# Policy Synthesis for Switched Linear Systems with Markov Decision Process Switching 

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#### Abstract

We study the synthesis of mode switching protocols for a class of discrete-time switched linear systems in which the mode jumps are governed by Markov decision processes (MDPs). We call such systems MDP-JLS for brevity. Each state of the MDP corresponds to a mode in the switched system. The probabilistic state transitions in the MDP represent the mode transitions. We focus on finding a policy that selects the switching actions at each mode such that the switched system that follows these actions is guaranteed to be stable. Given a policy in the MDP, the considered MDP-JLS reduces to a Markov jump linear system (MJLS). We consider both mean-square stability and stability with probability one. For mean-square stability, we leverage existing stability conditions for MJLSs and propose efficient semidefinite programming formulations to find a stabilizing policy in the MDP. For stability with probability one, we derive new sufficient conditions and compute a stabilizing policy using linear programming. We also extend the policy synthesis results to MDP-JLS with uncertain mode transition probabilities.


Index Terms-switched systems, Markov decision processes, optimization

## I. Introduction

Switched linear systems [1], [2] which consist of a set of modes with linear dynamics and a switching logic that describes the evolution of the modes, have recently found a broad range of applications, e.g., in robotics [3], [4], wireless sensor networks [5], [6], [7], networked control systems [8], security and privacy [9], [10].
The switching logic for switched systems can be autonomous or controlled [11]. The former may be the result of the system's own characteristics or an influence of its environment, and the latter may be due to the designer's deliberate intervention.
In this paper, we study a class of the switched linear systems, where the switching logic is characterized by Markov decision processes (MDPs) [12]. We name such systems MDPJLS for brevity. The MDP in an MDP-JLS includes a set of states that correspond to the modes in a switched system, a set of actions, and a transition relation that defines the probability of transiting from the current mode to the next under a particular action. The mode switching in this MDP captures both deliberate intervention through the action selection and the environment or system uncertainties through the corresponding probabilistic mode transitions. Given a policy that selects the switching actions at each mode, an MDP-JLS reduces to a Markov jump linear system (MJLS) [13], where the mode

[^0]switches in the system follow a discrete-time Markov chain (DTMC).
We are interested in synthesizing a stabilizing policy for an MDP-JLS and consider both mean-square stability and stability with probability one of the induced MJLS. We first show that policies that deterministically select an action in each mode of the MDP are not sufficient to stabilize an MDPJLS.
For mean-square stability, we introduce two approaches to compute the stabilizing policies based on existing stability conditions for MJLSs [13], [14], [15], [16]. The first approach provides a sufficient condition for a policy to stabilize an MDP-JLS and formulates the policy synthesis problem as a semidefinite programming problem that results in a simultaneous search for both a stabilizing policy and a diagonal Lyapunov function for each mode. In the second approach, we partition the variables for the policy and Lyapunov functions into two groups on which we perform coordinate descent [17], [18]. We alternate between searching for candidate Lyapunov functions and searching for a policy that satisfies the stability conditions by solving a semidefinite programming problem while fixing the other block of variables.
For stability with probability one, we find stabilizing polices based on new sufficient stability conditions. These conditions extend the average dwell-time constraints for the stability in non-stochastic switched linear systems [19], [11], [20] to MJLSs. More precisely, comparing with the traditional average dwell-time constraints, which require that the average time interval between any two consecutive mode switching is above a certain threshold, The proposed stability conditions establish a lower bound for the probability of mode switching. Such conditions translate into constraints on the stationary distribution of the induced DTMC, based on which we solve policy synthesis efficiently as a linear programming problem. We additionally extend the policy synthesis to MDP-JLSs with uncertain mode transition probabilities and the optimization of the expected state-dependent costs.
We illustrate the use of the proposed methodologies with two examples. For mean-square stability, we show that the method based on coordinate descent outperforms the semidefinite programming method. For stability with probability one, since mean-square stability implies stability with probability one, the coordinate descent method is also applicable but scales poorly with the dimension of the linear dynamical systems in the modes. On the other hand, we observe in the numerical examples that the computation time for the linear programming approach is not sensitive to the dimension of the dynamics in the modes and thus is more scalable. In oru experiments, the computation time of the linear programming approach is only a fraction (as low as $0.02 \%$ ) of that of the
coordinate descent method.
Related work. Stability and stabilization are major concerns for switched linear systems and have been extensively studied in the literature. The most popular stability analysis approaches for systems with arbitrary switching include common and multiple Lyapunov functions [21], [22] and (average) dwelltime conditions [23], [19], [24]. However, in many practical systems, switching between modes is often constrained due to physical limitations, and one usually has control over switching [25]. As a result, it is of interest to synthesize controllers that regulate the mode switching to stabilize the system while respecting constraints on switching. In [26], the authors studied a switched linear autonomous system where a finite automaton generates the mode switching sequences. Such a model considers non-stochastic mode switches governed by a finite automaton while the model proposed in this paper considers probabilistic mode switches governed by an MDP. Furthermore, the focus of [26] is only stability analysis rather than stabilization. References [27], [28], [29] considered switching controller synthesis for safety, reachability and temporal logic specifications, though only for switched systems with non-stochastic switching.

Recall that, for a given policy, an MDP-JLS reduces to an MJLS. For mean-square stability of an MJLS, there exist conditions that are both necessary and sufficient or only sufficient[13], [30]. There also exist results for MJLS stability analysis that involve average dwell-time [31], [32]. However, the average dwell-time in those papers refers to how frequently the transition probabilities of the DTMC change. While in MDP-JLSs, mode transition probabilities do not change over time and the average dwell-time represents the probability of mode switching.
Contributions. The contributions of this paper are three-fold.

- We propose MDP-JLS, a new modeling framework for a class of switched linear systems where MDPs describes the switching logic.
- To guarantee mean-square stability, we study policy synthesis problems for MDP-JLSs with existing stability conditions of MJLSs and formulate them as optimization problems based on semidefinite programming.
- For stability with probability one, we derive new stability conditions with constraints in the probability of mode switching. We also consider MDP-JLSs with uncertain mode transition probabilities and the optimization of the expected average cost incurred by the switching actions.
We organize the rest of this paper as follows. Section $\Pi$ introduces the modeling framework and necessary definitions. Section III formulates the policy synthesis problem. Solutions are proposed in Section IV and Section $\square$ for mean-square stability and stability with probability one, respectively. Section VI provides two examples to show the validity of the proposed solutions and compare their performances. Section VII concludes the paper and discusses future directions.


## II. Preliminaries

In this section, we describe preliminary notions and definitions used in the sequel.


Fig. 1: An MDP-JLS with three modes. Transition probabilities and actions are omitted.

Notations: $|S|$ denotes the cardinality of a set $S$. Given a real matrix $A \in \mathbb{R}^{m \times n}$, $A^{\prime}$ denotes its transpose. We use 1 to denote a column vector with all elements equal to 1 . $\|A\|_{\infty}:=\max _{1 \leq i \leq m} \sum_{j=1}^{n}\left|a_{i j}\right|$. If $m=n, \rho(A)$ represents the spectral radius of $A$, i.e., $\rho(A):=\max _{i}\left|\lambda_{i}\right|$ where $\lambda_{i}, i \in\{1, \ldots, n\}$ are eigenvalues of $A$. Furthermore, $A>0$ ( $A \geq 0$ ) denotes that the matrix $A$ is positive definite (positive semidefinite). $E[$.$] stands for computing the expectation. \otimes$ denotes the Kronecker product. For $A_{i} \in \mathbb{R}^{n \times n}, i \in\{1, \ldots, N\}$, $\operatorname{diag}\left(A_{i}\right) \in \mathbb{R}^{N n \times N n}$ represents the block diagonal matrix formed with $A_{i}$ at the diagonal and zero anywhere else, i.e.

$$
\operatorname{diag}\left(A_{i}\right):=\left[\begin{array}{ccc}
A_{1} & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & A_{N}
\end{array}\right]
$$

## A. Switched Linear Systems

Mathematically, a discrete-time switched linear system is described by

$$
\begin{equation*}
x(k+1)=A_{s_{k}} x(k), \tag{1}
\end{equation*}
$$

where $x(k) \in \mathbb{R}^{n}$ is the state vector, $A_{s_{k}} \in \mathbb{R}^{n \times n}$ implies a matrix $A \in\left\{A_{1}, \ldots, A_{|S|}\right\}$. The linear dynamics of 11 is given by matrices $A_{i}$ when $s_{k}=i$, i.e, the mode that the system is in at time $k$.

## B. Switched Linear Systems with Markov Decision Process Switching

The system in (1) in its general form is a hybrid system where the mode switches could depend on both the continuous dynamics and discrete mode [1], [33]. In this paper, we consider a class of switched linear systems, where the mode switches are governed by a Markov decision process (MDP) [12] defined as follows.

Definition 1. An $M D P$ is a tuple $\mathcal{M}=(S, \hat{s}, \Sigma, T)$ which includes a finite set $S$ of states, an initial state $\hat{s}$, a finite set $\Sigma$ of actions. $T: S \times \Sigma \times S \rightarrow[0,1]$ is the probabilistic transition function with $T\left(s, \sigma, s^{\prime}\right):=p\left(s^{\prime} \mid s, \sigma\right)$, for $s, s^{\prime} \in$ $S$ and $\sigma \in \Sigma$. We denote the number of modes, i.e., $|S|$ as $N$.

For simplicity, we denote $T_{\sigma} \in \mathbb{R}^{N \times N}$ as the transition probabilities induced by an action $\sigma \in \Sigma$ between state pairs. If $\sigma$ is not defined on a state $s_{i}, T\left(s_{i}, \sigma, s_{j}\right)=0$ for any $s_{j} \in$
$S$. For notational simplicity, in the sequel we write $T(i, \sigma, j)$ instead of $T\left(s_{i}, \sigma, s_{j}\right)$. Then we formally define the system that we study in this paper as follows.

Definition 2. An MDP-JLS is a switched system defined in (1) with the mode switches governed by an MDP $\mathcal{M}=$ $(S, \hat{s}, \Sigma, T)$.

We shown an example of an MDP-JLS in Figure 1 There are three linear dynamics corresponding to three modes, respectively, i.e.

$$
x(k+1)=A_{s_{i}} x(k), \text { for } s_{i} \in S,
$$

where each state $s$, there is a set of actions available to choose. The nondeterminism of the action selection is resolved by a policy $\pi$.

Definition 3. $A$ (randomized) policy $\pi: S \times \Sigma \rightarrow[0,1]$ of an MDP $\mathcal{M}$ is a function that maps every state action pair $(s, \sigma)$ where $s \in S$ and $\sigma \in \Sigma$ with a probability $\pi(s, \sigma)$.

By definition, the policy $\pi$ specifies the probability to take an action $\sigma$ at a state $s$. For notational simplicity, we use $\pi(i, \sigma)$ for $\pi\left(s_{i}, \sigma\right)$. As a result, given a policy $\pi$, the MDP $\mathcal{M}$ reduces to a discrete-time Markov chain (DTMC) $\mathcal{C}=$ $(S, \hat{s}, P)$. The matrix $P$ represents the transition probabilities and can be calculated by

$$
P\left(s_{i}, s_{j}\right):=P_{i j}=\sum_{\sigma \in \Sigma} T(i, \sigma, j) \pi(i, \sigma),
$$

where $P_{i j}$ is the $(i, j)$ element of $P$ that denotes the probability of transitioning from $s_{i}$ to $s_{j}$ in one step. We can naturally generalize this definition to $n$-step transition probabilities where $P_{i j}^{n}$ denotes the probability of transitioning from $s_{i}$ to $s_{j}$ with $n$ steps, and $P_{i j}^{n}$ is the $(i, j)$ element of $P^{n}:=\prod_{n} P$.

We denote the probability of being in a state $s_{i}$ at time step $n$ as $p^{n}\left(s_{i}\right)$ which can be computed by $P^{n}$ and the initial state. A stationary distribution (if it exists) over the states is a vector $\bar{p} \in \mathbb{R}^{|S|}$ that

$$
\bar{p}\left(s_{i}\right)=\sum_{s_{j}} \bar{p}\left(s_{j}\right) P_{j i} .
$$

In a DTMC $\mathcal{C}=(S, \hat{s}, P)$, state $s_{j}$ is accessible from $s_{i}$ if $P_{i j}^{n}>0$ for some non-negative $n<\infty$. Two states $s_{i}$ and $s_{j}$ are said to communicate if $s_{i}$ is accessible from $s_{j}$ and $s_{j}$ is accessible from $s_{i}$. A state $s_{i}$ is said to be recurrent if it is accessible from all states that are accessible from $s_{i}$. A state that is not recurrent is called transient. A state $s_{i}$ is said to be aperiodic if $\operatorname{gcd}\left(\left\{n \mid P_{i i}^{n}>0\right\}\right)=1$ where $g c d$ stands for the greatest common divisor. A class $\mathcal{X} \subseteq S$ is a non-empty set of states where each $s_{i} \in \mathcal{X}$ communicates with every other state $s_{j} \in \mathcal{X}$ and communicates with no state $s_{j} \notin \mathcal{X}$. An ergodic class is a class that consists of states that are both recurrent and aperiodic.

Definition 4. A DTMC is an ergodic unichain if it contains a single ergodic class and maybe some transient states. An MDP is an ergodic unichain if every policy $\pi$ induces a DTMC that is an ergodic unichain.

If a DTMC is an ergodic unichain, its stationary distribution $\bar{p}$ exists and is unique, and $\lim _{n \rightarrow \infty} p^{n}=\bar{p}$ [34].

## C. Markov Jump Linear Systems

The Markov jump linear system is defined as follows [13].
Definition 5. A Markov jump linear system (MJLS) is a switched system defined in (1) with the mode switches governed by a DTMC $\mathcal{C}=(S, \hat{s}, P)$.

Given an MDP-JLS with an MDP $\mathcal{M}=(S, \hat{s}, \Sigma, T)$ and a policy $\pi$, the resulting system is an MJLS whose mode switches can be characterized by the DTMC $\mathcal{C}$ induced from the policy $\pi$. If the system (1) is in mode $s_{i}$, then the probability that it switches to mode $s_{j}$ is given by $P_{i j}$.

For MJLS analysis, stability is one of the major concerns. Several notions of stability exist in the literature [15]. In this paper, we are interested in mean-square stability and stability with probability one as defined below.

Definition 6. [13] An MJLS with the dynamics in (1) is said to be mean-square stable if

$$
\lim _{k \rightarrow \infty}\left\|E\left[x(k) x^{\prime}(k)\right]\right\|_{\infty}=0
$$

for any initial condition $x_{0}$.
Definition 7. [13] An MJLS with the dynamics in (1) is said to be stable with probability one if

$$
\lim _{k \rightarrow \infty}\|x(k)\|_{\infty}=0 \text { with probability one }
$$

for any initial condition $x_{0}$.
Mean-square stability is shown to imply stability with probability one [13].

## D. Optimization basics

In this paper, we use optimization problems such as linear programs (LPs), semidefinite programs (SDPs) and bilinear matrix inequalities (BMIs) extensively in the policy synthesis. We briefly define them as follows.
An LP is an optimization problem with a linear objective and constraints on the variable $y \in \mathbb{R}^{n}$, which is given by

$$
\begin{equation*}
\operatorname{minimize} \quad c^{\prime} y \tag{2}
\end{equation*}
$$

subject to

$$
\begin{align*}
& A y=b,  \tag{3}\\
& x \geq 0 \tag{4}
\end{align*}
$$

where $A \in \mathbb{R}^{m \times n}$ is a given matrix, and $c \in \mathbb{R}^{n}, b \in \mathbb{R}^{m}$ are given vectors. The LP in this form is also referred to as the standard form [35, Chapter 4.3]. LPs are convex optimization problems and can be solved efficiently using interior point methods [36], [35].

An SDP is an optimization problem with a linear objective, linear equality constraints and a matrix nonnegativity constraint on the variable $y \in \mathbb{R}^{n}$, which can be written as

$$
\begin{align*}
\operatorname{minimize} & c^{\prime} y  \tag{5}\\
\text { subject to } & \\
& A y=b,  \tag{6}\\
& \sum_{i=1}^{n} y_{i} F_{i} \geq F_{0},
\end{align*}
$$

where $F_{0}, \ldots, F_{m} \in \mathbb{R}^{p \times p}$, are given symmetric matrices, $A \in \mathbb{R}^{m \times n}$ is a given matrix, and $c \in \mathbb{R}^{n}, b \in \mathbb{R}^{m}$ are given vectors. SDPs are convex optimization problems, and is a generalization of LPs. SDPs can also be solved efficiently [36], [35]. The constraint in (7) is named as a linear matrix inequality (LMI), and it is a convex constraint in $y$.

A BMI can be written as the following form:

$$
\sum_{i=1}^{n} y_{j} F_{i}+\sum_{j=1}^{m} z_{j} G_{j}+\sum_{i=1}^{n} \sum_{j=1}^{m} y_{i} z_{j} H_{i j} \geq F_{0}
$$

where $F_{i}, G_{j}, H_{i j} \in \mathbb{R}^{p \times p}$ for $i=1, \ldots, n$ and $j=1, \ldots, m$ are given symmetric matrices, and $x \in \mathbb{R}^{n}, y \in \mathbb{R}^{m}$ are a vector of variables. A BMI is an LMI in $y$ for fixed $z$ and an LMI in $z$ for fixed $y$. The bilinear terms in a BMI make the feasible set not jointly convex in $y$ and $z$ and it is generally hard to find a feasible solution to a BMI [37].

## III. Problem Formulation

In traditional MDP literature, finding a policy for an optimized expected cost [12] or to satisfy a specification in temporal logic [38] is of the primary concern. However, in this paper, we are interested in synthesizing a policy $\pi$ in an MDP that governs the switches of a dynamical system defined in (1). In this case, the objective is to stabilize an MDP-JLS.

Problem 1 (Synthesis for mean-square stable). Given an MDP-JLS, find a policy $\pi: S \times \Sigma \rightarrow[0,1]$ such that the resulting MJLS is mean-square stable.
Problem 2 (Synthesis for stability with probability one). Given an MDP-JLS, find a policy $\pi: S \times \Sigma \rightarrow[0,1]$ such that the resulting MJLS is stable with probability one.

## IV. MEAN-SQUARE Stability Guaranteed Policy Synthesis

This section solves Problem 1. Since an MDP-JLS will reduce to an MJLS with a policy, we first review some stability conditions in MJLS that we will leverage to synthesize policies.

## A. Stability Conditions

We give two necessary and sufficient stability conditions for an MJLS as the following.

Theorem 1. [13] Given an MJLS as defined in (1] whose mode $s \in S$ makes random transitions described by a DTMC $\mathcal{C}=(S, \hat{s}, P)$, the following assertions are equivalent.

1) The MJLS is mean-square stable.
2) $\rho(\mathcal{A})<1$, where

$$
\mathcal{A}=\left(P^{\prime} \otimes I\right) \operatorname{diag}\left(A_{i} \otimes A_{i}\right)
$$

and $I$ is the identity matrix of a proper dimension.
3) There exists a $V=\left(V_{1}, \ldots, V_{N}\right) \in \mathbb{R}^{n \times n}$ with $V>0$ such that

$$
\begin{equation*}
V-\mathcal{T}(V)>0 \tag{8}
\end{equation*}
$$

where

$$
\mathcal{T}_{j}(V)=\sum_{i=1}^{N} P_{i j} A_{i} V_{i} A_{i}^{\prime}
$$

Note that the stability conditions do not depend on either the initial state $\hat{s}$ of the MDP or the initial continuous state $x(0)$. For computational efficiency, we state a sufficient stability condition as follows.
Corollary 1. [13] Given an MJLS as defined in (1] whose mode $s \in S$ makes transitions following a DTMC $\mathcal{C}=$ ( $S, \hat{s}, P$ ), the MJLS is mean-square stable if there exists $\alpha_{i}>0$ such that the following is satisfied.

$$
\begin{equation*}
\alpha_{i} I-\sum_{j=1}^{N} P_{i j} \alpha_{j} A_{i} A_{i}^{\prime}>0, i \in\{1, \ldots, N\} \tag{9}
\end{equation*}
$$

The condition given in (8) can be checked by solving an SDP with $V_{i}$ as variables. However, the number of variables for this SDP is $n^{2} \cdot N$, and finding a feasible solution for the SDP can be time consuming for large $n$ and $N$. On the other hand, the condition in (9) can be checked by solving an SDP with $N$ variables, and the size of the optimization problem is smaller compared to the optimization problem in (8).

## B. Deterministic Policies Are Not Sufficient for Stability

We first show that a deterministic policy, i.e, $\pi: S \rightarrow A$ is not sufficient to guarantee the stability of an MDP-JLS. It means that there may not exist a deterministic policy to stabilize an MDP-JLS, but there exists an randomized policy that achieves stability.

We illustrate this fact by a counterexample. Consider a switched system with system dynamics in (1)

$$
A_{1}=\left[\begin{array}{cc}
0.99 & -0.56 \\
-0.19 & 0.73
\end{array}\right] \text { and } A_{2}=\left[\begin{array}{cc}
0.38 & -0.98 \\
-0.66 & -0.66
\end{array}\right]
$$

The MDP $\mathcal{M}=(S, \hat{s}, \Sigma, T)$ where $S=\left\{s_{1}, s_{2}\right\}$ and $\Sigma=$ $\left\{\sigma_{1}, \sigma_{2}\right\}$. The transition probabilities induced by action $\sigma_{1}$ and $\sigma_{2}$ are

$$
T_{\sigma_{1}}=\left[\begin{array}{ll}
0.21 & 0.79 \\
0.90 & 0.10
\end{array}\right] \text { and } T_{\sigma_{2}}=\left[\begin{array}{cc}
0.71 & 0.29 \\
0.13 & 0.87
\end{array}\right]
$$

The deterministic policy that induces a minimal spectral radius is selecting $\sigma_{1}$ in both mode 1 and 2 . The spectral radius $\rho(\mathcal{A})$ of the MJLS induced by this policy is $1.04>1$, which makes the overall system unstable. However, the policy that selects $\sigma_{1}$ in mode 1 , and selects $\sigma_{1}$ in mode 2 with a probability of 0.27 induces an MJLS that has a spectral radius of $\rho(\mathcal{A})=0.90<$ 1. So the system is stable according to Theorem 1 Therefore, we conclude that deterministic policies are not sufficient to stabilize an MDP-JLS.

## C. Policy Synthesis via Bilinear Matrix Inequalities

In this section, we formulate a condition based on bilinear matrix inequalities to synthesize a randomized policy to stabilize an MDP-JLS. The condition is a straightforward generalization of the linear matrix inequalities given in (8). The following result states that we can search for a stabilizing policy by finding a solution to a set of bilinear matrix inequalities.

Theorem 2. Consider an MDP-JLS whose mode $s \in S$ makes transitions following a MDP $\mathcal{M}=(S, \hat{s}, \Sigma, T)$ and the dynamics in each modes as in (1). If there exists matrices $V_{i} \in \mathbb{R}^{n \times n}$, and $\pi$ such that the following holds:

$$
\begin{align*}
& V_{i}>0  \tag{10}\\
& V-\mathcal{T}(V)>0,  \tag{11}\\
& \mathcal{T}_{j}(V)=\sum_{i=1}^{N} P_{i j} A_{i} V_{i} A_{i}^{\prime}  \tag{12}\\
& P_{i j}=\sum_{\sigma \in \Sigma} T(i, \sigma, j) \pi(i, \sigma),  \tag{13}\\
& \sum_{\sigma \in \Sigma} \pi_{i, \sigma}=1  \tag{14}\\
& \pi(i, \sigma) \geq 0 \tag{15}
\end{align*}
$$

for $i, j=\{1, \ldots, N\}$ and $\sigma \in \Sigma$, then the induced MJLS is mean-square stable.

Proof. Constraints (13, 14, (15) construct the induced DTMC $\mathcal{C}$ with transitions governed by $P_{i j}$. Using the result of Theorem 1, the constraints (10), (11), and (12) ensure that the MJLS is mean-square stable with the induced DTMC $\mathcal{C}$. Hence, the existence of a policy and matrices $V_{i}$ that satisfies the constraints $\sqrt{10}-(13)$ shows that the MJLS is mean-square stable.

Note that the constraints given in $\sqrt{10}-(15)$ are BMI constraints due to multiplication between variables $\pi$ and $V$ in (11)-(13), therefore it is hard in general to find a policy by solving the BMI directly. In the next section, we propose two approaches based on convex optimization that are easier to compute, and discuss their relationship with the BMI in (10)(15).

## D. Policy Synthesis via Convex Optimization

In this section, we propose two methods to synthesize a policy that stabilizes an MDP-JLS. The first method is based on checking feasibility of an SDP, which is an relaxation of the original stability condition. The second method is based on applying a coordinate descent on the variables $V$ and $\pi$. We can use coordinate descent in our case efficiently, as the constraints in (11-13) are LMI constraints if $V$ or $\pi$ is fixed.

1) Semidefinite Relaxation: In the following, we state the semidefinite relaxation to compute a policy that stabilizes an MDP-JLS. the relaxation extends the stability condition given in (9) for an MJLS to a switched system whose mode switches are governed by an MDP.

Theorem 3. Consider an MDP-JLS whose mode $s \in S$ makes transitions following a MDP $\mathcal{M}=(S, \hat{s}, \Sigma, T)$ and the dynamics in each modes as in (1). If there exists $K_{i, \sigma}, \alpha_{i} \in$
$\mathbb{R}>0$ such that

$$
\begin{align*}
& V_{i}=\alpha_{i} I>0  \tag{16}\\
& V-\mathcal{T}(V)>0  \tag{17}\\
& \mathcal{T}_{j}(V)=\sum_{i=1}^{N} \sum_{\sigma \in \Sigma} T(i, \sigma, j) K_{i, \sigma} A_{i} A_{i}^{\prime}  \tag{18}\\
& \sum_{\sigma \in \Sigma} K_{i, \sigma}=\alpha_{i}  \tag{19}\\
& K_{i, \sigma} \geq 0 \tag{20}
\end{align*}
$$

for $i, j=\{1, \ldots, N\}$ and $\sigma \in \Sigma$, then the MJLS is meansquare stable.

Proof. Suppose that the condition given by constraints 10 (15) is satisfied with $V_{i}=\alpha_{i} I>0, i=\{1, \ldots, N\}$. Then, the constraint (12) becomes

$$
\begin{equation*}
\mathcal{T}_{j}(V)=\sum_{i=1}^{N} \sum_{\sigma \in \Sigma} T(i, \sigma, j) \pi(i, \sigma) \alpha_{i} A_{i} A_{i}^{\prime} \tag{21}
\end{equation*}
$$

with variables $\alpha_{i}>0, i=\{1, \ldots, N\}$ and $\pi$. Note that for a given policy $\pi$ and the induced DTMC $\mathcal{C}$, the constraint in 21) is equivalent to the condition given by 9 in Corollary 1. By defining the change of variable $K_{i, \sigma}=\pi(i, \sigma) \cdot \alpha_{i}$ for $i=\{1, \ldots, N\}$ and $\sigma \in \Sigma$, the constraints (12)-15) are equivalent to the constraints in (18)-20). Finding a feasible solution that satisfies the constraints in (10) yields a policy $\pi(i, \sigma)=K_{i, \sigma} / \alpha_{i}$ for $i=\{1, \ldots, N\}$ and $\sigma \in \Sigma$, which by construction satisfies the constraints in 10)-15. Therefore, the policy $\pi$ and $V$ ensures that the induced MJLS is mean-square stable.

The constraints in (16) are LMIs in the variables $K$ and $\alpha$. Finding a feasible solution of a set of LMIs can be done by solving an SDP. However, this condition is only a sufficient as we restrict the structure of the matrix $V$, therefore we may not be able to certify the stability of an MJLS even though there may exists a policy that ensures that the induced MJLS is mean-square stable.
2) Coordinate Descent: In this section we discuss the coordinate descent (CD) approach and the differences with a basic CD algorithm. Recall that a BMI is an LMI if one the variables is fixed, and we can check if the constraints in (10)(15) are feasible for a fixed $V$ or $\pi$. However, applying the basic CD on $V$ and $\pi$ requires the problems to be feasible for a fixed $V$ or $\pi$, which is not necessarily true in our case. If the initial problem is feasible, then we know that $\pi$ stabilizes the induced MJLS. Therefore, we assume that the initial policy does not stabilize the system.

Our implementation differs from a basic coordinate descent algorithm in the addition of the slack variables to the constraint in (11), which ensures that the resulting LMI is feasible for a fixed set of variables, and we use a proximal update between the variables instead of the original update method between $V$ and $\pi$. Details about the proximal update and the convergence guarantees can be found in [39].

We start with an initial guess of the variables $V^{0}$ and $\pi^{0}$. Then in each iteration $k$, we solve the following SDP for a fixed $\pi^{k}$ :

$$
\begin{equation*}
\operatorname{minimize} \quad-\gamma+\sum_{i=1}^{N} L\left\|V_{i}-V_{i}^{k-1}\right\|_{2} \tag{22}
\end{equation*}
$$

subject to

$$
\begin{align*}
& V_{i}>0  \tag{23}\\
& V-\mathcal{T}(V) \geq \gamma I  \tag{24}\\
& \mathcal{T}_{j}(V)=\sum_{i=1}^{N} P_{i j} A_{i} V_{i} A_{i}^{\prime}  \tag{25}\\
& P_{i j}=\sum_{\sigma \in \Sigma} T(i, \sigma, j) \pi^{k}(i, \sigma) \tag{26}
\end{align*}
$$

where $V_{i} \in \mathbb{R}^{n \times n}, i=\{1, \ldots, N\}$ and $\gamma \in \mathbb{R}$ are variables, and $L \in \mathbb{R}$ is a small positive constant. After we get $V$ from (22)-(26), the SDP we solve for a fixed $V^{k}=V$ is given as follows:

$$
\begin{equation*}
\text { minimize } \quad-\gamma+\sum_{i=1}^{N} \sum_{\sigma \in \Sigma} L\left\|\pi(i, \sigma)-\pi^{k-1}(i, \sigma)\right\|_{2} \tag{27}
\end{equation*}
$$

subject to

$$
\begin{align*}
& V^{k}-\mathcal{T}\left(V^{k}\right) \geq \gamma I  \tag{28}\\
& \mathcal{T}_{j}(V)=\sum_{i=1}^{N} P_{i j} A_{i} V_{i}^{k} A_{i}^{\prime}  \tag{29}\\
& P_{i j}=\sum_{\sigma \in \Sigma} T(i, \sigma, j) \pi(i, \sigma),  \tag{30}\\
& \sum_{\sigma \in \Sigma} \pi_{i, \sigma}=1  \tag{31}\\
& \pi(i, \sigma) \geq 0 \tag{32}
\end{align*}
$$

with variables $\pi$ for $i=\{1, \ldots, N\}$ and $\sigma \in \Sigma$, and $\gamma \in$ $\mathbb{R}$. After solving each SDP, we update the variables until we converge to a solution or we obtain a solution with $\gamma>0$. If we can find a solution with $\gamma>0$, the conditions (24) and $(28)$ implies the condition given in 11 , and the rest of the conditions in (10)-(15) are already satisfied in either SDPs that we solve during CD. In this case, we stop the algorithm as the solution given by $V$ and $\pi$ guarantees that the MJLS is meansquare stable. Note that our method is guaranteed to converge as we use the update (1.3b) in [39], however the procedure can converge to a solution with $\gamma \leq 0$, which implies that the CD method cannot certify if the MJLS is mean-square stable.

## V. Policy Synthesis for Stability with Probability One

This section solves Problem 2. We assume that $\mathcal{M}$ is an ergodic unichain MDP which has a unique stationary distribution for every DTMC induced by a policy $\pi$. For a stationary distribution $p^{\infty}$, the probability to jump to state $s$ from a different state in one time step denoted as $\overrightarrow{p_{s}}$ is given by

$$
\begin{equation*}
\overrightarrow{p_{s}}=\sum_{s^{\prime} \neq s} P\left(s^{\prime}, s\right) p_{s^{\prime}}^{\infty} \tag{33}
\end{equation*}
$$

Furthermore, the event that there is a mode jump is Bernoulli distributed such that

$$
\begin{equation*}
P_{j u m p}=1-\sum_{s} p_{s}^{\infty} P(s, s)=\sum_{s} \overrightarrow{p_{s}} \tag{34}
\end{equation*}
$$

where $P_{\text {jump }}$ denotes the probability of a mode jump.

## A. Stability with Mode-Independent Conditions

Theorem 4. Consider an MDP-JLS whose mode $s \in S$ makes transitions following a MDP $\mathcal{M}=(S, \hat{s}, \Sigma, T)$ and the dynamics in each modes as in (1) with given constants $0<\alpha<1$ and $\mu>1$. Assume there exists a Lyapunov function candidate $V(x)=\left\{V_{s}(x), s \in S\right\}$ that satisfies the following for any pair of $s, s^{\prime} \in S$.

$$
\begin{align*}
& V_{s}\left(x_{k+1}\right)-V_{s}\left(x_{k}\right) \leq-\alpha V_{s}\left(x_{k}\right), \text { and }  \tag{35}\\
& V_{s}\left(x_{k}\right) \leq \mu V_{s^{\prime}}\left(x_{k}\right) \tag{36}
\end{align*}
$$

where $x_{k}:=x(k)$ for simplicity. Given a policy $\sigma$ and if the induced DTMC has a stationary distribution, then the system is stable with probability one if

$$
\begin{equation*}
P_{j u m p}<\frac{\ln \left(\frac{1}{1-\alpha}\right)}{\ln (\mu)} \tag{37}
\end{equation*}
$$

Proof. For a time horizon $k$, suppose there are $m$ mode jumps so far at time instants $k_{i}, i \in\{1, \ldots, m\}$ such that $s_{k_{i}} \neq s_{k_{i}-1}$, i.e., $k_{i}$ is the time instant that the $i$-th jump just occurred. Then we have

$$
\begin{equation*}
V_{s_{k}}\left(x_{k}\right) \leq(1-\alpha)^{k-k_{m}} V_{s_{k_{m}}}\left(x_{k_{m}}\right) \tag{38}
\end{equation*}
$$

from equation (35) since the mode remains the same during the time interval $\left[k_{m}, k\right]$, i.e. $s_{t}=s_{k}$ for all $t \in\left[k_{m}, k\right]$. Together with equations (35) and 36, we have

$$
\begin{align*}
V_{s_{k}}\left(x_{k}\right) & \leq(1-\alpha)^{k-k_{m}} \mu V_{s_{k_{m}-1}}\left(x_{k_{m}}\right) \\
& \leq(1-\alpha)^{k-k_{m}} \mu(1-\alpha)^{k_{m}-k_{m-1}} \mu V_{s_{k_{m-1}-1}}\left(x_{k_{m-1}}\right) \\
& \leq \cdots \leq(1-\alpha)^{k} \mu^{m} V_{\hat{s}}\left(x_{0}\right) \\
& =\left((1-\alpha) \mu^{\frac{m}{k}}\right)^{k} V_{\hat{s}}\left(x_{0}\right) . \tag{39}
\end{align*}
$$

To prove the system is stable with probability one, we need to prove that

$$
P\left(\lim _{k \rightarrow \infty}\left((1-\alpha) \mu^{\frac{m}{k}}\right)^{k}=0\right)=1
$$

which implies that

$$
\begin{aligned}
P\left(\lim _{k \rightarrow \infty}(1-\alpha) \mu^{\frac{m}{k}}<1\right) & =1 \\
\Longleftrightarrow & P\left(\lim _{k \rightarrow \infty} \frac{m}{k}<\frac{\ln \left(\frac{1}{1-\alpha}\right)}{\ln (\mu)}\right)=1 .
\end{aligned}
$$

Using the law of total probability and conditional probability, we have that

$$
\begin{align*}
& P\left(\left.\lim _{k \rightarrow \infty} \frac{m}{k}<\frac{\ln \left(\frac{1}{1-\alpha}\right)}{\ln (\mu)} \right\rvert\, \mathcal{B}\right) P(\mathcal{B}) \\
& +P\left(\left.\lim _{k \rightarrow \infty} \frac{m}{k}<\frac{\ln \left(\frac{1}{1-\alpha}\right)}{\ln (\mu)} \right\rvert\, \overline{\mathcal{B}}\right) P(\overline{\mathcal{B}})=1 \tag{40}
\end{align*}
$$

where $\mathcal{B}$ represents the event that $\lim _{k \rightarrow \infty} \frac{m}{k}=P_{\text {jump }}$ and $\overline{\mathcal{B}}$ represents the event that $\lim _{k \rightarrow \infty} \frac{m}{k} \neq P_{j u m p}$. According to the law of large numbers [40], we know that

$$
P(\mathcal{B})=P\left(\lim _{k \rightarrow \infty} \frac{m}{k}=P_{j u m p}\right)=1 \text { and } P(\overline{\mathcal{B}})=0
$$

Therefore we have that

$$
\begin{equation*}
P\left(\left.\lim _{k \rightarrow \infty} \frac{m}{k}<\frac{\ln \left(\frac{1}{1-\alpha}\right)}{\ln (\mu)} \right\rvert\, \overline{\mathcal{B}}\right) P(\overline{\mathcal{B}})=0 . \tag{41}
\end{equation*}
$$

Thus from 40 we require

$$
\begin{align*}
& P\left(\left.\lim _{k \rightarrow \infty} \frac{m}{k}<\frac{\ln \left(\frac{1}{1-\alpha}\right)}{\ln (\mu)} \right\rvert\, \mathcal{B}\right) P(\mathcal{B}) \\
& =P\left(\left.\lim _{k \rightarrow \infty} \frac{m}{k}<\frac{\ln \left(\frac{1}{1-\alpha}\right)}{\ln (\mu)} \right\rvert\, \lim _{k \rightarrow \infty} \frac{m}{k}=P_{\text {jump }}\right)  \tag{42}\\
& =P\left[P_{\text {jump }}<\frac{\ln \left(\frac{1}{1-\alpha}\right)}{\ln (\mu)}\right]=1
\end{align*}
$$

Since given a policy $\pi, P_{j u m p}$ is a constant, equivalently we require that

$$
P_{j u m p}<\frac{\ln \left(\frac{1}{1-\alpha}\right)}{\ln (\mu)}
$$

## B. Stability with Mode-Dependent Conditions

From Theorem 4 we find a sufficient condition for the policy of the MDP to satisfy, such that the stability can be guaranteed. However, the conditions (35) and (36) in Theorem 4 are mode-independent, which may introduce conservativeness because the same pair of $\alpha$ and $\mu$ has to be satisfied by all the modes. Therefore, inspired by [20] we introduce the following theorem where the parameters such as $\alpha$ and $\mu$ in (35) and (36) are mode-dependent.

Theorem 5. Consider an MDP-JLS whose mode $s \in S$ makes transitions following a MDP $\mathcal{M}=(S, \hat{s}, \Sigma, T)$ and the dynamics in each modes as in (1) with given constants $0<\alpha_{s}<1$ and $\mu_{s}>1$ for all $s \in S$. Assume there exists a Lyapunov function candidate $V(x)=\left\{V_{s}(x), s \in S\right\}$ that satisfies the following for any pair of $s, s^{\prime} \in S$.

$$
\begin{align*}
& V_{s}\left(x_{k+1}\right)-V_{s}\left(x_{k}\right) \leq-\alpha_{s} V_{s}\left(x_{k}\right), \text { and }  \tag{43}\\
& V_{s}\left(x_{k}\right) \leq \mu_{s} V_{s^{\prime}}\left(x_{k}\right) \tag{44}
\end{align*}
$$

where $x_{k}:=x(k)$ for simplicity. Given a policy $\sigma$ and if the induced DTMC has a unique stationary distribution, then the system is stable with probability one if

$$
\begin{equation*}
\sum_{s} \overrightarrow{p_{s}} \ln \mu_{s}+p_{s}^{\infty} \ln \left(1-\alpha_{s}\right)<0 \tag{45}
\end{equation*}
$$

Proof. We start the proof similar to that of Theorem 4. For a time horizon $k$, suppose there are $m$ mode jumps so far at time instants $k_{i}, i \in\{1, \ldots, m\}$ such that $s_{k_{i}} \neq s_{k_{i}-1}$, i.e., $k_{i}$ is the time instant that the $i$-th jump just occurred. Then we have

$$
\begin{equation*}
V_{s_{k}}\left(x_{k}\right) \leq\left(1-\alpha_{s_{k_{m}}}\right)^{k-k_{m}} V_{s_{k_{m}}}\left(x_{k_{m}}\right) \tag{46}
\end{equation*}
$$

from equation 43) since the mode remains the same during the time interval $\left[k_{m}, k\right]$, i.e. $s_{t}=s_{k}$ for all $t \in\left[k_{m}, k\right]$. Applying equations (43) and 44 recursively, we have

$$
\begin{align*}
& V_{s_{k}}\left(x_{k}\right) \leq\left(1-\alpha_{s_{k_{m}}}\right)^{k-k_{m}} \mu_{s_{k_{m}}} V_{s_{k_{m}-1}}\left(x_{k_{m}}\right) \\
& \leq\left(1-\alpha_{s_{k}}\right)^{k-k_{m}} \mu_{s_{k}} \\
& \quad \times\left(1-\alpha_{s_{k_{m}-1}}\right)^{k_{m}-k_{m-1}} \mu_{s_{k_{m}-1}} V_{s_{k_{m-1}-1}}\left(x_{k_{m-1}}\right) \\
& \quad \leq \cdots \leq V_{\hat{s}}\left(x_{0}\right) \prod_{i=0}^{m}\left(1-\alpha_{s_{k_{i}}}\right)^{k_{i+1}-k_{i}} \mu_{s_{k_{i}}} \tag{47}
\end{align*}
$$

where $k_{m+1}=k, k_{0}=0$ and $\mu_{s_{k_{0}}}=1$. Among the $m$ number of jumps, we denote $m_{s}$ as the number of jumps to mode $s$ from a different mode. It is immediate that $\sum_{s} m_{s}=m$. Furthermore, we denote the time that the system spends in mode $s$ up to $k$ as $k_{s}$. By definition, $\sum_{s} k_{s}=k$. Then from (47) we have

$$
\begin{align*}
V_{s_{k}}\left(x_{k}\right) & \leq V_{\hat{s}}\left(x_{0}\right) \prod_{i=0}^{m}\left(1-\alpha_{s_{k_{i}}}\right)^{k_{i+1}-k_{i}} \mu_{s_{k_{i}}} \\
& =V_{\hat{s}}\left(x_{0}\right) \prod_{s=1}^{N} \mu_{s}^{m_{s}}\left(1-\alpha_{s}\right)^{k_{s}}  \tag{48}\\
& =V_{\hat{s}}\left(x_{0}\right)\left(\prod_{s=1}^{N} \mu_{s}^{\frac{m_{s}}{k}}\left(1-\alpha_{s}\right)^{\frac{k_{s}}{k}}\right)^{k} .
\end{align*}
$$

Similar to the proof of Theorem 4 to prove the system is stable with probability one, we need to show that

$$
P\left[\lim _{k \rightarrow \infty}\left(\prod_{s=1}^{N} \mu_{s}^{\frac{m_{s}}{k}}\left(1-\alpha_{s}\right)^{\frac{k_{s}}{k}}\right)^{k}=0\right]=1
$$

which implies that

$$
\begin{align*}
& P\left[\lim _{k \rightarrow \infty} \prod_{s=1}^{N} \mu_{s}^{\frac{m_{s}}{k}}\left(1-\alpha_{s}\right)^{\frac{k_{s}}{k}}<1\right]=1  \tag{49}\\
\Longleftrightarrow & P\left[\lim _{k \rightarrow \infty} \sum_{s} \frac{m_{s}}{k} \ln \left(\mu_{s}\right)+\frac{k_{s}}{k} \ln \left(1-\alpha_{s}\right)<0\right]=1
\end{align*}
$$

According to the law of large numbers [40], we know that

$$
\begin{gathered}
P\left[\lim _{k \rightarrow \infty} \frac{m_{s}}{k}=\vec{p}_{s}\right]=1, \text { and } \\
P\left[\lim _{k \rightarrow \infty} \frac{k_{s}}{k}=p_{s}^{\infty}\right]=1
\end{gathered}
$$

Therefore, from (49) and similar derivations that reach 42), we prove that if (45) holds, the system is stable with probability one.

It is worth noticing that if $\mu_{s}=\mu$ and $\alpha_{s}=\alpha$ for any mode $s$, then condition (42) becomes

$$
\begin{equation*}
\sum_{s} \overrightarrow{p_{s}} \ln \mu+p_{s}^{\infty} \ln (1-\alpha)<0 \tag{50}
\end{equation*}
$$

Since $\sum_{s} \overrightarrow{p_{s}}=P_{j u m p}$ and $\sum_{s} p_{s}^{\infty}=1$, equation (50) reduces to

$$
P_{j u m p} \ln \mu+\ln (1-\alpha)<0
$$

Therefore, we recover the stability condition from Theorem 4

## C. Computation of $\alpha$ and $\mu$

The stability conditions in Theorem 4 and Theorem 5 hinge on the existence of multiple Lyapunov functions for each mode $s$ as well as constants $\alpha$ and $\mu$ or $\alpha_{s}$ and $\mu_{s}$. This subsection gives an algorithm to find these Lyapunov functions and constants.

The Lyapunov function in mode $s$ can be formed as follows:

$$
\begin{equation*}
V_{s}(x)=x^{\prime} M_{s} x \tag{51}
\end{equation*}
$$

where $M_{s}$ is a positive definite matrix.
We have

$$
\begin{align*}
V_{s}\left(x_{k+1}\right) & =x_{k+1}^{\prime} M_{s} x_{k+1} \\
& \leq\left(1-\alpha_{s}\right) V_{s}\left(x_{k}\right)=\left(1-\alpha_{s}\right) x_{k}^{\prime} M_{s} x_{k} \tag{52}
\end{align*}
$$

where $x_{k}:=x(k)$ for simplicity.
Therefore, $M_{s}$ and the largest $\alpha_{s}$ can be computed from the following bilinear optimization problem:

$$
\begin{array}{ll}
\max _{\alpha_{s}, M_{s}} & \alpha_{s} \\
\text { s.t. } & M_{s}^{\prime}=M_{s} \succeq I  \tag{53}\\
& A_{s}^{\prime} M_{s} A_{s} \preceq\left(1-\alpha_{s}\right) M_{s},
\end{array}
$$

where $I$ is the identity matrix.
When the state jump from mode $s$ to mode $s^{\prime}\left(s^{\prime} \neq s\right)$, we have

$$
\begin{equation*}
V_{s}\left(x_{k}\right)=x_{k}^{\prime} M_{s} x_{k} \leq \mu_{s, s^{\prime}} V_{s^{\prime}}\left(x_{s}\right)=\mu_{s, s^{\prime}} x_{k}^{\prime} M_{s^{\prime}} x_{k} \tag{54}
\end{equation*}
$$

After $M_{s}$ is computed for every mode $s$ using (53), the smallest $\mu_{s, s^{\prime}}$ can be computed from the following SDP problem:

$$
\begin{array}{ll}
\min _{\mu_{s, s^{\prime}}} & \mu_{s, s^{\prime}} \\
\text { s.t. } & \mu_{s, s^{\prime}}>1  \tag{55}\\
& M_{s} \preceq \mu_{s, s^{\prime}} M_{s^{\prime}} .
\end{array}
$$

## D. Policy Synthesis for Stability With Probability One

This subsection introduces computation approach to find a policy such that an MDP-JLS is guaranteed to be stable with probability one, based on stability conditions in Theorem 4 and Theorem 55. In addition to stability, we assign a cost function $c: S \rightarrow \mathbb{R}$ to each mode, such that the stabilizing policy should also minimize a cost $\sum_{s} c(s) p_{s}^{\infty}$, which represents an average cost over states when the underlying DTMC reaches its stationary distribution. We start with policy synthesis following stability conditions stated in Theorem 4

Theorem 6. In case of perfect model knowledge of the MDP, given stability constants $0<\alpha<1, \mu>1$ and given small $\epsilon>0$, Problem 2 can be formulated as the linear optimization problem

$$
\begin{array}{cl}
\underset{\hat{\pi}, p^{\infty}, \hat{P}}{\operatorname{minimize}} & \sum_{s} c(s) p_{s}^{\infty} \\
\text { subject to } & \hat{\pi} \mathbf{1}=p^{\infty}, \quad \mathbf{1}^{\prime} p^{\infty}=1 \\
& \hat{P}^{\prime} \mathbf{1}=p^{\infty}, \\
& \hat{P}_{i j}=\sum_{\sigma} T(i, \sigma, j) \hat{\pi}(i, \sigma) \quad \forall i, j \in S \\
& 1-\operatorname{Tr}(\hat{P})<\frac{\ln \left(\frac{1}{1-\alpha}\right)}{\ln (\mu)} \\
& \hat{\pi} \geq 0, \quad p^{\infty} \geq \epsilon \tag{61}
\end{array}
$$

where $\hat{P} \in \mathbb{R}^{|S| \times|S|}, p_{\hat{\infty}}^{\infty} \in \mathbb{R}^{|S|}$ and $\hat{\pi} \in \mathbb{R}^{|S| \times|\Sigma|}$. If we denote by $\hat{\pi}_{o p t}, p_{o p t}^{\infty}, \hat{P}_{\text {opt }}$ the optimal solutions, then the policy $\pi$, the induced Markov matrix $P$ and the stationary distribution $p^{\infty}$ solutions of problem 2 are given by

$$
\begin{gathered}
\pi=\operatorname{diag}\left(p_{o p t}^{\infty}\right)^{-1} \hat{\pi}_{o p t}, \\
P=\operatorname{diag}\left(p_{o p t}^{\infty}\right)^{-1} \hat{P}_{o p t}, \text { and } \\
p^{\infty}=p_{o p t}^{\infty}
\end{gathered}
$$

Proof. This LP formulation comes immediately from the change of variables

$$
\begin{aligned}
\hat{P} & =\operatorname{diag}\left(p^{\infty}\right) P, \text { and } \\
\hat{\pi} & =\operatorname{diag}\left(p^{\infty}\right) \pi
\end{aligned}
$$

Observe that $\forall i, j \in S, \hat{P}_{i j}=p_{i}^{\infty} P_{i j}$ and $\forall \sigma \in \Sigma, \hat{\pi}_{i k}=$ $p_{i}^{\infty} \pi_{i \sigma}$.

By definition $p^{\infty}>0$ and $\pi \geq 0$ is equivalent to $\hat{\pi} \geq 0$ and $p^{\infty} \geq \epsilon$ for a fixed small $\epsilon>0$. This gives constraint 61.
$\pi \mathbf{1}=1 \Longleftrightarrow \hat{\pi} \mathbf{1}=p^{\infty}$ by left multiplication with invertible matrix $\operatorname{diag}\left(p^{\infty}\right)$. Since $p^{\infty}$ is a probability distribution, $1^{\prime} p^{\infty}=1$. This proves constraint (57).

By definition of the stationary distribution $p^{\infty}$, we have

$$
\sum_{i} p_{i}^{\infty} P_{i j}=\sum_{i} \hat{P}_{i j}=p_{j}^{\infty} \text { for all } j \in S
$$

Therefore $\left(\pi^{\infty}\right)^{\prime} P=\left(\pi^{\infty}\right)^{\prime} \Longleftrightarrow \mathbf{1}^{\prime} \hat{P}=\left(\pi^{\infty}\right)^{\prime}$. This gives constraint (58).

Since $p^{\infty}>0$, the constraint 59 is given by

$$
\begin{aligned}
& P_{i j}=\sum_{\sigma} T(i, \sigma, j) \pi(i, \sigma) \\
\Longleftrightarrow & p_{i}^{\infty} P_{i j}=\sum_{\sigma} T(i, \sigma, j)\left(p_{i}^{\infty} \pi(i, \sigma)\right) \\
\Longleftrightarrow & \hat{P}_{i j}=\sum_{\sigma} T(i, \sigma, j) \hat{\pi}(i, \sigma),
\end{aligned}
$$

for all $i, j \in S$.
Finally, constraint 60) is given by Theorem 4

$$
P_{\text {jump }}=1-\sum_{i} p_{i}^{\infty} P_{i i}=1-\operatorname{Tr}(\hat{P})<\frac{\ln \left(\frac{1}{1-\alpha}\right)}{\ln (\mu)}
$$

Given the mode-dependent stability coefficients $\alpha_{s}$ and $\mu_{s}$, Theorem 5] can be formulated as another LP where a feasible
solution contains a policy that guarantees stability of the system with probability one.

Theorem 7. Consider an MDP-JLS whose mode $s \in S$ makes transitions following a MDP $\mathcal{M}=(S, \hat{s}, \Sigma, T)$ and the dynamics in each modes as in (1) with given mode-dependent constants $0<\alpha_{s}<1$ and $\mu_{s}>1$ for all $s \in S$. Given small $\epsilon>0$, Problem 2 can be formulated as the linear optimization problem

$$
\begin{array}{ll}
\underset{\hat{\pi}, \pi^{\infty}, \hat{P}}{\operatorname{minimize}} & \sum_{s} c(s) p_{s}^{\infty} \\
\text { s.t } & \hat{\pi} \mathbf{1}=p^{\infty}, \quad \mathbf{1}^{\prime} p^{\infty}=1 \\
& \hat{P}^{\prime} \mathbf{1}=p^{\infty} \\
& \hat{P}_{i j}=\sum_{\sigma} T(i, \sigma, j) \hat{\pi}(i, \sigma) \quad \forall i, j \in S \\
& \sum_{i}\left(\sum_{j \neq i} \hat{P}_{j i}\right) \ln \left(1-\mu_{i}\right)+p_{i}^{\infty} \ln \left(1-\alpha_{i}\right)<0 \\
& \hat{\pi} \geq 0, \quad p^{\infty} \geq \epsilon \tag{67}
\end{array}
$$

where $c: S \rightarrow \mathbb{R}$ is an optimization criteria over the set of solutions, $\hat{P} \in \mathbb{R}^{|S| \times|S|}, p^{\infty} \in \mathbb{R}^{|S|}$ and $\hat{\pi} \in \mathbb{R}^{|S| \times|\Sigma|}$. If $\hat{\pi}_{o p t}, p_{o p t}^{\infty}, \hat{P}_{\text {opt }}$ are optimal solutions, then the policy $\pi$, the induced Markov matrix $P$ and the stationary distribution $p^{\infty}$ solutions of problem 2 are given by

$$
\begin{gathered}
\pi=\operatorname{diag}\left(p_{o p t}^{\infty}\right)^{-1} \hat{\pi}_{o p t}, \\
P=\operatorname{diag}\left(p_{o p t}^{\infty}\right)^{-1} \hat{P}_{o p t}, \text { and } \\
p^{\infty}=p_{o p t}^{\infty} .
\end{gathered}
$$

Proof. This is an immediate consequence of Theorem 5 Observe that $\forall i, j \in S, \hat{P}_{i j}=p_{i}^{\infty} P_{i j}$. Therefore, $\overrightarrow{p_{s}}=$ $\sum_{s^{\prime} \neq s} P\left(s^{\prime}, s\right) p_{s^{\prime}}^{\infty}=\sum_{s^{\prime} \neq s} \hat{P}\left(s^{\prime}, s\right)$. The constraint 66) is immediately obtained from the last observation and Theorem (5)

The others inequalities constraints can be derived using the same proof sketch as in Theorem 6

If the state-based cost is not of interest, then a feasibility program with the constraints from (57) to (61) or from 63) to 67) can be solved instead.

## E. Stability Guarantee With Imperfect Model Knowledge

In many cases, the true $\operatorname{MDP} \mathcal{M}$ that governs the switch dynamics may not be precisely known but can only be estimated through statistical experiments. As a result, it may only be possible to obtain an approximated model $\overline{\mathcal{M}}$ such that the transition probabilities in $\mathcal{M}$ is known to lie in some neighborhood of that in $\overline{\mathcal{M}}$. To make such notion of approximation more precise, we first define $\Delta$-approximation in MDPs [41].

Definition 8. Let $\mathcal{M}=(S, \hat{s}, \Sigma, T)$ and $\overline{\mathcal{M}}=(\bar{S}, \overline{\hat{s}}, \bar{\Sigma}, \bar{T})$ be two MDPs. $\overline{\mathcal{M}}$ is a $\Delta$-approximation of $\mathcal{M}$ if

- $S=\bar{S}, \hat{s}=\overline{\hat{s}}, \Sigma=\bar{\Sigma}$. That is, they share the same state space, initial condition and action space;
- $\left|T\left(s, \sigma, s^{\prime}\right)-\bar{T}\left(s, \sigma, s^{\prime}\right)\right| \leq \Delta$ for any $s, s^{\prime}$ and $\sigma$;
- $T\left(s, \sigma, s^{\prime}\right)>0$ if and only if $\bar{T}\left(s, \sigma, s^{\prime}\right)>0$ for any $s, s^{\prime}$ and $\sigma$.

By definition, it is not hard to see that if $\overline{\mathcal{M}}$ is a $\Delta$-approximation of $\mathcal{M}$, then with the same policy $\pi$, we have that $\mathcal{C}$, the induced Markov chain from $\mathcal{M}$ is a $\Delta$-approximation of $\overline{\mathcal{C}}$, the induced Markov chain from $\overline{\mathcal{M}}$.

When the transition probabilities of two DTMCs are close to each other, their stationary distribution is also close as shown the following theorem.
Theorem 8. 42$]$ Let $\overline{\mathcal{C}}$ and $\mathcal{C}$ be two DTMCs both with $N$ states, and transition matrices are $\bar{P}$ and $P=\bar{P}-F$, respectively. Then the stationary distribution $p^{\infty}$ and $\bar{p}^{\infty}$ satisfy

$$
\begin{equation*}
\left|p_{i}^{\infty}-\bar{p}_{i}^{\infty}\right| \leq\|F\|_{\infty} \max _{j}\left|h_{i j}^{\#}\right|, \text { for each } i \in\{1, \ldots, N\} \tag{68}
\end{equation*}
$$

where $N=|S|, p_{i}^{\infty}:=p\left(s_{i}\right)^{\infty}, h_{i j}^{\#}$ is the $(i, j)$ entry of a matrix $H^{\#}$ which is the group inverse of $H=I-\bar{P}$.

The group inverse $H^{\#}$ of a matrix $H$, if it exists, can be uniquely determined by three equations $H H^{\#} H=$ $H, H^{\#} H H^{\#}=H^{\#}$, and $H H^{\#}=H^{\#} H$ [42]. For a transition matrix $P, H^{\#}$ always exists and can be computed as the following [43]. We can write $H$ as

$$
H=\left[\begin{array}{ll}
U & c \\
d^{\prime} & \alpha
\end{array}\right]
$$

where $U \in \mathbb{R}^{(N-1) \times(N-1)}$, and

$$
h^{\prime}=d^{\prime} U^{-1}, \delta=-h^{\prime} U^{-1} 1_{N-1}, \beta=1-h^{\prime} 1_{N-1}, G=U^{-1}-\frac{\delta}{\beta} I
$$

Here, $\delta$ and $\beta$ are nonzero scalars. Then $H^{\#}$ can be calculated by

$$
H^{\#}=\left[\begin{array}{cc}
U^{-1}+\frac{U^{-1} 1_{N-1} h^{\prime} U^{-1}}{\delta}-\frac{G 1_{N-1} h^{\prime} G}{\delta} & -\frac{G 1_{N-1}}{\beta}  \tag{69}\\
\frac{h^{\prime} G}{\beta} & \frac{\delta}{\beta^{2}}
\end{array}\right]
$$

The following lemma is a direct application of Theorem 8 and the definition of $\Delta$-approximation.
Lemma 1. Given two DTMCs $\mathcal{C}$ and $\overline{\mathcal{C}}$ with $N$ states, transition matrices $P$ and $\bar{P}$, respectively. If $\overline{\mathcal{C}}$ is a $\Delta$-approximation of $\mathcal{C}$, then their stationary distribution $p^{\infty}$ and $\bar{p}^{\infty}$ satisfy

$$
\begin{equation*}
\left|p_{i}^{\infty}-\bar{p}_{i}^{\infty}\right| \leq N \Delta \max _{j}\left|h_{i j}^{\#}\right| \tag{70}
\end{equation*}
$$

Suppose we have a $\Delta$-approximation $\overline{\mathcal{M}}$ of the underlying MDP $\mathcal{M}$. Since it is not possible to know $\mathcal{M}$ precisely, we can only find some policy $\pi$ based on the estimated model $\overline{\mathcal{M}}$ and apply it to the true MDP $\mathcal{M}$. As illustrated in Theorem 4. to be able to guarantee the stability of the system, we have to bound the difference between $P_{j u m p}$ and $\bar{P}_{j u m p}$, which are probabilities of mode jumps in the $\mathcal{M}$ and $\overline{\mathcal{M}}$, respectively.

Recall that $P_{j u m p}=1-\sum_{i} p_{i}^{\infty} P_{i i}$ and $\bar{P}_{j u m p}=1-$ $\sum_{i} \bar{p}_{i}^{\infty} \bar{P}_{i i}$. Let $P_{i i}=\bar{P}_{i i}+\delta_{i}$ and $p_{i}^{\infty}=\bar{p}_{i}^{\infty}+\epsilon_{i}$. Once we
have the estimated model $\overline{\mathcal{M}}$ which is a $\Delta$-approximation of $\mathcal{M}$, given any policy $\pi$, it is possible to obtain the transition matrix $\bar{P}$ and the stationary distribution $\bar{p}^{\infty}$. Furthermore, by the definition of $\Delta$-approximation and Lemma 11 we know that

$$
\begin{align*}
-\Delta & \leq \delta_{i} \leq \Delta, \text { and } \\
-N \Delta \max _{j}\left|h_{i j}^{\#}\right| & \leq \epsilon_{i} \leq N \Delta \max _{j}\left|h_{i j}^{\#}\right| . \tag{71}
\end{align*}
$$

Then we have

$$
\begin{align*}
& P_{j u m p}-\bar{P}_{j u m p}=\sum_{i} p_{i}^{\infty} P_{i i}-\bar{p}_{i}^{\infty} \bar{P}_{i i} \\
& =\sum_{i}\left(\bar{p}_{i}^{\infty}+\epsilon_{i}\right)\left(\bar{P}_{i i}+\delta_{i}\right)-\bar{p}_{i}^{\infty} \bar{P}_{i i}  \tag{72}\\
& =\sum_{i} \bar{p}_{i}^{\infty} \delta_{i}+\bar{P}_{i i} \epsilon_{i}+\epsilon_{i} \delta_{i} \\
& \leq \sum_{i} \bar{p}_{i}^{\infty} \Delta+\bar{P}_{i i} N \Delta \max _{j}\left|h_{i j}^{\#}\right|+N \Delta^{2} \max _{j}\left|h_{i j}^{\#}\right| .
\end{align*}
$$

Therefore, we have the following theorem to guarantee the policy we find from the estimated model $\overline{\mathcal{M}}$ is able to stabilize the switched system whose switching is governed by the true model $\mathcal{M}$.

Theorem 9. Let $\mathcal{M}=(S, \hat{s}, \Sigma, T)$ and $\overline{\mathcal{M}}=(S, \hat{s}, \Sigma, \bar{T})$ be two MDPs where $\overline{\mathcal{M}}$ is a $\Delta$-approximation of $\mathcal{M}$. A policy $\pi$ incurs two DTMCs $\mathcal{C}=(S, \hat{s}, P)$ and $\overline{\mathcal{C}}=(S, \hat{s}, \bar{P})$, respectively. Then the system is stable with probability one with the policy $\pi$ if conditions (35) and (36) are satisfied and

$$
\begin{align*}
& \bar{P}_{j u m p}+\sum_{i} \bar{p}_{i}^{\infty} \Delta+\bar{P}_{i i} N \Delta \max _{j}\left|h_{i j}^{\#}\right|+N \Delta^{2} \max _{j}\left|h_{i j}^{\#}\right| \\
&<\frac{\ln \left(\frac{1}{1-\alpha}\right)}{\ln (\mu)} \tag{73}
\end{align*}
$$

Proof. This is a direct result of Theorem 4 and equation 72 .

As a result, to synthesize a policy $\pi$ from the MDP $\overline{\mathcal{M}}$, the requirement for $\bar{P}_{\text {jump }}$ is changed to

$$
\begin{align*}
\bar{P}_{\text {jump }}< & \frac{\ln \left(\frac{1}{1-\alpha}\right)}{\ln (\mu)}-\sum_{i} \bar{p}_{i}^{\infty} \Delta  \tag{74}\\
& +\bar{P}_{i i} N \Delta \max _{j}\left|h_{i j}^{\#}\right|+N \Delta^{2} \max _{j}\left|h_{i j}^{\#}\right|
\end{align*}
$$

Similar theorem can also be derived for mode-dependent stability conditions in (43), (44) and (45). First we denote $\overrightarrow{p_{i}}$ as the probability to jump to $s_{i}$ for the estimated MDP $\mathcal{M}$ with some policy $\pi$, we observe from (33) that

$$
\begin{align*}
\overrightarrow{p_{i}}-\overrightarrow{p_{i}} & =\sum_{j \neq i} P_{j i} p_{j}^{\infty}-\bar{P}_{j i} \bar{p}_{j}^{\infty}, \\
& \leq \sum_{j \neq i}\left(\bar{P}_{j i}+\Delta\right)\left(\bar{p}_{j}^{\infty}+N \Delta \max _{j}\left|h_{i j}^{\#}\right|\right)-\bar{P}_{j i} \bar{p}_{j}^{\infty} \\
& =\sum_{j \neq i} \bar{P}_{j i} N \Delta \max _{j}\left|h_{i j}^{\#}\right|+\bar{P}_{j i} \Delta+N \Delta^{2} \max _{j}\left|h_{i j}^{\#}\right| . \tag{75}
\end{align*}
$$

We are now ready to give the following theorem similar to Theorem 9 for the mode-dependent parameters case.

Theorem 10. Let $\mathcal{M}=(S, \hat{s}, \Sigma, T)$ and $\overline{\mathcal{M}}=(S, \hat{s}, \Sigma, \bar{T})$ be two MDPs where $\overline{\mathcal{M}}$ is a $\Delta$-approximation of $\mathcal{M}$. $A$ policy $\pi$ incurs two DTMCs $\mathcal{C}=(S, \hat{s}, P)$ and $\overline{\mathcal{C}}=(S, \hat{s}, \bar{P})$, respectively. Consider an MDP-JLS whose mode $s \in S$ makes transitions following the $M D P \mathcal{M}=(S, \hat{s}, \Sigma, T)$ and the dynamics in each modes as in (1). The MDP-JLS is stable with probability one with the policy $\pi$ if conditions (43) and (44) are satisfied and

$$
\begin{array}{r}
\sum_{i}\left[\left(\overrightarrow{p_{i}}+\sum_{j \neq i} \bar{P}_{j i} N \Delta \max _{j}\left|h_{i j}^{\#}\right|+\bar{P}_{j i} \Delta+N \Delta^{2} \max _{j}\left|h_{i j}^{\#}\right|\right)\right. \\
\left.\ln \mu_{i}+\left(\bar{p}_{i}^{\infty}+N \Delta^{2} \max _{j}\left|h_{i j}^{\#}\right|\right) \ln \left(1-\alpha_{i}\right)\right]<0 \tag{76}
\end{array}
$$

Proof. This is a direct result of (45) and (75).
The counterpart of policy synthesis formulations for uncertain MDPs can be derived in a way similar to Theorem 6 and Theorem 7 but with conditions stated in Theorem 9 and Theorem 10, respectively.

## VI. Numerical Examples

We demonstrate the proposed approach on two examples: vehicle formation and transportation networks. The simulations were performed on a computer with an Intel Core i99900 K 3.60 GHz x 16 processors and 62.7 GB of RAM with MOSEK [44] as the SDP solver, GUROBI [45] as the LP solver and using the CVX [46] interface.

In each subsection, we show and compare the results of proposed methods with both mean-square stability and stability with probability one. For mean-square stability, we use both CD and SDP approaches. For the CD method, we initialize $V^{0}=I$ and $\pi^{0}$ to be uniform over all actions. CD methods could converge to a saddle point, and we add additional random term to each action uniformly selected over the interval $[-\delta, \delta]$, where $\delta>0$ is a small constant, to ensure that the procedure does not converge to a saddle point.

For stability with probability one, we apply two linear programming-based methods for mode-independent and mode-dependent coefficients. The mode-independent LP is the LP in 56 with mode-independent stability coefficients from Theorem 6 The mode-dependent LP is the LP in 62 with mode-dependent stability coefficients from Theorem 7 For the LP-based methods, $\epsilon$ is taken sufficiently small in order for the positive constraints to be satisfied. The modedependent/independent coefficients $\alpha_{s}$ and $\mu_{s}$ are generated with (53) and 55).

## A. Vehicle Formation Example

The example used in this section is adapted from [47]. We consider the vehicle formation example where the continuoustime dynamic is given by

$$
\left\{\begin{array}{l}
\dot{x}_{1}=-x_{1}+l_{13}\left(x_{3}-x_{1}\right)  \tag{77}\\
\dot{x}_{2}=l_{21}\left(x_{1}-x_{2}\right)+l_{23}\left(x_{3}-x_{2}\right) \\
\dot{x}_{3}=l_{32}\left(x_{2}-x_{3}\right)+l_{34}\left(x_{4}-x_{3}\right) \\
\dot{x}_{4}=-4 x_{4}+l_{43}\left(x_{3}-x_{4}\right)
\end{array}\right.
$$

The state $x_{i}$ represents the position of vehicle $i$ and the parameters $l_{i j}$ represent position adjustments based on distance measurements between the vehicles. We consider the discretetime version of the model (77) with sampling time $d t=0.1$.

The team of vehicle can be modeled as an MDP-JLS with three modes corresponding to three different position adjustment parameters. There are two actions that trigger transitions between modes probabilistically. We are interested in the MDP-JLS with the following characteristics.

| Mode | Dynamics | $\alpha, \mu$ |
| :---: | :--- | :--- |
| 1 | $l_{13}=l_{32}=l_{34}=3$ | $\alpha_{1}=0.21875$ |
|  | $l_{21}=l_{23}=5$ | $\mu_{1}=1.682$ |
|  | $l_{43}=2$ |  |
| 2 | $l_{13}=l_{43}=0$ | $\alpha_{2}=0.09375$ |
|  | $l_{21}=0.5$ | $\mu_{2}=1.885$ |
|  | $l_{23}=l_{32}=l_{34}=0.5$ |  |
| 3 | $l_{13}=l_{43}=1$ | $\alpha_{3}=0.21093$ |
|  | $l_{21}=l_{34}=3$ | $\mu_{3}=1.928$ |
|  | $l_{23}=l_{32}=5$ |  |

The parameters $\alpha, \mu$ for every modes have been found by solving the SDP in (53) and (55) for each modes. Moreover, we consider that the transitions between modes are governed by an MDP with two actions and the transitions probabilities given by

$$
T_{1}=\left[\begin{array}{ccc}
0.8 & 0.15 & 0.05 \\
0.03 & 0.95 & 0.02 \\
0.85 & 0.05 & 0.1
\end{array}\right] \text { and } T_{2}=\left[\begin{array}{ccc}
0.3 & 0.6 & 0.1 \\
0.9 & 0.05 & 0.05 \\
0.08 & 0.02 & 0.9
\end{array}\right]
$$

The goal is to generate a policy that guarantees the system stability with probability one and in the mean-square sense. Using the SDP, CD, the mode-independent LP, and the modedependent LP methods, we synthesize such policies and summarize the results in Fig. 2. The evolution of state $x(t)$ in Fig. 2 is of logarithmic scale to show how fast the policies generated by different methods converge.

## B. Transportation Example

Consider the linear transportation example [48] [49] given by the continuous time dynamic $\dot{x}=A x$ where

$$
A=\left[\begin{array}{cccc}
-1-l_{31} & l_{12} & 0 & 0  \tag{78}\\
0 & 2-l_{12}-l_{32} & l_{23} & 0 \\
l_{31} & l_{32} & 3-l_{23}-l_{43} & l_{34} \\
0 & 0 & l_{43} & -4-l_{34}
\end{array}\right]
$$

This model describes a transportation network connecting four buffers. The state $x$ represents the quantities of contents in the buffers and the parameter $l_{i j}$ determines the rate of transfer from buffer $j$ to buffer $i$.

We consider a discrete-time version of 78 with sample time $d t=0.1$. In particular, there are two actions that may affect the rate of transfer probabilistically which result in four different matrices $A$. This can be modeled as an MDP-JLS with transitions governed by an MDP with four discrete modes


Fig. 2: Log scale states evolution of the system w.r.t. the number of time steps for policies generated from the four methods explained in this sequel.
and two actions. The transition probabilities induced by the two actions are given by

$$
T_{1}=\left[\begin{array}{cccc}
0.1 & 0.7 & 0.1 & 0.1 \\
0.1 & 0.8 & 0.05 & 0.05 \\
0.2 & 0.6 & 0.1 & 0.1 \\
0.1 & 0.05 & 0.05 & 0.8
\end{array}\right]
$$

and

$$
T_{2}=\left[\begin{array}{cccc}
0.8 & 0.05 & 0.05 & 0.1 \\
0.3 & 0.15 & 0.4 & 0.15 \\
0.1 & 0.1 & 0.7 & 0.1 \\
0.1 & 0.7 & 0.1 & 0.1
\end{array}\right]
$$

We run the policy synthesis using the four different methods for 25 different switched linear systems with the same MDP as defined above but different continuous dynamics. The feasibility of finding a policy stabilizing each system is evaluated by the number of times the method can find a stabilizing policy and the average run time.

TABLE I: Linear transportation network results with 25 different systems. Each mode in each system has a spectral radius between $[0.63,0.98$ [ with a mean spectral radius of 0.8 .

|  | Successful cases | Mean Time |
| :---: | :---: | :---: |
| CD | 25 | 1.32 s |
| SDP | 23 | 0.044 s |
| Mode-independent LP | 2 | 0.027 s |
| Mode-dependent LP | 22 | 0.029 s |

For mean-square stability, Table $\Pi$ shows that the CD method always manages to find a policy that stabilizes the system in all cases given sufficient amount of time. The SDP approximation is not as reliable as CD, but can find a feasible policy with better performances in term of computational time. The LPbased methods may fail to find a stabilizing policy. When
they are able to find such a policy, the computational time for the task is significantly less than that of the CD-based method. The insight is that, the number of the variables of the approach based on LP only depends on the number of modes, but the number of the variables of the approach based on SDP depends on both the number of modes and the dimension of the continuous system.

For example, consider the linear transportation example (78) where instead of having just four buffers and four modes, we have 12 buffers and 8 modes. Running the CD method and the mode-dependent LP method on multiple instances of this example gives a mean execution time of 2.3 s for the CD method and 0.17 s for the LP method. If we add four more buffers ( 16 states per mode) with the same number of modes and actions, we obtain in average 0.17 s for the modedependent LP to find a feasible policy and 7.9s for the CD method. We perform the same experiment with 20 states per mode and obtain 0.17 s for the LP mean execution time and 19.1s for the CD method mean execution time. Finally, using the same settings as previously described but with 40 states per mode, the mode-dependent LP is able to find a feasible policy with an average computational time of 0.23 s when the CD method has an average of 1136.8 s . Table $\Pi$ summarizes the computation time results.

TABLE II: Evolution of mean computational time for the CD and LP (62) methods with the number of states per mode.

| Number of states | Mode-dependent LP | CD |
| :---: | :---: | :---: |
| 12 | 0.17 s | 2.3 s |
| 16 | 0.17 s | 7.9 s |
| 20 | 0.17 s | 19.1 s |
| 40 | 0.23 s | 1136.8 s |

Based on Table [I] to find a policy to guarantee stability with probability one with coefficients $\alpha_{s}$ and $\mu_{s}$, one could first try to solve the LP-based methods as they are much more faster than the other methods. If the LP methods fail to find a stabilizing policy, then one can try the CD-based method which is more reliable but could have much longer run time.

## VII. CONCLUSION

In this paper, we consider a class of switched linear systems whose mode switches are governed by a Markov decision process (MDP) and we name such systems MDP-JLS for brevity. The objective is to find a policy in the MDP to stabilize the MDP-JLS. Given a policy, an MDP reduces to a discrete-time Markov chain, and an MDP-JLS becomes a Markov jump linear system (MJLS). For mean-square stability, we leverage the existing stability conditions in MJLSs and propose semidefinite programming (SDP)-based approaches to compute the stabilizing policy. For stability with probability one, we derive new sufficient stability conditions based on which we formulate linear programs to find the stabilizing policy. We also extend the policy synthesis results to MDP-JLSs with uncertain transition probabilities and the optimization of the expected state-dependent cost. The numerical experiments validate the proposed approaches.

This paper opens the door to study a class of switched systems whose switches are governed by MDPs. For future work, we will continue to investigate how to incorporate additional temporal logic constraint on mode switches. We will also study policy synthesis with partially observable modes.

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