Parameter Estimation in Adaptive Control of Time-Varying Systems Under a Range of Excitation Conditions

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Abstract—This paper presents a new parameter estimation algorithm for the adaptive control of a class of time-varying plants. The main feature of this algorithm is a matrix of time-varying learning rates, which enables parameter estimation error trajectories to tend exponentially fast towards a compact set whenever excitation conditions are satisfied. This algorithm is employed in a large class of problems where unknown parameters are present and are time-varying. It is shown that this algorithm guarantees global boundedness of the state and parameter errors of the system, and avoids an often used filtering approach for constructing key regressor signals. In addition, intervals of time over which these errors tend exponentially fast toward a compact set are provided, both in the presence of finite and persistent excitation. A projection operator is used to ensure the boundedness of the learning rate matrix, as compared to a time-varying forgetting factor. Numerical simulations are provided to complement the theoretical analysis.

Index Terms—Adaptive control, time-varying learning rates, finite excitation, parameter convergence.

I. INTRODUCTION

Adaptive control is a well established sub-field of control which compensates for parametric uncertainties that occur online so as to lead to regulation and tracking [1]–[4]. This is accomplished by constructing estimates of the uncertainties in real-time and ensuring that the closed loop system is well behaved even when these uncertainties are learned imperfectly. Both in the adaptive control and system identification literature, numerous tools for ensuring that the parameter estimates converge to their true values have been derived over the past four decades [5]–[8]. While most of the current literature in these two topics makes an assumption that the unknown parameters are constants, the desired problem statement involves plants where the unknown parameters are varying with time. This paper proposes a new algorithm for such plants.

Parameter convergence in adaptive systems requires a necessary and sufficient condition, denoted as persistent excitation, which ensures that the convergence is uniform in time [9]–[11]. If instead, a weaker condition is enforced where the excitation holds only over a finite interval, then parameter errors decrease only over a finite time. It is therefore of interest to achieve a fast rate of convergence by leveraging any excitation that may be available so that the parameter estimation error remains as small as possible, even in the presence of time-variations. The algorithm proposed in this paper will be shown to lead to such a fast convergence under a range of excitation conditions.

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The underlying structure in many of the adaptive identification and control problems consists of a linear regression relation between two dominant errors in the system [3], [12]. Examples include adaptive observers [13]–[17] and certain classes of adaptive controllers [1]. The underlying algebraic relation is often leveraged in order to lead to a fast convergence through the introduction of a time-varying learning rate in the parameter estimation algorithm, which leads to the wellknown recursive least squares algorithm [18]. Together with the use of an outer product of the underlying regressor, a matrix of time-varying learning rates is often adjusted to enable fast convergence [13], [16], [19]-[23]. In many cases, however, additional dynamics are present in the underlying error model that relates the two dominant errors, which prevents the derivation of the corresponding algorithm and therefore a fast convergence of the parameter estimates. To overcome this roadblock, filtering has been proposed in the literature [19]–[21], [24]–[28]. This in turn leads to an algebraic regression, using which corresponding adaptive algorithms are derived in [13], [16], [19]-[23] with time-varying learning rates become applicable. In [24], [25], it is shown that parameter convergence can occur even with finite excitation for a class of adaptive control architectures considered in [26]–[28]. In all of these papers, the underlying unknown parameters are assumed to be constants. The disadvantage of such a filtering approach is that the convergence properties cannot be easily extended to the case when the unknown parameters are time-varying, as the filtering renders the problem intractable. The algorithm that we propose in this paper introduces no filtering of system dynamics, and is directly applied to the original error model with the dynamics intact. As such, we are able to establish conditions for fast decreases of errors, even in the presence of time-varying parameters under varied properties of excitation.

Adaptive control in the presence of time-varying parameters has been studied in [29], [30] using the σ -modification [31], and shown to lead to global boundedness. More recently, this problem has also been studied in [32], [33] using concurrent learning, but with the assumption that the state derivatives from previous time instances are available. While [34] relaxes this assumption, it is at the expense of restricting the parameters to be a constant. In contrast to [29], [30], our paper can be shown in certain cases to result in convergence to a smaller compact set. In contrast to [32]–[34], our paper does not require knowledge of state derivatives and considers time-varying unknown parameters. References [35], [36] further consider adaptive control for plants with a specific structure leveraging Nussbaum gains [35] and additional additive terms in the update law [36], while our proposed algorithm does not require such potentially high gain terms.

This paper focuses on the ultimate goal of all adaptive control and identification problems, which is to provide a tractable parameter estimation algorithm for problems where the unknown parameters are time-varying. We will derive such an algorithm that guarantees, in the presence of time-varying parameters, 1) exponentially fast tending of parameter errors and tracking errors to a compact set for a range of excitation conditions, 2) does not require filtering of system dynamics, and 3) is applicable to a large class of adaptive systems. An error model approach as in [1], [37], [38] is adopted to describe

the underlying class. The algorithm consists of time-varying learning rates in order to guarantee fast parameter convergence. Rather than use a forgetting factor, automatic corrections are provided by the algorithm to ensure that the learning rates are bounded. Additionally, fewer number of integrations are required to implement the algorithm as compared to the existing literature [24], [25], [32]-[34].

This paper proceeds as follows: Section II presents mathematical preliminaries regarding continuous projection-based operators and definitions of persistent and finite excitation. The underlying problem is introduced in Section III. The main algorithm with time-varying learning rates is presented in Section IV. Stability and convergence properties of this algorithm are established for a range of excitation conditions in Section V. Numerical simulations follow in Section VI. Concluding remarks are provided in Section VII.

II. PRELIMINARIES

In this paper we use $\|\cdot\|$ to represent the 2-norm. Definitions, key lemmas, and properties of the Projection Operator (c.f. [3], [4], [39]-[42]) are all presented in this section. Proofs of all lemmas can be found in the appendix, and omitted where it is straightforward. Core variables employed throughout are listed in Table III in the appendix.

We begin with a few definitions and properties of convex sets and convex, coercive functions.

Definition 1 ([39]): A set $E \subset \mathbb{R}^N$ is convex if $\lambda x + (1-\lambda)y \in E$ for all $x \in E$, $y \in E$, and $0 \le \lambda \le 1$.

Definition 2 ([39]): A function $f: \mathbb{R}^N \to \mathbb{R}$ is convex if $f(\lambda x + (1 - \lambda y)) \le \lambda f(x) + (1 - \lambda)f(y)$ for all $0 \le \lambda \le 1$.

Definition 3 ([39]): A function $f(x) : \mathbb{R}^N \to \mathbb{R}$ is said to be coercive if for all sequences $\{x_k\}, k \in \mathbb{N}$ with $||x_k|| \to \infty$ then $\lim_{k \to \infty} f(x_k) = \infty.$

Lemma 1 ([42]): For a convex function $f(x) : \mathbb{R}^N \to \mathbb{R}$ and any

constant $\delta > 0$, the subset $\Xi_{\delta} = \{x \in \mathbb{R}^N \mid f(x) \le \delta\}$ is convex. Lemma 2: For a coercive function $f(x) : \mathbb{R}^N \to \mathbb{R}$ and any constant $\delta > 0$, any nonempty subset $\Xi_{\delta} = \{x \in \mathbb{R}^N \mid f(x) \leq \delta\}$ is bounded.

Corollary 1: For a coercive, convex function $f(x) : \mathbb{R}^N \to \mathbb{R}$ and a constant $\delta > 0$, any nonempty subset $\Xi_{\delta} = \{x \in \mathbb{R}^N \mid f(x) \le \delta\}$ is convex and bounded.

Remark 1: Definitions 1, 2, 3, Lemmas 1, 2, and Corollary 1 hold by simple extension to functions of matrices i.e., $f : \mathbb{R}^{N \times N} \to \mathbb{R}$.

Lemma 3 ([42]): For a continuously differentiable convex function $f(x) : \mathbb{R}^N \to \mathbb{R}$ and any constant $\delta > 0$, let θ^* be an interior point of the subset $\Xi_{\delta} = \{x \in \mathbb{R}^N \mid f(x) \leq \delta\}$, i.e. $f(\theta^*) < \delta$, and let a boundary point θ be such that $f(\theta) = \delta$. Then $(\theta - \theta^*)^T \nabla f(\theta) \ge 0.$

We now present properties related to projection operators. While some of these properties have been previously reported (c.f. [3], [4], [40]-[42]), they are included here for the sake of completeness and to help discuss the main result of this paper.

Definition 4 ([42]): The Γ -projection operator for general matrices is defined as,

$$\operatorname{Proj}_{\Gamma}(\theta, Y, F) = \left[\operatorname{Proj}_{\Gamma}(\theta_1, y_1, f_1) \cdots \operatorname{Proj}_{\Gamma}(\theta_m, y_m, f_m)\right] (1)$$

where $\theta = [\theta_1, \dots, \theta_m] \in \mathbb{R}^{N \times m}$, $Y = [y_1, \dots, y_m] \in \mathbb{R}^{N \times m}$, $F(\theta) = [f_1, \dots, f_m]^T : \mathbb{R}^{N \times m} \to \mathbb{R}^m$, $f_j : \mathbb{R}^N \to \mathbb{R}$ are convex continuously differentiable functions, $0 < \Gamma = \Gamma^T \in \mathbb{R}^{N \times N}$ is a symmetric positive definite matrix and $\forall j \in 1, \ldots, m$,

$$\operatorname{Proj}_{\Gamma}(\theta_{j}, y_{j}, f_{j}) = \begin{cases} \Gamma y_{j} - \Gamma \frac{\nabla f_{j}(\theta_{j})(\nabla f_{j}(\theta_{j}))^{T}}{(\nabla f_{j}(\theta_{j}))^{T} \Gamma \nabla f_{j}(\theta_{j})} \Gamma y_{j} f_{j}(\theta_{j}), \\ f_{j}(\theta_{j}) > 0 \land y_{j}^{T} \Gamma \nabla f_{j}(\theta_{j}) > 0 \\ \Gamma y_{j}, \text{ otherwise} \end{cases}$$
(2)

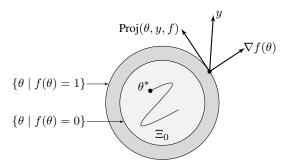


Fig. 1: (adapted from [42]) Γ -Projection operator in \mathbb{R}^2 with $\Gamma = I$. Uncertain parameter $\theta^*(t) \in \Xi_0 = \{\theta^* \in \mathbb{R}^2 \mid f(\theta^*) < 0\}.$

Definition 5: The projection operator for positive definite matrices is defined as,

$$\operatorname{Proj}(\Gamma, \mathcal{Y}, \mathcal{F}) = \begin{cases} \mathcal{Y} - \mathcal{F}(\Gamma) \mathcal{Y}, \, \mathcal{F}(\Gamma) > 0 \wedge Tr\left[\mathcal{Y}^T \nabla \mathcal{F}(\Gamma)\right] > 0\\ \mathcal{Y}, \quad \text{otherwise} \end{cases}$$

where $0 < \Gamma = \Gamma^T \in \mathbb{R}^{N \times N}$, $\mathcal{Y} \in \mathbb{R}^{N \times N}$ and $\mathcal{F} : \mathbb{R}^{N \times N} \to \mathbb{R}$ is a convex continuously differentiable function.

Remark 2: The projection operator in (3) may be expressed more compactly as $\operatorname{Proj}(\Gamma, \mathcal{Y}, \mathcal{F}) = \rho(t)\mathcal{Y}$ where

$$\rho(t) = \begin{cases} (1 - \mathcal{F}(\Gamma)), & \mathcal{F}(\Gamma) > 0 \wedge Tr\left[\mathcal{Y}^T \nabla \mathcal{F}(\Gamma)\right] > 0 \\ 1, & \text{otherwise} \end{cases}$$
(4)

From (3), (4), and as displayed in Figure 1, it can be seen that $\rho(t) =$ 1 on the inside of the projection boundary and $\rho(t) = 0$ on the outside edge of the boundary if $Tr \left| \mathcal{Y}^T \nabla \mathcal{F}(\Gamma) \right| > 0.$

Remark 3: An example of a coercive, continuously differentiable convex function commonly used in projection for adaptive control is given by [42]

$$f(\theta) = \frac{\|\theta\|^2 - \theta_{max}^{*2}}{2\varepsilon\theta_{max}^* + \varepsilon^2},$$
(5)

where θ_{max}^* and ε are positive scalars. It is easy to see that $f(\theta) = 0$ when $\|\theta\| = \theta_{max}^*$ and $f(\theta) = 1$ when $\|\theta\| = \theta_{max}^* + \varepsilon$. This function is commonly used in a projection-based parameter update law to result in a bounded parameter estimate (proven in this paper in Lemma 8). It should be noted that numerous choices other than the one in (5) exist for f.

Lemma 4: Let $\theta = [\theta_1, \ldots, \theta_m] \in \mathbb{R}^{N \times m}$, $\theta^* = [\theta_1^*, \ldots, \theta_m^*] \in \mathbb{R}^{N \times m}$, $Y = [y_1, \ldots, y_m] \in \mathbb{R}^{N \times m}$, $F(\theta) = [f_1, \ldots, f_m]^T$: $\mathbb{R}^{N \times m} \to \mathbb{R}^m$, where $f_j : \mathbb{R}^N \to \mathbb{R}$ are convex continuously differentiable functions, $0 < \Gamma = \Gamma^T \in \mathbb{R}^{N \times N}$ is a symmetric positive definite matrix, and $\theta_j^* \in \Xi_{0,j} = \{\theta_j^* \in \mathbb{R}^N \mid f_j(\theta_j^*) \leq 0\}$ $\forall j \in 1, \ldots, m$, then

$$Tr\left[(\theta - \theta^*)^T \Gamma^{-1}\left(\operatorname{Proj}_{\Gamma}(\theta, Y, F) - \Gamma Y\right)\right] \leq 0.$$

The following lemma lists a key property related to matrix inversion in the presence of time-variations.

Lemma 5: For a matrix $0 < \Gamma(t) = \Gamma^T(t) \in \mathbb{R}^{N \times N}$, the following holds: $\frac{d}{dt} \left(\Gamma^{-1}(t) \right) = -\Gamma^{-1}(t) \dot{\Gamma}(t) \Gamma^{-1}(t)$.

A central component of this paper is with regards to excitation of a regressor for which two definitions are provided.

Definition 6 ([1]): A bounded function $\phi : [t_0, \infty) \to \mathbb{R}^N$ is persistently exciting (PE) if there exists T > 0 and $\alpha > 0$ such that

$$\int_{t}^{t+T} \phi(\tau) \phi^{T}(\tau) d\tau \ge \alpha I, \quad \forall t \ge t_0$$

TABLE I: Adaptive Control Systems with a Common Structure

Name	Error Model	Y
State Feedback MRAC [1]	$ \dot{e} = Ae + B\tilde{\theta}^T \phi \\ e_y = e $	$-\phi e^T P B$
Output Feedback MRAC $W(s)$ A.S, SPR [1]	$ \dot{e} = Ae + B\tilde{\theta}^T \phi \\ e_y = Ce $	$-\phi e_y$
Output Feedback MRAC $W(s)$ A.S., not SPR [1]	$\epsilon = \tilde{\theta}^T \zeta$	$-\zeta\epsilon$
Nonlinear Adaptive Backstepping [2] Relative Degree ≤ 2	$ \begin{split} \dot{e} &= A_z(e,\theta,t) e \\ &+ \tilde{\theta}^T \phi(e,\theta,t) \\ e_y &= e \end{split} $	$-\phi e$

Definition 7 (adapted from [24], [25]): A bounded function ϕ : $[t_0, \infty) \to \mathbb{R}^N$ is finitely exciting (FE) on an interval $[t_1, t_1 + T]$ if there exists $t_1 \ge t_0$, T > 0, and $\alpha > 0$ such that

$$\int_{t_1}^{t_1+T} \phi(\tau) \phi^T(\tau) d\tau \ge \alpha I$$

In both Definitions 6^{1} and 7, the degree of excitation is given by α . It can be noted that the PE condition in Definition 6 pertains to a property over a moving window for all $t \ge t_0$, whereas the FE condition in Definition 7 pertains to a single interval $[t_1, t_1 + T]$.

III. ADAPTIVE CONTROL OF A CLASS OF PLANTS WITH TIME-VARYING PARAMETERS

Large classes of problems in adaptive identification and control can be represented in the form of differential equations containing two errors, $e(t) \in \mathbb{R}^n$ and $\tilde{\theta}(t) \in \mathbb{R}^{N \times m}$. The first is an error that represents an identification error or tracking error. The second is the underlying parameter error, either in estimation of the plant parameter or the control parameter. The parameter error is commonly expressed as the difference between a parameter estimate θ and the true unknown value θ^* as $\tilde{\theta}(t) = \theta(t) - \theta^*(t)$. The differential equations which govern the evolution of e(t) with $\tilde{\theta}(t)$ are referred to as error models [1], [37], [38], and provide insight into how stable adaptive laws for adjusting the parameter error can be designed for a large class of adaptive systems. The class of error models we focus on in this paper is of the form

$$\dot{e}(t) = g_1(e(t), \phi(t), \theta(t), \theta^*(t)),
e_y(t) = g_2(e(t), \phi(t), \theta(t), \theta^*(t)),$$
(6)

where the regressor $\phi(t) \in \mathbb{R}^N$ and $e_y(t) \in \mathbb{R}^p$ is a measurable error at each t. The corresponding adaptive law for adjusting θ is assumed to be of the form

$$\theta(t) = \Gamma_0 Y(e_y(t), \phi(t), \theta(t)), \tag{7}$$

where Y is a known function that is implementable at each t and $\Gamma_0 \in \mathbb{R}^{N \times N}$ is a symmetric positive definite matrix referred to as the learning rate. In addition, for a given g_1 and g_2 , Y is chosen so that e(t) = 0, $\tilde{\theta}(t) = 0$ is an equilibrium point of the system, when θ^* is a constant. We consider all classes of adaptive systems that can be expressed in the form of (6) and (7) where g_1 , g_2 , Y, and Γ_0 are such that all solutions are bounded for constant θ^* , and where $\lim_{t\to\infty} e(t) = 0$. In particular, we assume that g_1 , g_2 , and Y are such that a quadratic Lyapunov function candidate

$$V(t) = e^{T}(t)Pe(t) + Tr\left[\tilde{\theta}^{T}(t)\Gamma_{0}^{-1}\tilde{\theta}(t)\right],$$
(8)

yields a derivative for the case of constant θ^* as

$$\dot{V}(t) \le -e^{T}(t)Qe(t) - 2Tr\left[\tilde{\theta}^{T}(t)Y(t)\right] + 2Tr\left[\tilde{\theta}^{T}(t)\Gamma_{0}^{-1}\dot{\theta}(t)\right],$$
(9)

where P and Q are symmetric positive definite matrices. Due to the choice of the adaptive law in (7), it follows therefore $\dot{V}(t) \leq -e^{T}(t)Qe(t)$. Further conditions on g_1 , g_2 , and Y guarantee that $e(t) \to 0$ as $t \to \infty$. We formalize this assumption below:

Assumption 1 (Class of adaptive systems): For the case of a constant unknown parameter ($\dot{\theta}^*(t) = 0$), the error model in (6) and the adaptive law in (7) are such that they admit a Lyapunov function V as in (8) which satisfies the inequality in (9).

Several adaptive systems that have been discussed in the literature satisfy Assumption 1, some examples of which are shown in Table I. They include plants where state feedback is possible and certain matching conditions are satisfied, and where only outputs are accessible and a strictly positive real transfer function W(s) can be shown to exist. For a SISO plant that is minimum phase, the same assumption can be shown to hold as well. Finally, for a class of nonlinear plants, where the underlying relative degree does not exceed two, Assumption 1 once again can be shown to be satisfied.

A. Problem Formulation

The class of error models we consider is of the form (6), where $\theta^*(t)$, the time-varying unknown parameter, is such that if $\theta(t) \equiv \theta^*(t)$, then the solutions of (6) are globally bounded. This is formalized in the following assumption:

Assumption 2 (Uncertainty variation): The uncertainty, $\theta^*(t)$, in (6) is such that $||\theta^*(t)|| \leq \theta^*_{max}$, $\forall t \geq t_0$. In addition, its time derivative, $\dot{\theta}^*(t)$, is assumed to be bounded, i.e. $||\dot{\theta}^*(t)|| \leq \theta^*_{d,max}$, $\forall t \geq t_0$. Furthermore, if $\theta(t) = \theta^*(t)$, $\forall t \geq t_0$ then the solutions of (6) are globally bounded.

The problem that we address in this paper is the determination of an adaptive law similar to (7) for all error models of the form (6) where Assumptions 1 and 2 hold. Our goal is to ensure global boundedness of solutions of (6) and exponentially fast tending of both e(t) and $\tilde{\theta}(t)$ to a compact set with finite excitation.

IV. UPDATE LAW WITH A TIME-VARYING LEARNING RATE

The adaptive law that we propose is a modification of (7) with a *time-varying learning rate* $\Gamma(t)$ as $\dot{\theta}(t) = \Gamma(t)Y(e_y(t), \phi(t), \theta(t))$. To ensure a bounded $\theta(t)$, we include a projection operator in this adaptive law which is stated compactly as

$$\dot{\theta}(t) = \operatorname{Proj}_{\Gamma(t)} \left(\theta(t), Y(t), F \right), \quad \theta(t_0) \in \Xi_1.$$
(10)

where $\operatorname{Proj}_{\Gamma(t)}(\cdot,\cdot,\cdot)$ is defined as in Definition 4. The Γ -projection operator in (10) uses $F(\theta) = [f_1, \ldots, f_m]^T : \mathbb{R}^{N \times m} \to \mathbb{R}^m$, where $f_j(\theta_j) : \mathbb{R}^N \to \mathbb{R}$ are coercive, continuously differentiable convex functions. Define the subsets $\Xi_{\delta,j} = \{\theta_j \in \mathbb{R}^N \mid f_j(\theta_j) \leq \delta\}, \forall j \in 1, \ldots, m$, and $\Xi_{\delta} = \{M \in \mathbb{R}^{N \times m} \mid M_j \in \Xi_{\delta,j}, \forall j \in 1, \ldots, m\}$. Via Assumption 2, each f_j are chosen such that $||\theta^*(t)|| \leq \theta_{max}^*$ and $\delta = 0$ corresponds to $\theta_j^*(t) \in \Xi_{0,j}, \forall j \in 1, \ldots, m, \forall t \geq t_0$.

The time-varying learning rate $\Gamma(t)$ is adjusted using the projection operator for positive definite matrices (see Definition 5) as

$$\Gamma(t) = \lambda_{\Gamma} \operatorname{Proj} \left(\Gamma(t), \mathcal{Y}(t), \mathcal{F} \right), \quad \Gamma(t_0) = \Gamma_{t_0},$$

$$\mathcal{Y}(t) = \Gamma(t) - \kappa \Gamma(t) \Omega(t) \Gamma(t),$$
(11)

where λ_{Γ} , κ are positive scalars and $\Omega(t) \in \mathbb{R}^{N \times N}$. Γ_{t_0} is a symmetric positive definite constant matrix chosen so that $\Gamma_{t_0} \in \Upsilon_1 = \{\Gamma \in \mathbb{R}^{N \times N} \mid \mathcal{F}(\Gamma) \leq 1\}$, where $\mathcal{F}(\Gamma) : \mathbb{R}^{N \times N} \to \mathbb{R}$ is a coercive, continuously differentiable convex function. Lemma 2 implies there exists a constant $\Gamma_{max} > 0$ such that $\|\Gamma\| \leq \Gamma_{max}$ for all $\Gamma \in \Upsilon_1$. We assume that \mathcal{F} is chosen so that $\mathcal{F}(\Gamma) = 1$ for all $\|\Gamma\| = \Gamma_{max}$. It should be noted that a large $\Omega(t)$ contributes to a decrease in $\Gamma(t)$.

TABLE II: Norm of Signals in Phases of Excitation Propagation

$t\in$	$[t_1, t_2]$	t_2	$[t_2, t_3]$	$[t_3, t_4]$
$\ \phi(t)\phi^T(t)\ $	≥ 0			
$\left\ \int_{t_1}^{t_2} \phi(\tau) \phi^T(\tau) d\tau\right\ $		$\geq \alpha$		
$\ \Omega(t)\ $			$\geq \Omega_{FE}$	
$\ \Gamma(t)\ $				$\leq \Gamma_{FE}$
ho(t)				$\geq \rho_0$

Finally the matrix $\Omega(t)$ is adjusted as

$$\dot{\Omega}(t) = -\lambda_{\Omega}\Omega(t) + \lambda_{\Omega}\frac{\phi(t)\phi^{T}(t)}{1 + \phi^{T}(t)\phi(t)}, \quad \Omega(t_{0}) = \Omega_{0}, \quad (12)$$

where Ω_0 is a symmetric positive semi-definite matrix with $0 \leq$ $\Omega_0 \leq I$ and denotes a filtered normalized regressor matrix. λ_{Γ} and λ_{Ω} are arbitrary positive scalars and κ is chosen so that $\kappa > \Gamma_{max}^{-1}$. These scalars represent various weights of the proposed algorithm. The main contribution of this paper is the adaptive law in (10), (11), and (12), which will be shown to result in bounded solutions in Section V which tend exponentially fast to a compact set if $\phi(t)$ is finitely exciting. If in addition, $\phi(t)$ is persistently exciting, exponentially fast convergence to a compact set will occur $\forall t \geq t_0$.

Remark 4: It should be noted that while different aspects of the algorithm in (10), (11), and (12) have been explored in the literature, a combined algorithm as presented and analyzed here has not been reported thus far. For example, filtered regressor outer products are considered in [23]-[25], but parameters are assumed to be constant. It is the fact that we have the use of Γ in (11) in a specific manner (see choice of $\mathcal{Y}(t)$), the fact that we are using projections to contain Γ within a bounded set, and that we are using a filtered version of $\phi \phi^T$ together with normalization to adjust Ω as in (12) that enables our proposed algorithm to have desirable convergence properties, over a range of excitation conditions.

Remark 5: The projection operator employed in (11) is one method to bound the time-varying learning rate. Instead of (11), one can also use a time-varying forgetting factor, $(1 - \|\Gamma(t)\|/\Gamma_{max})$, to provide for $\|\Gamma(t)\| \leq \Gamma_{max}$ in an update of the form $\dot{\Gamma}(t) =$ $\lambda_{\Gamma}(1 - \|\Gamma(t)\|/\Gamma_{max})[\Gamma(t) - \kappa\Gamma(t)\Omega(t)\Gamma(t)]$. While the timevarying forgetting factor also achieves a bounded $\|\Gamma(t)\|$, it is more conservative than the projection operator in (11) as it is always active. In comparison, the projection operator as in (11) only provides limiting action if $\Gamma(t)$ is in a specified boundary region and the direction of evolution of $\Gamma(t)$ causes $\mathcal{F}(\Gamma)$ to increase. An equivalence between time-varying forgetting factors and projection operators may be drawn using the square root of the function in (5) with $\theta_{max}^* = 0$, $\varepsilon = \Gamma_{max}$, and the limiting action always active.

Remark 6: It can be noted that while the regressor normalization in (12) is optional for linear regression systems [19], [20], it is required for general adaptive control problems in the presence of system dynamics as the regressor cannot be assumed to be bounded.

Remark 7: The standard parameter update in (7) requires $N \times m$ integrations to adjust the $N \times m$ parameters θ . Given that the updates for both Γ and Ω result in symmetric matrices, an additional N(N+1)/2 integrations are required for each update for a total increase of N(N+1) integrations.

V. STABILITY AND CONVERGENCE ANALYSIS

We now state and prove the main result. The following assumption is needed for discussion of a finite excitation. We define an excitation level α_0 on an interval $[t_1, t_2]$ as

$$\alpha_0 = \frac{k_\Omega d}{\kappa \Gamma_{max} \rho_\Omega \lambda_\Omega \exp(-\lambda_\Omega (t_2 - t_1))},\tag{13}$$

where $\rho_{\Omega} \in (0, 1)$, $k_{\Omega} > 1$, and $d = \max_{\tau \in [t_1, t_2]} \{1 + \|\phi(\tau)\|^2\}$.

Assumption 3 (Finite excitation): There exists a time $t_1 \ge t_0$ and a time $t_2 > t_1$ such that the regressor $\phi(t)$ in (6) is finitely exciting over $[t_1, t_2]$, with excitation level $\alpha \geq \alpha_0$.

A. Propagation of Excitation and Boundedness of Information Matrix, Time-Varying Learning Rate

We first prove a few important properties of $\Omega(t)$ and $\Gamma(t)$ under different excitation conditions.

Lemma 6: For the algorithm in (12), it follows that for any $\phi(t)$, 1) $\Omega(t) \geq 0, \forall t \geq t_0,$

2) $\Omega(t) \leq I, \forall t \geq t_0.$

If in addition ϕ is finitely exciting as in Assumption 3, then

3) $\Omega(t) \geq \Omega_{FE}I > (1/(\kappa \Gamma_{max}))I, \forall t \in [t_2, t_3],$

where $\Omega_{FE} = (k_{\Omega}/(\kappa\Gamma_{max}))$ and $t_3 = t_2 - (\ln\rho_{\Omega})/\lambda_{\Omega}$. If in addition ϕ is persistently exciting $\forall t \geq t_1 \geq t_0$ (see Definition 6), with interval T and level $\alpha \ge \alpha'_0$, $t'_2 = t_1 + T$, then

4) $\Omega(t) \ge \Omega_{FE}I > (1/(\kappa \Gamma_{max}))I, \forall t \ge t'_2,$

- $\begin{aligned} \alpha_0' &= \alpha_0 \exp(-\lambda_\Omega (t_2 t_2')) d'/d, \ d' = \max_{\tau \geq t_1} \{1 + \|\phi(\tau)\|^2\}. \\ \text{Lemma 7: The solutions of (11) and (4) satisfy the following:} \end{aligned}$
 - 1) $\Gamma(t) \leq \Gamma_{max}I, \Gamma^{-1}(t) \geq \Gamma_{max}^{-1}I > 0, \forall t \geq t_0,$

2)
$$\rho(t) \in [0, 1], \forall t \ge t_0,$$

3) $\Gamma(t) \ge \Gamma_{min}I > 0, \ \Gamma^{-1}(t) \le \Gamma_{min}^{-1}I, \ \forall t \ge t_0,$

where $\Gamma_{min} = 1/(\max eig(\Gamma_{t_0}^{-1}) + \kappa)$. If in addition ϕ is finitely exciting as in Assumption 3, then there exists a $\rho_0 \in (0, 1]$ such that

4) $\Gamma(t) \leq \Gamma_{FE}I < \Gamma_{max}I, \Gamma^{-1}(t) \geq \Gamma_{FE}^{-1}I > 0, \forall t \in [t_3, t_4],$ 5) $\rho(t) \geq \rho_0 > 0, \forall t \in [t_3, t_4],$

where $\Gamma_{FE} = \rho_{\Gamma}^{-1} \Gamma_{t_3}$, $\rho_{\Gamma} \in ((\Gamma_{t_3}/\Gamma_{max}), 1)$, $\Gamma_{t_3} = \Gamma(t_3) < \Gamma(t_3)$ Γ_{max} , and $t_4 = t_3 - (\ln \rho_{\Gamma})/\lambda_{\Gamma}$. If in addition ϕ is persistently exciting $\forall t \geq t_1 \geq t_0$ (see Definition 6), with interval T and level $\alpha \geq \alpha_0'$, then there exists a $\rho_0 \in (0,1], t_3' > t_2'$, and $\Gamma(t_3') \leq$ $\Gamma_{PE} < \Gamma_{max}$ such that

- 6) $\Gamma(t) \leq \Gamma_{PE}I < \Gamma_{max}I, \ \Gamma^{-1}(t) \geq \Gamma_{PE}^{-1}I > 0, \ \forall t \geq t'_3,$ 7) $\rho(t) \ge \rho_0 > 0, \, \forall t \ge t'_3.$

The properties of Ω and Γ for a persistently exciting ϕ are relatively well known. For a finitely exciting ϕ , it should be noted that after a certain time elapses, the lower bound for Ω is realized. This propagation is illustrated in Table II.

The choice of the finite excitation level α_0 in Assumption 3 enables a fast convergence rate as follows: The denominator $\kappa \Gamma_{max}$ in α_0 ensures that the update in (11) pushes $\Gamma(t)$ away from Γ_{max} , ρ_Ω provides for a bound for Ω away from a minimum value, and $\lambda_{\Omega} \exp(-\lambda_{\Omega}(t_2 - t_1))$ accounts for excitation propagation through (12). The numerator scaling d accounts for the normalization in (12), and k_{Ω} provides for a bound away from a minimum excitation level.

B. Stability and Convergence Analysis

With the properties of the learning rate and filtered regressor above, we now proceed to the main theorem. The following lemma and corollary state important properties of the parameter estimate θ .

Lemma 8: The update for $\theta(t)$ in (10) guarantees that there exists a θ_{max} such that $\|\theta(t)\| \leq \theta_{max}, \forall t \geq t_0$.

Corollary 2: Under Assumption 2, the update for $\theta(t)$ in (10) provides for a constant $\tilde{\theta}_{max}$ such that $\|\tilde{\theta}(t)\| \leq \tilde{\theta}_{max}, \forall t \geq t_0$.

The following definitions are useful for stating the main result in Theorem 1. Define scalars v(t) and $\eta(t)$ as

$$\upsilon(t) = \lambda_{\Gamma} \rho(t) \kappa \|\Omega(t)\| \|\tilde{\theta}(t)\|^2 + 2\Gamma_{min}^{-1} \|\tilde{\theta}(t)\| \|\dot{\theta}^*(t)\|, \quad (14)$$

$$\eta(t) = \min\left\{q_0, \lambda_{\Gamma}\rho(t)\Gamma_{max}^{-1}\right\} / \max\left\{p_{max}, \Gamma_{min}^{-1}\right\}.$$
 (15)

It is easy to see that $0 \le \eta(t)$ and $0 \le v(t) \le v_{max}$, where

$$\upsilon_{max} = \lambda_{\Gamma} \kappa \tilde{\theta}_{max}^2 + 2\Gamma_{min}^{-1} \tilde{\theta}_{max} \theta_{d,max}^*.$$
(16)

Define η_0 as

$$\eta_0 = \min\left\{q_0, \lambda_{\Gamma} \rho_0 \Gamma_{max}^{-1}\right\} / \max\left\{p_{max}, \Gamma_{min}^{-1}\right\}, \qquad (17)$$

where $\rho_0 \in (0, 1]$. Define a compact set D as

$$D = \left\{ \left(e, \tilde{\theta}\right) \middle| \eta \left[p_{min} \|e\|^2 + \Gamma_{max}^{-1} \|\tilde{\theta}\|^2 \right] \le \upsilon \right\}, \quad (18)$$

and

$$D_{max} = \left\{ \left(e, \tilde{\theta}\right) \middle| \eta_0 \left[p_{min} \|e\|^2 + \Gamma_{max}^{-1} \|\tilde{\theta}\|^2 \right] \le v_{max} \right\}.$$
(19)

We now state the main theorem.

- *Theorem 1:* Under Assumptions 1 and 2, the update laws in (10), (11), and (12) for the error model in (6) guarantee for any $\phi(t)$,
- A) boundedness of the trajectories of e(t) and $\tilde{\theta}(t)$, $\forall t \ge t_0$.
- If in addition ϕ is finitely exciting as in Assumption 3, then
 - B) the trajectories of e(t), $\tilde{\theta}(t)$ tend exponentially fast towards a compact set $D \subset D_{max}$, $\forall t \in [t_3, t_4]$.

If in addition ϕ is persistently exciting $\forall t \ge t_1 \ge t_0$ as in Definition 6 with level $\alpha \ge \alpha'_0$ and interval *T*, then

C) exponential convergence of the trajectories follows, of e(t), $\tilde{\theta}(t)$ towards a compact set $D \subset D_{max}$, $\forall t \ge t'_3$,

where time instances and excitation levels are as in Lemmas 6, 7.

Proof: Let $q_0 = \min eig(Q)$, $p_{min} = \min eig(P)$, $p_{max} = \max eig(P)$. Consider a candidate Lyapunov function of the form

$$V(t) = e^{T}(t)Pe(t) + Tr\left[\tilde{\theta}^{T}(t)\Gamma^{-1}(t)\tilde{\theta}(t)\right].$$
 (20)

It follows that

$$\dot{V}(t) \leq \underbrace{-e^{T}(t)Qe(t) - 2Tr\left[\tilde{\theta}^{T}(t)Y(t)\right] + 2Tr\left[\tilde{\theta}^{T}(t)\Gamma^{-1}(t)\dot{\theta}(t)\right]}_{\text{due to Assumption I}} + \underbrace{Tr\left[\tilde{\theta}^{T}(t)\left\{\frac{d}{dt}\left(\Gamma^{-1}(t)\right)\right\}\tilde{\theta}(t)\right]}_{\text{due to time-varying }\Gamma(t)} - \underbrace{2Tr\left[\tilde{\theta}^{T}(t)\Gamma^{-1}(t)\dot{\theta}^{*}(t)\right]}_{\text{due to time-varying }\theta^{*}(t)}.$$

Using (10), (11), and Lemma 5, $\dot{V}(t)$ may be simplified as

$$\begin{split} \dot{V}(t) &\leq -e^{T}(t)Qe(t) - 2Tr\left[\tilde{\theta}^{T}(t)\Gamma^{-1}(t)\dot{\theta}^{*}(t)\right] \\ &+ 2Tr\left[\tilde{\theta}^{T}(t)\Gamma^{-1}(t)\left(\operatorname{Proj}_{\Gamma(t)}\left(\theta(t),Y(t),F\right) - \Gamma(t)Y(t)\right)\right] \\ &- \lambda_{\Gamma}Tr\left[\tilde{\theta}^{T}(t)\Gamma^{-1}(t)\operatorname{Proj}\left(\Gamma(t),\mathcal{Y}(t),\mathcal{F}\right)\Gamma^{-1}(t)\tilde{\theta}(t)\right]. \end{split}$$

Using Lemma 4 and $\rho(t)$ in (4), we obtain that

$$\dot{V}(t) \leq -e^{T}(t)Qe(t) - 2Tr\left[\tilde{\theta}^{T}(t)\Gamma^{-1}(t)\dot{\theta}^{*}(t)\right] - \lambda_{\Gamma}\rho(t)Tr\left[\tilde{\theta}^{T}(t)\left\{\Gamma^{-1}(t) - \kappa\Omega(t)\right\}\tilde{\theta}(t)\right].$$
(21)

Using (14), (15), Corollary 2, and Assumption 2, the inequality becomes

$$\dot{V}(t) \leq \begin{cases} -q_0 \|e(t)\|^2 + 2\Gamma_{min}^{-1} \tilde{\theta}_{max} \theta_{d,max}^*, & \text{if } \rho(t) = 0\\ -\eta(t)V(t) + \upsilon(t), & \text{if } \rho(t) \in (0,1] \end{cases}$$
(22)

From the first case of (22) it can be seen that $\dot{V}(t) \leq 0$ for $||e(t)|| \geq \sqrt{2\Gamma_{min}^{-1}\tilde{\theta}_{max}\theta_{d,max}^*/q_0}$. From Lemmas 6, 7, 8, and Corollary 2, each of $\Omega(t)$, $\Gamma(t)$, $\Gamma^{-1}(t)$, $\theta(t)$, $\theta^*(t)$ and $\tilde{\theta}(t)$ are bounded. Thus the trajectories of the closed loop system remain bounded. This proves Theorem 1-A).

From (20) and (22), it can be noted that $\dot{V} < 0$ in D^c , where the compact set D is defined in (18). Applying the Comparison Lemma (see [43], Lemma 3.4) for the second case of (22), we obtain that

$$V(t) \le \Phi(t, t_3) V(t_3) + \int_{t_3}^t \Phi(t, \tau) v(\tau) d\tau, \quad \forall t \in [t_3, t_4],$$
(23)

with transition function $\Phi(t,\tau) = \exp\left[-\int_{\tau}^{t} \eta(\tau)d\tau\right]$. It can be noted that from Lemma 7 it was shown that $\rho(t) \ge \rho_0$, $\forall t \in [t_3, t_4]$, and thus $\eta(t) \ge \eta_0$, $\forall t \in [t_3, t_4]$, which follows from (15), (17). Thus (23) is simplified using (16) as

$$V(t) \le \exp\left[-\eta_0(t - t_3)\right] \left(V(t_3) - \frac{\upsilon_{max}}{\eta_0}\right) + \frac{\upsilon_{max}}{\eta_0}, \,\forall t \in [t_3, \, t_4].$$
(24)

Furthermore given that $\eta(t) \geq \eta_0 \ \forall t \in [t_3, t_4]$ and $0 \leq \upsilon(t) \leq \upsilon_{max}$, it can be noted that $D \subset D_{max}$, $\forall t \in [t_3, t_4]$. Therefore it can be seen that over the interval of time $t \in [t_3, t_4]$, the state error e(t) and parameter error $\tilde{\theta}(t)$ tend exponentially fast towards the bounded set $D \subset D_{max}$. This proves Theorem 1-B).

If ϕ is persistently exciting with level $\alpha \ge \alpha'_0$, and interval T, then it follows from Lemma 7-7) that (23) and (24) hold for all $t \ge t'_3$, which proves Theorem 1-C).

Remark 8: $\eta(t)$ denotes the convergence rate of V(t). This in turn follows if $\rho(t) > 0$, i.e. if $\Gamma(t)$ is bounded away from Γ_{max} . The latter follows from Lemma 6-3) and 6-4) if $\phi(t)$ is either finitely exciting or persistently exciting, with an exponentially fast trajectory of V(t) towards a compact set occurring over a finite interval or for all $t \ge t_3$, respectively. This convergence rate however is upper bounded by q_0/p_{max} .

Remark 9: Theorem 1-C) guarantees convergence of V(t) to a compact set D, while Theorem 1-B) guarantees that V(t) approaches D. This set scales with the signal v(t) in (14), which contains contributions both from $\Omega(t)$ and $\dot{\theta}^*(t)$. For static parameters ($\dot{\theta}^*(t) \equiv 0$) and low excitation (i.e., $\rho(t) \neq 0$ and $||\Omega(t)|| < (1/(\kappa \Gamma_{max})))$, from (21) it can be shown that the trajectories of $(e, \tilde{\theta})$, tend towards the origin, i.e. the set $(e, \tilde{\theta}) = (0, 0)$.

Remark 10: Since we did not introduce any filtering of the dynamics, the bound $\theta_{d,max}^*$ on the uncertain parameters is explicit in the compact set D_{max} . It can be seen that D_{max} directly scales with $\theta_{d,max}^*$ from (16). Such an explicit bound cannot be derived using existing approaches in the literature which filter dynamics.

Remark 11: The dependence of v(t) on $\dot{\theta}^*(t)$ is reasonable. As the time-variations in the uncertain parameters grow, it should be expected that the residue will increase as well. The dependence of v(t) on the filtered regressor $\Omega(t)$ is introduced due to the structure of our algorithm in (10), (11), and (12). As a result, even with persistent excitation, we can only conclude convergence of V(t) to a compact set as opposed to convergence to the origin. This compact set will be present even in the absence of time-variations in θ^* . This disadvantage, however, is offset by the property of exponential convergence to the compact set, which is virtue of the fact that we have a time-varying $\Gamma(t)$.

A closer examination of the convergence properties of the proposed algorithm is worth carrying out for the case of constant parameters. It is clear from (21) that the negative contributions to $\dot{V}(t)$ come from the first term, while any positive contribution comes if $\kappa \Omega(t) > \Gamma^{-1}(t)$. That is, if there is a large enough excitation, then the third term can be positive. This in turn is conservatively reflected in the magnitude of v(t). It should however be noted that a large $\Omega(t)$ with persistent excitation, leads to a large e(t), which implies that as the third term in (21) becomes positive, it leads to a first term that is proportionately large and negative as well, thereby resulting in a net contribution that is negative. An analytical demonstration of this effect, however, is difficult to obtain. For this reason, the nature of our main result is convergence to a bounded set rather than to zero, in the presence of persistent excitation. Finally, we note that in our simulation studies, $\dot{V}(t)$ remained negative for almost the entire period of interest in the case of constant parameters, resulting in a steady convergence of the parameter estimation error to zero.

Remark 12: For a scalar adaptive system (for example in Figure 3), initial conditions can be easily found such that the compact set for the well-known σ -mod [31], *e*-mod [44] is larger than the compact set for the proposed method. This can be quantified but is beyond the scope of the present paper.

Remark 13: The bounds for $\Omega(t)$ and $\Gamma(t)$ derived in Lemmas 6 and 7 were crucial in establishing the lower bound for $\eta(t)$. That this occurs over an interval of time results in exponential and not just asymptotic properties. This has obvious implications of robustness (c.f. [43, §9.1]).

VI. NUMERICAL SIMULATIONS

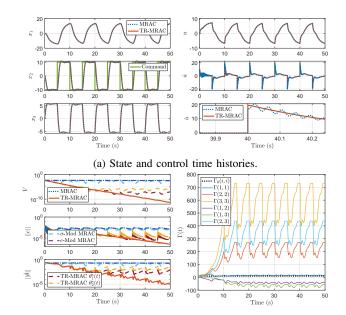
In this section we present two numerical simulations. In the first, we present results for F-16 longitudinal dynamics from [45], trimmed and linearized at a straight and level flying condition with a velocity of 500 ft/s and an altitude of 15,000 ft. We further include integral tracking of the command signal thus resulting in the extended model

$$\underbrace{ \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} }_{\dot{x}(t)} = \underbrace{ \begin{bmatrix} -0.6398 & 0.9378 & 0 \\ -1.5679 & -0.8791 & 0 \\ 0 & 1 & 0 \end{bmatrix} }_{A} \underbrace{ \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} }_{x(t)} + \underbrace{ \begin{bmatrix} -0.0777 \\ -6.5121 \\ 0 \\ B \end{bmatrix} }_{B} u(t) + \underbrace{ \begin{bmatrix} 0 \\ 0 \\ -1 \\ B_z \end{bmatrix} }_{B_z} z_{cmd}(t)$$

where z_{cmd} is the pitch rate command (dps), u is an elevator deflection (deg), and the state variables x_1, x_2, x_3 are the angle of attack (deg), pitch rate (dps), and integrated pitch rate tracking error (deg), respectively. A reference model, representing the desired dynamics, is designed as $\dot{x}_m(t) = A_m x_m(t) + B_z z_{cmd}(t)$, where $A_m = A - BK^T$ is Hurwitz, with $K = [0.1965, -0.3835, -1]^T$. The tracking error dynamics may be expressed in the form of (6), with error $e(t) = x_m(t) - x(t), e_y(t) = e(t), \phi(t) = x(t)$, and the control input selected as $u(t) = -K^T x(t) - \theta^T(t)\phi(t)$. With the parameter update argument selected as $Y(t) = -\phi(t)e^T(t)PB$ (as in Table I), Assumption 1 may be verified.

The adaptive parameter estimate θ is initialized at zero, to estimate the nominal unknown parameter $\theta^* = [0.1965, -0.03835, 0]^T$, which represents uncertainty as a function of angle of attack and pitch rate. For the algorithm in (10), (11), (12), we set $\lambda_{\Gamma} = 0.5, \kappa = 0.5, \lambda_{\Omega} = 10$, and find a matrix P which solves $A_m^T P + PA_m = -I$. The time-varying learning rate is initialized as $\Gamma(t_0) = \Gamma_0 = 10I$. For the projection algorithms in (10) and (11), we use the convex, coercive continuous function in (5) where the 2-norm is used for the θ update and the Frobenius norm is used for the Γ update.

We first present results for the case of a constant unknown parameter in order to demonstrate exponential convergence properties *towards the origin* with finite excitations, and then proceed to the case when the unknown parameters are time-varying. The results in Figure 2a demonstrate command tracking of pitch rate step responses using the standard MRAC update with a constant Γ (as in (7)) and the timevarying learning rate update (TR-MRAC) proposed in this paper in (10)-(12), for the case of a constant θ^* . It can be noted that while step responses contain some frequency content, the regressor $\phi(t)$ does not satisfy the PE condition in Definition 6 with a large α . This is reflected in Figure 2b, as the parameter errors remain constant for the static learning rate method. However, as time proceeds, with the help of (12), the excitation content is retained and increased, leading to a change in $\Gamma(t)$ for the time-varying learning rate update, which is immediately followed by a fast decrease in V(t) (note the log-scale



(b) Lyapunov function, tracking error norm, parameter error norm and time-varying learning rate time histories.

Fig. 2: Time histories of numerical simulation.

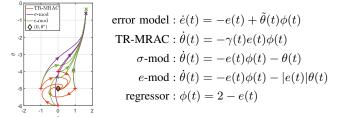


Fig. 3: Trajectories of a first order PE adaptive system. Initial conditions marked by red "+", equilibrium points marked by " \times ".

in Figure 2b). It is this interplay between excitation, change in $\Gamma(t)$, and a large decrease in V that ensures the exponential convergence of errors to a compact set.

Figure 2b (left) also includes results for the σ -mod [31], e-mod [44] adaptive update modifications. It can be noted that these modifications do not greatly improve the error convergence rate.

We further include, in Figure 2b (left), results obtained with two sets of time-varying parameters $\theta_1^*(t) = (1 + 1.1t/50)\theta^*$ (in dark red) and $\theta_2^*(t) = (1 + 1.5t/50)\theta^*$ (in yellow). As all plots show, the errors decrease despite the time-variations. The larger time-variation results in a larger error as shown in Figure 2b (left).

As a benchmark, we consider a first-order plant example with a single unknown parameter (see [31] and [44] for details of the setup), the corresponding phase plots in the (e, θ) space of which are shown in Figure 3, for the σ -mod [31], e-mod [44], and the proposed TR-MRAC which is designed with $0 < \gamma_{min} \leq \gamma(t) \leq 1$. Figure 3 illustrates that the σ -mod and e-mod create spurious equilibrium points to which all trajectories shown with these two algorithms converge. In contrast, TR-MRAC has a sole equilibrium point at (0, -5) to which all trajectories converge.

VII. CONCLUDING REMARKS

In this paper we presented a new parameter estimation algorithm for the adaptive control of a class of time-varying plants. The main feature of this algorithm is a matrix of time-varying learning rates, which enables exponentially fast trajectories of parameter estimation errors towards a compact set whenever excitation conditions are satisfied. It is shown that even in the presence of time-varying parameters, this algorithm guarantees global boundedness of the state and parameter errors of the system. The learning rate matrix is ensured to be bounded through the use of a projection operator. Since no filtering is employed and the original dynamic structure of the system is preserved, the bounds derived are tractable and are clearly related to the bounds on the time-variations of the unknown parameters as well as the excitation properties. Numerical simulations were provided to complement the theoretical analysis. Future work will focus on connecting these time-varying learning rates to accelerated learning in machine learning problems.

PROOFS OF LEMMAS

Proof of Lemma 1: Let $x_1, x_2 \in \Xi_{\delta}$ and thus $f(x_1) \leq \delta$ and $f(x_2) \leq \delta$. From the convexity of f, for any $0 \leq \lambda \leq 1$: $f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) \leq \lambda \delta + (1 - \lambda)\delta = \delta$. Therefore for all $x = \lambda x_1 + (1 - \lambda)x_2$: $f(x) \leq \delta$, and thus $x \in \Xi_{\delta}$. Therefore Ξ_{δ} is a convex set.

Proof of Lemma 2: Suppose there exists a constant $\delta > 0$ such that the subset $\Xi_{\delta} = \{x \in \mathbb{R}^N \mid f(x) \leq \delta\}$ is nonempty and unbounded. Thus there exists a sequence $\{x_k \mid k \in \mathbb{N}\} \in \Xi_{\delta}$ such that $||x_k|| \to \infty$. From Definition 3, given that f is coercive then $\lim_{k\to\infty} f(x_k) = \infty$. This contradicts $f(x) \leq \delta$, $\forall x \in \Xi_{\delta}$.

Proof of Lemma 3: $f(\theta)$ is convex, thus for any $0 < \lambda \leq 1$: $f(\lambda\theta^* + (1-\lambda)\theta) = f(\theta + \lambda(\theta^* - \theta)) \leq f(\theta) + \lambda(f(\theta^*) - f(\theta))$. Thus $(\theta - \theta^*)^T \nabla f(\theta) = \lim_{\lambda \to 0} (f(\theta) - f(\theta + \lambda(\theta^* - \theta))) / \lambda \geq f(\theta) - f(\theta^*) \geq \delta - \delta = 0$.

Proof of Lemma 4: For each $j \in 1, ..., m$, if $f_j(\theta_j) > 0 \land y_j^T \Gamma \nabla f_j(\theta_j) > 0$, then using (2) and Lemma 3,

$$Tr\left[\left(\theta - \theta^*\right)^T \Gamma^{-1} \left(\operatorname{Proj}_{\Gamma}(\theta, Y, F) - \Gamma Y\right)\right]$$

= $\sum_{j=1}^m (\theta_j - \theta_j^*)^T \Gamma^{-1} \left(\operatorname{Proj}_{\Gamma}(\theta_j, y_j, f_j) - \Gamma y_j\right)$
= $-\sum_{j=1}^m \underbrace{\frac{(\theta_j - \theta_j^*)^T \nabla f_j(\theta_j)}{\geq 0} \underbrace{(\nabla f_j(\theta_j))^T \Gamma \nabla f_j(\theta_j)}_{>0}}_{>0} \underbrace{f_j(\theta_j)}_{>0} \leq 0$

otherwise $(\theta_j - \theta_j^*)^T \Gamma^{-1} (\operatorname{Proj}_{\Gamma}(\theta_j, y_j, f_j) - \Gamma y_j) = 0.$

Proof of Lemma 5: It can be noticed that $0 = \frac{d}{dt}(I) = \frac{d}{dt}\left(\Gamma(t)\Gamma^{-1}(t)\right) = \dot{\Gamma}(t)\Gamma^{-1}(t) + \Gamma(t)\left[\frac{d}{dt}\left(\Gamma^{-1}(t)\right)\right]$. Lemma 5 follows by solving for $\frac{d}{dt}\left(\Gamma^{-1}(t)\right)$.

Proof of Lemma 6: Let $v \in \mathbb{R}^N$. Given the initial condition for (12), it can be noted that $0 \le v^T \Omega(t_0) v \le ||v||^2$. Furthermore

$$0 \le \frac{|v^T \phi(t)|^2}{1 + \|\phi(t)\|^2} \le \|v\|^2, \quad \forall v, t \ge t_0,$$
(25)

as multiplying through by $1 + \|\phi(t)\|^2 \ge 1$, the lower and upper bounds may be shown as $0 \le |v^T \phi(t)|^2$, and $|v^T \phi(t)|^2 \le ||v||^2 \|\phi(t)\|^2 \le ||v||^2 + ||v||^2 \|\phi(t)\|^2$, $\forall v, t \ge t_0$.

1) From the integral update in (12) we obtain

$$v^{T} \Omega(t) v = e^{-\lambda_{\Omega}(t-t_{0})} v^{T} \Omega(t_{0}) v$$
$$+ \int_{t_{0}}^{t} e^{-\lambda_{\Omega}(t-\tau)} \lambda_{\Omega} \frac{|v^{T} \phi(\tau)|^{2}}{1 + \|\phi(\tau)\|^{2}} d\tau.$$
(26)

Given that $\lambda_{\Omega} > 0$, using (25), all terms in (26) are non-negative, therefore $v^T \Omega(t) v \ge 0$, $\forall v, t \ge t_0$ and thus $\Omega(t) \ge 0$, $\forall t \ge t_0$.

2) From the integral update in equation (12) we obtain $v^T(I - \Omega(t))v = v^T(I - \exp(-\lambda_{\Omega}(t - t_0))\Omega(t_0))v - \int_{t_0}^t \exp(-\lambda_{\Omega}(t - \tau))\lambda_{\Omega}|v^T\phi(\tau)|^2/(1 + \|\phi(\tau)\|^2)d\tau$, which may be bounded using $\lambda_{\Omega} > 0$ and (25) to result in $v^T(I - \Omega(t))v \ge 0, \forall v, t \ge t_0$ and thus $\Omega(t) \le I, \forall t \ge t_0$.

3) The left hand side of (26) may be lower bounded as $v^T \Omega(t_2)v \ge \exp(-\lambda_\Omega(t_2-t_1))\lambda_\Omega v^T \int_{t_1}^{t_2} (\phi(\tau)\phi^T(\tau))/(1+\|\phi(\tau)\|^2)d\tau v$. Thus using the finite excitation condition in Assumption 3 (see Definition 7), $v^T \Omega(t_2)v \ge v^T (\lambda_\Omega \alpha/d)\exp(-\lambda_\Omega(t_2-t_1))Iv \ge (k_\Omega/(\rho_\Omega \kappa \Gamma_{max}))v^T Iv$. Furthermore from update (12): $v^T \Omega(t)v \ge \exp(-\lambda_\Omega(t-t_2))v^T \Omega(t_2)v$, $\forall v, t \ge t_2$. Therefore $v^T \Omega(t)v \ge \exp(-\lambda_\Omega(t-t_2))(k_\Omega/(\rho_\Omega \kappa \Gamma_{max}))v^T Iv$, $\forall v, t \ge t_2$. Therefore $\Omega(t) \ge (k_\Omega/(\kappa \Gamma_{max}))I > (1/(\kappa \Gamma_{max}))$, $\forall t \in [t_2, t_3]$.

4) Immediate from extension of the proof of case 3) $\forall t \ge t'_2$. *Proof of Lemma 7:* 1) The time derivative for $\mathcal{F}(\Gamma)$ in (11) is $\dot{\mathcal{F}}(\Gamma) = Tr[(\nabla \mathcal{F}(\Gamma))^T \dot{\Gamma}] = \lambda_{\Gamma} Tr[(\operatorname{Proj}(\Gamma, \mathcal{Y}, \mathcal{F}))^T \nabla \mathcal{F}(\Gamma)]$. With the projection equation in (3),

$$\dot{\mathcal{F}}(\Gamma) = \begin{cases} \lambda_{\Gamma} Tr[\mathcal{Y}^{T} \nabla \mathcal{F}(\Gamma)](1 - \mathcal{F}(\Gamma)), \ \mathcal{F}(\Gamma) > 0 \land Tr[\mathcal{Y}^{T} \nabla \mathcal{F}(\Gamma)] > 0\\ \lambda_{\Gamma} Tr[\mathcal{Y}^{T} \nabla \mathcal{F}(\Gamma)], & \text{otherwise} \end{cases}$$

Therefore within the limiting region $\mathcal{F}(\Gamma) > 0$,

$$\begin{cases} \mathcal{F}(\Gamma) > 0, \quad 0 < \mathcal{F}(\Gamma) < 1 \land Tr[\mathcal{Y}^{T} \nabla \mathcal{F}(\Gamma)] > 0\\ \dot{\mathcal{F}}(\Gamma) = 0, \quad \mathcal{F}(\Gamma) = 1 \land Tr[\mathcal{Y}^{T} \nabla \mathcal{F}(\Gamma)] > 0\\ \dot{\mathcal{F}}(\Gamma) \le 0, \quad Tr[\mathcal{Y}^{T} \nabla \mathcal{F}(\Gamma)] \le 0 \end{cases}$$
(27)

Thus given the initial condition for (11), $\mathcal{F}(\Gamma(t_0)) \leq 1$ and therefore $\mathcal{F}(\Gamma(t)) \leq 1$ for all $t \geq t_0$. Therefore from Lemma 2, there exists a constant denoted Γ_{max} such that $\|\Gamma\| \leq \Gamma_{max}$ for all $\Gamma \in \Upsilon_1$, i.e. all $\mathcal{F}(\Gamma) \leq 1$, $\forall t \geq t_0$, which implies $\Gamma(t) \leq \Gamma_{max}I$, $\forall t \geq t_0$.

2) From (27) and Remark 2, $\rho(t) \in [0, 1], \forall t \ge t_0$.

From (3), (11), Remark 2 and Lemma 5, the inverse of the timevarying learning rate may be expressed as

$$\frac{d}{dt}\left(\Gamma^{-1}(t)\right) = -\lambda_{\Gamma}\rho(t)\Gamma^{-1}(t) + \lambda_{\Gamma}\rho(t)\kappa\Omega(t).$$
(28)

Let $v \in \mathbb{R}^N$. From (28) we obtain

$$v^{T}\Gamma^{-1}(t)v = e^{-\lambda_{\Gamma}\int_{t_{0}}^{t}\rho(\tau)d\tau}v^{T}\Gamma^{-1}(t_{0})v + \int_{t_{0}}^{t}e^{-\lambda_{\Gamma}\int_{\tau}^{t}\rho(\nu)d\nu}\lambda_{\Gamma}\rho(\tau)\kappa v^{T}\Omega(\tau)vd\tau.$$
(29)

3) Using $\Omega(t) \leq I$, $\forall t \geq t_0$ from Lemma 6 and $\rho(t) \in [0, 1]$, $\forall t \geq t_0$, with equation (29) we obtain $v^T \left(\Gamma_{\min}^{-1} I - \Gamma^{-1}(t) \right) v \geq 0$, $\forall v, t \geq t_0$ and thus $\Gamma(t) \geq \Gamma_{\min} I > 0$, $\forall t \geq t_0$.

4) From the excitation lower bound: $\Omega(t) \geq (k_{\Omega}/(\kappa\Gamma_{max}))I > (1/(\kappa\Gamma_{max}))I$, $\forall t \in [t_2, t_3]$ from Lemma 6, it can be noted with equation (11) that $\Gamma_{max} - \kappa\Gamma_{max}\Omega(t)\Gamma_{max} \leq -\Gamma_{max}(k_{\Omega}-1) < 0$, $\forall t \in [t_2, t_3]$, as $k_{\Omega} > 1$. Thus even if $\Gamma(t)$ were to be at the limit Γ_{max} , the evolution of $\Gamma(t)$ is bounded away from Γ_{max} (towards Γ_{max}/k_{Ω}). Therefore there exists a $\rho_{t_3} \in (0, 1)$, such that $\rho(t) \geq \rho_{t_3}$, $\forall t \in [t_2, t_3]$, and a $\Gamma_{t_3} < \Gamma_{max}$, such that $\Gamma(t) \leq \Gamma_{t_3}$, $\forall t \in [t_2, t_3]$. Furthermore from (29): $v^T \Gamma^{-1}(t)v \geq \exp(-\lambda_{\Gamma}(\int_{t_3}^t \rho(\tau)d\tau)v^T \Gamma^{-1}(t_3)v, \forall v, t \geq t_3$. Thus $v^T \Gamma^{-1}(t) \geq \Gamma_{FE}^{-1}I$, $\forall t \in [t_3, t_4]$, thus $\Gamma(t) \leq \Gamma_{FE}I < \Gamma_{max}$, $\forall t \in [t_3, t_4]$.

5) Given that $\Gamma(t) \leq \Gamma_{FE}I < \Gamma_{max}$, $\forall t \in [t_3, t_4]$, it can be seen that $\exists \rho_0 \in (0, 1]$ such that $\rho(t) \geq \rho_0 > 0$, $\forall t \in [t_3, t_4]$.

- 6) Immediate from extension of the proof of case 4) $\forall t \ge t'_3$.
- 7) Immediate from extension of the proof of case 5) $\forall t \ge t_3^{'}$.

Proof of Lemma 8: For each $j \in 1, ..., m$, the time derivative for $f_j(\theta_j)$ in (10) may be expressed as $\dot{f}_j(\theta_j) = (\nabla f_j(\theta_j))^T \dot{\theta}_j =$

TABLE III: Core variable nomenclature

Learning Rate	Г	Projection Scalar	ρ
Information Matrix	Ω	Γ-Projection Operator	Proj _Γ
Tracking Error	e	User-Defined Gains	$\lambda_{\Gamma}, \kappa, \lambda_{\Omega}$
Parameter Estimation Error	$\tilde{ heta}$	Excitation Level	$lpha_0$
Parameter Estimate	θ	Compact set	D
True Unknown Parameter	θ^*	Compact Set Scalar	v
Regressor	ϕ	Convergence Rate	η

 $(\operatorname{Proj}_{\Gamma}(\theta_{j}, y_{j}, f_{j}))^{T} \nabla f_{j}(\theta_{j}).$ With the projection equation in (2),

$$\dot{f}_{j}(\theta_{j}) = \begin{cases} y_{j}^{*} \Gamma \nabla f_{j}(\theta_{j})(1 - f_{j}(\theta_{j})), & f_{j}(\theta_{j}) > 0 \land y_{j}^{*} \Gamma \nabla f_{j}(\theta_{j}) > 0 \\ y_{j}^{T} \Gamma \nabla f_{j}(\theta_{j}), & \text{otherwise} \end{cases}$$

Therefore within the limiting region $f_i(\theta_i) > 0$,

$$\begin{cases} \dot{f}_j(\theta_j) > 0, \quad 0 < f_j(\theta_j) < 1 \land y_j^T \Gamma \nabla f_j(\theta_j) > 0\\ \dot{f}_j(\theta_j) = 0, \quad f_j(\theta_j) = 1 \land y_j^T \Gamma \nabla f_j(\theta_j) > 0\\ \dot{f}_j(\theta_j) \le 0, \quad y_j^T \Gamma \nabla f_j(\theta_j) \le 0 \end{cases}$$

Thus given the initial condition for (10), $f_j(\theta_j(t_0)) \leq 1$ and therefore $f_j(\theta_j(t)) \leq 1$ for all $t \geq t_0$. Therefore from Lemma 2, there exists constants denoted $\theta_{j,max}$ such that $\|\theta_j(t)\| \leq \theta_{j,max}$ for all $\theta_j(t) \in \Xi_{1,j}$, i.e. all $f_j(\theta_j) \leq 1$, $\forall t \geq t_0$, as proven. Thus there exists a constant denoted θ_{max} such that $\|\theta(t)\| \leq \theta_{max}$, $\forall t \geq t_0$.

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