mu_1(\pi_1 A\alpha(t) + \pi_1 BH(\alpha(t)) + \pi_1 W)^{1/n_1}$$

and the equality [17]

$$\mu(A\alpha(t))^{1/n} = |\det A|^{1/n} \mu(\alpha(t))^{1/n}$$

together with $\mu_1(\pi_1 \alpha(t))^{1/n_1} = m_t$ and (25), to derive

$$|\det A|_{\mathbb{E}_1}|^{1/n_1} m_t + \mu_1(\pi_1 W)^{1/n_1} \leq m_{t+1} (\#P(\alpha|_{[0;t+1]})^{1/n_1}) \quad (26)$$

for all $t \in [0; \tau - 1[$. Note that we also used the fact that \mathbb{E}_1 is A invariant to show inequality (26). Also, for every $t \in [0; \tau[$ we have

$$|\det A|_{\mathbb{E}_1}|^{1/n_1} m_t + \mu_1(\pi_1 W)^{1/n_1} \leq \mu_1(\pi_1 Q)^{1/n_1} \quad (27)$$

since $A\alpha(t) + BH(\alpha(t)) + W \subseteq Q$ which follows from the fact that $\alpha(t) \in \mathcal{C}$ and (\mathcal{C}, H) is an invariant cover. To ease the notation, let us introduce $N_0 := (\#P(\alpha))^{1/n_1}$ and $N_t := (\#P(\alpha|_{[0;t]})^{1/n_1}$ for $t \in [1; \tau[$. We use induction over $\tau' \in [0; \tau[$ to show

$$\left(|\det A|_{\mathbb{E}_1}|^{1/n_1} \frac{\mu_1(\pi_1 Q)^{1/n_1}}{\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1}} \right)^{\tau'+1} \leq \prod_{t=0}^{\tau'} N_t. \quad (28)$$

Let us show (28) for $\tau' = 0$. Since $P(\alpha)$ is a cover of Q and $\#P(\alpha)^{1/n} = N_0$ we obtain

$$\mu_1(\pi_1 Q)^{1/n_1} \leq m_0 N_0. \quad (29)$$

From (27) we obtain $m_0 \leq (\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1}) / |\det A|_{\mathbb{E}_1}|^{1/n_1}$ and (28) follows for $\tau' = 1$.

If $\tau = 1$ we have shown (28) and subsequently we consider $\tau > 1$. We fix $\tau'' \in [1; \tau[$ and assume that (28) holds for all $\tau' \in [0; \tau''[$. We use (26) recursively to derive

$$m_0 \leq \frac{m_{\tau''}}{|\det A|_{\mathbb{E}_1}|^{\tau''/n_1}} \left(\prod_{t=1}^{\tau''} N_t \right) - \sum_{t=1}^{\tau''} \frac{\mu_1(\pi_1 W)^{1/n_1}}{|\det A|_{\mathbb{E}_1}|^{t/n_1}} \prod_{t'=1}^{t-1} N_{t'} \quad (30)$$

with the convention that $\prod_{t=a}^b x_t = 1$ for $b < a$. Using (29) and rearranging the terms in (30) we obtain

$$\mu_1(\pi_1 Q)^{1/n_1} + \sum_{t=1}^{\tau''} \frac{\mu_1(\pi_1 W)^{1/n_1}}{|\det A|_{\mathbb{E}_1}|^{t/n_1}} \prod_{t'=0}^{t-1} N_{t'} \leq \frac{m_{\tau''}}{|\det A|_{\mathbb{E}_1}|^{\tau''/n_1}} \prod_{t=0}^{\tau''} N_t. \quad (31)$$

We invoke the induction hypothesis and use the inequality

$\prod_{t'=0}^{t-1} N_{t'} \geq ((|\det A|_{\mathbb{E}_1}| \mu_1(\pi_1 Q))^{1/n_1} / (\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1}))^t$ to derive

$$\mu_1(\pi_1 Q)^{1/n_1} + \sum_{t=1}^{\tau''} \frac{\mu_1(\pi_1 W)^{1/n_1} \mu_1(\pi_1 Q)^{t/n_1}}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^t} \leq \frac{m_{\tau''}}{|\det A|_{\mathbb{E}_1}|^{\tau''/n_1}} \prod_{t=0}^{\tau''} N_t. \quad (32)$$

From Lemma 10 (given in the Appendix) it follows that the left-hand-side of (32) evaluates to

$$\mu_1(\pi_1 Q)^{1/n_1} + \sum_{t=1}^{\tau''} \frac{\mu_1(\pi_1 W)^{1/n_1} \mu_1(\pi_1 Q)^{t/n_1}}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^t} = \frac{\mu_1(\pi_1 Q)^{(\tau''+1)/n_1}}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^{\tau''}}. \quad (33)$$

We combine $m_{\tau''} \leq (\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})/|\det A|_{\mathbb{E}_1}|^{1/n_1}$ (that follows from (27)) with (32) and (33) to get

$$\frac{\mu_1(\pi_1 Q)^{(\tau''+1)/n_1}}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^{\tau''}} \leq \frac{\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1}}{|\det A|_{\mathbb{E}_1}|^{(\tau''+1)/n_1}} \prod_{t=0}^{\tau''} N_t \quad (34)$$

which shows that (28) holds for $\tau' = \tau''$. Hence, (28) holds for all $\tau' \in [0; \tau[$. In particular, for $\tau' = \tau - 1$ and we conclude that (24) holds.

Inequality (24) together with the definition of $N(\mathcal{S})$ yields

$$\left(|\det A|_{\mathbb{E}_1} \left| \frac{\mu_1(\pi_1 Q)}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^{n_1}} \right| \right)^\tau \leq N(\mathcal{S}) = r_{\text{inv}}(\tau, Q)$$

where the equality follows by our choice of \mathcal{S} . From (5) we get

$$\log_2 \left(|\det A|_{\mathbb{E}_1} \left| \frac{\mu_1(\pi_1 Q)}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^{n_1}} \right| \right) \leq h(\mathcal{C}, H) \leq h_{\text{inv}}(Q) + \varepsilon \quad (35)$$

which implies (23) since (35) holds for every $\varepsilon > 0$. \square

Remark 1. Let $\text{spec}(A)$ denote the spectrum of A , \mathbb{E}^λ denote the eigenspace of A associated with $\lambda \in \text{spec}(A)$ and $B \subseteq \text{spec}(A)$. In Theorem 7 if $\mathbb{E}_1 = \bigoplus_{\lambda \in B} \mathbb{E}^\lambda$, then a good choice of \mathbb{E}_1 will be the one that gives the largest lower bound in (23).

Remark 2. Note that the lower bound, i.e., the left-hand-side of inequality (23), is invariant under coordinate transformation. Let $z = Tx$ for some invertible matrix $T \in \mathbb{R}^{n \times n}$ so that the transition function \bar{F} of the system in the new coordinates is

$$\bar{F}(z, u) = TAT^{-1}z + TBU + TW \quad (36)$$

and $\bar{Q} = TQ$. Let $\bar{\mathbb{E}}_i = T\mathbb{E}_i$, $i \in \{1, 2\}$, $\bar{\pi}_1 : \mathbb{R}^n \rightarrow \bar{\mathbb{E}}_1$ be the projection on $\bar{\mathbb{E}}_1$ along $\bar{\mathbb{E}}_2$. Then we obtain

$$\begin{aligned} & |\det(TAT^{-1})|_{\bar{\mathbb{E}}_1} \left| \frac{\mu_1(\bar{\pi}_1 TQ)}{(\mu_1(\bar{\pi}_1 TQ)^{1/n_1} - \mu_1(\bar{\pi}_1 TW)^{1/n_1})^{n_1}} \right| = \\ & |\det A|_{\mathbb{E}_1} \left| \frac{\mu_1(T\pi_1 Q)}{(\mu_1(T\pi_1 Q)^{1/n_1} - \mu_1(T\pi_1 W)^{1/n_1})^{n_1}} \right| = \\ & |\det A|_{\mathbb{E}_1} \left| \frac{\mu_1(\pi_1 Q)}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^{n_1}} \right|. \end{aligned}$$

When W is a singleton set, by taking \mathbb{E}_1 as the unstable subspace, we get the largest lower bound in (23) which recovers the well-known value of the invariance entropy [13, Th. 3.1] for deterministic linear control systems, i.e., the invariance entropy equals $\log_2 |\det A|_{\mathbb{E}_1}$. This matches also other results known from stabilization with rate limited feedback [4].

6.2. Static coder-controllers. We restrict our attention to static coder-controllers and derive a lower bound of the data rate of such coder-controllers.

Let (\mathcal{C}, H) be an invariant cover of (3) and a nonempty set $Q \subseteq X$. We define the data rate of (\mathcal{C}, H) by

$$R(\mathcal{C}, H) := \log_2 \#\mathcal{C}. \quad (37)$$

The definition is motivated by the fact that any invariant cover (\mathcal{C}, H) immediately provides a *static* or *memoryless* coder-controller scheme: given $x \in Q$ at the coder side,

it is sufficient that the coder transmits one of the cover elements $C \in \mathcal{C}$ that contains the current state $x \in C$, to ensure that the controller is able to confine the successor states of x to Q , i.e.,

$$Ax + BH(C) + W \subseteq Q. \quad (38)$$

The number of different cover elements that need to be transmitted via the digital, noiseless channel at any time $t > 0$ is bounded by $\#\mathcal{C}$. Neither the coder nor the controller requires any past information for a correct functioning. Hence, we speak of (\mathcal{C}, H) as static or memoryless coder-controller for (X, U, F) .

The next result provides a lower bound on the data rate of any static coder-controller.

Theorem 8. *Consider the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and two nonempty sets $W, Q \subseteq \mathbb{R}^n$ with $W \subseteq Q$ and suppose that W is measurable and Q is compact. Let (3) be given by $X = \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$ with $U \neq \emptyset$, F according to (22), \mathbb{E}_1 , \mathbb{E}_2 , μ_1 , n_1 and π_1 as in Theorem 7 and $\mu_1(\pi_1 W) < \mu_1(\pi_1 Q)$. Then, we have*

$$\log_2 \left[|\det A|_{\mathbb{E}_1} \frac{\mu_1(\pi_1 Q)}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^{n_1}} \right] \leq \inf_{(\mathcal{C}, H)} R(\mathcal{C}, H) \quad (39)$$

where we take the infimum over all invariant covers (\mathcal{C}, H) of (3) and Q .

Proof. If $|\det A|_{\mathbb{E}_1} = 0$ the left-hand-side of (39) evaluates to $-\infty$ so that (39) holds. Let us consider $|\det A|_{\mathbb{E}_1} > 0$. If the right-hand-side of (39) evaluates to ∞ nothing needs to be shown and we consider $\inf_{(\mathcal{C}, H)} R(\mathcal{C}, H) < \infty$. Since $\inf_{(\mathcal{C}, H)} R(\mathcal{C}, H)$ is finite, there exists an invariant cover (\mathcal{D}, G) of (X, U, F) and Q . Let (\mathcal{C}, H) be the invariant cover with closed cover elements as constructed from (\mathcal{D}, G) in Theorem 5. Then (\mathcal{C}, H) is an invariant cover of (X, U, F) and Q and we have $R(\mathcal{C}, H) \leq R(\mathcal{D}, G)$.

As (\mathcal{C}, H) is an invariant cover of (X, U, F) and Q , we have for every $\Omega \in \mathcal{C}$ the inclusion

$$\pi_1(A\Omega + BH(\Omega) + W) \subseteq \pi_1 Q. \quad (40)$$

We use the Brunn-Minkowsky inequality for compact, measurable sets (see proof of Theorem 7) together with the identity [17] $\mu(A\Omega)^{1/n} = |\det A|^{1/n} \mu(\Omega)^{1/n}$ to derive $|\det A|_{\mathbb{E}_1}^{1/n_1} \mu_1(\pi_1 \Omega)^{1/n_1} + \mu_1(\pi_1 W)^{1/n_1} \leq \mu_1(\pi_1 Q)^{1/n_1}$ which yields the bound

$$\mu_1(\pi_1 \Omega)^{1/n_1} \leq \frac{\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1}}{|\det A|_{\mathbb{E}_1}^{1/n_1}}. \quad (41)$$

As $\#\mathcal{C}$ is an upper bound on the number of cover elements needed to cover $F(\Omega, H(\Omega))$, we have

$$\mu_1(\pi_1 Q)^{1/n_1} \leq (\#\mathcal{C})^{1/n_1} \max\{\mu_1(\pi_1 \Omega)^{1/n_1} \mid \Omega \in \mathcal{C}\}. \quad (42)$$

We use (41) (which holds for every $\Omega \in \mathcal{C}$) in (42) and rearrange the result to obtain

$$|\det A|_{\mathbb{E}_1}^{1/n_1} \frac{\mu_1(\pi_1 Q)^{1/n_1}}{\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1}} \leq (\#\mathcal{C})^{1/n_1}.$$

Since this inequality holds for every invariant cover (\mathcal{C}, H) , we obtain (39). \square

It is easy to bound the difference between the universal lower bound in (23) and the lower bound of data rates for static coder-controllers in (39) so that we arrive at the following corollary, which allows us to quantify the performance loss due to the restriction to static coder-controllers.

Corollary 1. *In the context and under the assumptions of Theorem 8, let $a \in \mathbb{R}_{\geq 0}$ be given by*

$$a := |\det A|_{\mathbb{E}_1} \frac{\mu_1(\pi_1 Q)}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^{n_1}}.$$

Suppose that $a < \infty$ and there exists an invariant cover (\mathcal{C}, H) of (3) and Q with $R(\mathcal{C}, H) = \log_2 \lceil a \rceil$. Then, the data rate R of (\mathcal{C}, H) satisfies

$$R \leq h_{\text{inv}}(Q) + 1. \quad (43)$$

Proof. Let $b \in [0, 1[$ be so that $a + b = \lceil a \rceil$. We use $a \leq 2^{h_{\text{inv}}(Q)}$ and $0 \leq h_{\text{inv}}(Q)$ to derive

$$R = \log_2(a + b) \leq \log_2(2^{h_{\text{inv}}(Q)} + b) \leq h_{\text{inv}}(Q) + \log_2(1 + 2^{-h_{\text{inv}}(Q)}) \leq h_{\text{inv}}(Q) + 1. \quad \square$$

6.3. Tightness of the lower bounds. We show for a particular class of scalar linear difference inclusions of the form

$$\xi(t+1) \in a\xi(t) + \nu(t) + [w_1, w_2] \quad (44)$$

with $a \in \mathbb{R}_{\neq 0}$, $w_1, w_2 \in \mathbb{R}$ and $w_1 \leq w_2$ that the lower bounds established in the previous subsections are tight.

Subsequently, we assume that Q is given as an interval containing $[w_1, w_2]$

$$Q := [q_1, q_2], \quad q_1, q_2 \in \mathbb{R}, q_1 < w_1, w_2 < q_2.$$

We are going to construct a static coder-controller (\mathcal{C}, H) and show that its data rate equals the lower bound in Theorem 8. To this end, we introduce

$$\begin{aligned} \Delta q &:= q_2 - q_1, & \Delta w &:= w_2 - w_1, \\ q_c &:= (q_2 + q_1)/2 & \text{and} & \quad w_c := (w_2 + w_1)/2 \end{aligned} \quad (45a)$$

and consider

$$m := \left\lceil |a| \frac{\Delta q}{\Delta q - \Delta w} \right\rceil \quad \text{and} \quad d := \frac{\Delta q}{m}. \quad (45b)$$

Given q_c and d , we introduce the intervals $\Lambda_i \subseteq \mathbb{R}$, $i \in \mathbb{Z}$

$$\Lambda_i := \begin{cases} q_c + [id, (i+1)d] & \text{if } m \text{ is even} \\ q_c + [(i - \frac{1}{2})d, (i + \frac{1}{2})d] & \text{if } m \text{ is odd} \end{cases} \quad (45c)$$

which we use to define

$$\mathcal{C} := \{\Lambda_i \cap Q \mid \Lambda_i \cap (\text{int}Q) \neq \emptyset\}. \quad (45d)$$

The control function follows for every $C_i \in \mathcal{C}$ by

$$H(C_i) := q_c - aq_c - w_c - \begin{cases} ad(i + \frac{1}{2}) & \text{if } m \text{ is even} \\ adi & \text{if } m \text{ is odd.} \end{cases} \quad (45e)$$

For this construction of (\mathcal{C}, H) , we have the following result.

Theorem 9. *Consider the scalars $a \in \mathbb{R}_{\neq 0}$, $w_1, q_1, w_2, q_2 \in \mathbb{R}$ with $q_1 < w_1 \leq w_2 < q_2$. Let (3) be given by $X = U = \mathbb{R}$ and F by $F(x, u) = ax + u + [w_1, w_2]$. Then, (\mathcal{C}, H) defined in (45) is an invariant cover of (3) and $[q_1, q_2]$ and we have*

$$\log_2 \left\lceil |a| \frac{\Delta q}{\Delta q - \Delta w} \right\rceil = R(\mathcal{C}, H). \quad (46)$$

Proof. We show the theorem for odd m . The case for even m , follows along the same arguments. It is rather straightforward to show that \mathcal{C} is a cover of Q and subsequently we show that $\#\mathcal{C} = m$. Note that $i > m/2 - 1/2$ implies that the left limit of Λ_i satisfies $q_c + (i - \frac{1}{2})d \geq q_c + m/2d = q_2$, which shows that $i > m/2 - 1/2$ implies $\Lambda_i \cap (\text{int}Q) = \emptyset$. Similarly, $i < -m/2 + 1/2$ implies $\Lambda_i \cap (\text{int}Q) = \emptyset$, and we see that $\Lambda_i \cap (\text{int}Q) \neq \emptyset$ implies $-m/2 + 1/2 \leq i \leq m/2 - 1/2$ so that $\#\mathcal{C} \leq m$ holds.

We continue to show that $F(C_i, H(C_i)) \subseteq [q_1, q_2]$ holds for every $C_i \in \mathcal{C}$. Given (45e) we obtain for $F(C_i, H(C_i))$ the interval

$$a((q_c + d[i - \frac{1}{2}, i + \frac{1}{2}]) \cap Q) + q_c - aq_c - w_c - adi + [w_1, w_2]$$

which is a subset of $I := q_c + |a|\frac{d}{2}[-1, 1] + \frac{\Delta w}{2}[-1, 1]$. Let us show that $I \subseteq Q$. Since I is centered at q_c , it is sufficient to show $|a|d/2 + \Delta w/2 \leq \Delta q/2$. Note that $m \geq |a|\Delta q/(\Delta q - \Delta w)$ so that $d \leq (\Delta q - \Delta w)/|a|$ follows and we obtain the desired inequality $|a|d/2 + \Delta w/2 \leq \Delta q/2$ which shows $F(C_i, H(C_i)) \subseteq [q_1, q_2]$. Hence (\mathcal{C}, H) is an invariant cover with $R(\mathcal{C}, H) \leq \log_2 m$, which together with the inequality in Theorem 8 shows the assertion. \square

Example 2 (Continued). Let us recall the linear system in Example 2 with $a = 1/2$, $W = [-3, 3]$ and $Q = [-4, 4]$. For this case, $m = 2$ and $d = 4$. The cover elements of \mathcal{C} are given according to (45c) by

$$C_{-1} = [-4, 0] \text{ and } C_0 = [0, 4].$$

The inputs follow according to (45e) by

$$H(C_{-1}) = 1 \text{ and } H(C_0) = -1.$$

The data rate of (\mathcal{C}, H) is given by $\log_2 2 = 1$ bits per time unit.

We can use Corollary 1 to conclude that the performance loss due to the restriction to static coder-controllers in Example 2 is no larger than 1 bit/time unit. However, for this example, and in general for scalar systems of the form (44) for which $|a|\Delta q/(\Delta q - \Delta w)$ is in \mathbb{N} , we see that the data rate of the proposed static coder-controller matches the best possible data rate $h_{\text{inv}}(Q)$ since in this case $R(\mathcal{C}, H)$ equals the lower bound in Theorem 7.

The construction of static coder-controllers whose data rate achieves the lower bound in Theorem 8 in a more general setting is currently under investigation.

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APPENDIX A.

A.1. Mean-Payoff Games: A *mean-payoff game* (MPG) [31] is played by two players, player 1 and player 2, on a finite, directed, edge-weighted graph $G = (V, E, w)$, where $V := V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ with V_i , $i \in \{1, 2\}$ being two nonempty sets, $E \subseteq V \times V$, $w : E \rightarrow \mathbb{Z}$ and for every $v \in V$ there exists $v' \in V$ so that $(v, v') \in E$. The vertices V are also referred to as *positions* of the game. Starting from an initial position $v_0 \in V$, player 1 and player 2 take turns in picking the next position depending on the current position of the game: given $v_0 \in V_i$ for $i \in \{1, 2\}$ player i picks the successor vertex $v_1 \in V$ so that $(v_0, v_1) \in E$ and the play continues with v_1 . The infinite sequence of edges $e = (e_k)_{k \in [0; \infty[}$ with $e_k = (v_k, v_{k+1}) \in E$ is called a play. Player 1 wants to minimize the payoff

$$\nu_{\min}(e_0 e_1 e_2 \dots) := \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{j=0}^{k-1} w(e_j)$$

while player 2 wants to maximize the payoff

$$\nu_{\max}(e_0 e_1 e_2 \dots) := \liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{j=0}^{k-1} w(e_j).$$

A *positional strategy* for player i is a function $\sigma_i : V_i \rightarrow V$ so that $(v, \sigma_i(v)) \in E$ holds for all $v \in V_i$. By $\mathcal{P}_i(v, \sigma_i) \subseteq E^{[0; \infty[}$ we denote the set of all plays that start from the position v and wherein the player i follows the positional strategy σ_i .

As it turns out, there exist *optimal positional strategies* σ_i^* for each player i and a function $\nu : V \rightarrow \mathbb{R}$ so that player 1 is able to secure a payoff of $\nu(v)$ against any other strategy of player 2 and vice versa, i.e., for all sequences $\check{e} \in \mathcal{P}_1(v, \sigma_1^*)$ and $\hat{e} \in \mathcal{P}_2(v, \sigma_2^*)$ we have

$$\nu_{\min}(\check{e}) \leq \nu(v) \leq \nu_{\max}(\hat{e}). \quad (47)$$

We call ν the *value function* of the MPG (V, E, w) , see e.g. [31] for details. Note that σ_1^* is optimal in the sense that any deviation of player 1 from σ_1^* can only lead to a larger or equal payoff than $\nu(v)$ considering the worst case with respect to possible strategies of player 2. Similarly, a deviation of player 2 from σ_2^* may only lead to suboptimal payoff. We exploit the following fact, which follows from the proof of [31, Lemma. 1]: there exist constants c_1 and c_2 , so that for every $\tau \in \mathbb{N}$, $\check{e} \in \mathcal{P}_1(v, \sigma_1^*)$ and $\hat{e} \in \mathcal{P}_2(v, \sigma_2^*)$

we have

$$\frac{1}{\tau} \sum_{j=0}^{\tau-1} w(\check{e}_j) \leq \nu(v) + \frac{c_1}{\tau} \quad (48)$$

and

$$\frac{1}{\tau} \sum_{j=0}^{\tau-1} w(\hat{e}_j) \geq \nu(v) + \frac{c_2}{\tau}. \quad (49)$$

We use the following lemma in Theorem 1 and Theorem 5.

Lemma 9. *Consider two systems $\Sigma_i = (X_i, U_i, F_i)$, $i \in \{1, 2\}$, a map $r : U_2 \rightarrow U_1$ and let Q_i be nonempty subsets of X_i . Suppose that $M : \wp(X_2) \rightarrow \wp(X_1)$ maps subsets of X_2 to subsets of X_1 and satisfies for every $u \in U_2$ and $A_2, A'_2 \subseteq Q_2$ the following conditions*

- (1) $M(Q_2) = Q_1$,
- (2) $A_2 \subseteq A'_2 \implies M(A_2) \subseteq M(A'_2)$,
- (3) $M(A_2 \cup A'_2) = M(A_2) \cup M(A'_2)$ and
- (4) $F_1(M(A_2), r(u)) \subseteq M(F_2(A_2, u))$.

Let (\mathcal{A}_2, G_2) be an invariant cover of Σ_2 and Q_2 and let

$$\mathcal{A}_1 := \{M(A) \mid A \in \mathcal{A}_2\}.$$

Then there exists a map $G_1^* : \mathcal{A}_1 \rightarrow U_1$ such that (\mathcal{A}_1, G_1^*) is an invariant cover of Σ_1 and Q_1 , and

$$h(\mathcal{A}_1, G_1^*) \leq h(\mathcal{A}_2, G_2). \quad (50)$$

Proof. Let us first point out that \mathcal{A}_1 is a cover of Q_1 . We use 1) and 3) to derive

$$Q_1 = M(Q_2) = M(\cup_{A_2 \in \mathcal{A}_2} A_2) = \cup_{A_2 \in \mathcal{A}_2} M(A_2)$$

and we see that \mathcal{A}_1 is a cover of Q_1 .

Consider the map $G_1 : \mathcal{A}_1 \rightrightarrows U_1$ defined by

$$G_1(A_1) := \{r(G_2(A_2)) \mid A_2 \in \mathcal{A}_2, M(A_2) = A_1\}$$

and let

$$\mathcal{V}(A_1) := \{(V, u) \mid V \subseteq \mathcal{A}_1, u \in G_1(A_1), F_1(A_1, u) \subseteq \cup_{A \in V} A\}.$$

We show that $\mathcal{V}(A_1)$ is nonempty for every $A_1 \in \mathcal{A}_1$. Let $A_1 \in \mathcal{A}_1$ and $u \in G_1(A_1)$. Then there exists $A_2 \in \mathcal{A}_2$ so that $A_1 = M(A_2)$ and $u = r(G_2(A_2))$. We use 4) to see that $F_1(A_1, u) \subseteq M(F_2(A_2, G_2(A_2)))$. Since (\mathcal{A}_2, G_2) is an invariant cover we have $F_2(A_2, G_2(A_2)) \subseteq Q_2$ and it follows from 2) that $F_1(A_1, u) \subseteq M(Q_2)$. Since \mathcal{A}_1 covers $M(Q_2) = Q_1$, we see that $F_1(A_1, u) \subseteq \cup_{A \in \mathcal{A}_1} A$, which ensures that $\mathcal{V}(A_1) \neq \emptyset$.

Given Σ_1 and (\mathcal{A}_1, G_1) we construct an MPG (V, E, w) . Let $V_1 := \mathcal{A}_1$ and $V_2 := \cup_{A \in V_1} \mathcal{V}(A)$ then the *positions* of the MPG follow by $V = V_1 \cup V_2$. We introduce the *edges* $E := E_1 \cup E_2$ of the MPG by

$$\begin{aligned} E_1 &:= \{(v_1, v_2) \in V_1 \times V_2 \mid v_2 \in \mathcal{V}(v_1)\} \\ E_2 &:= \{(v_2, v_1) \in V_2 \times V_1 \mid v_1 \in V', v_2 = (V', u)\}. \end{aligned}$$

For $v \in V_2$ with $v = (V', u)$ by $\#v$ we refer to $\#V'$. The weights for $(v_1, v_2) \in E_1$ and $(v_2, v_1) \in E_2$ are given by $w(v_1, v_2) := \log_2 \#v_2$ and $w(v_2, v_1) := \log_2 \#v_2$. We refer to (V, E, w) as the MPG associated with Σ_1 and (\mathcal{A}_1, G_1) . Subsequently, we use σ_i^* , $i \in \{1, 2\}$ to denote the optimal positional strategy for player i .

Fix $\tau \in \mathbb{N}$ and let $r_{2,\text{inv}}(\tau, Q_2)$ denote the smallest possible expansion number associated with the invariant cover (\mathcal{A}_2, G_2) at time τ . Let \mathcal{S}_2 be a (τ, Q) -spanning set in (\mathcal{A}_2, G_2) such that $N(\mathcal{S}_2) = r_{2,\text{inv}}(\tau, Q_2)$. We observe that $Q_1 = M(Q_2) = M(\cup_{\alpha \in \mathcal{S}_2} \alpha(0)) \subseteq \cup_{\alpha \in \mathcal{S}_2} M(\alpha(0))$. Thus $V_0 := \{M(\alpha(0)) \mid \alpha \in \mathcal{S}_2\}$ covers Q_1 . We pick $\bar{v} \in V_0$ so that $\nu(\bar{v}) = \max_{v \in V_0} \nu(v)$. We show by induction over $t \in [0; \tau - 1[$ the existence of an $\alpha \in \mathcal{S}_2$ and an $(v_k, v_{k+1})_{k \in [0; \infty[} \in \mathcal{P}_2(\bar{v}, \sigma_2^*)$ such that

$$v_{2k} = M(\alpha(k)) \text{ and } v_{2k+1} = (\{M(A) \mid A \in P(\alpha|_{[0;k]})\}, u_k) \quad (51)$$

with $u_k = r(G_2(\alpha(k)))$ holds for all $k \in [0; t]$. Let $t = 0$, then there exists $\alpha \in \mathcal{S}_2$ with $M(\alpha(0)) = \bar{v}$. As \mathcal{S}_2 is (τ, Q) -spanning we have $F_2(\alpha(0), G_2(\alpha(0))) \subseteq \cup_{A \in P(\alpha(0))} A$. For $u = G_2(\alpha(0))$ and $V' = \{M(A) \mid A \in P(\alpha(0))\}$ we use 4), 2) and 3) to derive

$$F_1(\bar{v}, r(u)) \subseteq M(F_2(\alpha(0), u)) \subseteq M(\cup_{A \in P(\alpha(0))} A) \subseteq \cup_{A \in V'} A. \quad (52)$$

Hence, for $v_1 := (V', r(u))$ we have $v_1 \in \mathcal{V}(\bar{v})$ and $(\bar{v}, v_1) \in E_1$ thus $e_0 = (\bar{v}, v_1)$ for some $e \in \mathcal{P}_2(\bar{v}, \sigma_2^*)$. Now suppose that the induction hypothesis (51) holds for $t \in [0; \tau - 2[$, $\alpha \in \mathcal{S}_2$ and $(v_k, v_{k+1})_{k \in [0; \infty[} \in \mathcal{P}_2(\bar{v}, \sigma_2^*)$. Let $v_{2t+1} = (V', u)$. From the definition of E_2 we have $v_{2t+2} = \sigma_2^*(v_{2t+1}) \in V'$. Hence, together with (51) we see that there exists $A \in P(\alpha|_{[0;t]})$ with $M(A) = v_{2t+2}$. Then we can pick $\hat{\alpha} \in \mathcal{S}_2$ such that $\hat{\alpha}|_{[0;t]} = \alpha|_{[0;t]}$ and $\hat{\alpha}(t+1) = A$. Further let $\hat{v}_{2t+3} = (V', r(u))$ with $u = G_2(\hat{\alpha}(t+1))$ and $V' = \{M(A) \mid A \in P(\hat{\alpha}|_{[0;t+1]})\}$. Then, by using the same arguments used to derive (52) with v_{2t+2} and $P(\hat{\alpha}|_{[0;t+1]})$ in place of \bar{v} and $P(\alpha(0))$ we obtain $F_1(v_{2t+2}, r(u)) \subseteq \cup_{A \in V'} A$. Thus $(v_{2t+2}, \hat{v}_{2t+3}) \in E_1$ and there exists $e \in \mathcal{P}_2(\bar{v}, \sigma_2^*)$ such that $e_k = (v_k, v_{k+1})$ for all $k \in [0; 2t+1]$ and $e_{2t+2} = (v_{2t+2}, \hat{v}_{2t+3})$ which completes the induction. Let α and $e := (v_k, v_{k+1})_{k \in [0; \infty[}$ satisfy (51) for all $k \in [0; \tau - 1[$, which implies $\#v_{2t+1} \leq \#P(\alpha|_{[0;t]})$ for every $t \in [0; \tau - 1[$. As $e \in \mathcal{P}_2(\bar{v}, \sigma_2^*)$ from (49) we have

$$\begin{aligned} \nu(\bar{v}) + \frac{c_2}{2\tau} &\leq \frac{1}{2\tau} \sum_{j=0}^{2\tau-1} w(e_j) \\ &\leq \frac{1}{\tau} \sum_{t=0}^{\tau-1} \log_2 \#P(\alpha|_{[0;t]}) + \frac{1}{\tau} \log_2 \#v_{2\tau-1} - \frac{1}{\tau} \log_2 \#V_0 \\ &\leq \frac{1}{\tau} \log_2 r_{2,\text{inv}}(\tau, Q_2) + \frac{\bar{c}_2}{\tau} \end{aligned} \quad (53)$$

where $\bar{c}_2 = \log_2 \max_{v \in V_2} \#v$.

We define $G_1^* : \mathcal{A}_1 \rightarrow U_1$ based on the value of $\sigma_1^*(A)$, i.e., $G_1^*(A) := u$ where $\sigma_1^*(A) = (V', u)$. For any $A_1 \in \mathcal{A}_1$ and $u = G_1^*(A_1)$ there exists $A_2 \in \mathcal{A}_2$ such that $A_1 = M(A_2)$ and $u = r(G_2(A_2))$. Hence, we use 4), 2) and 1) to derive $F_1(A_1, G_1^*(A_1)) \subseteq M(F_2(A_2, G_2(A_2))) \subseteq M(Q_2) = Q_1$. Thus (\mathcal{A}_1, G_1^*) is an invariant cover of Σ_1 and Q_1 .

Now consider the set $\mathcal{S}_1 \subseteq \mathcal{A}_1^{[0;\tau]}$ implicitly defined by $\alpha \in \mathcal{S}_1$ if and only if there exists $(v_k, v_{k+1})_{k \in [0; \infty[} \in \mathcal{P}_1(v_0, \sigma_1^*)$ with $v_0 \in V_0$ so that $\alpha(t) = v_{2t}$ holds for all $t \in [0; \tau[$. The set $\{\alpha(0) \mid \alpha \in \mathcal{S}_1\}$ equals V_0 therefore it covers Q_1 . Consider any $\alpha \in \mathcal{S}_1$ and a play $(v_k, v_{k+1})_{k \in [0; \infty[} \in \mathcal{P}_1(v_0, \sigma_1^*)$ such that $\alpha(t) = v_{2t}$ holds for all $t \in [0; \tau[$. For $k \in [0; \tau - 1[$ if $v_{2k+1} = (V', u)$ then from the definition of \mathcal{S}_1 we have $P(\alpha|_{[0;k]}) = V'$ and from the definition of the MPG we have that V' covers $F_1(v_{2k}, u)$. Thus

$$\forall \alpha \in \mathcal{S}_1 \forall t \in [0; \tau - 1[F(\alpha(t), G_1^*(\alpha(t))) \subseteq \cup_{A' \in P(\alpha|_{[0;t]})} A'.$$

Therefore \mathcal{S}_1 is a (τ, Q) -spanning set in (\mathcal{A}_1, G_1^*) . Let $\alpha \in \mathcal{S}_1$ such that $\prod_{t=0}^{\tau-1} \#P(\alpha|_{[0;t]}) = N(\mathcal{S}_1)$. Pick an $e \in \mathcal{P}_1(\alpha(0), \sigma_1^*)$ such that $\alpha(t) = v_{2t}$ holds for all $t \in [0; \tau[$. Then from (48) we have

$$\begin{aligned} \nu(v_0) + \frac{c_1}{2\tau} &\geq \frac{1}{2\tau} \sum_{j=0}^{2\tau-1} w(e_j) \\ &= \frac{1}{\tau} \sum_{t=0}^{\tau-1} \log_2 \#P(\alpha|_{[0;t]}) + \frac{1}{\tau} \log_2 \#v_{2\tau-1} - \frac{1}{\tau} \log_2 \#V_0 \\ &\geq \frac{1}{\tau} \log_2 r_{1,\text{inv}}(\tau, Q_1) + \frac{\bar{c}_1}{\tau} \end{aligned} \tag{54}$$

where $\bar{c}_1 = -\log_2 \#V_1$.

From (53) and (54) we get

$$\begin{aligned} \frac{1}{\tau} \log_2 r_{1,\text{inv}}(\tau, Q_1) + \frac{\bar{c}_1}{\tau} &\leq \nu(v_0) + \frac{c_1}{2\tau} \leq \nu(\bar{v}) + \frac{c_1}{2\tau} \\ &\leq \frac{1}{\tau} \log_2 r_{2,\text{inv}}(\tau, Q_2) + \frac{c_1 + 2\bar{c}_2 - c_2}{2\tau}. \end{aligned}$$

Since this inequality holds for every $\tau \in \mathbb{N}$, we get

$$h(\mathcal{A}_1, G_1^*) \leq h(\mathcal{A}_2, G_2). \quad \square$$

A.2. Other Lemmas and Proofs.

Proof of Lemma 1. We fix $\tau_1, \tau_2 \in \mathbb{N}$ and choose two minimal (τ_i, Q) -spanning sets \mathcal{S}_i , $i \in \{1, 2\}$ in (\mathcal{A}, G) so that $r_{\text{inv}}(\tau_i, Q) = N(\mathcal{S}_i)$. Let \mathcal{S} be the set of sequences $\alpha : [0; \tau_1 + \tau_2[\rightarrow \mathcal{A}$ given by $\alpha(t) := \alpha_1(t)$ for $t \in [0; \tau_1[$ and $\alpha(t) := \alpha_2(t - \tau_1)$ for $t \in [\tau_1; \tau_1 + \tau_2[$, where $\alpha_i \in \mathcal{S}_i$ for $i \in \{1, 2\}$. We claim that \mathcal{S} is $(\tau_1 + \tau_2, Q)$ -spanning in (\mathcal{A}, G) . It is easy to see that $\{A \in \mathcal{A} \mid \exists \alpha \in \mathcal{S} A = \alpha(0)\}$ covers Q , since $\{A \in \mathcal{A} \mid \exists \alpha \in \mathcal{S}_1 A = \alpha(0)\}$ covers Q . Let $t \in [0; \tau_1 + \tau_2[$ and $\alpha \in \mathcal{S}$. If $t \in [0; \tau_1 - 1[$, we immediately see that $F(\alpha(t), G(\alpha(t))) \subseteq \cup_{A' \in P(\alpha|_{[0;t]})} A'$ since $\alpha_1 := \alpha|_{[0;\tau_1[} \in \mathcal{S}_1$ and \mathcal{S}_1 satisfies (4). Similarly, if $t \in [\tau_1; \tau_1 + \tau_2 - 1[$, we have $F(\alpha(t), G(\alpha(t))) \subseteq \cup_{A' \in P(\alpha|_{[0;t]})} A'$ since $\alpha_2 := \alpha|_{[\tau_1;\tau_1+\tau_2[} \in \mathcal{S}_2$ and \mathcal{S}_2 satisfies (4). For $t = \tau_1 - 1$, we know that $P(\alpha|_{[0;\tau_1[})$ equals $\{A \mid \exists \alpha_2 \in \mathcal{S}_2 \alpha_2(0) = A\}$ which covers Q and the inclusion $F(\alpha(t), G(\alpha(t))) \subseteq \cup_{A' \in P(\alpha|_{[0;t]})} A'$ follows. Hence, \mathcal{S} satisfies (4) and we see that \mathcal{S} is (τ, Q) -spanning. Subsequently, for $i \in \{1, 2\}$ and $\alpha \in \mathcal{S}_i$, $t \in [0; \tau_i - 1[$, let us use $P_i(\alpha|_{[0;t]}) := \{A \in \mathcal{A} \mid \exists \hat{\alpha} \in \mathcal{S}_i \hat{\alpha}|_{[0;t]} = \alpha|_{[0;t]} \wedge A = \hat{\alpha}(t+1)\}$. Then we have $P(\alpha|_{[0;t]}) = P_1(\alpha_1|_{[0;t]})$ with $\alpha_1 := \alpha|_{[0;\tau_1[}$ if $t \in [0; \tau_1 - 1[$ and $P(\alpha|_{[0;t]}) = P_2(\alpha_2|_{[0;t-\tau_1]})$ with $\alpha_2 := \alpha|_{[\tau_1;\tau_1+\tau_2[}$ if $t \in [\tau_1; \tau_1 + \tau_2 - 1[$, while for $t = \tau_1 - 1$ we have $P(\alpha|_{[0;t]}) = P_2(\alpha_2)$ with $\alpha_2 := \alpha|_{[\tau_1;\tau_1+\tau_2[}$ and $P(\alpha) := P_1(\alpha_1)$ with $\alpha_1 := \alpha|_{[0;\tau_1[}$. Therefore, $N(\mathcal{S})$ is bounded by $N(\mathcal{S}_1) \cdot N(\mathcal{S}_2)$ and we have $r_{\text{inv}}(\tau_1 + \tau_2, Q) \leq r_{\text{inv}}(\tau_1, Q) \cdot r_{\text{inv}}(\tau_2, Q)$. Hence, $\tau \mapsto \log_2 r_{\text{inv}}(\tau, Q)$, $\mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ is a subadditive sequence of real numbers and (7) follows by [12, Lem. 2.1]. \square

Proof of Lemma 2. For every $t \in [0; \tau[$, we define the set $\mathcal{S}_t := \{\alpha \in \mathcal{A}^{[0;t]} \mid \exists \alpha' \in \mathcal{S} \alpha'|_{[0;t]} = \alpha\}$. By definition of P , we have for all $\alpha \in \mathcal{S}$ the equality $P(\alpha) = \mathcal{S}_0$, which shows the assertion for $\tau = 1$ since in this case we have $\mathcal{S}_0 = \mathcal{S}$. Subsequently, we assume $\tau > 1$. For $t \in [0; \tau[$ and $a_0 \dots a_t \in \mathcal{S}_t$, we use $Y(a_0 \dots a_t) := \{\alpha \in \mathcal{S} \mid a_0 \dots a_t = \alpha|_{[0;t]}\}$ to

denote the sequences in \mathcal{S} whose initial part is restricted to $a_0 \dots a_t$. For $t \in [0; \tau - 1[$ and $a_0 \dots a_t \in \mathcal{S}_t$, we have

$$\begin{aligned} \#Y(a_0 \dots a_t) &= \sum_{a_{t+1} \in P(a_0 \dots a_t)} \#Y(a_0 \dots a_{t+1}) \\ &\leq \#P(a_0 \dots a_t) \max_{a_{t+1} \in P(a_0 \dots a_t)} \#Y(a_0 \dots a_{t+1}). \end{aligned}$$

For every $a_0 \dots a_{\tau-2} \in \mathcal{S}_{\tau-2}$ we have $\#Y(a_0 \dots a_{\tau-2}) = \#P(a_0 \dots a_{\tau-2})$ and we obtain a bound for $\#Y(a_0)$ by

$$\#P(a_0) \max_{a_1 \in P(a_0)} \#P(a_0 a_1) \cdots \max_{a_{\tau-2} \in P(a_0 \dots a_{\tau-3})} \#P(a_0 \dots a_{\tau-2})$$

so that $\#Y(a_0) \leq \max_{\alpha \in \mathcal{S}} \prod_{t=0}^{\tau-2} \#P(\alpha|_{[0;t]})$ holds for any $a_0 \in \mathcal{S}_0$. As $\cup_{a_0 \in \mathcal{S}_0} Y(a_0) = \mathcal{S}$ we observe $\#\mathcal{S} = \sum_{a_0 \in \mathcal{S}_0} \#Y(a_0) \leq \#\mathcal{S}_0 \max_{\alpha \in \mathcal{S}} \prod_{t=0}^{\tau-2} \#P(\alpha|_{[0;t]})$. Since $\mathcal{S}_0 = P(\alpha) = P(\alpha|_{[0;\tau-1]})$, we obtain the desired inequality $\#\mathcal{S} \leq \max_{\alpha \in \mathcal{S}} \prod_{t=0}^{\tau-1} \#P(\alpha|_{[0;t]})$. \square

Lemma 10. For $a, b \in \mathbb{R}$ and $T \in \mathbb{N}$, it holds

$$a + \sum_{t=1}^T \frac{ba^t}{(a-b)^t} = \frac{a^{T+1}}{(a-b)^T}. \quad (55)$$

Proof. We show the identity by induction over T . For $T = 1$, equation (55) is easy to verify and subsequently, we assume that the equality holds for $T - 1$ with $T \in \mathbb{N}_{\geq 2}$. Now we obtain

$$\begin{aligned} a + \sum_{t=1}^T \frac{ba^t}{(a-b)^t} &= \frac{ba^T}{(a-b)^T} + a + \sum_{t=1}^{T-1} \frac{ba^t}{(a-b)^t} \\ &= \frac{ba^T}{(a-b)^T} + \frac{a^T}{(a-b)^{T-1}} = \frac{ba^T + a^T(a-b)}{(a-b)^T} = \frac{a^{T+1}}{(a-b)^T} \end{aligned}$$

which completes the proof. \square

M. S. TOMAR IS WITH THE COMPUTER SCIENCE DEPARTMENT, LUDWIG MAXIMILIAN UNIVERSITY OF MUNICH, GERMANY.

M. RUNGGER IS WITH THE HYBRID CONTROL SYSTEMS GROUP, DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING, TECHNICAL UNIVERSITY OF MUNICH, GERMANY.

M. ZAMANI IS WITH THE COMPUTER SCIENCE DEPARTMENT, UNIVERSITY OF COLORADO BOULDER, USA. M. ZAMANI IS WITH THE COMPUTER SCIENCE DEPARTMENT, LUDWIG MAXIMILIAN UNIVERSITY OF MUNICH, GERMANY.

E-mail address: mahendra.tomar@lmu.de, matthias.rungger@tum.de, majid.zamani@colorado.edu