

Arbitrary Order Fixed-Time Differentiators

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Abstract

Differentiation is an important task in control, observation and fault detection. Levant's differentiator is unique, since it is able to estimate exactly and robustly the derivatives of a signal with a bounded high-order derivative. However, the convergence time, although finite, grows unboundedly with the norm of the initial differentiation error, making it uncertain when the estimated derivative is exact. In this paper we propose an extension of Levant's differentiator so that the worst case convergence time can be arbitrarily assigned independently of the initial condition, i.e. the estimation converges in *Fixed-Time*. We propose also a family of continuous differentiators and provide a unified Lyapunov framework for analysis and design.

I. INTRODUCTION

Given a (Lebesgue-measurable) signal $f(t)$ defined on $[0, \infty)$ the objective of a differentiator is to estimate as close as possible some of its time derivatives. Usually, signal $f(t)$ is composed of the base signal $f_0(t)$, which we want to differentiate and is assumed to be n -times differentiable, and a noise signal $\nu(t)$, that we will assume to be uniformly bounded, i.e. $f(t) = f_0(t) + \nu(t)$.

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In order to estimate the derivatives $f_0^{(i)}(t) = \frac{d^i}{dt^i} f_0(t)$, for $i = 1, \dots, n-1$, we propose the following nonlinear family of differentiators ($i = 1, \dots, n-1$)

$$\begin{aligned} \dot{x}_i &= -k_i \phi_i(x_1 - f) + x_{i+1}, \\ \dot{x}_n &= -k_n \phi_n(x_1 - f), \end{aligned} \tag{1}$$

where the nonlinear output injection terms, given by

$$\phi_i(z) = \varphi_i \circ \dots \circ \varphi_2 \circ \varphi_1(z), \tag{2}$$

are the composition of the monotonic growing functions $\varphi_i : \mathbb{R} \rightarrow \mathbb{R}$ (note that $[z]^p = |z|^p \text{sign}(z)$)

$$\varphi_i(s) = \kappa_i [s]^{\frac{r_{0,i+1}}{r_{0,i}}} + \theta_i [s]^{\frac{r_{\infty,i+1}}{r_{\infty,i}}}. \tag{3}$$

φ_i is a sum of two (signed) power functions, with powers selected as $r_{0,n} = r_{\infty,n} = 1$, and for $i = 1, \dots, n+1$

$$\begin{aligned} r_{0,i} &= r_{0,i+1} - d_0 = 1 - (n-i)d_0, \\ r_{\infty,i} &= r_{\infty,i+1} - d_\infty = 1 - (n-i)d_\infty, \end{aligned} \tag{4}$$

which are completely defined by two parameters $-1 \leq d_0 \leq d_\infty < \frac{1}{n-1}$. With this selection the powers in (3) satisfy $\frac{r_{0,i+1}}{r_{0,i}} \leq \frac{r_{\infty,i+1}}{r_{\infty,i}}$, so that the first term in $\varphi_i(s)$ is dominating for small values of s , while the second is dominating for large values of s . This domination effect is naturally extended to the injection terms ϕ_i in (2). The (internal) gains $\kappa_i > 0$ and $\theta_i > 0$ can be selected as arbitrarily positive values, and correspond to the desired weighting of each of the terms in φ_i (and therefore in ϕ_i). One possible and simple selection is $\kappa_i = \mu$ and $\theta_i = 1 - \mu$ for $i = 1, \dots, n$, with $0 < \mu < 1$ giving the weight of the low-power and the high-power terms. Note that, since for $d_0 = -1$ system (1) has a discontinuous right hand side, their solutions are understood in the sense of Filippov [1].

Some well-known differentiators in the literature are homogeneous. For example, the High-Gain observer used as differentiator in [2], [3] (see also [4]), being linear, is

homogeneous of degree zero. The classical robust and exact differentiator proposed by Levant [5], [6], [7] (see also [8]), has discontinuous injection terms and is also homogeneous. A family of homogeneous differentiators, including the previous ones, has been also proposed recently [9], [10], [8] for non positive homogeneity degrees, and in [11] for arbitrary degrees.

Differentiator (1) is not homogeneous, but it is homogeneous in the bi-limit [12] (bl-homogeneous for short), that is, near to the origin it is approximated by a homogeneous system of degree d_0 and far from the origin it is approximated by a homogeneous system of degree d_∞ . Although the scaling properties of the homogeneous systems are lost, the design of bl-homogeneous differentiators is more flexible, since the properties near the origin and far from it can be assigned independently. In particular, by selecting $d_0 = d_\infty = d$ the differentiator (1) becomes homogeneous. For $d = 0$ one obtains the High-Gain differentiator, for $d = -1$ Levant's robust and exact differentiator is recovered and for other values of d the family of differentiators in [9], [10], [8], [11] is attained. Note that for $d < 0$ (resp. $d = 0$) the estimation converges in finite-time (resp. exponentially). For $d > 0$ the convergence is asymptotic, but it attains any neighborhood of zero in a time which is uniform in the initial conditions [12].

Of particular interest for a differentiator is a property that is only achieved when $d_0 = d_\infty = -1$ [5], [6], [7]. In that case ϕ_n is discontinuous and it induces a Higher-Order Sliding-Mode at the origin, allowing the estimation to converge (in the absence of noise) exactly, robustly and in finite-time to the actual values of the signal derivatives, when the n -th derivative of the signal is bounded by a non zero constant, i.e. $\left| f_0^{(n)}(t) \right| \leq \Delta$. For all other values of $d_0 = d_\infty > -1$, convergence is only achieved if $\Delta = 0$.

One disadvantage of homogeneous (including Levant's exact) differentiators with negative homogeneity degree, is that the convergence time, although finite, grows unboundedly (and faster than linearly) with the size of the initial estimation error. One of the nice features of the bl-homogeneous design in general [12], and of the proposed differentiator

(1) in particular, is that assigning a positive homogeneity degree to the ∞ -limit approximation $d_\infty > 0$ and a negative homogeneity degree to the 0-limit approximation $d_0 < 0$, it is possible to counteract this effect: Convergence of the estimation will be achieved in *Fixed-Time (FxT)* [13], that is, the estimation error converges globally, in finite-time and the settling-time function is globally bounded by a positive constant \bar{T} , independent of the initial estimation error. This is an important feature, since the differentiator can be designed such that we are sure that after an arbitrarily assigned time \bar{T} the estimation is correct no matter what the initial conditions are. Moreover, if $d_0 = -1$ exact and robust estimation is obtained for all signals having bounded Lipschitz constant $\left|f_0^{(n)}(t)\right| \leq \Delta$, and not only for time polynomial signals, for which $f_0^{(n)}(t) \equiv 0$. For the first order differentiator (i.e. $n = 2$) this property has been obtained in [14], [15], [16], [17], using quadratic-like Lyapunov functions [18], [19]. This approach for the discontinuous first-order differentiator has been extended and refined in [20], where a detailed gain scaling has been developed and a tight convergence time estimation has been obtained, using the results of [21]. For differentiators of arbitrary order this can be achieved by using a switching strategy between two homogeneous differentiators of positive and negative degrees, as it is proposed in [22]. In [13] also a switching strategy between homogeneous differentiators with restricted degrees is presented.

This work can be seen as an extension to an arbitrary order of the smooth strategy of combining two homogeneous differentiators proposed in [14], [15], [16], and in the recent work [20]. Our construction extends to the discontinuous case the recursive observer design developed for continuous homogeneous observers in [23], [24] and highly improved in [12], [25] for *continuous* bl-homogeneous observers. Although many combinations in the selection of $d_0 \leq d_\infty$ are possible, we are particularly interested in the cases $-1 \leq d_0 \leq 0 \leq d_\infty < \frac{1}{n-1}$, and especially when $d_0 = -1$.

In Section II, some necessary concepts on bl-homogeneous functions and systems are briefly recalled. Section III presents the main properties of the proposed differentiator.

Section IV contains all the proofs. A simulation example is presented in Section V. In Section VI we draw some conclusions.

II. PRELIMINARIES

Our notation is fairly standard. We recall briefly some definitions of homogeneity and homogeneity in the bi-limit. However, for precise definitions and properties, we refer the reader to [27], [28], [29] for homogeneity of continuous or discontinuous systems, and to [12], [30] for homogeneity in the bi-limit of continuous or discontinuous systems, respectively.

For a vector $x \in \mathbb{R}^n$, all real values $\epsilon > 0$, and n positive real numbers $r_i > 0$ the dilation operator is defined as $\Delta_\epsilon^{\mathbf{r}}x = [\epsilon^{r_1}x_1, \dots, \epsilon^{r_n}x_n]^\top$. Constants $r_i > 0$ are the weights of the coordinates x_i , and $\mathbf{r} := [r_1, \dots, r_n]$ is the vector of weights. A function $V : \mathbb{R}^m \mapsto \mathbb{R}^n$ (resp. a vector field $f : \mathbb{R}^n \mapsto \mathbb{R}^n$) is said to be \mathbf{r} -homogeneous of degree $l \in \mathbb{R}$, or (\mathbf{r}, l) -homogeneous for short, if for all $\epsilon > 0$ and for all $x \in \mathbb{R}^m \setminus \{0\}$ the equality $V(\Delta_\epsilon^{\mathbf{r}}x) = \epsilon^l V(x)$ (resp., $f(\Delta_\epsilon^{\mathbf{r}}x) = \epsilon^l \Delta_\epsilon^{\mathbf{r}}f(x)$) holds.

A function $\varphi : \mathbb{R}^n \mapsto \mathbb{R}$ is said to be homogeneous in the 0-limit with associated triple $(\mathbf{r}_0, l_0, \varphi_0)$, if it is approximated near $x = 0$ by the (\mathbf{r}_0, l_0) -homogeneous function φ_0 . It is said to be homogeneous in the ∞ -limit with associated triple $(\mathbf{r}_\infty, l_\infty, \varphi_\infty)$, if it is approximated near $x = \infty$ by the $(\mathbf{r}_\infty, l_\infty)$ -homogeneous function φ_∞ . Similar definitions apply for vector fields and set-valued vector fields. Finally, a function $\varphi : \mathbb{R}^n \mapsto \mathbb{R}$ (or a vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, or set-valued vector field $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$) is said to be homogeneous in the bi-limit, or *bl-homogeneous* for short, if it is homogeneous in the 0-limit and homogeneous in the ∞ -limit.

III. PROPERTIES OF THE DIFFERENTIATOR

The main result of this work states that the differentiator (1), in the absence of noise, is able to estimate asymptotically the first $n-1$ derivatives of the signal $f_0(t)$. Let $\mathcal{S}_0^n \triangleq$

$\{f^{(n)}(t) \equiv 0\}$ represent the class of polynomial signals, while $\mathcal{S}_\Delta^n \triangleq \{|f^{(n)}(t)| \leq \Delta\}$ corresponds to the the class of n -Lipschitz signals.

Theorem 1. *Assume that the signal $f(t)$ satisfy the stated conditions. Select $-1 \leq d_0 \leq d_\infty < \frac{1}{n-1}$ and choose arbitrary positive (internal) gains $\kappa_i > 0$ and $\theta_i > 0$, for $i = 1, \dots, n$. Assume further that $|f_0^{(n)}(t)| \leq \Delta$ for some non negative Lipschitz constant $\Delta > 0$ if $d_0 = -1$ or $\Delta = 0$ if $d_0 > 0$. Under these conditions and in the absence of noise ($\nu(t) \equiv 0$), there exist appropriate gains $k_i > 0$, for $i = 1, \dots, n$, such that the solutions of the bl-homogeneous differentiator (1) converge globally and asymptotically to the derivatives of the signal, i.e. $x_i(t) \rightarrow f_0^{(i-1)}(t)$ as $t \rightarrow \infty$. In particular, they converge in Fixed-Time, i.e. $x_i(t) \rightarrow f_0^{(i-1)}(t)$ as $t \rightarrow \bar{T}$, for $i = 1, \dots, n$, if either*

(a) $-1 < d_0 < 0 < d_\infty < \frac{1}{n-1}$ and $f(t) \in \mathcal{S}_0^n$, or

(b) $-1 = d_0 < 0 < d_\infty < \frac{1}{n-1}$ and $f(t) \in \mathcal{S}_\Delta^n$. □

All proofs are given in Section IV. The distinguishing feature of the differentiator (1), compared to their homogeneous counterparts, is that it converges within a Fixed-Time when $d_0 < 0 < d_\infty$. For the discontinuous differentiator ($d_0 = -1$) this is accomplished for a much larger class of signals, since $\mathcal{S}_0^n \subset \mathcal{S}_\Delta^n$, and \mathcal{S}_Δ^n is much larger than \mathcal{S}_0^n .

Remark 2. The selection of the function φ_i in (3) is dictated by the simplicity and concreteness of the presentation. However, a rather large family of functions can be selected if they satisfy similar appropriate conditions.

A. Differentiation error dynamics and Lyapunov function

Defining the differentiation error as $e_i \triangleq x_i - f_0^{(i-1)}$, their dynamics satisfy ($i = 1, \dots, n-1$)

$$\begin{aligned} \dot{e}_i &= -k_i \phi_i(e_1 - \nu) + e_{i+1}, \\ \dot{e}_n &= -k_n \phi_n(e_1 - \nu) + \delta(t), \end{aligned} \tag{5}$$

where $\delta(t) = -f_0^{(n)}(t)$. For the variables $z_1 = \frac{e_1}{1}$, $z_i = \frac{e_i}{k_{i-1}}$ for $i = 2, \dots, n$, the dynamics of (5) becomes

$$\begin{aligned}\dot{z}_i &= -\tilde{k}_i (\phi_i(z_1 - \nu) - z_{i+1}), \\ \dot{z}_n &= -\tilde{k}_n (\phi_n(z_1 - \nu) - \bar{\delta}(t)),\end{aligned}\tag{6}$$

where for $i = 1, \dots, n$,

$$\tilde{k}_i = \frac{k_i}{k_{i-1}}, \quad k_0 = 1, \quad \bar{\delta}(t) = -\frac{f^{(n)}(t)}{k_n}.$$

For the convergence proof we will use a (smooth) bi-homogeneous Lyapunov Function V . To define it we select for $n \geq 2$ two positive real numbers p_0 and p_∞ , corresponding to the homogeneity degrees of the 0-limit and the ∞ -limit approximations of V , such that

$$p_0 \geq \max_{i \in \{1, \dots, n\}} \left\{ \frac{r_{0,i}}{r_{\infty,i}} (2r_{\infty,i} + d_\infty) \right\},\tag{7}$$

$$p_\infty \geq 2 \max_{i \in \{1, \dots, n\}} \{r_{\infty,i}\} + d_\infty,$$

$$\frac{p_0}{r_{0,i}} < \frac{p_\infty}{r_{\infty,i}}.\tag{8}$$

For $i = 1, \dots, n$ choose arbitrary positive real numbers $\beta_{0,i} > 0$, $\beta_{\infty,i} > 0$ and define the functions

$$Z_i(z_i, z_{i+1}) = \sum_{j \in \{0, \infty\}}\tag{9}$$

$$\beta_{j,i} \left[\frac{r_{j,i}}{p_j} |z_i|^{\frac{p_j}{r_{j,i}}} - z_i [\xi]^{\frac{p_j - r_{j,i}}{r_{j,i}}} + \frac{p_j - r_{j,i}}{p_j} |\xi|^{\frac{p_j}{r_{j,i}}} \right]$$

where $\xi = \varphi_i^{-1}(z_{i+1})$, and for $i = 1, \dots, n-1$, φ_i^{-1} is the inverse function of φ_i (3). For $i = n$ take $\xi = z_{n+1} \equiv 0$, i.e. $Z_n(z_n) = \beta_{0,n} \frac{1}{p_0} |z_n|^{p_0} + \beta_{\infty,n} \frac{1}{p_\infty} |z_n|^{p_\infty}$. The Lyapunov Function candidate is then defined as

$$V(z) = \sum_{j=1}^{n-1} Z_j(z_j, z_{j+1}) + Z_n(z_n).\tag{10}$$

Proposition 3. *Let the hypothesis of Theorem 1 be satisfied, and select p_0 and p_∞ such that (7) and (8) are fulfilled. Under these conditions and in the absence of noise, there exist gains $k_i > 0$, for $i = 1, \dots, n$, such that $V(z)$ in (10) is a C^1 , bl-homogeneous Lyapunov function for the estimation error dynamics (6) for any selection $-1 \leq d_0 \leq d_\infty < \frac{1}{n-1}$ and if $\Delta = 0$ in case $d_0 \neq -1$. Moreover, V satisfies the differential inequality (11) for some positive constants η_0, η_∞*

$$\dot{V}(z) \leq -\eta_0 V^{\frac{p_0+d_0}{p_0}}(z) - \eta_\infty V^{\frac{p_\infty+d_\infty}{p_\infty}}(z). \quad (11)$$

Thus, $z = 0$ is a Globally Asymptotically Stable equilibrium point of (6). In particular, if $d_0 < 0 < d_\infty$ then $z = 0$ is Fixed-Time Stable (FxTS) [26], that is, it is globally FTS and the settling-time function $T(z_0)$ is globally bounded by a positive constant \bar{T} , independent of z_0 , i.e., $\exists \bar{T} \in \mathbb{R}_{>0}$ such that $\forall z_0 \in \mathbb{R}^n$, $T(z_0) \leq \bar{T}$. $T(\cdot)$ is continuous at zero and locally bounded. Moreover, the Fixed-Time \bar{T} can be estimated from (11) as

$$\bar{T} \leq \frac{p_0}{d_0 \eta_\infty} \left(\frac{p_\infty d_0}{p_0 d_\infty} - 1 \right) \left(\frac{\eta_0}{\eta_\infty} \right)^{\frac{1}{\left(\frac{p_\infty d_0}{p_0 d_\infty} - 1 \right)}}. \quad (12)$$

□

Theorem 1 is in fact a consequence of this proposition.

B. Gain Calculation

Stabilizing gains $k_i > 0$, $i = 1, \dots, n$, for the differentiator (6) can be calculated using $V(z)$.

Proposition 4. *Let the hypothesis of Proposition 3 be satisfied. A sequence of stabilizing gains $k_i > 0$, for $i = 1, \dots, n$, can be calculated backwards as follows:*

- (a) Select $k_n \kappa_n > \Delta$ and $\tilde{k}_n > 0$.
- (b) For $i = n - 1, n - 2, \dots, 1$ select

$$\tilde{k}_i > \omega_i \left(\tilde{k}_{i-1}, \dots, \tilde{k}_n \right).$$

Functions ω_i are given by (18), in Section IV-A, and are obtained from V and \dot{V} .

Each function ω_i depends on the previous gains $(\tilde{k}_{i-1}, \dots, \tilde{k}_n)$, $\beta_{0,j}$, $\beta_{\infty,j}$, d_0 , d_∞ , p_0 , p_∞ , κ_j , and θ_j . Due to the recursive nature of the process, gains $(\tilde{k}_j, \dots, \tilde{k}_n)$ are appropriate for the differentiator of order $n - j + 1$.

C. Convergence acceleration and scaling the Lipschitz constant Δ

Perform on system (1), for arbitrary constants $\alpha > 0$ and $L > 0$, the following scaling of the gains,

$$\kappa_i \rightarrow \left(\frac{L^n}{\alpha}\right)^{\frac{d_0}{r_{0,i}}} \kappa_i, \theta_i \rightarrow \left(\frac{L^n}{\alpha}\right)^{\frac{d_\infty}{r_{\infty,i}}} \theta_i, k_i \rightarrow L^i k_i. \quad (13)$$

It is easy to show that the linear state transformation

$$e_i \rightarrow \frac{L^{n-i+1}}{\alpha} e_i,$$

together with a time scaling $t \rightarrow Lt$, transforms the scaled error system to system (5).

This means that the convergence is accelerated and the Lipschitz constant increased as

$$T(z_0) \rightarrow \frac{1}{L} T(z_0), \quad \Delta \rightarrow \alpha \Delta.$$

Using the scaling (13), it is possible to assign an *arbitrary* pair of (worst case) *convergence time* \bar{T}^* and Lipschitz constant Δ^* to the differentiator, following the procedure:

(i) Given $d_0 < 0 < d_\infty$, $\kappa_j > 0$ and $\theta_j > 0$, fix a set of stabilizing gains k_i and the corresponding supported perturbation size Δ , using e.g. Proposition 4.

(ii) Calculate the corresponding fixed-convergence time \bar{T} , either by means of (12) or by simulations.

(iii) Select the scaling gains (α, L) of (13) as $\alpha \geq \Delta^*/\Delta$ and $L \geq \bar{T}^*/\bar{T}$.

This procedure generalizes to an arbitrary order and arbitrary degrees that proposed in [20] for the first order differentiator with $d_0 = -1$. Note that this scaling, using two parameters, is novel also for the homogeneous case.

D. Effect of noise and the perturbation $\delta(t)$

In the presence of noise the estimation error cannot be zero asymptotically, but it is uniformly and ultimately bounded. Moreover, when $d_0 > -1$ and $\Delta > 0$ the estimation error is also only uniformly and ultimately bounded. This also happens when $d_0 = -1$, $d_\infty > -1$ and the differentiator gains are not sufficiently large to fully compensate the effect of $\delta(t)$.

Proposition 5. *Let the hypothesis of Theorem 1 be satisfied and select stabilizing gains k_i for the differentiator (1). If $-1 < d_0 \leq d_\infty < \frac{1}{n-1}$ or $-1 = d_0 < d_\infty < \frac{1}{n-1}$ then the estimation error system (6) is Input-to-State Stable (ISS), considering $\nu(t)$ and $\delta(t)$ as inputs. \square*

It follows from Proposition 5 that if noise and perturbation are bounded, then the estimation error z will be also bounded, and if $(\nu(t), \delta(t)) \rightarrow 0$ then $e(t) \rightarrow 0$. The precision for small noise signals is determined by the 0-limit approximation and it is therefore identical to the one of the homogeneous differentiator of homogeneity degree d_0 [6], [7], [8], [10]. In particular, when $d_0 = -1$, $|\nu(t)| \leq \epsilon$, $|\delta(t)| \leq \Delta$ the following inequalities are achieved in finite-time

$$\left| x_i(t) - f_0^{(i-1)}(t) \right| \leq \lambda_i \Delta^{\frac{i-1}{n}} |\epsilon|^{\frac{n-i+1}{n}}, \forall t \geq T.$$

IV. PROOF OF THE RESULTS: A LYAPUNOV APPROACH

We write (6) in compact form as $\dot{z} \in F(z) + b\bar{\delta}(t)$, where $b = [0, \dots, 0, 1]^T \in \mathbb{R}^n$. Since for $d_0 = -1$ the function $\phi_n(z_1)$ is set-valued due to the sign function, $F(z)$ is in general a set-valued vector field which satisfies standard assumptions.

Since for $i = 1, \dots, n-1$, $r_{0,i} > 0$ and $r_{\infty,i} > 0$, each function $\varphi_i(z_i)$ in (3) is \mathcal{C} on \mathbb{R} , \mathcal{C}^1 on $\mathbb{R} \setminus \{0\}$, strictly increasing and surjective. Its inverse $\varphi_i^{-1}(z_i)$ is well-defined, \mathcal{C} on \mathbb{R} , \mathcal{C}^1 on $\mathbb{R} \setminus \{0\}$, and also strictly increasing. For $\varphi_n(z_n)$ the same is true if

$d_0 > -1$. If $d_0 = -1$ function $\varphi_n(z_n) = \kappa_n [z_n]^0 + \theta_n [z_n]^{1+d_\infty}$ is discontinuous in $z_n = 0$, and \mathcal{C}^1 on $\mathbb{R} \setminus \{0\}$.

Since $d_\infty > d_0$, $\varphi_i(z_i)$ in (3) is homogeneous in the 0-limit and in the ∞ -limit, with approximating functions $\varphi_{i,0}(z_i) = \kappa_i [z_i]^{\frac{r_{0,i+1}}{r_{0,i}}}$ and $\varphi_{i,\infty}(z_i) = \theta_i [z_i]^{\frac{r_{\infty,i+1}}{r_{\infty,i}}}$, respectively. For $i = 1, \dots, n-1$, the inverse $\varphi_i^{-1}(s)$ is also homogeneous in the 0-limit and in the ∞ -limit, with approximating functions $\varphi_{i,0}^{-1}(s) = \frac{1}{\theta_i} [s]^{\frac{r_{\infty,i}}{r_{\infty,i+1}}}$ and $\varphi_{i,\infty}^{-1}(s) = \frac{1}{\kappa_i} [s]^{\frac{r_{0,i}}{r_{0,i+1}}}$, respectively. Note also that when $d_\infty > 0$, for $i = 1, \dots, n-1$, $\varphi_{i,0}^{-1}(s)$ is homogeneous of negative degree, and therefore $\varphi_i^{-1}(s)$ not differentiable at $s = 0$. However, $[\varphi_i^{-1}(s)]^\mu$ is differentiable at $s = 0$ for every $\mu \geq \frac{r_{\infty,i}+d_\infty}{r_{\infty,i}}$, and $|\varphi_i^{-1}(s)|^\mu$ for every $\mu > \frac{r_{\infty,i}+d_\infty}{r_{\infty,i}}$.

For $i = 1, \dots, n-1$, functions ϕ_i in (2), being compositions of φ_j , are \mathcal{C} on \mathbb{R} , \mathcal{C}^1 on $\mathbb{R} \setminus \{0\}$, strictly increasing and surjective. In case $d_0 = -1$, function $\phi_n(z_1) = \kappa_n [z_1]^0 + \theta_n [\varphi_{n-1} \circ \dots \circ \varphi_2 \circ \varphi_1(z_1)]^{\frac{r_{\infty,n+1}}{r_{\infty,n}}}$ is discontinuous. Since it is used in the Differential Inclusion (6), using the Filippov's regularization procedure [1] it will become an upper semi-continuous set-valued function, where the sign function $[s]^0$ is defined as usually for $s \neq 0$, but for $s = 0$ its values are an interval, i.e. $[0]^0 = [-1, 1] \in \mathbb{R}$. ϕ_i 's are also homogeneous in the 0-limit and in the ∞ -limit, with approximating functions

$$\phi_{i,0}(s) = K_{i,0} [s]^{\frac{r_{0,i+1}}{r_{0,1}}}, \phi_{i,\infty}(s) = K_{i,\infty} [s]^{\frac{r_{\infty,i+1}}{r_{\infty,1}}},$$

where $K_{i,0} = \prod_{j=1}^i \kappa_j^{\frac{r_{0,i+1}}{r_{0,j+1}}}$, $K_{i,\infty} = \prod_{j=1}^i \theta_j^{\frac{r_{\infty,i+1}}{r_{\infty,j+1}}}$. For $\delta(t) \equiv 0$ system (6) is bl-homogeneous with homogeneity degrees d_0 and d_∞ and weights $\mathbf{r}_0 = [r_{0,1}, \dots, r_{0,n}]$ and $\mathbf{r}_\infty = [r_{\infty,1}, \dots, r_{\infty,n}]$ as in (4).

Since $\varphi_n(z_n)$ is not involved in the definition of Z_i , it has to satisfy weakened conditions compared to the other functions φ_i . From the properties of functions φ_i it follows that Z_i is \mathcal{C} on \mathbb{R} . For Z_i to be \mathcal{C}^1 on \mathbb{R} the powers in (9) have to be sufficiently large, what is the case if (7) is fulfilled. Note that if (8) is met, Z_i is also bl-homogeneous with approximations $Z_{i,0}(z_i, z_{i+1})$, given by the first term in (9)

with $\xi = \frac{1}{\theta_i} [z_{i+1}]^{\frac{r_{\infty,i}}{r_{\infty,i+1}}}$; and $Z_{i,\infty}(z_i, z_{i+1})$, given by the second term in (9) with $\xi = \frac{1}{\kappa_i} [z_{i+1}]^{\frac{r_{0,i}}{r_{0,i+1}}}$. Moreover, functions $Z_i(z_i, z_{i+1})$ are nonnegative.

Lemma 6. $Z_i(z_i, z_{i+1}) \geq 0$ for every $i = 1, \dots, n$ and $Z_i(z_i, z_{i+1}) = 0$ if and only if $\varphi_i(z_i) = z_{i+1}$.

Proof: From Young's inequality it follows that

$$z_i [\xi]^{\frac{p_0 - r_{0,i}}{r_{0,i}}} \leq \frac{r_{0,i}}{p_0} |z_i|^{\frac{p_0}{r_{0,i}}} + \left(1 - \frac{r_{0,i}}{p_0}\right) |\xi|^{\frac{p_0}{r_{0,i}}}.$$

From this and (9) it follows that $Z_i(z_i, z_{i+1}) \geq 0$. ■

The partial derivatives of $Z_i(z_i, z_{i+1})$, for which we introduce the symbols σ_i and s_i , are given by

$$\sigma_i(z_i, z_{i+1}) \triangleq \frac{\partial Z_i(z_i, z_{i+1})}{\partial z_i} = \tag{14}$$

$$\sum_{j=\{0,\infty\}} \beta_{j,i} \left([z_i]^{\frac{p_j - r_{j,i}}{r_{j,i}}} - [\varphi_i^{-1}(z_{i+1})]^{\frac{p_j - r_{j,i}}{r_{j,i}}} \right)$$

$$s_i(z_i, z_{i+1}) \triangleq \frac{\partial Z_i(z_i, z_{i+1})}{\partial z_{i+1}} = \tag{15}$$

$$\sum_{j=\{0,\infty\}} -\beta_{j,i} \frac{p_j - r_{j,i}}{r_{j,i}} (z_i - \xi_i) |\xi_i|^{\frac{p_j - 2r_{j,i}}{r_{j,i}}} \frac{\partial \xi_i}{\partial z_{i+1}}$$

where $\xi_i = \varphi_i^{-1}(z_{i+1})$. Note that $s_n(z_n, z_{n+1}) \equiv 0$, and that functions $\sigma_i(z_i, z_{i+1})$ and $s_i(z_i, z_{i+1})$ are \mathcal{C} on \mathbb{R} , bl-homogeneous of degrees $p_0 - r_{0,i}$, $p_0 - r_{0,i+1}$ for the 0-approximation and $p_\infty - r_{\infty,i}$, $p_\infty - r_{\infty,i+1}$ for the ∞ -approximation, respectively. Furthermore, for $i = 1, \dots, n-1$, $\{\sigma_i = 0\} = \{s_i = 0\}$, i.e. σ_i, s_i are zero where Z_i achieves its minimum $Z_i = 0$.

V is bl-homogeneous of degrees p_0 and p_∞ and \mathcal{C}^1 on \mathbb{R} . It is also non negative, since it is a positive combination of non negative terms. Moreover, V is positive definite since $V(z) = 0$ only if all $Z_i = 0$, what only happens at $z = 0$. Due to bl-homogeneity it is radially unbounded [28].

For calculation of the time derivative of V along the trajectories of (6), we consider first the nominal situation in which $\delta(t) \equiv 0$ if $d_0 \neq -1$ and $|\delta(t)| \leq \Delta$ when $d_0 = -1$. In that case

$$\dot{V}(z) \in W(z), \quad (16)$$

where

$$\begin{aligned} W(z) = & -\tilde{k}_1 \sigma_1 (\phi_1(z_1) - z_2) \\ & - \sum_{j=2}^{n-1} \tilde{k}_j [s_{j-1} + \sigma_j] (\phi_j(z_1) - z_{j+1}) \\ & - \tilde{k}_n [s_{n-1} + \sigma_n] \left(\phi_n(z_1) - \frac{\Delta}{k_n} [-1, 1] \right) \end{aligned} \quad (17)$$

where $[-1, 1] \in \mathbb{R}$ is an interval and we omit the variable dependence of σ_i and s_i . Since the set-valued vector field $F(z)$ on the right-hand side of (6) with $\bar{\delta}(t) \equiv 0$ is upper semi-continuous and the gradient $\nabla V(z)$ of $V(z)$ is continuous, by [30, Lemma 6] the set-valued function $W(z) = \nabla V(z) F(z)$ is also upper semi-continuous, and the single-valued function $W^*(z) = \max\{W(z)\}$ is upper semi-continuous and bl-homogeneous of degrees $p_0 + d_0$ and $p_\infty + d_\infty$, respectively. Moreover, $W(0) = 0$ since $\nabla V(0) = 0$. We want to show that there exist values of $\tilde{k}_i > 0$ such that $W(z) < 0$, i.e. W is negative definite. This is equivalent to showing that $W^*(z) < 0$. We note that when $-1 < d_0$ function W is indeed single-valued and continuous, and thus $W^*(z) = W(z)$ (recall that in this case $\Delta = 0$). When $d_0 = -1$, W is set-valued because the term $\phi_n(z_1) - \frac{\Delta}{k_n} [-1, 1] = \kappa_n [z_1]^0 + \theta_n [\phi_{n-1}(z_1)]^{\frac{r_{\infty, n+1}}{r_{\infty, n}}} - \frac{\Delta}{k_n} [-1, 1]$ is set-valued. To simplify the development in what follows, we will write simply $\phi_n(z_1)$ but it is meant $\phi_n(z_1) - \frac{\Delta}{k_n} [-1, 1]$.

For the proof we will use the following property of upper semi-continuous, bl-homogeneous single-valued functions, proven in [30].

Lemma 7. *Let $\eta : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}_{\leq 0}$ be two upper semicontinuous (u.s.c.) single-valued bl-homogeneous functions, with the same weights \mathbf{r}_0 and \mathbf{r}_∞ , degrees m_0 and m_∞ , and approximating functions η_0, η_∞ and γ_0, γ_∞ , which are u.s.c. Suppose that $\forall x \in \mathbb{R}^n, \gamma(x) \leq 0, \gamma_0(x) \leq 0, \gamma_\infty(x) \leq 0$. If $\gamma(x) = 0 \wedge x \neq 0 \Rightarrow \eta(x) < 0$, $\gamma_0(x) = 0 \wedge x \neq 0 \Rightarrow \eta_0(x) < 0$, $\gamma_\infty(x) = 0 \wedge x \neq 0 \Rightarrow \eta_\infty(x) < 0$ then there are constants $\lambda^* \in \mathbb{R}$, $c_0 > 0$, and $c_\infty > 0$ such that for all $\lambda \geq \max\{\lambda_0, \lambda_\infty\}$, $\lambda_0 \geq \lambda^*$, $\lambda_\infty \geq \lambda^*$ and for all $x \in \mathbb{R}^n \setminus \{0\}$,*

$$\begin{aligned} \eta(x) + \lambda\gamma(x) &\leq -c_0 \|x\|_{\mathbf{r}_0, p}^{m_0} - c_\infty \|x\|_{\mathbf{r}_\infty, p}^{m_\infty}, \\ \eta_0(x) + \lambda\gamma_0(x) &\leq -c_0 \|x\|_{\mathbf{r}_0, p}^{m_0}, \\ \eta_\infty(x) + \lambda\gamma_\infty(x) &\leq -c_\infty \|x\|_{\mathbf{r}_\infty, p}^{m_\infty}. \quad \square \end{aligned}$$

To show that $W(z) < 0$ we exploit its structure. So consider the values of W restricted to some hypersurfaces: for $i = 1, \dots, n-1$

$$\mathcal{Z}_1 = \{\varphi_1(z_1) = z_2\} \cdots \mathcal{Z}_i = \mathcal{Z}_{i-1} \cap \{\varphi_i(z_i) = z_{i+1}\}.$$

These sets are clearly related as $\mathcal{Z}_{n-1} \subset \cdots \subset \mathcal{Z}_i \subset \cdots \subset \mathcal{Z}_1 \subset \mathbb{R}^n$. Note that on \mathcal{Z}_i functions σ_i and s_i vanish, i.e. $\sigma_i = s_i = 0$, and therefore they also vanish on \mathcal{Z}_j , for every $j > i$. Let $W_i = W_{\mathcal{Z}_i}$ represent the value of $W(z)$ restricted to the manifold \mathcal{Z}_i . We can obtain the value of W_1 by replacing in $W(z)$ the variable z_1 by $z_1 = \varphi_1^{-1}(z_2)$, so that W_1 becomes a function of (z_2, \dots, z_n) . In general, we obtain the value of W_i , for $i = 1, \dots, n-1$, by replacing in $W(z)$ the variables (z_1, \dots, z_i) by its values in terms of z_{i+1} , so that W_i becomes a function of $\bar{z}_{i+1} \triangleq (z_{i+1}, \dots, z_n)$.

The first term in $W(z)$ (17) is non positive, i.e., $\sigma_1(z_1, z_2)(\phi_1(z_1) - z_2) \leq 0$, and it vanishes on \mathcal{Z}_1 . Evaluating $W(z)$ on \mathcal{Z}_1 we obtain (recall that $s_1 = 0$, $\phi_i(z_1) =$

$\varphi_i \circ \cdots \circ \varphi_2 \circ \varphi_1 (z_1)$ and $z_{n+1} \equiv 0$)

$$W_1(\bar{z}_2) = -\tilde{k}_2 \sigma_2(z_2, z_3) (\varphi_2(z_2) - z_3) \\ - \sum_{j=3}^n \tilde{k}_j [s_{j-1} + \sigma_j] (\varphi_j \circ \cdots \circ \varphi_2(z_2) - z_{j+1}) .$$

Note that $W_1(\bar{z}_2)$ has the same structure as $W(z)$. Its first term is non positive, i.e., $\sigma_2(z_2, z_3) (\varphi_2(z_2) - z_3) \leq 0$, and it vanishes on \mathcal{Z}_2 . Evaluating $W_1(\bar{z}_2)$ on \mathcal{Z}_2 we obtain $W_2(\bar{z}_3)$. Applying this procedure recursively, and using the facts that $s_i = 0$ on \mathcal{Z}_i and $\phi_i(z_1) = \varphi_i \circ \cdots \circ \varphi_2 \circ \varphi_1(z_1)$, we find that, for $i = 1, \dots, n-1$,

$$W_i(\bar{z}_{i+1}) = -\tilde{k}_{i+1} \sigma_{i+1}(\varphi_{i+1}(z_{i+1}) - z_{i+2}) \\ - \sum_{j=i+2}^n \tilde{k}_j [s_{j-1} + \sigma_j] (\varphi_j \circ \cdots \circ \varphi_{i+1}(z_{i+1}) - z_{j+1}) .$$

Note that the first term of $W_i(\bar{z}_{i+1})$ is non positive, i.e., $\sigma_{i+1}(z_{i+1}, z_{i+2}) (\varphi_{i+1}(z_{i+1}) - z_{i+2}) \leq 0$, and it vanishes on \mathcal{Z}_{i+1} . For $i = n-1$ the value of $W_{n-1}(\bar{z}_n)$ is given by (recall that $z_{n+1} \equiv 0$)

$$W_{n-1}(\bar{z}_n) = -\tilde{k}_n (\beta_{0,n} [z_n]^{p_0-1} + \beta_{\infty,n} [z_n]^{p_\infty-1}) \\ \times \left(\kappa_n [z_n]^{1+d_0} + \theta_n [z_n]^{1+d_\infty} - \frac{\Delta}{k_n} [-1, 1] \right) ,$$

where we have used (14) and (3). Here we distinguish two cases: (i) $-1 < d_0$ and (ii) $d_0 = -1$.

When $-1 < d_0$, $\Delta = 0$, W is single-valued and continuous, $W_{n-1}(\bar{z}_n)$ is bi-homogeneous and it is negative for any $\tilde{k}_n > 0$.

In case $d_0 = -1$, if $\tilde{\Delta} \triangleq \frac{\Delta}{\kappa_n k_n} < 1$ the following equality is satisfied for the set-valued map

$$\kappa_n \left([z_n]^0 - \tilde{\Delta} [-1, 1] \right) = \kappa_n [z_n]^0 \left[1 - \tilde{\Delta}, 1 + \tilde{\Delta} \right] ,$$

so that it is clear that for any $\nu > 0$, $\kappa_n \left([z_n]^0 - \tilde{\Delta}[-1, 1] \right) [z_n]^\nu > 0$ for $z_n \neq 0$ and $\kappa_n \left([z_n]^0 - \tilde{\Delta}[-1, 1] \right) [z_n]^\nu = 0$ for $z_n = 0$. Therefore, with $W_{n-1}^*(\bar{z}_n) = \max \{W_{n-1}(\bar{z}_n)\}$,

$$W_{n-1}^*(\bar{z}_n) = -\tilde{k}_n \left(\beta_{0,n} |z_n|^{p_0-1} + \beta_{\infty,n} |z_n|^{p_\infty-1} \right) \times \left(\kappa_n \left(1 - \frac{\Delta}{\kappa_n k_n} \right) + \theta_n |z_n|^{1+d_\infty} \right).$$

Function W_{n-1}^* is single-valued, upper semi-continuous, bl-homogeneous and negative definite for any $\tilde{k}_n > 0$.

In all cases, the same is true for its homogeneous approximations (as shown in [9], [10]). $W_{n-2}(\bar{z}_{n-1})$ is also bl-homogeneous. According to Lemma 7 we conclude that $W_{n-2}(\bar{z}_{n-1})$ can be rendered negative definite (in \mathcal{Z}_{n-2}) by selecting $\tilde{k}_{n-1} > 0$ sufficiently large. Since $W_i(\bar{z}_{i+1})$ is bl-homogeneous and the conditions of Lemma 7 are satisfied for $W_i(\bar{z}_{i+1})$ and its homogeneous approximations (as shown in [9], [10]), we conclude that $W_{i-1}(\bar{z}_i)$ can be rendered negative definite (in \mathcal{Z}_{i-1}) selecting $\tilde{k}_i > 0$ sufficiently large. Applying the argument recursively, we conclude that there exist positive values of $(\tilde{k}_1, \dots, \tilde{k}_n)$ such that $W(z) < 0$.

Moreover, using [12, Corollary 2.15] (see also [30, Lemma 10]) the inequality (11) follows. Using inequality (11) satisfied by the Lyapunov function we obtain, as a direct consequence of [30, Lemma 3], the estimation of the convergence time given by (12).

A. Gain calculation

The gains \tilde{k}_i are calculated backwards from $i = n, n-1, \dots, 2, 1$ such that $W_i(\bar{z}_{i+1}) > 0$. This will be the case if they are chosen as given in Proposition 4, where functions

ω_i are defined as (recall that $\varphi_n(s)$ should be replaced by $\varphi_n(s) - \frac{\Delta}{k_n}[-1, 1]$) :

$$\omega_i \left(\tilde{k}_{i+1}, \dots, \tilde{k}_n \right) \triangleq \max_{(z_i, \dots, z_n) \in \mathbb{R}^{n-i+1}} \left\{ \frac{\sum_{j=i+1}^n \tilde{k}_j [s_{j-1} + \sigma_j] (z_{j+1} - \varphi_j \circ \dots \circ \varphi_i(z_i))}{\sigma_i(z_i, z_{i+1}) (\varphi_i(z_i) - z_{i+1})} \right\}. \quad (18)$$

These maximizations are well-posed, as it is shown in the previous steps of the proof.

B. ISS and effect of noise

We prove the Proposition 5 using function $V(z)$ (10), although it is not a Lyapunov Function for all possible stabilizing gains k_i . However, the converse Lyapunov theorem for bl-homogeneous differential inclusions [30, Theorem 1] assure the existence of an appropriate one. For it, the calculations are similar to those performed with (10). If we consider the effect of noise, in the continuous case, i.e. $d_0 > -1$, \dot{V} can be written as

$$\begin{aligned} \dot{V}(z) &= \frac{1}{2}W(z) + R(z, \nu, \delta) \\ R(\cdot) &\triangleq \frac{1}{2}W - \sum_{j=1}^n \tilde{k}_j [s_{j-1} + \sigma_j] (\phi_j(z_1 - \nu) - \phi_j(z_1)) \\ &\quad + \tilde{k}_n [s_{n-1}(z_{n-1}, z_n) + \sigma_n(z_n)] \bar{\delta}(t). \end{aligned}$$

Define $Q(\nu, \bar{\delta}) = |\nu|^{\frac{p_0+d_0}{r_{0,1}}} + |\nu|^{\frac{p_\infty+d_\infty}{r_{\infty,1}}} + |\bar{\delta}|^{\frac{p_0+d_0}{1+d_0}} + |\bar{\delta}|^{\frac{p_\infty+d_\infty}{1+d_\infty}}$. W , R and Q are continuous bl-homogeneous functions of degrees $p_0 + d_0$ and $p_\infty + d_\infty$, and Q is non negative. Furthermore, the function $R(z, \nu, \bar{\delta}) - \gamma Q(\nu, \bar{\delta})$ is also continuous and bl-homogeneous and $R(z, 0, 0) < 0$ for $z \neq 0$. The same is true for the homogeneous approximations. And therefore, using [12, Corollary 2.15] (see also [30, Lemma 10]) we conclude that there exists $\gamma > 0$ such that $R(z, \nu, \bar{\delta}) \leq \gamma Q(\nu, \bar{\delta})$. And thus $\dot{V}(z) \leq -\frac{1}{2}\eta_0 V^{\frac{p_0+d_0}{p_0}}(z) - \frac{1}{2}\eta_\infty V^{\frac{p_\infty+d_\infty}{p_\infty}}(z) + \gamma Q(\nu, \bar{\delta})$, implying ISS by standard arguments. For the discontinuous case, $d_0 = -1$, we can use the procedure used in [10].

V. EXAMPLE

We perform some simulations with the bi-homogeneous second order differentiator ($n = 3$), with $d_0 = -1$ and $d_\infty = \frac{1}{5}$. The signal to be differentiated is $f_0(t) = \frac{1}{2} \sin(\frac{1}{2}t) + \frac{1}{2} \cos(t)$, for which $\Delta = \frac{5}{8}$, the internal gains $\kappa_i = \theta_i = 1$ for $i = 1, 2, 3$ and the gains $k_1 = 3$, $k_2 = 1.5\sqrt{3}$, $k_3 = 1.1$. Two values of the scaling parameters were selected $(\alpha, L) = (1, 1)$ and $(\alpha, L) = (1, 2)$.

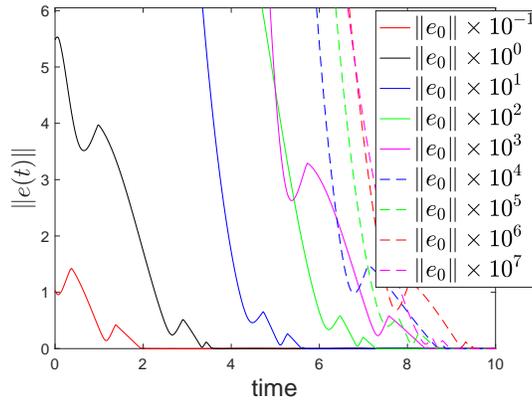
In Figures 1a and 1b the norm $\|e(t)\|$ of the estimation error is presented for different initial conditions $e_0 = [1, -5, 1] \times 10^p$, for $p = -1, 0, 1, \dots, 7$, with $L = 1$ in Figure 1a and $L = 2$ in Figure 1b. It is apparent from these graphs, that despite of a change in the initial conditions of 8 orders of magnitude, the convergence time does not increase accordingly, and it approaches an asymptote. This is illustrated in Figure 1c, where the convergence time versus the (logarithmic) value of the initial condition is shown. The figures also show that by doubling the scaling parameter L from $L = 1$ to $L = 2$, the convergence time is halved. In fact, using the parameter L any arbitrary convergence time can be attained.

VI. CONCLUSIONS

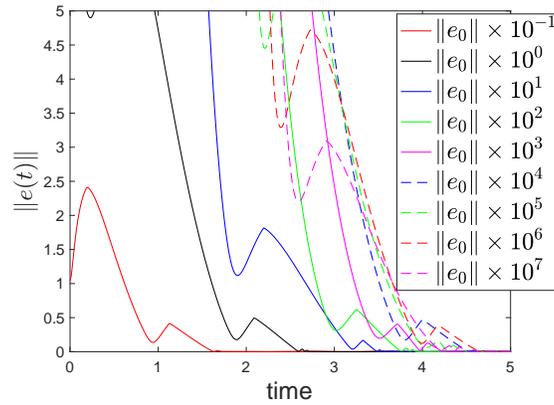
We have proposed Fixed-Time converging exact and robust differentiators. In particular, Levant's discontinuous differentiator is extended with higher order terms, so that its convergence time is independent of the initial estimation error and can be arbitrarily assigned. Moreover, a full family of continuous differentiators are also studied, in a unified Lyapunov framework. We use the concept of homogeneity in the bi-limit, developed in [12], and the recursive observer design, for our objective.

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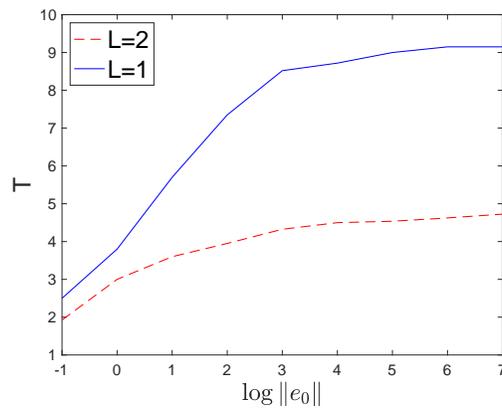
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(a) $\|e(t)\|$ for different $\|e_0\|$ in a logarithmic succession and $L = 1$.



(b) $\|e(t)\|$ for different $\|e_0\|$ in a logarithmic succession and $L = 2$.



(c) Convergence time versus the logarithmus of the initial condition $\|e_0\|$.

Fig. 1: Time behavior of the estimation error norm $\|e(t)\|$ and Convergence time.

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