Diagonal Stability of Discrete-time k-Positive linear Systems with Applications to Nonlinear Systems

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Abstract—A linear dynamical system is called k-positive if its dynamics maps the set of vectors with up to k-1 sign variations to itself. For k = 1, this reduces to the important class of positive linear systems. Since stable positive linear time-invariant (LTI) systems always admit a *diagonal* quadratic Lyapunov function, i.e. they are diagonally stable, we may expect that this holds also for stable k-positive systems. We show that, in general, this is not the case both in the continuous-time (CT) and discrete-time (DT) case. We then focus on DT k-positive linear systems and introduce the new notion of DT k-diagonal stability. It is shown that this is a necessary condition for standard DT diagonal stability. We demonstrate an application of this new notion to the analysis of a class of DT nonlinear systems.

Keywords: Sign variation, compound matrix, stability, diagonal Lyapunov function, wedge product, cyclic systems.

I. INTRODUCTION

Lyapunov functions are a powerful tool for stability analysis and control synthesis. For linear time-invariant (LTI) systems, stability is equivalent to the existence of a quadratic Lyapunov function, i.e. $V(x) = x^T Q x$, with Q positive-definite, that can be obtained constructively based on the eigenvectors of an associated Hamiltonian matrix [1]. An LTI is called *diagonally stable* if it is possible to find a diagonal Lyapunov function (DLF), i.e. $V(x) = x^T D x$, with D positive-definite and diagonal.

Diagonal stability of LTIs has attracted considerable attention in the systems and control community (see e.g. the monograph [2]). Due to its simplicity, diagonal stability can facilitate control synthesis, and it plays an important role in many fields including mathematical economics [3], ecology [4], numerical analysis [5], biochemistry [6], and networked systems [7].

The existence of a DLF has important implications to certain *nonlinear* systems associated with the LTI [8], [9]. This is true for both continuous-time (CT) and discrete-time (DT) nonlinear systems. We now briefly explain this. For $P \in \mathbb{R}^{n \times n}$, we write $P \succ 0$ [$P \prec 0$] to denote that P is symmetric and positive-definite [negative-definite]. Consider the CT nonlinear system:

$$\dot{x}(t) = Af(x(t)), \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $f(x) = \begin{bmatrix} f_1(x_1) & \cdots & f_n(x_n) \end{bmatrix}^T$, f_i is continuous and $f_i(z)z > 0$ for all $z \neq 0$ (so $f_i(0) = 0$). Such a

dynamics is called a Persidskii system (see, e.g. [10], [11] and the references therein). Suppose that A satisfies the Lyapunov inequality $DA + A^T D \prec 0$ with a *diagonal* matrix $D \succ 0$. Let

$$V(z) := 2\sum_{i=1}^{n} d_i \int_0^{z_i} f_i(\tau) \,\mathrm{d}\tau$$

where d_i is the *i*th diagonal entry of *D*. Then the derivative of V(x(t)) along solutions of (1) is

$$\dot{V}(x(t)) = f^T(x(t))(DA + A^T D)f(x(t))$$

so $\dot{V}(x(t)) < 0$ whenever $x(t) \neq 0$. If $\int_0^{x_i} f_i(\tau) d\tau \to \infty$ as $|x_i| \to \infty$, i = 1, ..., n, then we can conclude that the nonlinear system (1) is globally asymptotically stable (GAS).

Note that (1) can also be interpreted as a networked system. Indeed, assume that A is nonsingular and let $y := A^{-1}x$. Then (1) becomes

$$\dot{y}_i(t) = f_i\left(\sum_{j=1}^n a_{ij}y_j(t)\right), \ i = 1, \dots, n,$$
 (2)

where a_{ij} is the (i, j)-th entry of A. This can be viewed as a networked system with the weighted adjacency matrix A. In this case, diagonal stability of the LTI implies GAS of an associated nonlinear networked system. This idea was used in [6] to show that diagonal stability of a cyclic LTI implies the stability of a cyclically interconnected network of output strictly passive systems [12].

A similar construction holds for the DT nonlinear system:

$$x(j+1) = A\phi(x(j)), \tag{3}$$

where $\phi(x) := \begin{bmatrix} \phi_1(x_1) & \cdots & \phi_n(x_n) \end{bmatrix}^T$, with $\phi_i(z)$ continuous and $0 < |\phi_i(z)| \le |z|$ for all $z \ne 0$ (so $\phi(0) = 0$). If A in (3) satisfies the Stein inequality $A^T D A \prec D$, with a *diagonal* matrix $D \succ 0$, then $V(z) := z^T D z$ is a Lyapunov function for the nonlinear system (3). Similar to the CT case, the DT nonlinear system (3) can also be interpreted as a networked system using a suitable change of coordinates.

Stable LTIs always admit a quadratic Lyapunov function, but not necessarily a DLF [3]. It is well-known however that stable *positive* LTIs do admit a DLF (see, e.g., [7]).

Recently, the notion of positive linear systems was generalized to k-positive linear systems. For the theory and applications of such systems, see [13], [14] and also [15], [16], [17], [18]. For k = 1, this reduces to positive linear systems. This naturally raises the question of whether stable k-positive LTIs also admit a DLF. Here, we show that the answer is in general no.

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We then focus on the DT case. We show that k-positive DT LTI always satisfy a property that we call DT k-diagonal stability. It is showed that DT k-diagonal stability is a necessary condition for DT diagonal stability. We then describe an application to a class of DT nonlinear systems in a form similar to (3). By using wedge products and their geometric interpretation, we show that the asymptotic behavior of these systems can be analyzed using k-positivity and DT k-diagonal stability. These result generalize the construction described above when k = 1.

The remainder of this note is organized as follows. The next section briefly reviews some basic definitions and known results from the theory of diagonal stability, positive LTIs, compound matrices, and k-positive systems. Section III shows that in general stable k-positive systems, with k > 1, are *not* diagonally stable. Section IV introduces the notion of DT k-diagonal stability, and explains its relation to the standard DT diagonal stability. An application to DT *nonlinear* systems is described in Section V.

We use standard notation. A matrix $X \in \mathbb{R}^{n \times m}$ is called non-negative [positive], denoted $X \ge 0$ [$X \gg 0$], if all its entries are non-negative [positive]. The determinant of $A \in \mathbb{R}^{n \times n}$ is denoted by $\det(A)$. The eigenvalues of A are denoted by $\lambda_i(A)$, $i = 1, \ldots, n$, ordered such that

$$|\lambda_1(A)| \ge |\lambda_2(A)| \ge \dots \ge |\lambda_n(A)|. \tag{4}$$

The spectral radius of A is $\rho(A) := |\lambda_1(A)|$. For two integers $i \leq j$, we let $[i, j] := \{i, i + 1, \dots, j\}$. The non-negative orthant in \mathbb{R}^n is $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \dots, n\}$.

II. PRELIMINARIES

In this section, we review several known topics that are needed later on.

A. Diagonal stability of positive DT LTIs

If $A \in \mathbb{R}^{n \times n}$ is non-negative, then x(j + 1) = Ax(j) is called a positive DT LTI. The dynamics of positive DT LTIs leaves the proper cone \mathbb{R}^n_+ invariant [19]. The following result shows that stable positive DT LTIs are diagonally stable. Let $\mathbb{D}^{n \times n}$ denote the set of $n \times n$ positive diagonal matrices.

Lemma 1 (see e.g. [7, Prop. 2]). If $A \in \mathbb{R}^{n \times n}$ with $A \ge 0$ then the following statements are equivalent:

- (a) The matrix A is Schur, i.e., $\rho(A) < 1$;
- (b) There exists $\xi \in \mathbb{R}^n$ with $\xi \gg 0$ such that $A\xi \ll \xi$;
- (c) There exists $z \in \mathbb{R}^n$ with $z \gg 0$ such that $A^T z \ll z$;
- (d) There exists $D \in \mathbb{D}^{n \times n}$ such that $A^T D A \prec D$;
- (e) The matrix (I A) is nonsingular and $(I A)^{-1} \ge 0$.

Remark 1 (see e.g. [7]). Let $A \in \mathbb{R}^{n \times n}$ be non-negative and Schur. Pick $x, y \in \mathbb{R}^n$ with $x, y \gg 0$. Then $\xi := (I - A)^{-1}x$, $z := (I - A^T)^{-1}y$, and $D := \operatorname{diag}(\frac{z_1}{\xi_1}, \ldots, \frac{z_n}{\xi_n})$ satisfy conditions (b), (c), and (d) in Lemma 1, respectively. This provides a constructive procedure to obtain a DLF for positive DT LTIs. Note that if $A \in \mathbb{R}^{n \times n}$ is Schur and $A \leq 0$, then (-A) is Schur and non-negative. In this case, Lemma 1 also guarantees the existence of a $D \in \mathbb{D}^{n \times n}$ such that $A^T DA \prec D$.

B. k-positive systems

We recall two definitions for the number of sign variations in a vector. Define s^-, s^+ : $\mathbb{R}^n \to \{0, 1, ..., n-1\}$ as follows. First, $s^-(0) = 0$. Second, for $x \neq 0$, $s^-(x)$ is the number of sign variations in x after deleting all its zero entries. Let $s^+(x)$ denote the maximal possible number of sign variations in x after each zero entry is replaced by either 1 or -1. For example, for n = 4 and $x = \begin{bmatrix} 1.3 & 0 & 0 & -\pi \end{bmatrix}^T$, we have $s^-(x) = 1$ and $s^+(x) = 3$. Obviously,

$$0 \le s^-(x) \le s^+(x) \le n-1$$
 for all $x \in \mathbb{R}^n$.

For any $k \in [1, n] := \{1, ..., n\}$, define the sets:

$$P_{-}^{k} := \{ x \in \mathbb{R}^{n} : s^{-}(x) \le k - 1 \},$$

$$P_{+}^{k} := \{ x \in \mathbb{R}^{n} : s^{+}(x) \le k - 1 \}.$$
(5)

For example, $P_{-}^{1} = \mathbb{R}^{n}_{+} \cup (-\mathbb{R}^{n}_{+}).$

A linear dynamical system is called *k*-positive if its flow maps P_{-}^{k} to P_{-}^{k} , and strongly *k*-positive if its flow maps $P_{-}^{k} \setminus \{0\}$ to P_{+}^{k} [13], [14]. For example, the dynamics of positive LTIs maps the non-negative orthant \mathbb{R}_{+}^{n} to itself (and also $-\mathbb{R}_{+}^{n}$ to itself), so they are 1-positive systems.

Multiplying a vector by a non-zero scalar does not change the number of sign variations in the vector. This implies that P_{-}^{k} , P_{+}^{k} are cones. However, they are not convex cones. For example, the vectors $x := \begin{bmatrix} 4 & 2 & 4 \end{bmatrix}^{T}$ and $y := \begin{bmatrix} -2 & -4 & -2 \end{bmatrix}^{T}$ satisfy $x, y \in P_{-}^{1}$ and $x, y \in P_{+}^{1}$, but $z := (x+y)/2 = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^{T}$ satisfies $z \notin P_{-}^{1}$ and $z \notin P_{+}^{1}$.

The analysis of k-positive systems is based on compound matrices.

C. Multiplicative compound matrices

For an integer $n \ge 1$ and $k \in [1, n]$, let $Q_{k,n}$ denote the ordered set of all strictly increasing sequences of k integers chosen from [1, n]. We denote the $r := \binom{n}{k}$ elements of $Q_{k,n}$ by $\kappa_1, \ldots, \kappa_r$, with the κ_i s ordered lexicographically. For example, $Q_{2,3} = \{\kappa_1, \kappa_2, \kappa_3\}$, with $\kappa_1 = \{1, 2\}, \kappa_2 = \{1, 3\}$, and $\kappa_3 = \{2, 3\}$.

Given $A \in \mathbb{R}^{n \times n}$ and $\kappa_i, \kappa_j \in Q_{k,n}$, let $A[\kappa_i | \kappa_j] \in \mathbb{R}^{k \times k}$ denote the submatrix of A consisting of the rows [columns] indexed by $\kappa_i [\kappa_j]$. Let $A(\kappa_i | \kappa_j) := \det(A[\kappa_i | \kappa_j])$, i.e., the k-minor of A determined by the rows [columns] in $\kappa_i [\kappa_j]$.

The *kth multiplicative compound (MC)* of A is the matrix $A^{(k)} \in \mathbb{R}^{r \times r}$, whose entries, written in lexicographic order, are $A(\kappa_i | \kappa_j)$, see e.g. [20], [13] for more detailed explanations and examples. Note that this implies that $A^{(1)} = A$ and $A^{(n)} = \det(A)$. The MC satisfies the following properties (see, e.g., [21]).

Lemma 2. Let $A, B \in \mathbb{R}^{n \times n}$ and pick $k \in [1, n]$. Then

- (a) $(AB)^{(k)} = A^{(k)}B^{(k)};$
- (b) if A is nonsingular then $(A^{-1})^{(k)} = (A^{(k)})^{-1}$;
- (c) $(A^T)^{(k)} = (A^{(k)})^T;$
- (d) if $A^{\frac{1}{2}}$ exists then $(A^{\frac{1}{2}})^{(k)} = (A^{(k)})^{\frac{1}{2}}$;
- (e) the product of every k eigenvalues of A is an eigenvalue of $A^{(k)}$;
- (f) if A is Schur, then $A^{(k)}$ is Schur;

(g) if A is a diagonal matrix, then $A^{(k)}$ is a diagonal matrix. (h) if $A \succ 0$, then $A^{(k)} \succ 0$.

Note that Property (a) justifies the term multiplicative compound. For k = n, this property becomes the familiar formula det(AB) = det(A) det(B).

D. Necessary and sufficient conditions for k-positivity

A matrix $A \in \mathbb{R}^{n \times m}$ is called *sign-regular of order* k, denoted SR_k , if either $A^{(k)} \leq 0$ or $A^{(k)} \geq 0$. It is called *strictly sign-regular of order* k, denoted SSR_k , if either $A^{(k)} \ll 0$ or $A^{(k)} \gg 0$. In other words, all minors of order k of A have the same [strict] sign.¹ To refer to the common sign of the entries of $A^{(k)}$, we use the *signature* $\epsilon_k \in \{-1, 1\}$. That is, if $A^{(k)}$ is SSR_k [SR_k] with signature $\epsilon_k = 1$, then all the k-minors of A are positive [non-negative].

The next result provides a necessary and sufficient condition for a nonsingular matrix to map P_{-}^{k} to itself.

Proposition 1 ([22]). Let $T \in \mathbb{R}^{n \times n}$ be a nonsingular matrix and pick $k \in [1, n]$. Then

- (a) $TP_{-}^{k} \subseteq P_{-}^{k}$ if and only if T is SR_{k} ;
- (b) $T(P^k_{-} \setminus \{0\}) \subseteq P^k_{+}$ if and only if T is SSR_k .

For example, for k = 1 this implies that $T(\mathbb{R}^n_+ \cup (-\mathbb{R}^n_+)) \subseteq (\mathbb{R}^n_+ \cup (-\mathbb{R}^n_+))$ if and only if (iff) the entries of T are all non-negative or all non-positive, and that $T(\mathbb{R}^n_+ \cup (-\mathbb{R}^n_+)) \subseteq int(\mathbb{R}^n_+ \cup (-\mathbb{R}^n_+))$ iff the entries of T are all positive or all negative.

Remark 2. The assumption that T is nonsingular is not restrictive in our setting. Indeed, if x(j + 1) = Ax(j), with A singular, then the dynamics can be reduced to a lower-dimensional DT LTI with a nonsingular matrix.

The next result gives a necessary and sufficient condition for a DT LTI to be k-positive.

Proposition 2 ([14, Thm. 1]). Let $A \in \mathbb{R}^{n \times n}$ be nonsingular and pick $k \in [1, n]$. The DT LTI

$$x(j+1) = Ax(j) \tag{6}$$

is k-positive iff A is SR_k , and strongly k-positive iff A is SSR_k .

E. Wedge products

Fix an integer $n \ge 1$ and $k \in [1, n]$. The wedge product of the k vectors $a^1, \ldots, a^k \in \mathbb{R}^n$ is defined as

$$a^1 \wedge \dots \wedge a^k := \begin{bmatrix} a^1 & \dots & a^k \end{bmatrix}^{(k)}$$
. (7)

We also use the notation $\wedge_{i=1}^{k} a^{i} := a^{1} \wedge \cdots \wedge a^{k}$. Note that the right-hand side of (7) has dimensions $\binom{n}{k} \times \binom{k}{k}$, that is, it is a column vector of dimension $\binom{n}{k}$. In the special case k = n, Eq. (7) yields

$$\wedge_{i=1}^{n} a^{i} = \begin{bmatrix} a^{1} & \dots & a^{n} \end{bmatrix}^{(n)} = \det(\begin{bmatrix} a^{1} & \dots & a^{n} \end{bmatrix}).$$

¹We note that the terminology in this field is not uniform and some authors refer to such matrices as *sign-consistent of order* k.

The wedge product has an important geometric meaning. The value $|\wedge_{i=1}^{k} a^{i}|$ is the *k*-content [21] of the parallelotope whose edges are the given vectors. For k = 2 and k = 3, the *k*-content reduces to the standard notion of area and volume. For example, consider the case n = 3 and k = 2. Pick $a, b \in \mathbb{R}^{3}$. Then

$$a \wedge b = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}^{(2)}$$
$$= \begin{bmatrix} a_1 b_2 - b_1 a_2 & a_1 b_3 - b_1 a_3 & a_2 b_3 - b_2 a_3 \end{bmatrix}^T.$$

The entries here are the same as those in the cross product $a \times b$, up to a minus sign. Thus, $|a \wedge b| = |a \times b|$, and when $|\cdot|$ is the Euclidean norm this is the area of the parallelogram having a and b as sides.

F. Necessary conditions for diagonal stability

Recall that $A(\kappa_i|\kappa_j)$ is a *principal minor* of A if $\kappa_i = \kappa_j$. We briefly review necessary conditions for diagonal stability of a matrix A in terms of its principal minors.

Proposition 3 ([3, Thm. 2]). Let $A \in \mathbb{R}^{n \times n}$. If there exists $a \ D \in \mathbb{D}^{n \times n}$ such that $DA + A^T D \prec 0$, then every principal minor of (-A) is positive.

Combining this with the Cayley transform [23, Thm. 3] yields the following result.

Proposition 4 (see e.g. [2]). Let $A \in \mathbb{R}^{n \times n}$. If there exists $a \ D \in \mathbb{D}^{n \times n}$ such that $A^T D A \prec D$, then every principal minor of $-(A + I)(A - I)^{-1}$ is positive.

The next three sections describe our main results.

III. *k*-positivity does not imply diagonal stability

Since stable 1-positive systems (i.e. positive systems) are diagonally stable, a natural question is: are stable k-positive systems diagonally stable? This section shows that in general the answer is no, both in the DT and CT case.

Consider the DT LTI (6) with

$$A = \frac{1}{7} \begin{bmatrix} -4 & -2 & 1\\ 1 & -3 & -5\\ 7 & 1 & -2 \end{bmatrix}.$$
 (8)

It is straightforward to verify that A is Schur, and that $A^{(2)}$ is SSR_2 with $\epsilon_2 = 1$. Let $B := -(A + I)(A - I)^{-1}$. Then,

$$B(\{1,3\}|\{1,3\}) = \det\left(\frac{1}{461} \begin{bmatrix} 204 & 140\\ 497 & 323 \end{bmatrix}\right) = -\frac{8}{461} < 0.$$

Hence, Proposition 4 implies that although the DT LTI is stable and strongly 2-positive, it does *not* admit a DLF.

Remark 3. We focus on DT systems, but here we also briefly discuss the CT case. The CT LTI $\dot{x} = Ax$ is called strongly k-positive if its flow maps $P_{-}^{k} \setminus \{0\}$ to P_{+}^{k} that is, $\exp(At)(P_{-}^{k} \setminus \{0\}) \subseteq P_{+}^{k}$ for all t > 0. By using Proposition 3, we can

also prove that k-positive CT LTIs are not diagonally stable in general. Consider $\dot{x} = Ax$ with

$$A = \begin{bmatrix} -21 & 11 & -14\\ 18 & -19 & 37\\ -49 & 21 & -33 \end{bmatrix}$$

This system is strongly 2-positive (see [13]), and A is Hurwitz. Let B := -A. Then

$$B(\{2,3\}|\{2,3\}) = \det\left(\begin{bmatrix} 19 & -37\\ -21 & 33 \end{bmatrix}\right) = -150 < 0.$$

Thus, Proposition 3 implies that this system is not diagonally stable.

Summarizing, stable k-positive LTIs are in general not diagonally stable. A natural question then is what can be said about the diagonal stability of such systems.

IV. DT k-diagonal stability

We begin with defining a new notion called k-diagonal stability.

Definition 1. Given $A \in \mathbb{R}^{n \times n}$ and $k \in [1, n-1]$, let $r := \binom{n}{k}$. We say that A is DT k-diagonally stable if there exists $D \in \mathbb{D}^{r \times r}$ such that

$$(A^{(k)})^T D A^{(k)} \prec D. \tag{9}$$

Note that Definition 1 reduces to standard DT diagonal stability for k = 1, as then $A^{(1)} = A$ and $r = {n \choose 1} = n$. The next result is a generalization of Lemma 1. It shows that a k-positive DT LTI is k-diagonally stable iff $A^{(k)}$ is Schur.

Corollary 1. Suppose that $A \in \mathbb{R}^{n \times n}$ is SR_k for some $k \in [1, n - 1]$, with $\epsilon_k = 1$. Let $r := \binom{n}{k}$. Then the following statements are equivalent:

- (a) The matrix $A^{(k)}$ is Schur;
- (b) There exists $\xi \in \mathbb{R}^r$ with $\xi \gg 0$ such that $A^{(k)}\xi \ll \xi$;
- (c) There exists $z \in \mathbb{R}^r$ with $z \gg 0$ such that $(A^{(k)})^T z \ll z$;
- (d) There exists $D \in \mathbb{D}^{r \times r}$ such that (9) holds;

(e) $(I - A^{(k)})$ is nonsingular and $(I - A^{(k)})^{-1} \ge 0$.

Remark 4. Note that when these conditions hold we can use the idea described in Remark 1 to get an explicit matrix $D \in \mathbb{D}^{r \times r}$ such that (9) holds.

To demonstrate an application of Corollary 1, we revisit the class of *cyclic DT LTIs*, whose diagonal stability has been analyzed in [24].

Definition 2. The matrix $A \in \mathbb{R}^{n \times n}$ is called cyclic if

$$A = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0\\ 0 & \alpha_2 & \beta_2 & \cdots & 0\\ 0 & 0 & \alpha_3 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \beta_{n-1}\\ (-1)^{\ell+1}\beta_n & 0 & 0 & \cdots & \alpha_n \end{bmatrix}, \quad (10)$$

with $\alpha_i, \beta_i \geq 0$, i = 1, ..., n, and $\ell \geq 0$ is an integer.

We say that the DT LTI x(j + 1) = Ax(j) is cyclic if A is cyclic. Then the dynamics represents a linear chain such

that $x_i(j+1)$ depends only on $x_i(j), x_{i+1}(j)$, and $x_n(j+1)$ also depends on a feedback connection from $x_1(j)$. The feedback is negative [positive] if ℓ is even [odd]. The next result shows that such systems are DT k-diagonally stable.

Theorem 1. Suppose that A is cyclic for some $\ell \in [1, n-1]$. Then A is SR_{ℓ} with signature $\epsilon_{\ell} = 1$. Furthermore, if ℓ is odd, then A is DT diagonally stable iff A is Schur. If ℓ is even, then A is DT ℓ -diagonally stable iff $A^{(\ell)}$ is Schur.

Proof: Pick $\kappa_i, \kappa_j \in Q_{\ell,n}$. By the Leibniz formula,

$$A(\kappa_i|\kappa_j) = \det(A[\kappa_i|\kappa_j])$$
$$= \sum_{\sigma \in \mathrm{pt}(\kappa_j)} \left(\mathrm{sgn}(\sigma) \prod_{s=1}^{\ell} a_{\kappa_{is},\sigma_s} \right), \qquad (11)$$

where κ_{is} is the *s*th element of κ_i , σ_s is the *s*th element of the permutation $\sigma \in pt(\kappa_j)$, $sgn(\sigma) \in \{-1, 1\}$ denotes the signature of σ , and $pt(\kappa_j)$ denotes the set of all $\ell!$ permutations of the indexes in κ_j . For example, if n = 7, $\ell = 3$, and $\kappa_j = \{2, 5, 7\}$, then

$$pt(\kappa_j) = \{\{2, 5, 7\}, \{2, 7, 5\}, \{5, 2, 7\}, \\ \{5, 7, 2\}, \{7, 5, 2\}, \{7, 2, 5\}\}.$$

The cyclic structure (10) implies that $\operatorname{sgn}(\sigma) \prod_{s=1}^{\ell} a_{\kappa_{is},\sigma_s}$ can be non-zero only in the following cases:

- (i) $\kappa_{i\ell} \leq n-1$, and either $\sigma_s = \kappa_{is}$ or $\sigma_s = \kappa_{is} + 1$, for all $s \in \{1, \dots, \ell\}$;
- (ii) $\kappa_{i\ell} = n, \sigma_{\ell} = n$, and either $\sigma_s = \kappa_{is}$ or $\sigma_s = \kappa_{is} + 1$ for all $s \in \{1, \dots, \ell - 1\}$;
- (iii) $\kappa_{i\ell} = n, \sigma_{\ell} = 1$, and either $\sigma_s = \kappa_{is}$ or $\sigma_s = \kappa_{is} + 1$ for all $s \in \{1, \dots, \ell - 1\}$.

Additionally, the elements of σ are distinct, as $\sigma \in pt(\kappa_j)$. Since κ_i is an increasing sequence, in Cases (*i*) and (*ii*) the number of inversions in σ is zero, so $sgn(\sigma) = 1$. If Case (iii) holds, then $\sigma_{\ell} = 1 < \sigma_2 < \cdots < \sigma_{\ell-1}$, so σ has $\ell - 1$ inversions.

Assume that ℓ is even. Then all the entries of A are non-negative, except perhaps for a_{n1} . In Cases (i) or (ii) we have $\operatorname{sgn}(\sigma) \prod_{s=1}^{\ell} a_{\kappa_{is},\sigma_s} \geq 0$ since $\operatorname{sgn}(\sigma) = 1$, and all the a_{ij} s in $\prod_{s=1}^{\ell} a_{\kappa_{is},\sigma_s}$ are non-negative. If Case (iii) holds, then the number of inversions in σ is $\ell - 1$, which is odd, so $\operatorname{sgn}(\sigma) = -1$. Furthermore, $a_{n1} \leq 0$ appears in the term $\prod_{s=1}^{\ell} a_{\kappa_{is},\sigma_s}$. Thus, in this case we also have $\operatorname{sgn}(\sigma) \prod_{s=1}^{\ell} a_{\kappa_{is},\sigma_s} \geq 0$. Now (11) implies that $A(\kappa_i|\kappa_j) \geq 0$. Since $\kappa_i, \kappa_j \in Q_{\ell,n}$ are arbitrary, we conclude that A is SR_ℓ with signature $\epsilon_\ell = 1$. The proof for ℓ odd is similar.

Furthermore, If ℓ is even, the results in [24] show that A may be Schur yet not necessarily diagonally stable. However, since A is SR_{ℓ} , Corollary 1 ensures that A is DT ℓ -diagonally stable iff $A^{(\ell)}$ is Schur (which is weaker than the condition A is Schur). If ℓ is odd, then every entry of A in (10) is non-negative, and Lemma 1 implies that it is DT diagonally stable iff A is Schur.

Example 1. Consider the case n = 3, that is, A =

$$\begin{bmatrix} \alpha_1 & \beta_1 & 0\\ 0 & \alpha_2 & \beta_2\\ (-1)^{\ell+1}\beta_3 & 0 & \alpha_3 \end{bmatrix} \cdot A \text{ calculation gives}$$
$$A^{(2)} = \begin{bmatrix} \alpha_1\alpha_2 & \alpha_1\beta_2 & \beta_1\beta_2\\ (-1)^{\ell}\beta_1\beta_3 & \alpha_1\alpha_3 & \alpha_3\beta_1\\ (-1)^{\ell}\alpha_2\beta_3 & (-1)^{\ell}\beta_2\beta_3 & \alpha_2\alpha_3 \end{bmatrix}$$

If $\ell = 1$ then all the entries of A are non-negative, so A is SR_1 with signature $\epsilon_1 = 1$. If $\ell = 2$ then all the entries of $A^{(2)}$ are non-negative, so A is SR_2 with signature $\epsilon_2 = 1$.

The next result shows that DT k-diagonal stability, with k > 1, is a necessary condition for DT diagonal stability. Let I_s denote the $s \times s$ identity matrix.

Theorem 2. If $A \in \mathbb{R}^{n \times n}$ is DT diagonally stable, then A is DT k-diagonally stable for any $k \in [1, n - 1]$.

Proof: Since A is DT diagonally stable, there exists $P \in \mathbb{D}^{n \times n}$ such that $A^T P A \prec P$. Hence, $P^{-\frac{1}{2}} A^T P A P^{-\frac{1}{2}} \prec I_n$, so $P^{-\frac{1}{2}} A^T P A P^{-\frac{1}{2}}$ is Schur. Pick $k \in [1, n-1]$, and let $r := \binom{n}{k}$ and $D := P^{(k)}$. Note that $D \in \mathbb{D}^{r \times r}$. Lemma 2 implies that

$$(P^{-\frac{1}{2}}A^{T}PAP^{-\frac{1}{2}})^{(k)} = D^{-\frac{1}{2}}(A^{(k)})^{T}DA^{(k)}D^{-\frac{1}{2}}$$

is also Schur, i.e., $D^{-\frac{1}{2}}(A^{(k)})^T D A^{(k)} D^{-\frac{1}{2}} \prec I_r$. We conclude that $(A^{(k)})^T D A^{(k)} \prec D$.

Corollary 1 guarantees that a stable k-positive DT LTI is always k-diagonally stable with a matrix $D \in \mathbb{D}^{r \times r}$. If there exists $P \in \mathbb{D}^{n \times n}$ such that $P^{(k)} = D$, then the proof of Thm. 2 suggests that $V(z) := z^T P z$ is a candidate for a DLF for the original DT LTI x(j + 1) = Ax(j). However, for any k = [2, n - 2] and $D \in \mathbb{D}^{r \times r}$, the equation $P^{(k)} = D$ generally does not admit a solution $P \in \mathbb{D}^{n \times n}$. The next result shows that for k = n - 1 this equation is always solvable.

Theorem 3. For any $D \in \mathbb{D}^{n \times n}$, there exists a $P \in \mathbb{D}^{n \times n}$ such that $P^{(n-1)} = D$.

Proof: The proof is constructive. The equation $P^{(n-1)} = D$ can be written as

$$\prod_{s \in \kappa_q} p_s = d_q, \quad q = 1, \dots, n \tag{12}$$

where $\kappa_1, \ldots, \kappa_n \in Q_{n-1,n}$, and p_i, d_i denote the *i*th diagonal entry of *P* and *D*, respectively. For any $s \in [1, n]$, let j(s) be the single element in the set of indexes $[1, n] \setminus \kappa_s$. A length but straightforward computation shows that the solution of (12) is

$$p_s = \frac{\prod_{q \in \kappa_s} d_q^{\frac{1}{n-1}}}{d_{\hat{j}(s)}^{\frac{n-2}{n-1}}}.$$
(13)

Since $d_i > 0$ for any *i*, this implies that $p_s > 0$ for any *s*.

The following example shows that how the above results can be utilized to construct a DLF for an (n-1)-positive DT LTI.

Example 2. Consider the DT LTI x(j + 1) = Ax(j) with

$$A = \frac{1}{8} \begin{bmatrix} -4 & -2 & 0\\ 0 & -3 & -5\\ 7 & 0 & -2 \end{bmatrix}.$$
 (14)

A calculation shows that A is Schur. Since the entries of A have different signs, we cannot use Lemma 1 to conclude that A admits a DLF. However, A is SSR_2 with $\epsilon_2 = 1$. Hence, Corollary 1 implies that there exists $D \in \mathbb{D}^{3\times3}$ such that $(A^{(2)})^T DA^{(2)} \prec D$. According to Remark 4, one such D can be obtained as $D = \text{diag}\left(\frac{23}{21}, \frac{13}{8}, \frac{7}{13}\right)$. Using Theorem 3 to solve $P^{(2)} = D$ gives $P = \text{diag}\left(\sqrt{\frac{3887}{1176}}, \sqrt{\frac{184}{507}}, \sqrt{\frac{147}{184}}\right)$. It is straightforward to verify that $A^T PA \prec P$. Thus, we were able to build a DLF for A.

V. APPLICATIONS TO NONLINEAR DYNAMICAL SYSTEMS

As mentioned in the introduction, DT diagonal stability of A implies that certain nonlinear DT systems are also stable. A natural question is what are the implications of DT k-diagonal stability for nonlinear systems? In this section, we describe a new class of DT nonlinear system whose dynamics can be analyzed by exploiting k-positivity and wedge products. We first define a special kind of nonlinear mappings.

Definition 3. Let $\mathbb{S} \subseteq \mathbb{R}$ with $0 \in \text{int } \mathbb{S}$. Define $\phi : \mathbb{S}^n \to \mathbb{R}^n$ by $\phi(x) := \begin{bmatrix} \phi_1(x_1) & \dots & \phi_n(x_n) \end{bmatrix}^T$, where every $\phi_i : \mathbb{S} \to \mathbb{R}$ is a continuous scalar function such that $\phi_i(s) = 0$ holds only for s = 0. Pick $k \in [1, n - 1]$ and let $r := \binom{n}{k}$. We say that ϕ is k-content preserving if for any $a^1, \dots, a^k \in \mathbb{S}^n$ we have that

$$\begin{cases} q_i = 0, & \text{if } p_i = 0, \\ |q_i| \in (0, |p_i|] & \text{if } p_i \neq 0, \end{cases}$$
(15)

for all i = 1, ..., r, where $q := \wedge_{j=1}^k \phi(a^j)$ and $p := \wedge_{j=1}^n a^j$.

Example 3. For k = 1 we have p = a, $q = \phi(a)$, so (15) reduces to $\phi_i(0) = 0$, and $0 < |\phi_i(a_i)| \le |a_i|$ for $s \ne 0$. For k = 2, pick $a, b \in \mathbb{R}^n$, and let $p := a \land b$, $q := \phi(a) \land \phi(b)$. Then $p_1 = a_1b_2 - a_2b_1$ and $q_1 = \phi_1(a_1)\phi_2(b_2) - \phi_2(a_2)\phi_1(b_1)$. Thus, for i = 1, (15) yields

$$|\phi_1(a_1)\phi_2(b_2) - \phi_2(a_2)\phi_1(b_1)| \le |a_1b_2 - a_2b_1|.$$
(16)

(The equations for other values of *i* are similar.) For $a_2 = 0$ this gives $(\phi_1(a_1)\phi_2(b_2))^2 \leq a_1^2b_2^2$. If $a = \alpha b$, with $\alpha \in \mathbb{R} \setminus \{0\}$, then (16) becomes

$$(\phi_1(\alpha b_1)\phi_2(b_2) - \phi_2(\alpha b_2)\phi_1(b_1))^2 \le 0,$$

that is,

$$\phi_1(\alpha b_1)\phi_2(b_2) = \phi_2(\alpha b_2)\phi_1(b_1),$$

and for $b_2 \neq 0$ this becomes the homogeneity condition

$$\frac{\phi_1(\alpha b_1)}{\phi_2(\alpha b_2)} = \frac{\phi_1(b_1)}{\phi_2(b_2)}.$$

As a specific example, take $\mathbb{S} = [-1/2, 1/2]$ and $\phi_i(s) = s^2$, for all *i*. Then it is not difficult to show that (16) holds for any $a_i, b_i \in \mathbb{S}$, so this function is 2-content preserving on \mathbb{S} .

We can now state the main result in this section.

Theorem 4. Suppose that $A \in \mathbb{R}^{n \times n}$ is DT k-diagonally stable for some $k \in [1, n - 1]$. Consider the DT nonlinear system

$$x(j+1) = A\phi(x(j)), \tag{17}$$

where $\phi(x) = \begin{bmatrix} \phi_1(x_1) & \dots & \phi_n(x_n) \end{bmatrix}^T$ is k-content preserving on the state-space \mathbb{S}^n of (17). For $a^1, \dots, a^k \in \mathbb{S}^n$, let

$$y(j) = y(j; a^1, \dots, a^k) := \wedge_{i=1}^k x(j, a^i),$$
 (18)

where x(j, a) is the solution of (17) at time j with x(0) = a. Then

$$y(j+1) = A^{(k)} \wedge_{i=1}^{k} \phi(x(j,a^{i})),$$
(19)

and this nonlinear dynamical system is diagonally stable.

$$y(j+1) = \wedge_{i=1}^{k} x(j+1, a^{i})$$

= $A\phi(x(j, a^{1})) \wedge \dots \wedge A\phi(x(j, a^{k}))$
= $[A\phi(x(j, a^{1})) \dots A\phi(x(j, a^{k}))]^{(k)}$
= $A^{(k)} [\phi(x(j, a^{1})) \dots \phi(x(j, a^{k}))]^{(k)}$,

and this proves (19). Since A is DT k-diagonally stable, there exists $D \in \mathbb{D}^{r \times r}$ such that (9) holds. Define $V : \mathbb{R}^r \to \mathbb{R}_+$ by $V(z) := z^T Dz$, and let $\Delta V(j) := V(y(j+1)) - V(y(j))$. Then

$$\Delta V(j) = (\wedge_{i=1}^{k} \phi(x(j,a^{i})))^{T} (A^{(k)})^{T} D A^{(k)} \wedge_{i=1}^{k} \phi(x(j,a^{i})) - (\wedge_{i=1}^{k} x(j,a^{i}))^{T} D \wedge_{i=1}^{k} x(j,a^{i}).$$

$$(20)$$

Definition 3 implies that:

 $D_{\rm HO}$ of $D_{\rm H}$ (19)

$$(\wedge_{i=1}^{k}\phi(x(j,a^{i})))^{T}D\wedge_{i=1}^{k}\phi(x(j,a^{i}))$$

$$\leq (\wedge_{i=1}^{k}x(j,a^{i}))^{T}D\wedge_{i=1}^{k}x(j,a^{i}),$$

and combining this with (20) gives $\triangle V(j) \leq (\wedge_{i=1}^k \phi(x(j,a^i)))^T ((A^{(k)})^T DA^{(k)} - D) \wedge_{i=1}^k \phi(x(j,a^i))$. We conclude that $V(y(j+1)) - V(y(j)) \leq 0$, with equality only when y(j) = 0.

Note that the existence of a $D \in \mathbb{D}^{r \times r}$ that satisfies (9) plays a crucial role in the proof.

Theorem 4 implies that the k-content of the parallelotope induced by $x(j, a^i)$, i = 1, ..., k, converges to zero asymptotically. For k = 2 this means that any two trajectories of (17) converge to a line, i.e., to a one-dimensional subspace. In particular, this ensures that the dynamics of (17) has no nontrivial limit cycles.

Corollary 2. Consider the DT nonlinear system:

$$x(j+1) = A\phi(x(j)), \tag{21}$$

where A is cyclic for some $\ell \in [1, n - 1]$, and $\phi(x) = [\phi_1(x_1) \dots \phi_n(x_n)]^T$ is ℓ -content preserving on the statespace \mathbb{S}^n of (17). For any $a^1, \dots, a^\ell \in \mathbb{S}^n$, let $y(j) = y(j; a^1, \dots, a^\ell) := \wedge_{i=1}^\ell x(j, a^i)$. Then

$$y(j+1) = A^{(\ell)} \wedge_{i=1}^{\ell} \phi(x(j,a^i)),$$
(22)

and if $|\prod_{i=1}^{\ell} \lambda_i(A)| < 1$ then (22) is diagonally stable.

Proof: By Theorem 1, A is SR_{ℓ} with $\epsilon_{\ell} = 1$. By Corollary 1, A is DT diagonally stable iff $A^{(\ell)}$ is Schur, that is, iff $|\prod_{i=1}^{\ell} \lambda_i(A)| < 1$. Applying Theorem 4 completes the proof.

Example 4. Consider the DT nonlinear system (17) with n =

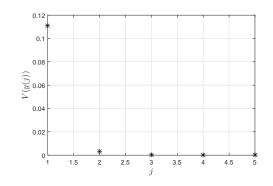


Fig. 1. V(y(j)) as a function $j \in [1, 5]$ in Example 4.

3, $A = \begin{bmatrix} 0.1 & 1.9 & 0 \\ 0 & 0.05 & 1.95 \\ -0.01 & 0 & 2.01 \end{bmatrix}$, and $\phi_i(s) = s^2$, i = 1, 2, 3, that is, $\phi(x) = \begin{bmatrix} x_1^2 & x_2^2 & x_3^2 \end{bmatrix}^T$. Let $\mathbb{S} := \begin{bmatrix} -1/2, 1/2 \end{bmatrix}$. It is

that is, $\phi(x) = \begin{bmatrix} x_1^2 & x_2^2 & x_3^2 \end{bmatrix}^T$. Let $\mathbb{S} := \begin{bmatrix} -1/2, 1/2 \end{bmatrix}$. It is not difficult to show that \mathbb{S}^3 is an invariant set of the dynamics. For example, $x_1(j+1) = 0.1x_1^2(j) + 1.9x_2^2(j)$. If $x(j) \in \mathbb{S}^3$ then $x_i^2(j) \in [0, 1/4]$ and this implies that $x_1(j+1) \in \mathbb{S}$. The matrix A is not Schur, as $\rho(A) = 2$. However, $A^{(2)}$ is Schur, and also $A^{(2)} \ge 0$, i.e. A is SR_2 with $\epsilon_2 = 1$. We use the idea described in Remark 1 to get a D such that (9) holds. Here n = 3 and k = 2, so $r = {n \choose k} = 3$. Denote $1_3 := \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, and let

$$\xi := (I - A^{(2)})^{-1} 1_3,$$

$$z := (I - (A^{(2)})^T)^{-1} 1_3,$$

$$D := \operatorname{diag}(z_1/\xi_1, z_2/\xi_2, z_3/\xi_3).$$

Fig. 1 depicts $V(y(j)) = y^T(j)Dy(j)$, as a function of j, where $y(j) := x(j, a^1) \wedge x(j, a^2)$, for the initial conditions $a^1 = (1/2)1_3$, $a^2 = \begin{bmatrix} -1/2 & 1/2 & 0.4 \end{bmatrix}^T$. Note that $a^1, a^2 \in \mathbb{S}^3$. As expected, V(y(j)) decreases with j.

If we take $a := (1/2)1_3$, $b \in \mathbb{S}^3$, then

$$y(j) = x(j, a) \land x(j, b) = a \land x(j, b) = (1/2) [x_2 - x_1 \quad x_3 - x_1 \quad x_3 - x_2]^T,$$

where $x_i := x_i(j, b)$. Thus, 4V(y(j)) is equal to

$$d_1(x_2 - x_1)^2 + d_2(x_3 - x_1)^2 + d_3(x_3 - x_2)^2,$$

where d_i is the *i*th diagonal entry of *D*. Since we already know that this function converges to zero, every trajectory converges to the line spanned by 1_3 .

VI. CONCLUSION

Diagonal stability is an important property of positive LTIs. k-positive LTIs are a generalization of positive LTIs and so a natural question is whether stable k-positive LTIs are also diagonally stable. We showed that in general the answer is no.

We then defined the new notion of DT *k*-diagonal stability and showed how it can be used to generalize the idea that diagonal stability of an LTI implies the stability of a certain nonlinear dynamical system. These results admit a clear geometric interpretation using the wedge product. We demonstrated our results for a class of nonlinear systems that include a cyclic matrix in their dynamics.

Due to space limitations, we focused here on DT systems. The CT case may be an interesting topic for further research.

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