# Structured preconditioning of conjugate gradients for path-graph network optimal control problems 

Armaghan Zafar, Michael Cantoni, and Farhad Farokhi


#### Abstract

A structured preconditioned conjugate gradient (PCG) solver is developed for the Newton steps in second-order methods for a class of constrained network optimal control problems. Of specific interest are problems with discrete-time dynamics arising from the path-graph interconnection of $N$ heterogeneous sub-systems. The computational complexity of each PGC step is shown to be $O(N T)$, where $T$ is the length of the time horizon. The proposed preconditioning involves a fixed number of block Jacobi iterations per PCG step. A decreasing analytic bound on the effective conditioning is given in terms of this number. The computations are decomposable across the spatial and temporal dimensions of the optimal control problem, into sub-problems of size independent of $N$ and $T$. Numerical results are provided for a mass-spring-damper chain.


Index Terms-Optimal control of networks; Structured secondorder solver; System chains.

## I. Introduction

CONSIDER the path-graph interconnection of $N$ heterogeneous sub-systems with dynamics given by

$$
\begin{equation*}
x_{j, t+1}=A_{j, t} x_{j, t}+B_{j, t} u_{j, t}+E_{j, t} x_{j-1, t}+F_{j, t} x_{j+1, t} \tag{1}
\end{equation*}
$$

where $x_{j, t} \in \mathbb{R}^{n_{j}}$ and $u_{j, t} \in \mathbb{R}^{m_{j}}$ are the state and input of sub-system $j \in \mathcal{N}=\{1,2, \ldots, N\}$ at time $t \in \mathcal{T}=$ $\{0,1, \ldots, T\}$, respectively. The initial conditions are given by $x_{j, 0}=\boldsymbol{\xi}_{j} \in \mathbb{R}^{n_{j}}$ for $j \in \mathcal{N}$ and the spatial boundary conditions are given by $x_{0, t}=\chi_{t} \in \mathbb{R}^{n_{0}}$ and $x_{N+1, t}=\boldsymbol{\zeta}_{t} \in \mathbb{R}^{n_{N+1}}$ for $t \in \mathcal{T}$. The constrained finite-horizon linear-quadratic (LQ) optimal control problem of interest is the following:

$$
\begin{equation*}
\min _{\substack{\left(x_{j, t}\right) \\\left(u_{j, t}(j, t) \in(\{0, N+1\} \cup \mathcal{N}) \in \mathcal{N} \times \mathcal{T}\right.}} \frac{1}{2} \sum_{j \in \mathcal{N}} \sum_{t \in \mathcal{T}} \ell_{j, t}\left(x_{j, t}, u_{j, t}\right) \tag{2a}
\end{equation*}
$$

subject to

$$
\begin{align*}
& \text { for }(j, t) \in \mathcal{N} \times(\mathcal{T} \backslash\{T\}),  \tag{2b}\\
x_{0, t}=\chi_{t}, x_{N+1, t}=\boldsymbol{\zeta}_{t} & \text { for } t \in \mathcal{T}  \tag{2c}\\
x_{j, 0}=\boldsymbol{\xi}_{j} & \text { for } j \in \mathcal{N}  \tag{2~d}\\
C_{j, t} x_{j, t}+D_{j, t} u_{j, t} \leq \boldsymbol{\kappa}_{j, t} & \text { for }(j, t) \in \mathcal{N} \times \mathcal{T} \tag{2e}
\end{align*}
$$

where $\ell_{j, t}(x, u)=x^{\prime} Q_{j, t} x+2 x^{\prime} S_{j, t} u+u^{\prime} R_{j, t} u, C_{j, t} \in$ $\mathbb{R}^{\nu_{j} \times n_{j}}, D_{j, t} \in \mathbb{R}^{\nu_{j} \times m_{j}}$ and $\kappa_{j, t} \in \mathbb{R}^{\nu_{j}}$. For $j \in \mathcal{N}$ and $t \in \mathcal{T} \backslash\{T\}$, it is assumed that $Q_{j, t}=Q_{j, t}^{\prime} \succeq 0$, $R_{j, t}=R_{j, t}^{\prime} \succ 0$, and $Q_{j, t}-S_{j, t}^{\prime} R_{j, t}^{-1} S_{j, t} \succeq 0$. Moreover, for every $j \in \mathcal{N}, Q_{j, T} \succeq 0$, but $S_{j, T}=0, R_{j, T}=0$, and $D_{j, T}=0$, so that $u_{j, T}$ plays no role (i.e., it can be removed as

[^0]a decision variable.) Under these assumptions the problem (2) is a convex quadratic program with $O(N T)$ decision variables and $O(N T)$ constraints.

While the cost (2a) and inequality constraints (2e) are separable across the sub-systems and time horizon, there is coupling in the equality constraint 2b). Specifically, there is spatial coupling between states of adjacent sub-systems, and inter-temporal coupling. Path-graph network dynamics of this kind are relevant in the operation of irrigation channels [1], vehicle platoons [2], supply chains [3], and radial power networks [4]. The structure also arises from the discretization of one-dimensional partial differential equations [5].

This note is about the computation of second-order search directions for solving the quadratic program (2). Specifically, a preconditioned conjugate gradient (PCG) solver (e.g., see [6]) is developed for the Newton steps in second-order methods, such as the interior point method [7]. The main innovation pertains to the $O(N T)$ computational complexity of each PCG iteration, and decomposability of the preconditioning computations across both the temporal and spatial dimensions, into sub-problems of sizes that are independent of $N$ and $T$. The computations are amenable to implementation as $\lceil N / 2\rceil$ parallel threads each comprising a sequence of $2 T$ (possibly dense but small) sub-problems.

Structure in second-order methods for optimal control problems was studied in [8], [9], where the so-called Riccatifactorization approach was originally developed, and more recently in [10]-[14]. These papers all focus on the structure associated with localized coupling in the temporal dimension of optimal control problems. Following the underlying approach for problem (2) results in solvers with $O\left(T N^{3}\right)$ computational complexity for each of the moderate number of Newton steps needed for second-order methods to converge (typically $10-20$ steps). The computations are decomposable across the temporal dimension, but not the spatial dimension. The resulting sub-problems, of size $O(N)$, are amenable to distribution across parallel processors in a tree type communication network, leading to $O\left(\log (T) N^{3}\right)$ time complexity [14].

In [15], the aforementioned approach is pursued in the special case of (2) with directed spatial coupling, by interchanging the role of the time and space indexes to develop a Newton step solver with computational complexity $O\left(N T^{3}\right)$. The computations are decomposable across the spatial dimension of the problem, but not the temporal dimension. Again, parallel processing can lead to $O\left(\log (N) T^{3}\right)$ time complexity.

All of the approaches described above constitute direct methods for solving the Newton steps. In particular, all are related, in some way, to structured block-LU factorization
for a permutation of variables that yields a block tri-diagonal structure in the linear system of equations to be solved. With direct methods, it appears to be difficult to leverage both the spatial and the temporal structure in (2).

The proposed PCG method is an iterative solver, of the kind used for large sparse problems [6]. For (2), the size of the linear equation to solve at each Newton step is $O(N T)$. Thus, in the worst case, it may take $O(N T)$ iterations to terminate. It is well-known that preconditioning can significantly reduce the number of PCG iterations needed. In this note, it is proposed to use a fixed number of block Jacobi iterations for preconditioning. In principle, this fixed number can be selected to achieve preconditioning specifications, in that a decreasing analytic bound on the conditioning of the outcome is provided. For the numerical example presented, it is observed that as few as two Jacobi iterations can result in a much smaller number of PCG steps than the worst-case bound described above. Importantly, the preconditioning steps are decomposable across both the spatial and temporal dimension of (2). The size of the resulting $O(N T)$ parallelizable sub-problems is independent of $N$ and $T$. As such, the computational complexity of PCG steps is $O(N T)$. In the worst-case of $O(N T)$ iterations, the computational complexity of a Newton step becomes $O\left(N^{2} T^{2}\right)$. So for $T \approx N$, as perhaps required for the optimal control problem to be meaningful, the proposed approach is (at the least) no worse than the structured direct methods discussed, and potentially much better for large problems.

First-order methods can also lead to structured solvers for separable-in-cost quadratic programs like (2). For example, methods based on dual decomposition [16], and operator splitting methods such as ADMM [17] and FAMA [18] can lead to simple parallelizable computations. For the structure in (2), the dual decomposition technique of [19] leads to local computations for each sub-system. Similarly, the ADMM approach presented in [20], and projected sub-gradient algorithm of [21], also yield decomposable computations. However, these first-order methods typically require a huge number of iterations to converge. The issue is exacerbated within the pathgraph context of this note, since the algebraic connectivity of the underlying sparsity pattern, which influences the rate of convergence [22], [23], tends to zero as $N$ grows. This motivates the consideration of second-order methods. The challenge is to maintain structure in the computations.

The note is organized as follows. An equivalent reformulation of problem (2) is presented in Section [I) including the structure of corresponding Newton steps in Section II-A PCG methods are overviewed in Section III) and the structured preconditioner based on fixed block Jacobi iterations is developed in Section IV The proposed PCG algorithm is explored numerically for mass-spring-damper chain example in Section (V) Concluding remarks are provided in Section VI.

## Notation

Identity matrices are denoted by $I$. blkdiag $(\cdot)$ denotes the matrix with block diagonal elements given by the arguments, which are the only non-zero elements, and $\operatorname{col}(\cdot)$ denotes the concatenation of the input arguments into a column vector.

Every block tri-diagonal matrix is parameterized by sequences $\Phi=\left(\Phi_{k}\right)_{k=1}^{m} \in \prod_{k=1}^{m} \mathbb{R}^{l_{k} \times l_{k}}$ and $\Omega=\left(\Omega_{k}\right)_{k=2}^{m} \in$ $\prod_{k=2}^{m} \mathbb{R}^{l_{k-1} \times l_{k}}$ for appropriate $\left(l_{k}\right)_{k=1}^{m} \subset \mathbb{N}^{m}$ and $m \in \mathbb{N}$. Given such sequences $\Phi$ and $\Omega$, the corresponding block tridiagonal matrix is denoted by

$$
\operatorname{blktrid}(\Phi, \Omega)=\left[\begin{array}{cccc}
\Phi_{1} & \Omega_{2}^{\prime} & & \\
\Omega_{2} & \Phi_{2} & \ddots & \\
& \ddots & \ddots & \Omega_{m}^{\prime} \\
& & \Omega_{m} & \Phi_{m}
\end{array}\right] \in \mathbb{R}^{\bar{l} \times \bar{l}}
$$

where $\bar{l}=\sum_{k=1}^{m} l_{k}$.

## II. Problem Re-Formulation

Defining $u_{j}=\operatorname{col}\left(u_{j, 0}, \ldots, u_{j, T-1}\right) \in \mathbb{R}^{m_{j} T}, x_{j}=$ $\operatorname{col}\left(x_{j, 0}, \ldots, x_{j, T}\right) \in \mathbb{R}^{n_{j}(T+1)}$, and slack variables $\theta_{j}=$ $\operatorname{col}\left(\theta_{j, 0}, \ldots, \theta_{j, T}\right) \in \mathbb{R}^{\nu_{j}(T+1)}$, problem (2) can be reformulated as the following quadratic program:

$$
\min _{\substack{\left(x_{j}\right)_{j \in\{0, N+1\} \cup \mathcal{N}}\left(u_{j}\right)_{j \in \mathcal{N}}}} \frac{1}{2} \sum_{j \in \mathcal{N}}\left[\begin{array}{l}
x_{j}  \tag{3a}\\
u_{j}
\end{array}\right]^{\prime}\left[\begin{array}{ll}
Q_{j} & S_{j}^{\prime} \\
S_{j} & R_{j}
\end{array}\right]\left[\begin{array}{l}
x_{j} \\
u_{j}
\end{array}\right]
$$

subject to $x_{0}=\boldsymbol{\chi}, x_{N+1}=\boldsymbol{\zeta}$, and

$$
\begin{align*}
& 0=A_{j} x_{j}+B_{j} u_{j}+E_{j} x_{j-1}+F_{j} x_{j+1}+H_{j} \boldsymbol{\xi}_{j}, \quad j \in \mathcal{N} \\
& 0=C_{j} x_{j}+D_{j} u_{j}+\theta_{j}-\kappa_{j}, \quad j \in \mathcal{N}  \tag{3b}\\
& 0 \leq \theta_{j}, \quad j \in \mathcal{N} \tag{3c}
\end{align*}
$$

where

$$
\begin{aligned}
& Q_{j}=\operatorname{blkdiag}\left(Q_{j, 0}, \ldots, Q_{j, T}\right) \in \mathbb{R}^{n_{j}(T+1) \times n_{j}(T+1)}, \\
& R_{j}=\operatorname{blkdiag}\left(R_{j, 0}, \ldots, R_{j, T-1}\right) \in \mathbb{R}^{m_{j} T \times m_{j} T}, \\
& S_{j}=\left[\operatorname{blkdiag}\left(S_{j, 0}, \ldots, S_{j, T-1}\right) 0\right] \in \mathbb{R}^{m_{j} T \times n_{j}(T+1)}, \\
& C_{j}=\operatorname{blkdiag}\left(C_{j, 0}, \ldots, C_{j, T}\right) \in \mathbb{R}^{\nu_{j}(T+1) \times n_{j}(T+1)} \text {, } \\
& D_{j}=\left[\operatorname{blkdiag}\left(D_{j, 0}, \ldots, D_{j, T-1}\right)^{\prime} 0\right]^{\prime} \in \mathbb{R}^{\nu_{j}(T+1) \times m_{j} T}, \\
& H_{j}=\left[\begin{array}{llll}
I & 0 & \cdots & 0
\end{array}\right]^{\prime} \in \mathbb{R}^{n_{j}(T+1) \times n_{j}} \text {, } \\
& \boldsymbol{\kappa}_{j}=\operatorname{col}\left(\boldsymbol{\kappa}_{j, 0}, \ldots, \boldsymbol{\kappa}_{j, T}\right) \in \mathbb{R}^{\nu_{j}(T+1)}, \\
& \chi=\operatorname{col}\left(\chi_{0}, \ldots, \chi_{T}\right) \in \mathbb{R}^{n_{0}(T+1)}, \\
& \boldsymbol{\zeta}=\operatorname{col}\left(\boldsymbol{\zeta}_{0}, \ldots, \boldsymbol{\zeta}_{T}\right) \in \mathbb{R}^{n_{N+1}(T+1)}, \\
& A_{j}=\left[\begin{array}{ccc}
-I & & \\
A_{j, 0}-I & \\
& \ddots & \ddots \\
& & A_{j, T-1}-I
\end{array}\right], B_{j}=\left[\begin{array}{ccc}
0 & \cdots & 0 \\
B_{j, 0} & \ddots & \vdots \\
& \ddots & 0 \\
& & B_{j, T-1}
\end{array}\right], \\
& E_{j}=\left[\begin{array}{cccc}
0 & & & \\
E_{j, 0} & 0 & & \\
& \ddots & \ddots & \\
& & E_{j, T-1} & 0
\end{array}\right], \text { and } F_{j}=\left[\begin{array}{cccc}
0 & & & \\
F_{j, 0} & 0 & & \\
& \ddots & \ddots & \\
& & & F_{j, T-1}
\end{array}\right] .
\end{aligned}
$$

Note that $A_{j} \in \mathbb{R}^{n_{j}(T+1) \times n_{j}(T+1)}, B_{j} \in \mathbb{R}^{n_{j}(T+1) \times m_{j} T}$, $E_{j} \in \mathbb{R}^{n_{j}(T+1) \times n_{j-1}(T+1)}$, and $F_{j} \in \mathbb{R}^{n_{j}(T+1) \times n_{j+1}(T+1)}$. The block bi-diagonal structure of the matrices $A_{j}$ arises from the temporal structure of the system dynamics in the optimal control problem (2).

For the quadratic program (3), the Karush-Kuhn-Tucker (KKT) conditions for optimality are given by

$$
\begin{align*}
& Q_{1} x_{1}+S_{1}^{\prime} u_{1}+A_{1}^{\prime} p_{1}+C_{1}^{\prime} \lambda_{1}+E_{2}^{\prime} p_{2}=0,  \tag{4a}\\
& Q_{j} x_{j}+S_{j}^{\prime} u_{j}+A_{j}^{\prime} p_{j}+C_{j}^{\prime} \lambda_{j}+F_{j-1}^{\prime} p_{j-1} \\
& \quad+\quad E_{j+1}^{\prime} p_{j+1}=0, \quad j \in \mathcal{N} \backslash\{1, N\},  \tag{4b}\\
& Q_{N} x_{N}+S_{N}^{\prime} u_{N}+A_{N}^{\prime} p_{N}+C_{N}^{\prime} \lambda_{N}+F_{N-1}^{\prime} p_{N-1}=0,  \tag{4c}\\
& S_{j} x_{j}+R_{j}^{\prime} u_{j}+B_{j}^{\prime} p_{j}+D_{j}^{\prime} \lambda_{j}=0, \quad j \in \mathcal{N},  \tag{4d}\\
& A_{1} x_{1}+B_{1} u_{1}+E_{1} \boldsymbol{\chi}+F_{1} x_{2}+H_{1} \boldsymbol{\xi}_{1}=0,  \tag{4e}\\
& A_{j} x_{j}+B_{j} u_{j}+E_{j} x_{j-1}+F_{j} x_{j+1}+H_{j} \boldsymbol{\xi}_{j}=0, \\
& \quad j \in \mathcal{N} \backslash\{1, N\},  \tag{4f}\\
& A_{N} x_{N}+B_{N} u_{N}+E_{N} x_{N-1}+F_{N} \boldsymbol{\zeta}+H_{N} \boldsymbol{\xi} \boldsymbol{\xi}_{N}=0,  \tag{4~g}\\
& C_{j} x_{j}+D_{j} u_{j}-\boldsymbol{\kappa}_{j}+\theta_{j}=0, \quad j \in \mathcal{N},  \tag{4h}\\
& \Lambda_{j} \Theta_{j} \mathbf{1}=0, \quad \text { and } \quad\left[\lambda_{j}^{\prime} \theta_{j}^{\prime}\right]^{\prime} \geq 0, \quad j \in \mathcal{N}, \tag{4i}
\end{align*}
$$

where $p_{j}=\operatorname{col}\left(p_{j, 0}, \ldots, p_{j, T}\right) \in \mathbb{R}^{n_{j}(T+1)}$ and $\lambda_{j}=$ $\operatorname{col}\left(\lambda_{j, 0}, \ldots, \lambda_{j, T}\right) \in \mathbb{R}^{\nu_{j}(T+1)}$ are Lagrange multipliers, $\Lambda_{j}=\operatorname{blkdiag}\left(\lambda_{j, 0}, \ldots, \lambda_{j, T}\right) \in \mathbb{R}^{\nu_{j}(T+1) \times \nu_{j}(T+1)}, \Theta_{j}=$ $\operatorname{blkdiag}\left(\theta_{j, 0}, \ldots, \theta_{j, T}\right) \in \mathbb{R}^{\nu_{j}(T+1) \times \nu_{j}(T+1)}$, and 1 denotes a vector of all ones. Since (3) is convex, the KKT conditions are necessary and sufficient for optimality [7].

## A. Newton's Method

Various second-order optimization algorithms can be understood in terms of Newton's method for solving the KKT conditions (e.g., see [7].) Typically, only a moderate number of Newton steps is required for convergence, and this is the main advantage over first-order optimization algorithms. The benefit comes from the use of second-order information, which can be constructed explicitly for quadratic programs. For the problem (3), the Newton steps in an interior point method (e.g., see [7]) take the form of the update

$$
\begin{equation*}
s^{(n+1)}=s^{(n)}+\alpha^{(n)} \delta^{(n)} \tag{5}
\end{equation*}
$$

where $\alpha^{(n)}>0$ is a step size, $s^{(n)}=\operatorname{col}\left(s_{1}^{(n)}, \ldots, s_{N}^{(n)}\right)$, $s_{j}^{(n)}=\operatorname{col}\left(x_{j}^{(n)}, u_{j}^{(n)}, p_{j}^{(n)}, \lambda_{j}^{(n)}, \theta_{j}^{(n)}\right)$, and the secondorder search direction $\delta^{(n)}=\operatorname{col}\left(\delta_{1}^{(n)}, \ldots, \delta_{N}^{(n)}\right)$ is obtained by solving the linearized KKT conditions, given by

$$
\begin{equation*}
\operatorname{blktrid}\left(\Phi^{(n)}, \Omega\right) \delta^{(n)}=b^{(n)} \tag{6}
\end{equation*}
$$

with $\Phi^{(n)}=\left(\Phi_{j}^{(n)}\right)_{j \in \mathcal{N}}, \Omega=\left(\Omega_{j}\right)_{j \in \mathcal{N} \backslash\{1\}}, b^{(n)}=$ $\operatorname{col}\left(b_{1}^{(n)}, \ldots, b_{N}^{(n)}\right)$,

$$
\begin{align*}
\Phi_{j}^{(n)} & =\left[\begin{array}{ccccc}
Q_{j} & S_{j}^{\prime} & A_{j}^{\prime} & C_{j}^{\prime} & 0 \\
S_{j} & R_{j} & B_{j}^{\prime} & D_{j}^{\prime} & 0 \\
A_{j} & B_{j} & 0 & 0 & 0 \\
C_{j} & D_{j} & 0 & 0 & I \\
0 & 0 & 0 & \Theta_{j}^{(n)} & \Lambda_{j}^{(n)}
\end{array}\right], j \in \mathcal{N},  \tag{7a}\\
\Omega_{j} & =\left[\begin{array}{ccccc}
0 & 0 & F_{j-1}^{\prime} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
E_{j} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right], j \in \mathcal{N} \backslash\{1\},  \tag{7b}\\
b_{1}^{(n)} & =\operatorname{col}\left(0,0,-H_{1} \boldsymbol{\xi}_{1}-E_{1} \chi, \kappa_{1}, \eta_{1}^{(n)}\right)-\Phi_{1}^{(n)} s_{1}^{(n)}-\Omega_{2}^{\prime} s_{2}^{(n)}, \tag{7c}
\end{align*}
$$

$$
\begin{align*}
b_{N}^{(n)}= & \operatorname{col}\left(0,0,-H_{N} \boldsymbol{\xi}_{N}-F_{N} \boldsymbol{\zeta}, \boldsymbol{\kappa}_{N}, \eta_{N}^{(n)}\right) \\
& -\Omega_{N} s_{N-1}^{(n)}-\Phi_{N}^{(n)} s_{N}^{(n)}  \tag{7d}\\
b_{j}^{(n)}= & \operatorname{col}\left(0,0,-H_{j} \boldsymbol{\xi}_{j}, \boldsymbol{\kappa}_{j}, \eta_{j}^{(n)}\right) \\
& -\Omega_{j} s_{j-1}^{(n)}-\Phi_{j}^{(n)} s_{j}^{(n)}-\Omega_{j+1}^{\prime} s_{j+1}^{(n)}, j \in \mathcal{N} \backslash\{1, N\},(7 \mathrm{e})  \tag{7e}\\
\eta_{j}^{(n)}= & \Theta_{j}^{(n)} \lambda_{j}^{(n)}+\Lambda_{j}^{(n)} \theta_{j}^{(n)}-\Lambda_{j}^{(n)} \Theta_{j}^{(n)} \mathbf{1}+\sigma^{(n)} \mu^{(n)} \mathbf{1}, j \in \mathcal{N} . \tag{7f}
\end{align*}
$$

In (7f), the scalar $\mu^{(n)}=\sum_{j=1}^{N}\left(\left(\lambda_{j}^{(n)}\right)^{\prime} \theta_{j}^{(n)}\right) / \sum_{j=1}^{N}\left(\nu_{j}(T+1)\right)$ is a measure of the duality gap and $\sigma^{(n)} \in(0,1)$ is a centering parameter. The step-size scalar $\alpha^{(n)}>0$ in (5) is selected (online) to ensure the components of $\lambda_{j}^{(n+1)}$ and $\theta_{j}^{(n+1)}$ remain positive for $j \in \mathcal{N}$. The coefficient matrix $\operatorname{blktrid}\left(\Phi^{(n)}, \Omega\right)$ in (6) is non-singular because, the matrices $A_{j}$ are non-singular for all $j \in \mathcal{N}$ (see, [15, Lemma A.1].)

## B. Structure-Preserving Block Elimination

$\Lambda_{j}, \Theta_{j}$ and $R_{j}$ in 7a are block diagonal, with block sizes that are independent of $N$ and $T$. For $j \in \mathcal{N}$, let $\delta_{j}^{(n)}=\operatorname{col}\left(\delta_{x_{j}}^{(n)}, \delta_{u_{j}}^{(n)}, \delta_{p_{j}}^{(n)}, \delta_{\lambda_{j}}^{(n)}, \delta_{\theta_{j}}^{(n)}\right)$ and $b_{j}^{(n)}=$ $\operatorname{col}\left(b_{x_{j}}^{(n)}, b_{u_{j}}^{(n)}, b_{p_{j}}^{(n)}, b_{\lambda_{j}}^{(n)}, b_{\theta_{j}}^{(n)}\right)$ be partitions aligned with the structure of $s_{j}^{(n)}$ noted below (5). Dropping the Newton iteration index $(n)$, the ordered elimination of

$$
\begin{align*}
\delta_{\theta_{j}} & =\Lambda_{j}^{-1}\left(b_{\theta_{j}}-\Theta_{j} \delta_{\lambda_{j}}\right),  \tag{8}\\
\delta_{\lambda_{j}} & =-\left(\Theta_{j}^{-1} \Lambda_{j}\right)\left(b_{\lambda_{j}}-C_{j} \delta_{x_{j}}-D_{j} \delta_{u_{j}}-\Lambda_{j}^{-1} b_{\theta_{j}}\right),  \tag{9}\\
\delta_{u_{j}} & =\hat{R}_{j}^{-1}\left(\hat{b}_{u_{j}}-\hat{S}_{j} \delta_{x_{j}}-B_{j}^{\prime} \delta_{p_{j}}\right), \tag{10}
\end{align*}
$$

from (6), for $j \in \mathcal{N}$, yields the smaller symmetric system

$$
\begin{equation*}
\operatorname{blktrid}(\tilde{\Phi}, \tilde{\Omega}) \tilde{\delta}=\tilde{b} \tag{11}
\end{equation*}
$$

where $\tilde{\Phi}=\left(\tilde{\Phi}_{j}\right)_{j \in \mathcal{N}}, \tilde{\Omega}=\left(\tilde{\Omega}_{j}\right)_{j \in \mathcal{N} \backslash\{1\}}, \tilde{\delta}=\operatorname{col}\left(\tilde{\delta}_{1}, \ldots, \tilde{\delta}_{N}\right)$, $\tilde{b}=\operatorname{col}\left(\tilde{b}_{1}, \ldots, \tilde{b}_{N}\right)$, and $\tilde{\delta}_{j}=\operatorname{col}\left(\delta_{x_{j}}, \delta_{p_{j}}\right) \in \mathbb{R}^{2 n_{j}(T+1)}$, $\tilde{b}_{j}=\operatorname{col}\left(\tilde{b}_{x_{j}}, \tilde{b}_{p_{j}}\right) \in \mathbb{R}^{2 n_{j}(T+1)}$,

$$
\begin{align*}
& \tilde{\Phi}_{j}=\left[\begin{array}{ll}
\tilde{Q}_{j} & \tilde{A}_{j}^{\prime} \\
\tilde{A}_{j} & \tilde{R}_{j}
\end{array}\right] \in \mathbb{R}^{2 n_{j}(T+1) \times 2 n_{j}(T+1)},  \tag{12a}\\
& \tilde{\Omega}_{j}=\left[\begin{array}{cc}
0 & F_{j-1}^{\prime} \\
E_{j} & 0
\end{array}\right] \in \mathbb{R}^{2 n_{j}(T+1) \times 2 n_{j}(T+1)},  \tag{12b}\\
& {\left[\begin{array}{cc}
\tilde{Q}_{j} & \tilde{A}_{j}^{\prime} \\
\tilde{A}_{j} & \tilde{R}_{j}
\end{array}\right]=\left[\begin{array}{cc}
\hat{Q}_{j} & A_{j}^{\prime} \\
A_{j} & 0
\end{array}\right]-\left[\begin{array}{c}
\hat{S}_{j}^{\prime} \\
B_{j}
\end{array}\right] \hat{R}_{j}^{-1}\left[\begin{array}{l}
\hat{S}_{j}^{\prime} \\
B_{j}
\end{array}\right]^{\prime},}  \tag{12c}\\
& {\left[\begin{array}{cc}
\hat{Q}_{j} & \hat{S}_{j}^{\prime} \\
\hat{S}_{j} & \hat{R}_{j}
\end{array}\right]=\left[\begin{array}{cc}
Q_{j} & S_{j}^{\prime} \\
S_{j} & R_{j}
\end{array}\right]+\left[\begin{array}{c}
C_{j}^{\prime} \\
D_{j}^{\prime}
\end{array}\right]\left(\Theta_{j}^{-1} \Lambda_{j}\right)\left[\begin{array}{c}
C_{j}^{\prime} \\
D_{j}^{\prime}
\end{array}\right]^{\prime},}  \tag{12d}\\
& {\left[\begin{array}{l}
\tilde{b}_{x_{j}} \\
\tilde{b}_{p_{j}}
\end{array}\right]=\left[\begin{array}{l}
\hat{b}_{x_{j}} \\
\hat{b}_{p_{j}}
\end{array}\right]-\left[\begin{array}{c}
\hat{S}_{j}^{\prime} \\
B_{j}
\end{array}\right] \hat{R}_{j}^{-1} \hat{b}_{u_{j}},}  \tag{12e}\\
& {\left[\begin{array}{l}
\hat{b}_{x_{j}} \\
\hat{b}_{u_{j}}
\end{array}\right]=\left[\begin{array}{l}
b_{x_{j}} \\
b_{u_{j}}
\end{array}\right]+\left[\begin{array}{ll}
C_{j}^{\prime} & C_{j}^{\prime} \\
D_{j}^{\prime} & D_{j}^{\prime}
\end{array}\right]\left[\begin{array}{c}
\left(\Theta_{j}^{-1} \Lambda_{j}\right) b_{\lambda_{j}} \\
\Theta_{j}^{-1} b_{\theta_{j}}
\end{array}\right],} \tag{12f}
\end{align*}
$$

for $j \in \mathcal{N}$. Note that the computations required to form (12c)-(12f) are decomposable. Only the manipulation of block diagonal matrices, with block sizes independent of $N$ and $T$, is required. Moreover, the structure of (6) is preserved in (11). Further, it is of note that in (12a), $\tilde{Q}_{j}$ and $\tilde{R}_{j}$ are block diagonal, and $\tilde{A}_{j}$ is block bi-diagonal.

Next, an iterative algorithm based on the PCG method is developed to solve (11). The number of iterations required depends on the quality of the preconditioner used. In the worst case, the maximum number of iterations is $O(N T)$, i.e., the size of the problem. The worst case computational complexity of the proposed approach is thus $O\left(N^{2} T^{2}\right)$, since the computational complexity of each PCG step is shown to be $O(N T)$ for the structured problem at hand. This is (at the least) no worse than the previously discussed direct methods when $T \approx N$. However, good preconditioning can substantially reduce the number of PCG iterations needed. The properties of a structured preconditioner are detailed in Section IV This is the main contributions of the work.

## III. PCG Solvers

The conjugate gradient (CG) method is an iterative Krylov subspace method. It is used for solving linear systems of equations with positive-definite coefficient matrix [24]. While non-singular, the block tri-diagonal matrix $\operatorname{blktrid}(\tilde{\Phi}, \tilde{\Omega})$ in (11) has both positive and negative eigenvalues. This indefinite system can be solved using other Krylov methods, like MINRES [25] or GMRES [26]. However, the computations for these are more involved than the CG method, with reduced scope for decomposability in the case of structured problems. Transforming both sides of (11) by blktrid $(\tilde{\Phi}, \tilde{\Omega})$ from the left yields the positive-definite system of equations

$$
\begin{equation*}
\Psi \tilde{\delta}=\breve{b} \tag{13}
\end{equation*}
$$

where $\Psi=(\operatorname{blktrid}(\tilde{\Phi}, \tilde{\Omega}))^{2}$ and $\breve{b}=\operatorname{blktrid}(\tilde{\Phi}, \tilde{\Omega}) \tilde{b}$. The positive-definite matrix $\Psi$ is now block penta-diagonal, but (13) now is amenable to the CG method.

Let $e^{(i)}=\tilde{\delta}^{(i)}-\tilde{\delta}^{*}$ be the error between $i$-th iterate $\tilde{\delta}^{(i)}$ of the CG method and the exact solution $\tilde{\delta}^{*}$ of (13). It can be shown that $e^{(i)}$ satisfies the following [27, Thm. 6.29]:

$$
\begin{equation*}
\left\|e^{(i)}\right\|_{\Psi} \leq 2((\sqrt{\kappa(\Psi)}-1) /(\sqrt{\kappa(\Psi)}+1))^{i}\left\|e^{(0)}\right\|_{\Psi} \tag{14}
\end{equation*}
$$

where $\|e\|_{\Psi}=e^{\prime} \Psi e, \kappa(\Psi)=\lambda_{\max }(\Psi) / \lambda_{\min }(\Psi)$ is the condition number, and $\lambda_{\max }(\Psi)$ (resp. $\lambda_{\min }(\Psi)$ ) is the maximum (resp. minimum) eigenvalue of $\Psi$. As such, the CG method converges faster for $\kappa(\Psi)$ closer to 1 . To improve the condition number, problem (13) can be transformed into

$$
\begin{equation*}
P^{-1 / 2} \Psi P^{-1 / 2} \breve{\delta}=P^{-1 / 2} \breve{b} \tag{15}
\end{equation*}
$$

where $\breve{\delta}=P^{1 / 2} \tilde{\delta}$ and $P=P^{\prime} \succ 0$. The CG method is then applied to (15). An efficient implementation of this PCG (i.e., preconditioned CG) method is given in Algorithm (1) [6].

The preconditioner $P=\Psi$ would give $P^{-1 / 2} \Psi P^{-1 / 2}=$ $I$. But steps 3 and 12 of Algorithm 1 are then the original problem. Incomplete sparse LU factorization of $\Psi$ can be used for $P$ instead. Such preconditioners are considered in [28], [29]. However, for the resulting preconditioner to be positive definite and effective, it may be necessary to use incomplete LU factors that are denser (i.e., have less structure) than $\Psi$.

In the next two sections, a structured approach is developed for the preconditioning steps. Specifically, it is proposed to use a fixed number of block Jacobi iterations (e.g., see [6]) to approximately solve steps 3 and 12 with $P=\Psi$. The approach builds on ideas borrowed from [30]-[32].

```
Algorithm 1 PCG for (13) with preconditioner \(P\).
    Initialize \(\tilde{\delta}^{(0)}, \epsilon\), iter \({ }_{\text {max }}\)
    \(r^{(0)}=\breve{b}-\Psi \tilde{\delta}^{(0)}\)
    Solve \(P d^{(0)}=r^{(0)}\)
    \(\beta^{(0)}=\left(d^{(0)}\right)^{\prime} r^{(0)}\)
    Set \(i=0\)
    while \(i<\) iter \(_{\text {max }}\) do
        \(y^{(i)}=\Psi d^{(i)}\)
        \({\underset{\gamma}{\gamma}}^{(i)}=\beta^{(i)} /\left(\left(y^{(i)}\right)^{\prime} d^{(i)}\right)\)
        \(\tilde{\delta}^{(i+1)}=\tilde{\delta}^{(i)}+\gamma^{(i)} d^{(i)}\)
        \(r^{(i+1)}=r^{(i)}-\gamma^{(i)} y^{(i)}\)
        if \(\left\|r^{(i+1)}\right\|_{\infty}<\epsilon\) exit
        Solve \(P q^{(i+1)}=r^{(i+1)}\)
        \(\beta^{(i+1)}=\left(q^{(i+1)}\right)^{\prime} r^{(i+1)}\)
        \(d^{(i+1)}=r^{(i+1)}+\left(\beta^{(i+1)} / \beta^{(i)}\right) d^{(i)}\)
        \(i=i+1\)
    end while
```


## IV. Block Jacobi Preconditioning

Let $\mathcal{K}=\{1,2, \ldots, K\}$, with $K=\lceil N / 2\rceil$, i.e., $K=N / 2$ when $N$ is even, and $K=(N+1) / 2$ otherwise. Also define

$$
\begin{gather*}
\Delta_{k}=\left[\begin{array}{cc}
Z_{2 k-1} & Y_{2 k}^{\prime} \\
Y_{2 k} & Z_{2 k}
\end{array}\right], k \in \mathcal{K} \backslash\{K\},  \tag{16a}\\
\Delta_{K}=\left\{\begin{array}{cc}
{\left[\begin{array}{cc}
Z_{N-1} & Y_{N}^{\prime} \\
Y_{N} & Z_{N}
\end{array}\right],} & N \text { even } \\
Z_{N}, & N \text { odd }
\end{array}\right.  \tag{16b}\\
\Upsilon_{k}=\left[\begin{array}{cc}
V_{2 k-1} & Y_{2 k-1} \\
0 & V_{2 k}
\end{array}\right], k \in \mathcal{K} \backslash\{1, K\}  \tag{16c}\\
\Upsilon_{K}=\left\{\begin{array}{cc}
{\left[\begin{array}{cc}
V_{N-1} & Y_{N-1} \\
0 & V_{N}
\end{array}\right],} & N \text { even } \\
{\left[\begin{array}{ll}
V_{N} & Y_{N}
\end{array}\right],} & N \text { odd }
\end{array}\right. \tag{16~d}
\end{gather*}
$$

where referring to (12),

$$
\begin{align*}
Z_{j} & =\tilde{\Phi}_{j}^{2}+\tilde{\Omega}_{j} \tilde{\Omega}_{j}^{\prime}+\tilde{\Omega}_{j+1}^{\prime} \tilde{\Omega}_{j+1}  \tag{17a}\\
Y_{j} & =\tilde{\Omega}_{j} \tilde{\Phi}_{j-1}+\tilde{\Phi}_{j} \tilde{\Omega}_{j}  \tag{17b}\\
V_{j} & =\tilde{\Omega}_{j} \tilde{\Omega}_{j-1} \tag{17c}
\end{align*}
$$

for $j \in \mathcal{N}$, with $\tilde{\Omega}_{N+1}=0$. Given this, $\Psi=\operatorname{blktrid}(\Delta, \Upsilon)$, where $\Delta=\left(\Delta_{k}\right)_{k \in \mathcal{K}}$ and $\Upsilon=\left(\Upsilon_{k}\right)_{k \in \mathcal{K} \backslash\{1\}}$. Moreover, the preconditioning steps 3 and 12 with $P=\Psi$, can be re-written in the form

$$
\begin{equation*}
\operatorname{blktrid}(\Delta, \Upsilon) \zeta=\tau \tag{18}
\end{equation*}
$$

Let $\Delta=\operatorname{blkdiag}\left(\Delta_{1}, \ldots, \Delta_{K}\right)$ and $\Sigma=\Delta-\Psi$.
The block Jacobi method for solving (18) involves the following iterations:

$$
\begin{equation*}
\Delta \zeta^{(l+1)}=\tau+\Sigma \zeta^{(l)} \tag{19}
\end{equation*}
$$

Since $\Psi=\operatorname{blktrid}(\Delta, \Upsilon) \succ 0$ is block tri-diagonal, it is known that these iterations converge [6]. The proposal is to apply just a fixed number of Jacobi iterations for the preconditioning steps of Algorithm 1. Characteristics of this approach are discussed in the next three sub-sections.

## A. Positive definiteness of the preconditioner

Executing a fixed number of block Jacobi steps from zero is equivalent to the use of a positive-definite preconditioner.

Theorem IV.1: Given $L \in \mathbb{N}$ and $\zeta^{(0)}=0$, the $L$-th iterate of (19) satisfies $P_{L} \zeta^{(L)}=\tau$ with $P_{L}=W_{L}^{-1}$, where $W_{L}=$ $\sum_{l=0}^{L-1}\left(\Delta^{-1} \Sigma\right)^{l} \Delta^{-1} \succ 0$.

Proof: Noting that $\Delta \succ 0$ is invertible, it follows from (19) that $\zeta^{(L)}=W_{L} \tau+\left(\Delta^{-1} \Sigma\right)^{L} \zeta^{(0)}=W_{L} \tau$. It is established below that $W_{L}$ is positive definite, and thus, invertible. As such, $P_{L} \zeta^{(L)}=W_{L}^{-1} \zeta^{(L)}=\tau$.

Positive definiteness of $W_{L}$ is a consequence of the known property $\Psi=\operatorname{blktrid}(\Delta, \Upsilon) \succ 0$. With $U=$ $\operatorname{blk} \operatorname{diag}\left(U_{K}, \ldots, U_{1}\right)$, and $U_{k}=(-I)^{k}$ for $k=1, \ldots, K$, first note that $2 \Delta-\Psi=U^{\prime} \Psi U \succ 0$. Then note that $W_{1}=\Delta^{-1} \succ 0$, and using $\left(\Delta^{-1} \Sigma\right)^{l} \Delta^{-1}=\Delta^{-1}\left(\Sigma \Delta^{-1}\right)^{l}$, that

$$
\begin{aligned}
W_{2 M}= & \sum_{l=0}^{M-1}\left(\Delta^{-1} \Sigma\right)^{l} \Delta^{-1}(\Delta+\Sigma) \Delta^{-1}\left(\Sigma \Delta^{-1}\right)^{l} \\
= & \sum_{l=1}^{M-1}\left(\Delta^{-1} \Sigma\right)^{l} \Delta^{-1}(2 \Delta-\Psi) \Delta^{-1}\left(\Sigma \Delta^{-1}\right)^{l} \\
& +\Delta^{-1}(2 \Delta-\Psi) \Delta^{-1} \succ 0
\end{aligned}
$$

and

$$
\begin{aligned}
W_{2 M+1}= & \sum_{l=0}^{2 M-1}\left(\Delta^{-1} \Sigma\right)^{l} \Delta^{-1}+\left(\Delta^{-1} \Sigma\right)^{2 M} \Delta^{-1} \\
= & \sum_{l=0}^{M-1}\left(\Delta^{-1} \Sigma\right)^{l} \Delta^{-1}(\Delta+\Sigma) \Delta^{-1}\left(\Sigma \Delta^{-1}\right)^{i} \\
& +\left(\Delta^{-1} \Sigma\right)^{M} \Delta^{-1}\left(\left(\Delta^{-1} \Sigma\right)^{M}\right)^{\prime} \succ 0
\end{aligned}
$$

for $M \in \mathbb{N}$. Therefore, $W_{L} \succ 0$, as claimed.

## B. An analytic bound on achieved conditioning

The iterations (19) converge to the solution of (18) if and only if

$$
\begin{equation*}
\varrho\left(\Delta^{-1} \Sigma\right)<1 \tag{20}
\end{equation*}
$$

where $\varrho(\cdot)$ denotes spectral radius [6, Thm 2.16]. For $\Psi=$ $\operatorname{blktrid}(\Delta, \Upsilon) \succ 0$, and the split $\Psi=\Delta-\Sigma$, condition (20) holds [6, Lem 4.7, Thm. 4.18].

Theorem IV.2: With $P_{L}=\left(\sum_{l=0}^{L-1}\left(\Delta^{-1} \Sigma\right)^{l} \Delta^{-1}\right)^{-1}$ for given $L \in \mathbb{N}$,

$$
\begin{equation*}
\kappa\left(P_{L}^{-1 / 2} \Psi P_{L}^{-1 / 2}\right) \leq \frac{1+\left(\varrho\left(\Delta^{-1} \Sigma\right)\right)^{L}}{1-\left(\varrho\left(\Delta^{-1} \Sigma\right)\right)^{L}} \tag{21}
\end{equation*}
$$

Proof: By Theorem IV.1, $P_{L} \succ 0$. Using $\Psi=\Delta-\Sigma$,

$$
\begin{equation*}
P_{L}^{-1} \Psi=\sum_{l=0}^{L-1}\left(\Delta^{-1} \Sigma\right)^{l}\left(I-\Delta^{-1} \Sigma\right)=I-\left(\Delta^{-1} \Sigma\right)^{L} . \tag{22}
\end{equation*}
$$

Furthermore, $P_{L}^{-1} \Psi=P_{L}^{-1 / 2}\left(P_{L}^{-1 / 2} \Psi P_{L}^{-1 / 2}\right) P_{L}^{1 / 2}$, whereby $\operatorname{spec}\left(P_{L}^{-1 / 2} \Psi P_{L}^{-1 / 2}\right)=\operatorname{spec}\left(P_{L}^{-1} \Psi\right)$. So the result holds as $\lambda_{\max }\left(I-\left(\Delta^{-1} \Sigma\right)^{L}\right) \leq 1+\left(\varrho\left(\Delta^{-1} \Sigma\right)\right)^{L}$ and $\lambda_{\min }(I-$ $\left.\left(\Delta^{-1} \Sigma\right)^{L}\right) \geq 1-\left(\varrho\left(\Delta^{-1} \Sigma\right)\right)^{L}>0$.

By Theorem IV.2 the number $L$ of block Jacobi iterations can be selected to achieve desired conditioning.

## C. Decomposable computations

Note that explicit construction of the preconditioner $P_{L}$ is not needed. At each PCG iteration, $L$ iterations of (19) are performed from $\zeta^{(0)}=0$. Since $\Delta$ is block diagonal, the computations required to implement each Jacobi iteration can be decomposed into $K=\lceil N / 2\rceil$ smaller problems

$$
\begin{equation*}
\Delta_{k} \zeta_{k}^{(l+1)}=\omega_{k} \tag{23}
\end{equation*}
$$

where $\omega_{k}=\tau_{k}+\Upsilon_{k} \zeta_{k-1}^{(l)}+\Upsilon_{k+1}^{\prime} \zeta_{k+1}^{(l)}$ for $k \in \mathcal{K}$, with $\Upsilon_{K+1}=0$. Each $\Delta_{k}$ is a block $2 \times 2$ matrix, with inner blocks that are structured. To see this structure, consider

$$
\Delta_{k}=\left[\begin{array}{cc}
Z_{2 k-1} & Y_{2 k}^{\prime}  \tag{24}\\
Y_{2 k} & Z_{2 k}
\end{array}\right]
$$

Note that

$$
\begin{align*}
Z_{j} & =\tilde{\Phi}_{j}^{2}+\tilde{\Omega}_{j} \tilde{\Omega}_{j}^{\prime}+\tilde{\Omega}_{j+1}^{\prime} \tilde{\Omega}_{j+1} \\
& =\left[\begin{array}{cc}
\tilde{Q}_{j}^{2}+\tilde{A}_{j}^{\prime} \tilde{A}_{j}+F_{j-1} F_{j-1}^{\prime}+E_{j}^{\prime} E_{j} & \tilde{Q}_{j} \tilde{A}_{j}^{\prime}+\tilde{A}_{j}^{\prime} \tilde{R}_{j} \\
\tilde{A}_{j} \tilde{Q}_{j}+\tilde{R}_{j} \tilde{A}_{j} & \tilde{A}_{j} \tilde{A}_{j}^{\prime}+\tilde{R}_{j}^{2}+F_{j} F_{j}^{\prime}+E_{j+1}^{\prime} E_{j+1}
\end{array}\right]  \tag{25a}\\
Y_{j} & =\tilde{\Omega}_{j} \tilde{\Phi}_{j-1}+\tilde{\Phi}_{j} \tilde{\Omega}_{j}=\left[\begin{array}{cc}
F_{j-1}^{\prime} \tilde{A}_{j-1}+\tilde{A}_{j}^{\prime} E_{j} & F_{j-1}^{\prime} \tilde{R}_{j-1}+\tilde{Q}_{j} F_{j-1}^{\prime} \\
E_{j} \tilde{Q}_{j-1}+\tilde{R}_{j-1} E_{j} & E_{j} \tilde{A}_{j-1}^{\prime}+\tilde{A}_{j} F_{j-1}^{\prime}
\end{array}\right] \tag{25b}
\end{align*}
$$

All blocks components of (25) are block diagonal, except for the block bi-diagonal $\tilde{A}_{j}$ for $j \in \mathcal{N}$. The sub-block sizes are all independent of both $N$ and $T$. The diagonal blocks of $Z_{j}$ are block tri-diagonal, while off-diagonal blocks are block bi-diagonal for $j \in \mathcal{N}$. Similarly, the diagonal blocks of $Y_{j}$ are block tri-diagonal, and the off-diagonal blocks are block diagonal for $j \in \mathcal{N} \backslash\{1\}$. To summarize, the matrices $\Delta_{k}$ have block-banded structure. In particular, there exists a permutation of variables such that (23) takes the form

$$
\begin{equation*}
\operatorname{blktrid}\left(\Xi_{k}, \Pi_{k}\right) \hat{\zeta}_{k}^{(l+1)}=\hat{\omega}_{k} \tag{26}
\end{equation*}
$$

where $\hat{\omega}_{k}=\hat{\tau}_{k}+\hat{\Upsilon}_{k} \hat{\zeta}_{k-1}^{(i)}+\hat{\Upsilon}_{k+1}^{\prime} \hat{\zeta}_{k+1}^{(i)}, \Xi_{k}=\left(\Xi_{k, t}\right)_{t \in \mathcal{T}}$, $\Pi_{k}=\left(\Pi_{k, t}\right)_{t \in \mathcal{T} \backslash\{T\}}$,

$$
\Xi_{k, t}=\left[\begin{array}{cccc}
\check{Q}_{2 k-1, t} & \Omega_{2 k-1, t} & \check{E}_{2 k, t}^{\prime} & 0  \tag{27a}\\
\Omega_{2 k-1, t} & \check{R}_{2 k-1, t-1} & 0 & \check{F}_{2 k, t-1}^{\prime} \\
\check{E}_{2 k, t} & 0 & \check{Q}_{2 k, t} & \Omega_{2 k, t} \\
0 & \check{F}_{2 k, t-1} & \Omega_{2 k, t} & \check{R}_{2 k, t-1}
\end{array}\right]
$$

with

$$
\begin{align*}
\check{Q}_{j, t} & =\tilde{Q}_{j, t}^{2}+I+\tilde{A}_{j, t}^{\prime} \tilde{A}_{j, t}+E_{j, t}^{\prime} E_{j, t}+F_{j-1, t}^{\prime} F_{j-1, t},  \tag{27b}\\
\check{R}_{j, t} & =\tilde{R}_{j, t}^{2}+I+\tilde{A}_{j, t} \tilde{A}_{j, t}^{\prime}+E_{j, t} E_{j, t}^{\prime}+F_{j, t} F_{j, t}^{\prime}  \tag{27c}\\
\Omega_{j, t} & =-\tilde{Q}_{j, t}-\tilde{R}_{j, t-1}  \tag{27d}\\
\check{E}_{j, t} & =\tilde{A}_{j, t}^{\prime} E_{j, t}+F_{j-1, t}^{\prime} \tilde{A}_{j-1, t},  \tag{27e}\\
\check{F}_{j, t} & =\tilde{A}_{j, t} F_{j-1, t}^{\prime}+E_{j, t} \tilde{A}_{j-1, t}^{\prime} \tag{27f}
\end{align*}
$$

for $j \in \mathcal{N}$, and

$$
\Pi_{k, t}=\left[\begin{array}{cccc}
-\tilde{A}_{2 k-1, t} & 0 & -F_{2 k-1, t}^{\prime} & 0  \tag{28a}\\
\check{A}_{2 k-1, t} & -\tilde{A}_{2 k-1, t} & G_{2 k-1, t} & -F_{2 k-1, t}^{\prime} \\
-E_{2 k, t} & 0 & -\tilde{A}_{2 k, t} & 0 \\
X_{2 k, t} & -E_{2 k, t} & \tilde{A}_{2 k, t} & -\tilde{A}_{2 k, t}
\end{array}\right]
$$

with

$$
\begin{equation*}
\check{A}_{j, t}=\tilde{A}_{j, t} \tilde{Q}_{j, t}+\tilde{R}_{j, t} \tilde{A}_{j, t}, \quad j \in \mathcal{N} \tag{28b}
\end{equation*}
$$

|  | Single Thread | N/2 Parallel Threads |  |
| :---: | :---: | :---: | :---: |
| PCG Steps | Computations | Computations per thread | Data xchg. per thread |
| Step 7 | $O\left(N T \bar{n}^{2}\right)$ | $O\left(T \bar{n}^{2}\right)$ | $O(T \bar{n})$ |
| Step 8 | $O(N T \bar{n})$ | $O(T \bar{n})$ | $O(1)$ |
| Step 9 | $O(N T \bar{n})$ | $O(T \bar{n})$ | 0 |
| Step 10 | $O(N T \bar{n})$ | $O(T \bar{n})$ | 0 |
| Step 11 | $O(N T \bar{n})$ | $O(T \bar{n})$ | $O(1)$ |
| Step 12 | $O\left(L T \bar{n}^{3}\right)$ | $O\left(L T \bar{n}^{3}\right)$ | $O(L T \bar{n})$ |
| Step 13 | $O(N T \bar{n})$ | $O(T \bar{n})$ | $O(1)$ |
| Step 14 | $O(N T \bar{n})$ | $O(T \bar{n})$ | 0 |

Complexity analysis of proposed PCG Algorithm
$\bar{n}=\max _{j}\left(n_{j}\right)$, WHERE $n_{j}$ IS THE SIZE OF $x_{j, t}$; AND $L$ IS THE FIXED number of Jacobi iterations.

$$
\begin{align*}
G_{j, t} & =F_{j-1, t}^{\prime} \tilde{R}_{j-1, t}+\tilde{Q}_{j, t} F_{j-1, t}^{\prime}, \quad j \in \mathcal{N} \backslash\{1\}  \tag{28c}\\
X_{j, t} & =E_{j, t} \tilde{Q}_{j-1, t}+\tilde{R}_{j-1, t} E_{j, t}, \quad j \in \mathcal{N} \backslash\{1\} \tag{28d}
\end{align*}
$$

Note that $\Xi_{k, t}, \Pi_{k, t} \in \mathbb{R}^{\hat{n}_{k, t} \times \hat{n}_{k, t}}$, where $\hat{n}_{k, t}=2\left(n_{2 k-1}+\right.$ $n_{2 k}$ ) for all $k \in \mathcal{K}$ and $t \in \mathcal{T}$. That is, the sizes of the subblocks of $\operatorname{blktrid}\left(\Xi_{k}, \Pi_{k}\right)$ are independent of $N$ and $T$.

For each $k \in \mathcal{K}$, the block tri-diagonal system (26) can be solved by backward-forward recursions, with computational complexity $O(2 T)$, that effectively implement an LDL factorization method [33]. In this way, the preconditioning computations decompose into a collection of $\lceil N / 2\rceil$ parallel threads each comprising computations for $2 T$ sequential (possibly dense) problems of size that is independent of $N$ and $T$. Table $\square$ provides a complexity analysis of each step of Algorithm 1 including the inter-thread data exchange overhead for an implementation with parallelism.

Remark IV.1: The per PCG iteration computational complexity is dominated by step 12, i.e., $O\left(L N T \bar{n}^{3}\right)$. With the number $L$ of block Jacobi preconditioning iterations fixed, and fixed bound $\bar{n}$ on the size of sub-system states, the overall computational complexity of PCG steps is $O(N T)$.

Remark IV.2: Note that steps 8,11 and 13 require sequential computations, to accumulate in forming dot-products and to test the stopping condition. For the $\lceil N / 2\rceil$ parallel thread implementation, these can be carried out using a backwardforward sweep with path-graph data exchange. Further, the parallel implementation of steps 7 and 12 requires the exchange of vectors of size less than $T \bar{n}$, between the neighbouring threads on this path-graph, since the partition of $\Psi$ is block tri-diagonal. As such, the overall inter-thread scalar data exchange overhead is $O(L N T \bar{n})$ per PCG iteration.

## V. Numerical Results

Numerical experiments are performed for an optimal control problem involving a one-dimensional mass-spring-damper chain of varying length of $N>0$ masses, taken from [34]. Each sub-system $j \in \mathcal{N}$ has dynamics of the form (1) with $n_{j}=2, m_{j}=1$, and $\nu_{j}=4$. The corresponding cost has $Q_{j, t}=\operatorname{diag}(1,0)$ and $R_{j, t}=1$ for $t \in \mathcal{T}$. The model parameters such as mass, spring constant, damping coefficient
are selected randomly between 0.8 to 1.5 to generate heterogeneous sub-systems. The experiments are performed by taking $N=T$ and varying this value from 10 to 1000 . The number of scalar variables in the largest problem is in the order of $10^{7}$, and there are a similar number of constraints. The linear system of equations at each Newton-step is solved in the following ways:

- Algorithm 1 to solve 18 with $L=2$;
- The block Jacobi method to solve (18) via iterations of the form (19);
- The direct method [15], via backward-forward recursions (BFR) to effectively solve (11) by LDL factorization;
- Solution of (11) via MATLAB's backslash.

In order to gauge the overall computational complexity a single thread implementation is used for all methods. The duality-gap based stopping criterion for the interior point method is set to $\epsilon_{\text {IPM }}=10^{-6}$. The stopping criterion for the infinity norm of the residuals in Algorithm 1, and in the pure block Jacobi iterations based implementation, is set to $\epsilon=10^{-9}$. For all experiments, IPM converged to specified tolerance within 15 to 20 Newton steps.

Fig. 1 shows the maximum/average number of iterations for the pure block Jacobi method, and the PCG method with $L=2$, taken across IPM iterations. The pure block Jacobi method consistently involves a large number of iterations, in the order of thousands. By contrast, the proposed PCG method consistently requires far fewer iterations, in the order of hundreds. This demonstrates effectiveness of proposed approach to preconditioning.

Fig. 2] shows the normalized average processor time for a single thread implementation as proxy for the per-IPM iteration computational complexity. Along the line $N=T$, the average time is $O\left(N^{2}\right)$ for the PCG method, compared to $O\left(N^{4}\right)$ for the direct method [15]. While the Jacobi method is also $O\left(N^{2}\right)$, the time is an order of magnitude greater than the PCG method. The average time for MATLAB's backslash, based on MA-57 [35], is provided as a base line. Note, that backslash is able to permute matrices in ways that does not respect the spatio-temporal structure of problem (2), which is by contrast preserved in the proposed PCG method.

Finally, the effect of increasing $L$ is shown in Fig. 3, as the value of $N=T$ is varied from 10 to 50 . It can be seen that the maximum number of PCG iterations decreases as $L$ is increased, with considerable decrease as $L$ is increased from 1 to 2 for this example.

## VI. Conclusions

A decomposable PCG method is proposed for computing second-order search directions for optimal control problems with path-graph network structure. The proposed algorithm exhibits per PCG iteration computational complexity that scales linearly with the number of sub-systems $N$ and the length of time horizon $T$. The computations at each iteration can be distributed across parallel processing agents in a network with path-graph structured information exchange. Future work includes extending the results for tree networks, where structure is manifest in three dimensions.


Fig. 1. Max/average iterations per Newton step


Fig. 2. Normalized average time per Newton step


Fig. 3. Maximum PCG iterations vs. $L$

## REFERENCES

[1] H. A. Nasir, M. Cantoni, Y. Li, and E. Weyer, "Stochastic model predictive control based reference planning for automated open-water channels," IEEE Transactions on Control Systems Technology, 2019.
[2] Y. Zheng, S. E. Li, K. Li, F. Borrelli, and J. K. Hedrick, "Distributed model predictive control for heterogeneous vehicle platoons under unidirectional topologies," IEEE Transactions on Control Systems Technology, vol. 25, no. 3, pp. 899-910, 2016.
[3] E. Perea-López, B. E. Ydstie, and I. E. Grossmann, "A model predictive control strategy for supply chain optimization," Computers and Chemical Engineering, vol. 27, no. 8-9, pp. 1201-1218, 2003.
[4] A. Giannitrapani, S. Paoletti, A. Vicino, and D. Zarrilli, "Optimal allocation of energy storage systems for voltage control in LV distribution networks," IEEE Transactions on Smart Grid, vol. 8, no. 6, pp. 2859-2870, 2017.
[5] T. Rees, H. S. Dollar, and A. J. Wathen, "Optimal solvers for PDE-constrained optimization," SIAM Journal on Scientific Computing, vol. 32, no. 1, pp. 271-298, 2010.
[6] W. Hackbusch, Iterative Solution of Large Sparse Systems of Equations, ser. Applied Mathematical Sciences, vol. 95, Springer, 2016.
[7] J. Nocedal and S. J. Wright, Numerical Optimization, 2nd ed., Springer, 2000.
[8] S. J. Wright, "Interior point methods for optimal control of discrete time systems," Journal of Optimization Theory and Applications, vol. 77, no. 1, pp. 161-187, 1993.
[9] C. V. Rao, S. J. Wright, and J. B. Rawlings, "Application of interiorpoint methods to model predictive control," Journal of Optimization Theory and Applications, vol. 99, no. 3, pp. 723-757, 1998.
[10] A. G. Wills and W. P. Heath, "Interior-point methods for linear model predictive control," Technical Report EE03016, University of Newcastle, NSW, 2003.
[11] Y. Wang and S. Boyd, "Fast model predictive control using online optimization," IEEE Transactions on Control Systems Technology, vol. 18, no. 2, pp. 267-278, 2010.
[12] A. Shahzad, E.C. Kerrigan, and G.A. Constantinides, "A stable and efficient method for solving a convex quadratic program with application to optimal control," SIAM Journal on Optimization, vol. 22, no. 4, pp. 1369-1393, 2012.
[13] A. Domahidi, A. U. Zgraggen, M. N. Zeilinger, M. Morari, and C. N. Jones, "Efficient interior point methods for multistage problems arising in receding horizon control," in Proc. 51st IEEE Conference on Decision and Control (CDC), pp. 668-674, 2012.
[14] I. Nielsen and D. Axehill, "Direct parallel computations of second-order search directions for model predictive control," IEEE Transactions on Automatic Control, vol. 64, no. 7, pp. 2845-2860, 2019.
[15] M. Cantoni, F. Farokhi, E. Kerrigan, and I. Shames, "Structured computation of optimal controls for constrained cascade systems," International Journal of Control, vol. 93, no. 1, pp. 30-39, 2020.
[16] D.P. Bertsekas, and J.N. Tsitsiklis, Parallel and Distributed Computation: Numerical Methods. Athena Scientific, 1997.
[17] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, "Distributed optimization and statistical learning via the alternating direction method of multipliers," Foundations and Trends®in Machine Learning, vol. 3, no. 1, pp. 1-122, 2011.
[18] G. Stathopoulos, H. A. Shukla, A. Szuecs, Y. Pu, and C. Jones, "Operator splitting methods in control," Foundations and Trends in Systems and Control, vol. 3, pp. 249-362, 2016.
[19] A. Falsone, K. Margellos, S. Garatti, and M. Prandini, "Dual decomposition for multi-agent distributed optimization with coupling constraints," Automatica, vol. 84, pp. 149-158, 2017.
[20] M. Cantoni, A. Zafar, and F. Farokhi, "Scalable iterations for solving constrained LQ control problems with cascade dynamics," in Proc. 23rd International Symposium on Mathematical Theory of Networks and Systems, 2018.
[21] A. Nedic, A. Ozdaglar, and P. A. Parrilo, "Constrained consensus and optimization in multi-agent networks," IEEE Transactions on Automatic Control, vol. 55, no. 4, pp. 922-938, 2010.
[22] G. França and J. Bento, "How is Distributed ADMM Affected by Network Topology?", 2017. [Online]. Available: http://arxiv.org/abs/1710.00889
[23] A. Nedic, A. Olshevsky, and M. G. Rabbat, "Network topology and communication-computation tradeoffs in decentralized optimization," Proceedings of the IEEE, vol. 106, no. 5, pp. 953-976, 2018.
[24] M. M. R. Hestenes and E. Stiefel, Methods of Conjugate Gradients for Solving Linear Systems. NBS, vol. 49, no. 1, 1952.
[25] C. C. Paige and M. A. Saunders, "Solution of sparse indefinite systems of linear equations," SIAM Journal on Numerical Analysis, vol. 12, no. 4, pp. 617-629, 1975.
[26] Y. Saad and M. H. Schultz, "GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems," SIAM Journal on Scientific and Statistical Computing, vol. 7, no. 3, pp. 856-869, 1986.
[27] Y. Saad, Iterative Methods for Sparse Linear Systems. SIAM, 2003.
[28] M. Benzi and M. Tůma, "A robust incomplete factorization preconditioner for positive definite matrices," Numerical Linear Algebra with Applications, vol. 10, no. 5-6, pp. 385-400, 2003.
[29] J. Xia and Z. Xin, "Effective and robust preconditioning of general SPD matrices via structured incomplete factorization," SIAM Journal on Matrix Analysis and Applications, vol. 38, no. 4, pp. 1298-1322, 2017.
[30] P. Concus, G. H. Golub, and D. P. O'Leary, "A generalized conjugate gradient method for the numerical solution of elliptic partial differential equations," in Sparse Matrix Computations. Elsevier, 1976, pp. 309-332.
[31] O. G. Johnson, C. A. Micchelli, and G. Paul, "Polynomial preconditioners for conjugate gradient calculations," SIAM Journal on Numerical Analysis, vol. 20, no. 2, pp. 362-376, 1983.
[32] L. Adams, "m-Step preconditioned conjugate gradient methods," SIAM Journal on Scientific and Statistical Computing, vol. 6, no. 2, pp. 452-463, 1985.
[33] G. Meurant, "A review on the inverse of symmetric tridiagonal and block tridiagonal matrices," SIAM Journal on Matrix Analysis and Applications, vol. 13, no. 3, pp. 707-728, 1992.
[34] M. Guo, A. Lang, and M. Cantoni, "Structured moving horizon estimation for linear system chains," 18th European Control Conference (ECC), vol. 2, pp. 1830-1835, 2019.
[35] I. S. Duff, "MA57-a code for the solution of sparse symmetric definite and indefinite systems," ACM Transactions on Mathematical Software (TOMS), vol. 30, no. 2, pp. 118-144, 2004.


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    AZ, MC, and FF are with the Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville VIC 3010, Australia. (emails: \{armaghanz@student; cantoni@; farhad.farokhi@ \}.unimelb.edu.au)

