# An Operator-Theoretic Approach to Robust Event-Triggered Control of Network Systems with Frequency-Domain Uncertainties

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Abstract—In this paper, we study the robustness of the event-triggered consensus algorithms against frequencydomain uncertainties. It is revealed that the sampling errors resulted by event triggering are essentially images of linear finite-gain  $\mathcal{L}_2$ -stable operators acting on the consensus errors of the sampled states and the event-triggered mechanism is equivalent to a negative feedback loop introduced additionally to the feedback system. In virtue of this, the robust consensus problem of the event-triggered network systems subject to additive dynamic uncertainties and network multiplicative uncertainties are considered, respectively. In both cases, quantitative relationships among the parameters of the controllers, the Laplacian matrix of the network topology, and the robustness against aperiodic event triggering and frequency-domain uncertainties are unveiled. Furthermore, the event-triggered dynamic average consensus (DAC) problem is also investigated, wherein the sampling errors are shown to be images of nonlinear finite-gain operators. The robust performance of the proposed DAC algorithm is analyzed, which indicates that the robustness and the performance are negatively related to the eigenratio of the Laplacian matrix. Simulation examples are also provided to verify the obtained results.

Index Terms— Event-triggered control, operator theory, frequency-domain uncertainties, robust control, distributed control.

## I. INTRODUCTION

**E** Vent-triggered control, whose basic idea is to replace the continuous or periodic sampling mechanism by the aperiodic and sporadic one in the control algorithms [1], [2], [3], [4], [5], [6], originates form the aperiodic sampling problem [7] and shows great effectivity when applied in controlling continuous-time systems with digital controllers. In the last decade or so, it has been further introduced to distributed control of network systems [8], [9], [10], [11], [12], [13], [14], wherein not only the sampling mechanism but also the communicating mechanism among agents is event-based.

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Compared to the continuous or time-driven distributed control algorithms, event-triggered ones release informationexchange burden and thus have lower communication cost. Moreover, for some environments in which continuous communication is restricted, forbidden or impossible, the eventtriggered algorithms are more practical [30]. Typical works on distributed event-triggered control include [18], [20], [21], [23] for integrator networks and [35], [36], [38] for general linear systems. Reference [44] introduced the event-triggered mechanism to dynamic average consensus (DAC) algorithms. [11], [13], [14], [39], [40], [41] considered the event-triggered leader-following tracking problems and [12] further considered distributed dynamic event-triggered control for nonlinear multi-agent systems.

In the aforementioned works, the effectivity of the eventtriggered control algorithms relies on the assumption that the system dynamics are accurately known. This assumption, however, is too stringent in reality due to the ubiquitous unmodeled dynamics, the omnipresent communication constraints and the universal parametric uncertainties. Therefore, it is quite an imperative task to examine the robustness of the eventtriggered control algorithms in the presence of uncertainties. To the knowledge of the authors, there are few works along this line, except [16], [17]. Reference [16] considered the robust event-triggered stabilization problem for discrete-time systems and [17] studied the continuous-time case. These works are fairly important in the sense that they provide conditions under which the event-triggered control algorithms can still work in the presence of time-domain uncertainties.

Apart from time-domain parametric uncertainties, frequency-domain uncertainties, including unmodeled dynamics, and modeling errors, are a more general class of uncertainties [15], which may be encountered and need to be dealt with in the event-triggered control problem. In network systems, communication delays, package dropping and network uncertainties are also very prevalent phenomena and thus put forward new challenges to the distributed eventtriggered control algorithms. In virtue of these observations, in this paper, we intend to handle the robustness of distributed event-triggered control algorithms against frequency-domain uncertainties.

Distributed control algorithms with continuous and ideal communications have been proved robust to various kinds of frequency-domain uncertainties such as additive dynamic uncertainties [19], [32], network multiplicative uncertainties

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[24], [25], [26], coprime factor uncertainties [33] and so on. It is a natural question that whether distributed event-triggered algorithms also possess the robustness against frequencydomain uncertainties. To answer this question, it is necessary to adopt the frequency-domain robust control tools such as the small gain theorem and the  $\mu$  analysis. However, essentially speaking, the event-triggered algorithms belong to a special branch of the aperiodic sampling algorithms, whose definition, modeling, and methodology are all based on the time-domain analysis. More specifically, the triggering function, which decides whether the certain agent updates its state estimation and broadcasts it to its neighbors, is expressed in a timedomain form. Moreover, it characterizes a time-domain pointwise inequality constraint of the sampling error. Therefore, the analysis and design of the event-triggered control problem have almost always been based on the Lyapunov stability analysis, which is severely different from the robust analysis and synthesis tools mentioned above. In a word, because of the systematic gap between the time-domain event triggering and the frequency-domain uncertainties, the robustness of event-triggered control of network systems against frequencydomain uncertainties still remains an open and challenging problem.

To solve this problem, one of the main difficulties is how to build a bridge between the time-domain sampling mechanism and frequency-domain uncertainties, or in other words, how to 'translate' the sampling mechanism into the frequency-domain language. In this paper, we utilize the operator theory as our basic tool to unify them. By studying the frequency-domain properties of the sampling errors, we find that in classical event-triggered consensus algorithms, the sampling errors are images of the linear operators acting on the consensus error of the sampled states. It is worth noting that these linear operators are neither generally rational transfer matrices in  $\mathcal{RH}_{\infty}$  nor sector bound uncertainties of logarithmic quantizers as in [31]. Nevertheless, we can ascertain from the triggering function that the operators are finite gain  $\mathcal{L}_2$  stable. The operator gain, depending on the sampling parameters and the Laplacian matrix of the topology graph, characterizes the extent of the sampling error introduced by event triggering. In light of this, the event-triggered mechanism is equivalent to additionally introducing a negative feedback loop consisting of the linear operators.

One of the crucial advantages of the proposed operatortheoretic approach is that we can handle the robustness of the event-triggered consensus problem of network systems with respect to various kinds of frequency-domain uncertainties. In this paper, we consider additive dynamic uncertainties and network topology uncertainties for illustration. Inherent constraints on the robustness, imposed by the parameters of the controllers, the network topology, the bounds of the additive/multiplicative uncertainties, and the gains of the operators representing the event-triggered sampling, are unveiled. Moreover, the results can be extended to the event-triggered dynamic average consensus (DAC) problem, where the sampling errors are found to be images of nonlinear but finitegain operators acting on the consensus errors of the sampled states. Especially, we consider the event-triggered robust DAC problem and examine the performance of the proposed DAC algorithm under additive dynamic uncertainties. It is shown that the smaller the eigenratio is, the better the robustness and the tracking performance will be under event triggering and additive dynamic uncertainties.

The remaining part of this paper is organized as follows: In Section II, we introduce some necessary mathematical preliminaries. In Section III, we revisit the event-triggered consensus algorithm from a an operator-theoretic perspective. In Section IV, we study the robustness of the event-triggered consensus algorithms against frequency-domain uncertainties under the operator-theoretic framework. In Section V, we extend the results to the event-triggered DAC problem. Section VI provides some simulation examples for illustration and Section VII concludes this paper.

*Notations:* The notations used in this paper are fairly standard.  $\mathbb{R}^{n \times m}$  denotes the linear space of all  $n \times m$ -dimensional matrices.  $I_N$  represents the N-dimensional identity matrix and  $1_N$  represents the N-dimensional vector whose elements are all equal to 1. diag $\{a_1, \dots, a_n\}$  denotes the diagonal matrix whose diagonal elements are equal to  $a_1, \dots, a_n$ . The set of all real rational stable transfer matrices is denoted by  $\mathcal{RH}_{\infty}$ .

## **II. MATHEMATICAL PRELIMINARIES**

This section reviews some useful results and conclusions from the operator theory in Subsection II-A, from the robust control theory in Subsection II-B and from the graph theory in Subsection II-C, respectively.

## A. Operator Theory and Hilbert Space

*Definition 1:* [29] Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two Banach spaces. An operator  $\phi(\cdot) : \mathcal{H}_1 \mapsto \mathcal{H}_2$  is called a linear operator if the following two conditions hold:

1) 
$$\phi(x+y) = \phi(x) + \phi(y)$$
, for  $\forall x, y \in \mathcal{H}_1$ ;

2)  $\phi(\lambda x) = \lambda \phi(x)$ , for  $\forall \lambda \in \mathbf{R}$ .

Definition 2: The  $\mathcal{L}_2$  norm of a signal f is defined as

$$\|f\|_2 = \sqrt{\int_0^\infty f^*(t)f(t)dt} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^\infty f^*(j\omega)f(j\omega)d\omega},$$

where  $\mathcal{L}_2$  denotes the Banach space with  $\mathcal{L}_2$  norm well defined.

Definition 3: [29] Letting  $\phi(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  be an operator such that for  $\forall x \in \mathcal{L}_2$ ,  $\|\phi(x)\|_2 \leq \gamma \|x\|_2 + \beta$ , where  $\gamma > 0$ and  $\beta > 0$  are positive constants, then this operator is called a finite-gain operator with operator norm  $\|\phi\|_{\infty} \leq \gamma$ .

Lemma 1: [29] Let  $\phi(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  denote a linear (finite-gain) operator in the time domain and suppose that  $y(t) = \phi(x(t))$ , where y(t) and x(t) are vectors in the  $\mathcal{L}_2$  space. Denote by y(s) and x(s) the Laplace transformation of y(t) and x(t). It then follows that  $y(s) = \Delta(x(s))$ , where  $\Delta(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  denotes a linear (finite-gain) operator in the frequency domain.

#### B. Robust Control Theory

*Lemma 2:* [29] (Small Gain Theorem) Supposing that  $G(\cdot), \Delta(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  are finite-gain operators with operator



Fig. 1. System interconnection.

norms  $||G||_{\infty} = \gamma_1$  and  $||\Delta||_{\infty} = \gamma_2$ , the system interconnection shown in Fig. 1 is internally stable, if  $\gamma_1 \gamma_2 < 1$ .

Definition 4 ([15]): Let  $\Delta$  represent the set of structured finite-gain  $\mathcal{L}_2$ - stable operators. For  $M \in \mathbf{R}^{m \times n}$ ,  $\mu_{\Delta}(M)$  is defined as

$$\mu_{\Delta}(M) = \frac{1}{\min\{\bar{\sigma}(\Delta) : \Delta \in \Delta, \det(I - M\Delta) = 0\}}$$

unless no  $\Delta \in \mathbf{\Delta}$  makes  $\det(I - M\Delta)$  singular, in which case  $\mu_{\mathbf{\Delta}}(M) := 0.$ 

Lemma 3 ([15]): Let  $\Delta$  represent the set of structured finite-gain  $\mathcal{L}_2$ - stable operators. The loop shown in Fig. 1 is well-posed and internally stable for all  $\Delta \in \Delta$  with operator norm  $\|\Delta\|_{\infty} \leq \gamma$  if and only if  $\sup_{\omega \in \mathbf{R}} \mu_{\Delta}(G(j\omega)) < \frac{1}{\gamma}$ , where  $\mu_{\Delta}(\cdot)$  denotes the structured singular value.

Lemma 4 ([15]): Assume that  $G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$ and  $\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$  is the set of all finite-gain  $\mathcal{L}_2$  stable block diagonal operators with compatible dimensions with G(s). Then we have  $\mu_{\Delta}(G(j\omega)) \leq \sqrt{\|G_{11}(j\omega)\|^2 + \|G_{22}(j\omega)\|^2 + 2\|G_{12}(j\omega)\|\|\|G_{21}(j\omega)\|}, \\ \forall \omega \in \mathbf{R}.$  Moreover, suppose that  $\Delta = \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix}$  and  $G = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix}$ . Then,  $\mu_{\Delta}^2(G(j\omega)) \leq \|G_{11}(j\omega)\|^2 + \|G_{22}(j\omega)\|^2 + \|G_{33}(j\omega)\|^2 + 2\|G_{12}(j\omega)\|\|G_{21}(j\omega)\| + 2\|G_{13}(j\omega)\| \|G_{31}(j\omega)\| + 2\|G_{23}(j\omega)\|\|G_{32}(j\omega)\|.$ 

Lemma 5 ([15]): Let  $\gamma > 0$  and  $\mathcal{F}_u(M, \Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}$  denote the upper linear fractional transformation with respect to  $\Delta$ . For all  $\Delta \in \Delta$  with  $\|\Delta\|_{\infty} \leq \gamma$ , the transfer function  $\mathcal{F}_u(G_p, \Delta)$  is internally stable and  $\|\mathcal{F}_u(G_p, \Delta)\|_{\infty} < \frac{1}{\gamma}$  if and only if  $\sup_{\omega \in \mathbf{R}} \mu_{\Delta_p}(G_p(j\omega)) < \frac{1}{\gamma}$ , where  $\Delta_p = \begin{bmatrix} \Delta_{\Delta_f} \end{bmatrix}$ .

# C. Graph Theory

An undirected graph  $\mathcal{G}{\mathcal{V}, \mathcal{E}, \mathcal{W}}$  describes the network topology among the agents, where  $\mathcal{V} = \{1, 2, \dots, N\}$  denotes the set of vertices,  $\mathcal{E} = \{1, 2, \dots, m\}$  denotes the set of the edges, and  $\mathcal{W} = \{\dots, w_{(i,j)}, \dots\}$  or equivalently  $\{w_1, w_2, \dots, w_m\}$  denotes the set of the weights corresponding to the edges. The adjacency matrix of the graph  $\mathcal{G}$  is denoted as  $\mathcal{A}$  and  $a_{ij}$  is the (i, j)-th element of  $\mathcal{A}$  defined as  $a_{ij} = w_{ij}$  if  $(i, j) \in \mathcal{E}$  and  $a_{ij} = 0$  otherwise. Letting  $d_i = \sum_{j=1}^N a_{ij}$  be the degree of the node i and H =diag $\{d_1, \dots, d_N\}$ . The Laplacian matrix of the graph  $\mathcal{G}$  is then defined as L = H - A. Let  $D \in \mathbb{R}^{N \times m}$  be the incidence matrix of  $\mathcal{G}$  such that  $d_{ij} = -1$  if *i* is the tail of the edge (i, j)and  $d_{ij} = 1$  if *i* is the head of the edge (i, j) and  $d_{ij} = 0$ otherwise. It is easy to see that the sum of each column of Dis equal to 0.

Lemma 6: [27] For an undirected graph  $\mathcal{G}$ , the Laplacian matrix  $L = DWD^T$ , where  $W = \text{diag}\{w_1, w_2, \cdots, w_m\}$ .

# III. REVISIT OF THE EVENT-TRIGGERED ALGORITHM FROM A ROBUST CONTROL PERSPECTIVE

Consider a network consisting of N single-input-singleoutput agents with scalar states. The dynamics of each agent can be described by a single integrator:

$$\dot{x}_i = u_i, \quad i = 1, \cdots, N,\tag{1}$$

with  $x_i$  as the state variable of agent i and  $u_i$  the control input. It is assumed that the communication among the agents is depicted by a graph  $\mathcal{G}$ . The control objective of distributed algorithms is to ensure that the states of the agents reach consensus, i.e.,  $x_i - x_j \to 0$ ,  $\forall i, j \in \{1, \dots, N\}$  as  $t \to \infty$ . Throughout this paper, the following assumption holds.

Assumption 1: The communication graph G is undirected and connected.

In this paper, we consider the event-triggered mechanism. Under this mechanism, instead of continuous communication between agents, each agent (say agent *i*) only updates the estimate of its state  $(\hat{x}_i)$  to the real value of  $x_i$  and send it to its neighbors at the triggering instants, between which the estimate is calculated locally by itself and its neighbors. We set the initial time  $t_0$  as the first triggering instant of each agent and define a triggering function:

$$f_i = e_i^2 - \alpha \sum_{j=1}^N a_{ij} (\hat{x}_i - \hat{x}_j)^2 - \mu e^{-\nu t}, \qquad (2)$$

where  $\alpha, \mu, \nu$  are positive constants and  $e_i$  denotes the gap between the estimate  $\hat{x}_i$  and the real state  $x_i$ , i.e.,  $e_i = \hat{x}_i - x_i$ . The next event will happen whenever the triggering condition  $f_i \ge 0$  is satisfied, i.e.,  $t_{k+1}^i = \inf\{t | t > t_k^i, f_i \ge 0\}$ .

In this section, we consider the following distributed control law:

$$u_i = -\beta \sum_{j=1}^{N} a_{ij} (\hat{x}_i - \hat{x}_j), \quad i = 1, \cdots, N.$$
 (3)

During the time interval between two triggering instants, the estimate  $\hat{x}_i$  used by all its neighbors is held to be a constant, i.e.,  $\hat{x}_i(t) = x_i(t_k^i), \forall t \in [t_k^i, t_{k+1}^i)$ . This estimating mechanism is called zero-order holder (ZOH) [30].

The closed-loop system derived from (1) and (3) is

$$\dot{x}_i = -\beta \sum_{j=1}^N a_{ij}(\hat{x}_i - \hat{x}_j), \quad i = 1, \cdots, N.$$
 (4)

Define  $z_i = \frac{1}{N} \sum_{j=1}^{N} (\hat{x}_i - \hat{x}_j)$ . Then it follows that

$$\dot{x}_{i} = -\beta \sum_{j=1}^{N} a_{ij} \left[ (x_{i} - x_{j}) + (e_{i} - e_{j}) \right],$$

$$z_{i} = \frac{1}{N} \sum_{j=1}^{N} \left[ (x_{i} - x_{j}) + (e_{i} - e_{j}) \right].$$
(5)

We can further rewrite (5) in a compact form as

$$\dot{x} = -\beta L x - \beta L e,$$
  

$$z = M x + M e,$$
(6)

where  $M = I - \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^T$  and L is the Laplacian matrix of the graph  $\mathcal{G}$ .

It is clear from the triggering mechanism and the triggering function (2) that  $e_i$  is reset to be zero at each triggering instant and increases from 0 to some positive value during two triggering instants, and then drops again to zero at the next triggering instant. The square of the sampling error  $e_i$  is bounded from above by a quadratic form of z(t) and an exponential decaying term at any time instant. While this bound relationship is described by the time domain terminology, we discover a frequency-domain relationship between e(s) and z(s), as will be unveiled in the next theorem.

Theorem 1: For the closed-loop network (6) with the triggering function (2), it follows that  $e(s) = \Delta(z(s))$ , where  $\Delta(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  is a linear operator in the frequency domain with  $\|\Delta(z)\|_2 \leq \sqrt{2\alpha\lambda_N} \|z\|_2 + \sqrt{\frac{N\mu}{\nu}}$ , i.e.,  $\|\Delta\|_{\infty} \leq \sqrt{2\alpha\lambda_N}$ , where  $\lambda_N$  denotes the largest eigenvalue of L.

*Proof:* From (4) and the ZOH mechanism of  $\hat{x}_i$ , it is not difficult to obtain that

$$e_i(t) = \hat{x}_i(t) - x_i(t)$$
  
=  $\beta \sum_{j=1}^N a_{ij} \int_{t_k^i}^t (\hat{x}_i(\tau) - \hat{x}_j(\tau)) d\tau$   
=  $\beta L_i^T \int_{t_k^i}^t \hat{x}(\tau) d\tau$   
=  $\beta L_i^T M \int_{t_k^i}^t \hat{x}(\tau) d\tau$   
=  $\beta L_i^T \int_{t_k^i}^t 1(t - \tau) z(\tau) d\tau$ ,

where  $L_i^T$  denotes the *i*-th row of the Laplacian matrix and 1(t) = 1 when  $t \ge 0$  and 1(t) = 0 when t < 0. Therefore,  $e_i(t) = \phi_i(z)$  where  $\phi_i(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  is a linear operator in the time domain. In light of Lemma 1, we have  $e_i(s) = \Delta_i(z(s))$ , where  $\Delta_i(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  denotes a linear operator in the frequency domain and thus we have  $e(s) = \Delta(z(s))$ .

Note that

$$||e||_2^2 = \sum_{i=1}^N ||e_i||_2^2 = \sum_{i=1}^N \int_0^\infty e_i^*(t)e_i(t)dt.$$

According to the triggering condition (2), for any time instant t, we have

$$e_i^2 \le \alpha \sum_{j=1}^N a_{ij} (\hat{x}_i - \hat{x}_j)^2 + \mu e^{-\nu t}$$



Fig. 2. An operator theoretic reformulation of the distributed eventtriggered consensus algorithm.

Therefore,

$$\|e\|_{2}^{2} \leq \sum_{i=1}^{N} \int_{0}^{\infty} \left( \alpha \sum_{j=1}^{N} a_{ij} (\hat{x}_{i} - \hat{x}_{j})^{2} + \mu e^{-\nu t} \right) dt$$
  
$$= \int_{0}^{\infty} \left( \alpha \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} (\hat{x}_{i} - \hat{x}_{j})^{2} + N \mu e^{-\nu t} \right) dt$$
  
$$= \int_{0}^{\infty} \left( 2\alpha z^{T} L z + N \mu e^{-\nu t} \right) dt$$
  
$$\leq 2\alpha \lambda_{N} \|z\|_{2}^{2} + \frac{N \mu}{\nu}$$
  
$$\leq \left( \sqrt{2\alpha \lambda_{N}} \|z\|_{2} + \sqrt{\frac{N \mu}{\nu}} \right)^{2}.$$

This completes the proof.

Remark 1: The importance of this theorem lies in the following aspects. Firstly, it illustrates that the sampling errors can be seen as the images of linear finite-gain operators in the frequency domain acting on the consensus errors of the sampled states. This paves the way to examine the robustness of the event-triggered algorithm to the frequency-domain uncertainties. These operators, generally speaking, are not rational transfer matrices in  $\mathcal{RH}_{\infty}$  in the classic robust control. Nevertheless, these operators are finite-gain  $\mathcal{L}_2$  stable. Secondly, it uncovers a quantitative relationship among the sampling parameter  $\alpha$ , the largest eigenvalue of the Laplacian matrix, and the  $\mathcal{L}_2$  gain of the operators which quantifies the effect of aperiodic event-triggering. More specifically, the larger  $\lambda_N$  and  $\alpha$  are, the larger the  $\mathcal{L}_2$  gain of the operators will be.

*Remark 2:* The event-triggered consensus algorithm has an equivalent block structure shown in Fig. 2. Note that the effect of the event-triggered mechanism is actually equivalent to introducing the virtual additional feedback loop in the dotted green block. The transfer function from x to  $\hat{x}$  is  $R = (I - \Delta M)^{-1}$ , i.e.,  $\hat{x} = (I - \Delta M)^{-1}x$ . When there

is no event triggering, the operator  $\Delta$  is equal to zero and thus R = I. The event-triggered consensus algorithm (4) then reduces to the classical one with continuous communication as in [28].

In the next theorem, a frequency-domain robust control framework will be utilized to find the condition under which the network system reaches consensus by the event-triggered protocol (3).

Theorem 2: The network system (1) reaches consensus under event-triggered control law (3), if the parameter  $\alpha$  in the triggering function  $f_i$  are selected to satisfy  $2\alpha\lambda_N < 1$ . Moreover, the closed-loop system does not exhibit the Zeno behavior as long as  $\mu > 0$  and  $\nu > 0$ .

**Proof:** Letting U be the unitary matrix such that  $U^T L U = \Lambda = \text{diag}\{0, \lambda_2, \dots, \lambda_N\} = \text{diag}\{0, \bar{\Lambda}\}$ , and denoting  $\tilde{x} = U^T x$ ,  $\tilde{z} = U^T z$ ,  $\tilde{e} = U^T e$ , it is easy to find that consensus is reached if and only if  $\tilde{x}_i$  is asymptotically stable  $\forall i = 2, \dots, N$ . Setting  $\bar{x} = \tilde{x}_{2:N}$ ,  $\bar{z} = \tilde{z}_{2:N}$ ,  $\bar{e} = \tilde{e}_{2:N}$ , where  $(\cdot)_{2:N}$  denotes the subvector that takes the second to the N-th elements of the original vector. We can then derive from (5) that

$$\begin{aligned} \dot{\bar{x}} &= -\beta \bar{\Lambda} \bar{x} - \beta \bar{\Lambda} \bar{e}, \\ \bar{z} &= \bar{x} + \bar{e}. \end{aligned} \tag{7}$$

Note also that U can be written as  $\begin{bmatrix} \frac{1_N}{\sqrt{N}} & Y \end{bmatrix}$  with  $Y^T Y = I_{N-1}$  and  $YY^T = M$ . It is not difficult to see that

$$\tilde{e} = U^T \Delta (UU^T z)$$

$$= \begin{bmatrix} \frac{1^T}{\sqrt{N}} \\ Y^T \end{bmatrix} \Delta (\frac{1}{\sqrt{N}} \tilde{z}_1 + Y \bar{z})$$

$$= \begin{bmatrix} \frac{1^T}{\sqrt{N}} \\ Y^T \end{bmatrix} \Delta (Y \bar{z}),$$

where we use  $\tilde{z}_1 \equiv 0$  to get the last equality. Therefore, we have

$$\bar{e} = Y^T \Delta(Y\bar{z}) = \bar{\Delta}(\bar{z}), \tag{8}$$

where  $\overline{\Delta}$  is a linear operator, and it is easy to verify that

$$\|\bar{e}\|_{2}^{2} = \int_{0}^{\infty} \Delta^{*}(Y\bar{z}) M \Delta(Y\bar{z}) dt$$
  

$$\leq \|\Delta(Y\bar{z})\|_{2}^{2}$$
  

$$\leq \left(\sqrt{2\alpha\Lambda_{N}} \|Y\bar{z}\|_{2} + \sqrt{\frac{N\mu}{\nu}}\right)^{2}$$
  

$$= \left(\sqrt{2\alpha\lambda_{N}} \|\bar{z}\|_{2} + \sqrt{\frac{N\mu}{\nu}}\right)^{2}.$$

Therefore,

$$\|\bar{\Delta}(\bar{z})\|_2 \le \sqrt{2\alpha\lambda_N} \|\bar{z}\|_2 + \sqrt{\frac{N\mu}{\nu}},$$

which is equivalent to saying that  $\|\bar{\Delta}\|_{\infty} \leq \sqrt{2\alpha\lambda_N}$ . In light of Lemma 2, the system is internally stable if  $\|T_{\bar{e}\bar{z}}\|_{\infty} \|\bar{\Delta}\|_{\infty} < 1$ , where  $T_{\bar{e}\bar{z}}$  denotes the transfer matrix from  $\bar{e}$  to  $\bar{z}$ , calculated by

$$T_{\bar{e}\bar{z}} = -(sI + \beta\bar{\Lambda})^{-1}\beta\bar{\Lambda} + I = (sI + \beta\bar{\Lambda})^{-1}sI.$$

Note that

$$\|T_{\bar{e}\bar{z}}\|_{\infty} = \max_{i=2,\cdots,N} \left\|\frac{s}{s+\beta\lambda_i}\right\|_{\infty} = 1.$$

Therefore, the system (7) is internally stable, if  $\|\bar{\Delta}\|_{\infty} < 1$ , which is satisfied if  $2\alpha\lambda_N < 1$ .

Next, we exclude the Zeno behavior. Notice that during each time interval between any two consecutive triggering instants, i.e.,  $[t_k^i, t_{k+1}^i)$ ,  $\dot{e}_i = -\dot{x}_i = \beta \sum_{j=1}^N a_{ij}(\hat{x}_i - \hat{x}_j)$  is bounded by a positive real number, say H. Suppose that there exists Zeno behavior. Then there exists an agent i, such that  $\lim_{k\to\infty} t_k^i = T < \infty$ . Thus, for a small positive number  $\delta < \frac{\sqrt{\mu}e^{-\frac{\nu T}{2}}}{H}$ , there exists a positive integer K such that for  $\forall k \ge K$ ,  $t_k^i \in (T - \delta, T]$ . Notice that at the triggering instant  $t_K^i$ ,  $e_i = 0$ . And the next triggering time is the first time when  $e_i^2$  reaches  $\alpha \sum_{j=1}^N a_{ij}(\hat{x}_i - \hat{x}_j)^2 + \mu e^{-\nu t}$ . Then there must exist some time instant  $T_0$  when  $e_i^2(T_0) = \mu e^{-\nu T_0}$ . Since

$$e_i^2(T_0) = (x_i(t_K^i) - x_i(T_0))^2 = \left(\int_{t_K^i}^{T_0} \beta \sum_{j=1}^N (\hat{x}_i - \hat{x}_j) dt\right)^2$$

we have

$$e_i^2(T_0) \le H^2(T_0 - t_K^i)^2.$$

On the other hand,  $\mu e^{-\nu T_0} \ge \mu e^{-\nu T}$ . Thus we have

$$H^2(T_0 - t_K^i)^2 \ge \mu e^{-\nu T},$$

which implies

$$t_{K+1}^{i} - t_{K}^{i} \ge T_{0} - t_{K}^{i} \ge \frac{\sqrt{\mu}e^{-\frac{\nu T}{2}}}{H}$$

Since  $\delta < \frac{\sqrt{\mu}e^{-\frac{\nu T}{2}}}{H}$ , it follows that the K + 1-th triggering instant  $t_{K+1} > t_K^i + \delta > T$ , which leads to a contradiction.

*Remark 3:* This theorem unveils some essential requirements to achieve consensus under event-triggered protocol (3) that the gain (norm) of the operator should not be too large. The operator gain characterizes the extent of the sampling error introduced by the aperiodic event triggering. The larger the operator gain is, the larger the  $\mathcal{L}_2$  norm of the sampling errors will be. From the quantitative relationship shown in this theorem, if  $\lambda_N$  is larger, then it is more reliable to trigger more frequently in the sense that  $\alpha$  should be smaller.

*Remark 4:* Different from most of the previous works, e.g., [18], [20], [21], [22], [23], where time-domain Lyapunov stability analysis is used, in this section an operator is constructed to characterize the relationship between the sampling error e and the variable z. A robust control method based on the small gain theorem is utilized to get the consensus condition. Interestingly, since  $\lambda_N \leq 2d_i$ , the consensus condition in Theorem 2 is less conservative than that of [18]. More importantly, this method provides a feasible way to handle the robustness of event-triggered control when the network systems are subject to frequency-domain uncertainties, as will be shown in the next section.

# IV. ROBUST CONSENSUS CONTROL VIA EVENT-TRIGGERED PROTOCOLS

In the last section, we analyze the event-triggered consensus problem of the integrator network without uncertainties in a frequency domain approach. One of the major merits of this approach is that it can handle various kinds of frequencydomain uncertainties in a unified framework. In this section we take additive dynamic uncertainties and network topology uncertainties as two illustrating examples.

## A. Additive Dynamic Uncertainties

In this subsection, we consider the event-triggered consensus problem for the integrator network subject to additive dynamic uncertainties. The robust synchronization of linear multi-agent systems under such additive dynamic uncertainties with continuous communications was previously considered in [19]. Instead of measuring and exchanging the state information directly at the triggering instants, each agent can only fetch an output variable  $y_i$  that consists of the state variable  $x_i$  and the disturbance signal  $d_i$  caused by the dynamic perturbation  $\Delta_i^a$ . The agent dynamics are described by

$$\begin{aligned} \dot{x}_i &= u_i, \\ y_i &= x_i + d_i, \\ d_i(s) &= \Delta_i^a(u_i(s)), \quad i = 1, \cdots, N \end{aligned} \tag{9}$$

where  $\Delta_i^a$  is a linear finite-gain  $\mathcal{L}_2$  stable operator with operator gain  $\|\Delta_i^a\|_{\infty} \leq \eta$ . Note that this definition include transfer matrices in  $\mathcal{RH}_{\infty}$  as special cases.

We consider the following distributed output feedback protocol:

$$u_i = -\beta \sum_{j=1}^{N} a_{ij} (\hat{y}_i - \hat{y}_j), \quad i = 1, \cdots, N,$$
 (10)

where  $\hat{y}_i$  denotes the estimate of the output  $y_i$ , which is updated to the real value  $y_i(t_k^i)$  and broadcasted to all its neighbors at the k-th triggering instant of the agent *i*, i.e.,  $t_k^i$  and keeps constant (ZOH) during two triggering instants. Define  $\epsilon_i = \hat{y}_i - y_i$ . Moreover, the triggering function of agent *i* is set to be

$$f_i = \epsilon_i^2 - \alpha \sum_{j=1}^N a_{ij} (\hat{y}_i - \hat{y}_j)^2 - \mu e^{-\nu t}.$$
 (11)

According to (9) and (10), we have

$$\dot{x}_{i} = -\beta \sum_{j=1}^{N} a_{ij} \left[ (x_{i} - x_{j}) + (d_{i} - d_{j}) + (\epsilon_{i} - \epsilon_{j}) \right].$$
(12)

Denote  $w_i = \frac{1}{N} \sum_{j=1}^{N} (\hat{y}_i - \hat{y}_j)$ . We can obtain the closed-loop network dynamics as follows:

$$\dot{x} = -\beta Lx - \beta Ld - \beta L\epsilon,$$
  

$$u = -\beta Lx - \beta Ld - \beta L\epsilon,$$
  

$$w = Mx + Md + M\epsilon.$$
(13)

*Theorem 3:* For the network (9) under the control law (10) with the triggering function (11), it follows that  $\epsilon(s) =$ 



Fig. 3. Event-triggered network system with additive dynamic uncertainties.

 $\Delta^b(w(s))$ , where  $\Delta^b(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  is a linear operator with the operator gain  $\|\Delta^b\|_{\infty} \leq \sqrt{2\alpha\lambda_N}$ .

*Proof:* From the definition of  $\epsilon_i$  we know that  $\epsilon_i(t) = \hat{y}_i - y_i = y(t_k^i) - y_i(t) = x_i(t_k^i) + d_i(t_k^i) - x_i(t) - d_i(t)$ . According to (9) and (10), when  $t \in [t_k^i, t_{k+1}^i)$ ,

$$\begin{aligned} x_i(t) - x_i(t_{k_i}^i) &= -\beta \int_{t_{k_i}^i}^t \sum_{j=1}^N a_{ij}(\hat{y}_i(\tau) - \hat{y}_j(\tau)) d\tau \\ &= -\beta \int_{t_k^i}^t L_i^T \hat{y}(\tau) d\tau \\ &= -\beta L_i^T \int_{t_k^i}^t w(\tau) d\tau. \end{aligned}$$

Thus, we have  $x_i(t) - x_i(t_{k_i}^i) = \psi_i(w(t))$ , where  $\psi_i(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  is a linear operator. Since  $d_i(s) = \Delta_i^a(u_i(s))$ , we let  $d_i(t) = \tilde{\Delta}_i^a(u_i(t))$  and  $d(t_k^i) = \bar{\Delta}_i^a(u_i(t))$ , and we have that  $\tilde{\Delta}_i^a(\cdot)$  and  $\bar{\Delta}_i^a(\cdot)$  are linear operators in the time domain. It is then not difficult to find that  $d_i(t_{k_i}^i) - d_i(t)$  is a linear operator of  $u_i(t)$ .

Denote 
$$d_i(t_k^i) - d_i(t) = \phi_i(u_i(t))$$
 and note that  $u_i(t) = -\beta \sum_{j=1}^N a_{ij}(\hat{y}_i(t) - \hat{y}_j(t)) = -\beta L_i^T \hat{y}(t) = -\beta L_i^T w(t)$ . Thus  $d_i(t_k^i) - d_i(t) = \phi_i(-\beta L_i^T w(t)) = \phi_i'(w(t)),$ 

and it is evident that  $\phi'_i(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  is a linear operator. Therefore,  $\epsilon_i = -\psi_i(w(t)) + \phi'_i(w(t)) = \Psi_i(w(t))$ , where  $\Psi'_i(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  is a linear operator. According to Lemma 1,  $\epsilon_i(s) = \Delta_i^b(w(s))$ , where  $\Delta_i^b(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  is a linear operator in the frequency domain. Moreover,  $\epsilon = \Delta^b(w(s))$ , where  $\Delta : \mathcal{L}_2 \mapsto \mathcal{L}_2$  is a linear operator. The operator gain  $\|\Delta^b\|_{\infty}$  can be similarly determined as in Theorem 1 and is omitted here for brevity.

*Remark 5:* It is worth noting that two blocks of operators  $\Delta^a$  and  $\Delta^b$  appearing in Fig. 3 are essentially different. The first block of operator represents the uncertainties of the agent dynamics in  $\mathcal{RH}_{\infty}$ , possibly caused by model uncertainties, unmodeled dynamics, or nonlinear behavior of the agent itself.

The second block of operator  $\Delta^b$  is caused by the eventtriggered sampling mechanism and in general does not belong to  $\mathcal{RH}_{\infty}$ . Quite interestingly, these two blocks of operators generated by totally different mechanisms can be unified in an operator-theoretic framework, since the two blocks of uncertainties are both linear finite-gain  $\mathcal{L}_2$  stable operators in the frequency domain.

As will shown in the the next theorem, the two blocks of operators in the Fig. 3 cannot be too large in order to guarantee the robust consensus of the network system.

Theorem 4: Let  $\gamma = \max\{\eta, \sqrt{2\alpha\lambda_N}\}\)$ . The uncertain network (9) reaches robust consensus under event-triggered control law (10) with the triggering function (11), if the positive constants  $\beta$ ,  $\alpha$ ,  $\mu$ , and  $\nu$  satisfy that  $(\beta\lambda_N + 1)\gamma < 1$ . Moreover, the closed-loop system does not exhibit the Zeno behavior.

*Proof:* Notice that the interconnecting system can be rewritten into the following form:

$$\dot{x} = -\beta L x + \begin{bmatrix} -\beta L & -\beta L \end{bmatrix} \begin{bmatrix} d \\ \epsilon \end{bmatrix},$$

$$\begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} -\beta L \\ M \end{bmatrix} x + \begin{bmatrix} -\beta L & -\beta L \\ M & M \end{bmatrix} \begin{bmatrix} d \\ \epsilon \end{bmatrix}, \quad (14)$$

$$d(s) = \Delta^{a}(u(s)),$$

$$\epsilon(s) = \Delta^{b}(w(s)).$$

Here  $\Delta^a(\cdot) = \text{diag}\{\Delta_1^a, \cdots, \Delta_N^a\}$  is also a linear operator with operator gain  $\|\Delta^a\|_{\infty} \leq \eta$ . Similarly to Theorem 2, denoting  $\tilde{\epsilon} = U^T \epsilon$ ,  $\tilde{w} = U^T w$ ,  $\bar{\epsilon} = \tilde{\epsilon}_{2:N}$ ,  $\bar{w} = \tilde{w}_{2:N}$ , we can get that the system (14) reaches robust consensus if and only if the following system interconnection:

$$\dot{\bar{x}} = -\beta \bar{\Lambda} \bar{x} + \begin{bmatrix} -\beta \bar{\Lambda} & -\beta \bar{\Lambda} \end{bmatrix} \begin{bmatrix} \bar{d} \\ \bar{\epsilon} \end{bmatrix}, \begin{bmatrix} \bar{u} \\ \bar{w} \end{bmatrix} = \begin{bmatrix} -\beta \bar{\Lambda} \\ I \end{bmatrix} \bar{x} + \begin{bmatrix} -\beta \bar{\Lambda} & -\beta \bar{\Lambda} \\ I & I \end{bmatrix} \begin{bmatrix} \bar{d} \\ \bar{\epsilon} \end{bmatrix},$$
(15)  
$$\bar{d}(s) = Y^T \Delta^a (Y \bar{u}(s)) = \bar{\Delta}^a (\bar{u}(s)), \bar{\epsilon}(s) = Y^T \Delta^b (Y \bar{w}(s)) = \bar{\Delta}^b (\bar{w}(s))$$

is internally stable, where  $\bar{\Delta}^a(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  and  $\bar{\Delta}^b(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  are linear operators with operator gains  $\|\bar{\Delta}^a\|_{\infty} \leq \eta$  and  $\|\bar{\Delta}^b\|_{\infty} \leq \sqrt{2\alpha\lambda_N}$ , respectively. Note that we can further derive that

$$\begin{bmatrix} \bar{d}(s)\\ \bar{\epsilon}(s) \end{bmatrix} = \mathbf{\Delta} \begin{bmatrix} \bar{u}(s)\\ \bar{w}(s) \end{bmatrix},\tag{16}$$

where  $\Delta$  is a linear block diagonal operator defined as

$$\mathbf{\Delta} = \begin{bmatrix} \bar{\Delta}^a(\cdot) & 0(\cdot) \\ 0(\cdot) & \bar{\Delta}^b(\cdot) \end{bmatrix}$$

and  $\|\mathbf{\Delta}\|_{\infty} \leq \gamma$ . In light of Lemma 3, the system (15) reaches internal stability if  $\mu_{\mathbf{\Delta}}(G(j\omega)) < \frac{1}{\gamma}$ , where G is the transfer matrix from  $\begin{bmatrix} \overline{d}^T & \overline{\epsilon}^T \end{bmatrix}^T$  to  $\begin{bmatrix} \overline{u}^T & \overline{w}^T \end{bmatrix}^T$ . From (15), it is easy to derive that

$$G(s) = \begin{bmatrix} -\beta s \bar{\Lambda} (sI + \beta \bar{\Lambda})^{-1} & -\beta s \bar{\Lambda} (sI + \beta \bar{\Lambda})^{-1} \\ s(sI + \beta \bar{\Lambda})^{-1} & s(sI + \beta \bar{\Lambda})^{-1} \end{bmatrix}$$
$$\triangleq \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix},$$

where

$$G_{11}(j\omega)\|^{2} = \|G_{12}(j\omega)\|^{2}$$
  
=  $\|j\omega\beta\bar{\Lambda}(j\omega I + \beta\bar{\Lambda})^{-1}\|^{2}$   
=  $\max_{i=2,\cdots,N} \frac{\beta^{2}\lambda_{i}^{2}\omega^{2}}{\omega^{2} + \beta^{2}\lambda_{i}^{2}} = \frac{\beta^{2}\lambda_{N}^{2}\omega^{2}}{\omega^{2} + \beta^{2}\lambda_{N}^{2}}$ 

and

$$\|G_{22}(j\omega)\|^{2} = \|G_{21}(j\omega)\|^{2}$$
$$= \|j\omega(j\omega I + \beta\bar{\Lambda})^{-1}\|^{2} = \frac{\omega^{2}}{\omega^{2} + \beta^{2}\lambda_{2}^{2}}.$$

According to Lemma 4, the system interconnection (15) is internally stable, if  $\frac{\beta\lambda_N|\omega|}{\sqrt{\omega^2+\beta^2\lambda_N^2}} + \frac{|\omega|}{\sqrt{\omega^2+\beta^2\lambda_2^2}} < \frac{1}{\gamma}$ , which holds if  $\beta\lambda_N+1 < \frac{1}{\gamma}$ . Next, it remains to rule out the Zeno behavior. This procedure is quite similar to the counterpart of Theorem 2, and is omitted here for conciseness.

*Remark 6:* Though the time-domain sampling mechanism and the frequency-domain uncertainties are incompatible seemingly, they can indeed be tackled in a unified way under our operator-theoretic framework. This theorem quantitatively characterizes the relationship among Laplacian matrix, the output feedback gain and the robustness of the event-triggered consensus algorithm against event triggering and frequencydomain uncertainties. From this theorem, it is not difficult to find that the smaller  $\lambda_N$  and the parameter  $\beta$  are, the larger  $\gamma$ can be, meaning that the more robust the network is against frequency-domain uncertainties and sparser triggering.

*Remark 7:* To the our best knowledge, in most of the literatures in this realm, e.g., [18], [20], [21], [22], [23], frequency-domain uncertainties have not been addressed yet. The operator-theoretic approach proposed here links for the first time the time-domain sampling mechanism to the frequency-domain uncertainties, and lays the foundation of the robust analysis and synthesis of event-triggered control.

#### B. Network Topology Uncertainties

In the previous parts of this paper, it is assumed that the agents interact via a fixed and known graph  $\mathcal{G}$ . In practice, however, the topology graph may be subject to various kinds of perturbations, which will render the network graph uncertain and time-varying [24]. In this subsection, we consider a network graph with uncertain communication strengths. Specifically, each element  $a_{ij}$  in the adjacency matrix cannot be known exactly but is rather perturbed to a bounded region that can be expressed in the form of  $a_{ij}(1 + \Delta_{ij}^c(\cdot))$ , where  $\Delta_{ij}^c(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  is a linear finite-gain operator in the time domain with operator norm  $\|\Delta_{ij}^c\|_{\infty} \leq \delta < 1$ . In this case, distributed control law (3) then becomes

$$u_{i} = -\beta \sum_{j=1}^{N} a_{ij}(\hat{x}_{i} - \hat{x}_{j}) - \beta \sum_{j=1}^{N} a_{ij} \Delta_{ij}^{c}(\hat{x}_{i} - \hat{x}_{j})$$

$$= -\beta \sum_{j=1}^{N} a_{ij}(\hat{x}_{i} - \hat{x}_{j}) - \beta \sum_{j=1}^{N} (a_{ij}^{\frac{1}{2}} \Delta_{ij}^{c}(a_{ij}^{\frac{1}{2}} \hat{x}_{i}) \quad (17)$$

$$- a_{ij}^{\frac{1}{2}} \Delta_{ij}^{c}(a_{ij}^{\frac{1}{2}} \hat{x}_{j}))$$

$$= -\beta L_{i}^{T} \hat{x} - \beta D_{i} W^{\frac{1}{2}} \Delta^{c}(W^{\frac{1}{2}} D^{T} \hat{x}),$$

where  $D_i$  is the *i*-th row of the incidence matrix of the graph  $\mathcal{G}$ and  $\Delta^c(\cdot)$  is the block diagonal linear finite-gain operator with  $\Delta_{ij}^c(\cdot)$  as its diagonal elements. It is easy to see that  $\|\Delta^c\|_{\infty} \leq \delta$ . Letting  $v = \Delta^c(W^{\frac{1}{2}}D^T\hat{x})$  and  $w = W^{\frac{1}{2}}D^T\hat{x}$ , we have  $v = \Delta^c(w)$ . According to Lemma 1,  $v(s) = \overline{\Delta}^c(w(s))$ . Define the sampling error  $e_i = \hat{x}_i - x_i$ . The triggering function of the agent *i* is

$$f_i = e_i^2 - \alpha \sum_{j=1}^N a_{ij} (\hat{x}_i - \hat{x}_j)^2 - \mu e^{-\nu t}.$$
 (18)

Recalling (1), then the system interconnection can be rewritten in the following compact form:

$$\begin{aligned} \dot{x} &= -\beta Lx - \beta Le - \beta DW^{\frac{1}{2}}v, \\ w &= W^{\frac{1}{2}}D^{T}x + W^{\frac{1}{2}}D^{T}e, \\ z &= Mx + Me, \\ v(s) &= \bar{\Delta}^{c}(w(s)). \end{aligned}$$
(19)

Similarly, we have

$$\begin{split} e_{i}(t) = &\beta \int_{t_{k}^{i}}^{t} L_{i}^{T} \hat{x}(\tau) d\tau + \beta \int_{t_{k}^{i}}^{t} D_{i} W^{\frac{1}{2}} \Delta^{c} (W^{\frac{1}{2}} D^{T} \hat{x}(\tau)) d\tau \\ = &\beta \int_{t_{k}^{i}}^{t} D_{i} W z(\tau) d\tau + \beta \int_{t_{k}^{i}}^{t} D_{i} W^{\frac{1}{2}} \Delta^{c} (W^{\frac{1}{2}} z(\tau)) d\tau \\ = &\Delta^{d} (z(t)), \end{split}$$

where  $\Delta^d(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  is a linear operator in the time domain, following the proof of Theorem 1. In virtue of Lemma 1,  $e_i(s) = \overline{\Delta}^d(z(s))$ , where  $\overline{\Delta}^d(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  is a linear operator in the frequency domain. Then, in light of Theorem 1, it is easy to see that  $\|\overline{\Delta}\|_{\infty}^d \leq \sqrt{2\alpha\lambda_N}$ . Moreover,

$$\begin{split} \|\bar{\Delta}^{c}\|_{\infty}^{2} &= \sup_{\|w\|_{2}=1} \frac{1}{2\pi} \int_{-\infty}^{\infty} v^{*}(j\omega)v(j\omega)d\omega \\ &= \sup_{\|w\|_{2}=1} \int_{0}^{\infty} v^{*}(t)v(t)dt \\ &= \|\Delta^{c}\|_{\infty}^{2} \leq \delta^{2}. \end{split}$$

We have  $\|\bar{\Delta}^c\|_{\infty} \leq \delta$ .

Theorem 5: The network (1) reaches consensus under the event-triggered control law (17) with the triggering function (18), if  $\gamma < \sqrt{\frac{\lambda_2^2}{\lambda_2^2 + \lambda_N^2}}$  where  $\gamma = \max\{\delta, \sqrt{2\alpha\lambda_N}\}$ . Moreover, the closed-loop system does not exhibit the Zeno behavior.

*Proof:* Letting  $\xi = Mx$ , the system interconnection can be written in the following form:

$$\dot{\xi} = -\beta L \xi + \begin{bmatrix} -\beta D W^{\frac{1}{2}} & -\beta L \end{bmatrix} \begin{bmatrix} v \\ e \end{bmatrix}, \\ \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} W^{\frac{1}{2}} D^T \\ M \end{bmatrix} \xi + \begin{bmatrix} 0 & W^{\frac{1}{2}} D^T \\ 0 & M \end{bmatrix} \begin{bmatrix} v \\ e \end{bmatrix}, \qquad (20) \\ \begin{bmatrix} v(s) \\ e(s) \end{bmatrix} = \begin{bmatrix} \bar{\Delta}^c \\ & \bar{\Delta}^d \end{bmatrix} \begin{bmatrix} w(s) \\ z(s) \end{bmatrix}.$$

It is also worth noting that the state x of the system reaches consensus if and only if  $\xi$  is asymptotically stable, i.e., the system interconnection (20) is internally stable. According to Lemma 3, it is enough to let  $\mu_{\Delta}(G(j\omega)) < \frac{1}{\gamma}$ , where G is the transfer matrix from  $\begin{bmatrix} v^T & e^T \end{bmatrix}^T$  to  $\begin{bmatrix} w^T & z^T \end{bmatrix}^T$  and

$$G \triangleq \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$$
$$= \begin{bmatrix} -\beta W^{\frac{1}{2}} D^{T} (sI + \beta L)^{-1} DW^{\frac{1}{2}} & sW^{\frac{1}{2}} D^{T} (sI + \beta L)^{-1} \\ -\beta M (sI + \beta L)^{-1} DW^{\frac{1}{2}} & sM (sI + \beta L)^{-1} \end{bmatrix}$$

Note that

$$\begin{split} \|G_{11}(j\omega)\|^2 &= \rho(\beta^2 W^{\frac{1}{2}} D^T (-j\omega I + \beta L)^{-1} L(j\omega I + \beta L)^{-1} DW^{\frac{1}{2}}) \\ &= \rho(\beta^2 (-j\omega I + \beta \Lambda)^{-1} \Lambda (j\omega I + \beta \Lambda)^{-1} \Lambda) \\ &= \frac{\beta^2 \lambda_N^2}{\omega^2 + \beta^2 \lambda_N^2} \le \frac{\beta^2 \lambda_N^2}{\omega^2 + \beta^2 \lambda_2^2}. \end{split}$$

Similarly,

$$\|G_{12}(j\omega)\|^2 \le \frac{\omega^2 \lambda_N}{\omega^2 + \beta^2 \lambda_2^2},$$
$$\|G_{21}(j\omega)\|^2 = \max_{i=2,\cdots,N} \frac{\beta^2 \lambda_i}{\omega^2 + \beta^2 \lambda_i^2} \le \frac{\beta^2 \lambda_N}{\omega^2 + \beta^2 \lambda_2^2}.$$

and

$$|G_{22}(j\omega)||^2 = \frac{\omega^2}{\omega^2 + \beta^2 \lambda_2^2}.$$

According to Lemma 4, the system interconnection is internally stable, if

$$\frac{(\beta\lambda_N + |\omega|)^2}{\beta^2\lambda_2^2 + \omega^2} < \frac{1}{\gamma^2}, \quad \forall \omega \in \mathbf{R}.$$
 (21)

It is not difficult to find that a sufficient condition for (21) to hold is  $\gamma < \sqrt{\frac{\lambda_2^2}{\lambda_2^2 + \lambda_N^2}}$ . The excluding of the Zeno behavior is quite similar to the counterpart of Theorem 2 and is omitted here for brevity.

*Remark 8:* Different from the additive dynamic uncertainty case considered in the previous subsection, the robustness of the event-triggered consensus algorithm against the network topology uncertainties is closely related to the eigenratio  $\frac{\lambda_N}{\lambda_2}$  of the Laplacian matrix. When the eigenratio  $\frac{\lambda_N}{\lambda_2}$  is smaller, the robust margin  $\gamma$  is larger in the sense that the network can tolerant larger network uncertainties and larger sampling errors.

*Remark 9:* Note that the uncertainties of the network topology can be either time-domain or frequency-domain, which are equivalent according to Lemma 1. Actually, it is not necessary for the uncertainties to be in  $\mathcal{RH}_{\infty}$ . As long as the uncertainties are finite-gain  $\mathcal{L}_2$ -stable operators, e.g., bounded communication delays or nonlinearities, the robust event-triggered consensus problem can be solved in our operator-theoretic framework.

#### V. EXTENSION TO DYNAMIC AVERAGE CONSENSUS

In the previous sections, the control objective is to reach static average consensus, i.e., each state variable  $x_i$  converges to the average of the initial states of all agents. In this section, we generalize the results to the event-based dynamic average consensus (DAC) problem. Different from the static average consensus problem, the DAC problem aims to make the state

of each agent converge to the average of the reference signals [43]. Till now, there have been a lot of DAC algorithms proposed in the literatures for first-order network systems, e.g., [42], [43]. An event-triggered DAC algorithm is provided in [44], where the agent model is assumed to be be nominal and there are no uncertainties. Here, we consider the following network subject to additive dynamic uncertainties:

$$x_{i} = u_{i} + r_{i}, 
 y_{i} = x_{i} + d_{i}, 
 d_{i}(s) = \Delta_{i}^{a}(u_{i}(s)), \quad i = 1, \cdots, N,$$
(22)

where  $r_i$  denote the reference signals and  $\Delta_i^a$  is defined as in (9). The objective of the robust DAC problem considered in this section is to ensure that for the uncertain agents in (22),  $x_i \rightarrow \frac{1}{N} \sum_{i=1}^{N} r_i, \forall i = 1, \dots, N$  as  $t \rightarrow \infty$ . Throughout this section, we suppose that the following assumption holds.

Assumption 2: The reference signals  $r_i$  are bounded.

To achieve DAC, the following event-triggered algorithm is utilized:

$$\dot{w}_{i} = -\theta(w_{i} - y_{i}) - \beta \sum_{j=1}^{N} a_{ij}(\hat{w}_{i} - \hat{w}_{j}),$$

$$u_{i} = -\beta \sum_{j=1}^{N} a_{ij}(\hat{w}_{i} - \hat{w}_{j}),$$
(23)

where  $w_i$  is the augmented state variable. To save the communication cost, the event-triggered scheme is used. At the *k*-th triggering instant  $t_k^i$  of the agent *i*, the *i*-th agent updates its estimate of its own states  $\hat{w}_i$  to  $w(t_k^i)$  and its estimate of its own output  $\hat{y}_i$  to  $y(t_k^i)$  and broadcasts them to all its neighbors. During two triggering instants of the agent *i*,  $\hat{w}_i$  is calculated in the following way:

$$\hat{w}_i(t) = e^{-\theta(t-t_k^i)}w(t_k^i) + \int_{t_k^i}^t e^{-\theta(t-\tau)}\theta\hat{y}_i(\tau)d\tau,$$

and

$$\hat{y}_i(t) = y(t_k^i).$$

The triggering function of the agent i is given by

$$f_i = \epsilon_i^2 - \alpha \sum_{j=1}^N a_{ij} (\hat{w}_i - \hat{w}_j)^2 - \mu e^{-\nu t}, \qquad (24)$$

where  $\epsilon_i = \hat{w}_i - w_i$  denotes the sampling error. Define  $z_i = \frac{1}{N} \sum_{j=1}^{N} (\hat{w}_i - \hat{w}_j), \ z = \begin{bmatrix} z_1 & \cdots, z_N \end{bmatrix}^T$  and  $\epsilon = \begin{bmatrix} \epsilon_1 & \cdots, \epsilon_N \end{bmatrix}^T$ . It then follows that  $z = M\hat{w}$ . Similarly, we have the following claim about the relationship between  $\epsilon$  and z.

Theorem 6: For the triggering function (24), it follows that  $\epsilon(s) = \Delta^e(z(s))$ , where  $\Delta^e(\cdot)$  is a nonlinear finite-gain  $\mathcal{L}_2$  stable operator with the operator norm  $\|\Delta^e\|_{\infty} \leq \sqrt{2\alpha\lambda_N}$ .

*Proof:* It follows from (23) that for  $t \in [t_k^i, t_{k+1}^i)$ ,

$$w_i(t) = e^{-\theta(t-t_k^i)} w(t_k^i) + \int_{t_k^i}^t e^{-\theta(t-\tau)} \theta y_i(\tau) d\tau$$
$$- \int_{t_k^i}^t e^{-\theta(t-\tau)} \beta \sum_{j=1}^N a_{ij}(\hat{w}_i(\tau) - \hat{w}_j(\tau)) d\tau.$$

Therefore,

$$\begin{split} \epsilon_{i} = & \hat{w}_{i} - w_{i} \\ = & \int_{t_{k}^{i}}^{t} e^{-\theta(t-\tau)} \beta \sum_{j=1}^{N} a_{ij} (\hat{w}_{i}(\tau) - \hat{w}_{j}(\tau)) d\tau \\ & + \int_{t_{k}^{i}}^{t} e^{-\theta(t-\tau)} \theta(\hat{y}_{i}(\tau) - y_{i}(\tau)) d\tau \\ = & \int_{t_{k}^{i}}^{t} e^{-\theta(t-\tau)} \theta((\hat{y}_{i}(\tau) - y_{i}(\tau)) - (\hat{r}_{i}(\tau) - r_{i}(\tau))) d\tau \\ & + \int_{t_{k}^{i}}^{t} e^{-\theta(t-\tau)} \theta(\hat{r}_{i}(\tau) - r_{i}(\tau)) d\tau \\ & + \int_{t_{k}^{i}}^{t} e^{-\theta(t-\tau)} \beta \sum_{j=1}^{N} a_{ij} (\hat{w}_{i}(\tau) - \hat{w}_{j}(\tau)) d\tau, \end{split}$$

where  $\hat{r}_i(t) = r_i(t_k^i), \forall t \in [t_k^i, t_{k+1}^i)$ . On the other hand,

$$\begin{aligned} \hat{y}_i - y_i &= x(t_k^i) + d(t_k^i) - (x(t) + d(t)) \\ &= d(t_k^i) - d(t) + \beta \int_{t_k^i}^t \sum_{j=1}^N a_{ij}(\hat{w}_i - \hat{w}_j) d\tau \\ &- r_i(t) + r_i(t_k^i). \end{aligned}$$

Since  $d_i(s) = \Delta_i^a(u_i(s))$ , we have  $d(t) = \tilde{\Delta}_i^a(u_i(t))$ and  $d(t_k^i) = \bar{\Delta}_i^a(u_i(t))$ , where  $\tilde{\Delta}_i^a(\cdot)$  and  $\bar{\Delta}_i^a(\cdot)$  are linear operators in the time domain. Notice that

$$\begin{aligned} d_i(t_k^i) - d_i(t) &= \Delta_i^a(u_i(t)) - \Delta_i^a(u_i(t)) \\ &= \Theta_i(u_i(t)) \\ &= \Theta_i(-\beta \sum_{j=1}^N a_{ij}(\hat{w}_i - \hat{w}_j)) \\ &= \Xi_i(z(t)) \end{aligned}$$

where  $\Theta, \Xi(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  are linear operators. Therefore,

$$\hat{y}_{i} - y_{i} - (\hat{r}_{i} - r_{i}) = \beta \int_{t_{k}^{i}}^{t} \sum_{j=1}^{N} a_{ij}(\hat{w}_{i} - \hat{w}_{j}) d\tau + d(t_{k_{i}}^{i}) - d(t) = \beta L_{i}^{T} \int_{t_{k}^{i}}^{t} z(\tau) d\tau + \Xi_{i}(z(t)) = \Phi_{i}(z(t)),$$

where  $\Phi_i(\cdot): \mathcal{L}_2 \mapsto \mathcal{L}_2$  is a linear operator. Note also that

$$\begin{split} \int_{t_k^i}^t e^{-\theta(t-\tau)} \beta \sum_{j=1}^N a_{ij}(\hat{w}_i(\tau) - \hat{w}_j(\tau)) d\tau \\ &= \beta L_i^T \int_{t_k^i}^t e^{-\theta(t-\tau)} z(\tau) d\tau = \Gamma_i(z(t)) \end{split}$$

where  $\Gamma_i(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  is a linear operator. We then have

$$\begin{aligned} \epsilon_i(t) &= \int_{t_k^i}^t e^{-\theta(t-\tau)} \theta \Phi_i(z(\tau)) d\tau + \Gamma_i(z(t)) \\ &+ \int_{t_{k_i}^i}^t e^{-\theta(t-\tau)} \theta(\hat{r}_i(\tau) - r_i(\tau)) d\tau \\ &= \Psi_i(z(t)) + \int_{t_k^i}^t e^{-\theta(t-\tau)} \theta(\hat{r}_i(\tau) - r_i(\tau)) d\tau, \end{aligned}$$

where  $\Psi_i(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  is a linear operator. Since  $r_i$  is bounded  $\forall t > 0$ , then we have  $\epsilon_i(s) = \Delta_i^e(z(s))$  where  $\Delta_i^e(\cdot) : \mathcal{L}_2 \mapsto \mathcal{L}_2$  is a nonlinear operator in the frequency domain. It then remains to determine the operator gain  $\|\Delta^e\|_{\infty}$ . This procedure is quite similar to the counterpart of Theorem 1, and is omitted here for brevity.

*Remark 10:* Different from the static event-triggered average consensus problem in the previous sections, due to the introducing of the exogenous reference signals, the sampling errors in the DAC problem are no longer images of linear operators acting on the consensus error of the sampled states. Nevertheless, the triggering function ensures that these operators are finite-gain  $\mathcal{L}_2$  stable and thus can be handled together with the additive dynamic uncertainties using the operator-theoretic approach.

Define the DAC tracking error of the agent *i* as  $e_i = x_i - \frac{1}{N} \sum_{i=1}^{N} r_i$ . Denoting further  $p_i = x_i - r_i$ ,  $p = [p_1, \dots, p_N]^T$ ,  $e = [e_1, \dots, e_N]^T$ ,  $d = [d_1, \dots, d_N]^T$ ,  $r = [r_1, \dots, r_N]^T$ ,  $u = [u_1, \dots, u_N]^T$  and  $w = [w_1, \dots, w_N]^T$ , the DAC system can then be rewritten in a compact form as follows:

$$\begin{bmatrix} \dot{p} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & -\beta L \\ \theta I & -\theta I - \beta L \end{bmatrix} \begin{bmatrix} p \\ w \end{bmatrix} + \begin{bmatrix} 0 & -\beta L & 0 \\ \theta I & -\beta L & \theta I \end{bmatrix} \begin{bmatrix} d \\ \epsilon \\ r \end{bmatrix}, \qquad (25)$$
$$\begin{bmatrix} u \\ z \\ e \end{bmatrix} = \begin{bmatrix} 0 & -\beta L \\ 0 & M \\ I & 0 \end{bmatrix} \begin{bmatrix} p \\ w \end{bmatrix} + \begin{bmatrix} 0 & -\beta L & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{bmatrix} \begin{bmatrix} d \\ \epsilon \\ r \end{bmatrix}.$$

Denoting  $\tilde{u} = U^T u$ ,  $\bar{u} = \tilde{u}_{2:N}$ ,  $\tilde{z} = U^T z$ ,  $\bar{z} = \tilde{z}_{2:N}$ ,  $\tilde{d} = U^T d$ ,  $\bar{d} = \tilde{d}_{2:N}$ ,  $\tilde{\epsilon} = U^T \epsilon$ ,  $\bar{\epsilon} = \tilde{\epsilon}_{2:N}$ ,  $\tilde{p} = U^T p$ ,  $\bar{p} = \tilde{p}_{2:N}$ ,  $\tilde{w} = U^T w$ ,  $\bar{w} = \tilde{w}_{2:N}$ ,  $\tilde{e} = U^T e$ ,  $\tilde{r} = U^T r$ , we have

$$\begin{bmatrix} \dot{\bar{p}} \\ \dot{\bar{w}} \end{bmatrix} = \begin{bmatrix} 0 & -\beta\bar{\Lambda} \\ \theta I & -\theta I - \beta\bar{\Lambda} \end{bmatrix} \begin{bmatrix} \bar{p} \\ \bar{w} \end{bmatrix} + \begin{bmatrix} 0 & -\beta\bar{\Lambda} & 0 \\ \theta I & -\beta\bar{\Lambda} & \theta I \end{bmatrix} \begin{bmatrix} \vec{d} \\ \bar{\epsilon} \\ \tilde{r} \end{bmatrix}, \qquad (26)$$
$$\begin{bmatrix} \bar{u} \\ \bar{z} \\ \bar{e} \end{bmatrix} = \begin{bmatrix} 0 & -\beta\bar{\Lambda} \\ 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \bar{p} \\ \bar{w} \end{bmatrix} + \begin{bmatrix} 0 & -\beta\bar{\Lambda} & 0 \\ 0 & I & 0 \\ 0 & 0 & M \end{bmatrix} \begin{bmatrix} \vec{d} \\ \bar{\epsilon} \\ \tilde{r} \end{bmatrix}.$$

Note that

$$\bar{d} = Y^T \Delta^a (Y\bar{u}) = \bar{\Delta}^a(\bar{u}),$$
  
$$\bar{\epsilon} = Y^T \Delta^e (Y\bar{z}) = \bar{\Delta}^e(\bar{z}),$$

where  $\bar{\Delta}^a, \bar{\Delta}^b : \mathcal{L}_2 \mapsto \mathcal{L}_2$  are nonlinear finite-gain operators with  $\|\bar{\Delta}^a\|_{\infty} \leq \eta$  and  $\|\bar{\Delta}^e\|_{\infty} \leq \sqrt{\alpha\lambda_N}$ . In this section, we aim to examine the robustness of the event-triggered DAC algorithm (22) and (23) under additive dynamic uncertainties, which lies in two aspects: First, when there is no external reference signal r, the states x and w in the DAC algorithm achieve robust consensus; Second, when there is a reference signal r, we want to see how small the  $H_{\infty}$  norm of the transfer function from the reference signal r to the average tracking error e, i.e.,  $||T_{re}||_{\infty}$ , could be.

We now address the first problem and assume that the reference signal r does not exist temporarily. It is obvious that x and w reach consensus if and only if  $\bar{p}$  and  $\bar{w}$  are both asymptotically stable. It is easy to derive from (26) that

$$\begin{bmatrix} d\\ \bar{\epsilon} \end{bmatrix} = \begin{bmatrix} \Delta^a & 0\\ 0 & \bar{\Delta}^e \end{bmatrix} \begin{bmatrix} \bar{u}\\ \bar{z} \end{bmatrix},$$
$$\begin{bmatrix} \bar{u}\\ \bar{z} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12}\\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} \bar{d}\\ \bar{\epsilon} \end{bmatrix},$$
(27)

where

$$G_{11} = -\theta\beta s\Lambda\Pi, \ G_{12} = -\beta(s^2 + s\theta)\Lambda\Pi,$$
  

$$G_{21} = s\theta\Pi, \ G_{22} = (s^2 + s\theta)\Pi,$$
  

$$\Pi = (s^2I + s(\theta I + \beta\bar{\Lambda}) + \theta\beta\bar{\Lambda})^{-1}.$$

Following similar lines in proving Theorem 4, we can obtain the following theorem which gives a sufficient condition for the event-triggered DAC system to reach robust consensus.

Theorem 7: Let  $\gamma = \max\{\eta, \sqrt{2\alpha\lambda_N}\}$ . The uncertain network system (22) reaches robust consensus under the event-triggered DAC algorithm (23), if  $\frac{\theta\beta\lambda_N}{\theta+\beta\lambda_N} + 1 < \frac{1}{\gamma}$ .

*Remark 11:* This theorem unveils the quantitative relationship among the robustness of the event-triggered DAC algorithm against aperiodic event triggering and additive dynamic uncertainties, the parameters  $\theta$  and  $\beta$  in the DAC algorithm and the largest eigenvalue of the Laplacian matrix.

Next, we move on to consider the DAC tracking performance, quantified by  $||T_{re}||_{\infty}$ . Given the reference signal r, the smaller the  $||T_{re}||_{\infty}$  is, the smaller the tracking error would be. From (26), we have

$$\begin{bmatrix} \vec{d} \\ \vec{\epsilon} \end{bmatrix} = \begin{bmatrix} \bar{\Delta}^a & 0 \\ 0 & \bar{\Delta}^e \end{bmatrix} \begin{bmatrix} \vec{u} \\ \vec{z} \end{bmatrix},$$
$$\begin{bmatrix} \bar{u} \\ \bar{z} \\ \vec{e} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} \vec{d} \\ \vec{\epsilon} \\ \tilde{r} \end{bmatrix},$$
(28)

where

$$G_{13} = \begin{bmatrix} 0_{N-1} & -\beta s \theta \Lambda \Pi \end{bmatrix},$$
  

$$G_{23} = \begin{bmatrix} 0_{N-1} & s \theta \Pi \end{bmatrix},$$
  

$$G_{31} = \begin{bmatrix} 0_{N-1} & -\Pi \theta \beta \bar{\Lambda} \end{bmatrix}^{T},$$
  

$$G_{32} = \begin{bmatrix} 0_{N-1} & -\Pi (s \beta \bar{\Lambda} + \theta \beta \bar{\Lambda}) \end{bmatrix},$$
  

$$G_{33} = \begin{bmatrix} M - \theta \beta \tilde{\Pi} \Lambda \end{bmatrix},$$
  

$$\tilde{\Pi} = (s^{2}I + s(\theta I + \beta \Lambda) + \theta \beta \Lambda)^{-1}.$$

Note that (28) is an upper linear fractional transformation form and the closed-loop transfer function is  $T_{\tilde{r}\tilde{e}} = \mathcal{F}_u(G, \Delta)$ . Therefore, the performance index  $||T_{re}||_{\infty} = ||T_{\tilde{r}\tilde{e}}||_{\infty} =$  $||\mathcal{F}_u(G, \Delta)||_{\infty}$ . *Theorem 8:* Suppose that the parameter  $\alpha$ ,  $\theta$ ,  $\beta$ ,  $\mu$  and  $\nu$  are chosen such that

$$\left(\frac{\theta}{\beta\lambda_2}+1\right)^2 + \left(\frac{\theta\beta\lambda_N}{\theta+\beta\lambda_N}+1\right)^2 + 2\frac{\theta\beta\lambda_N}{\theta+\beta\lambda_N} + \frac{2}{\beta\lambda_2} < \frac{1}{\gamma^2},$$
(29)

where  $\gamma = \max\{\eta, \sqrt{2\alpha\lambda_N}\}\)$ . Then the DAC tracking performance  $||T_{re}||_{\infty} \leq \frac{1}{\gamma}$  is achieved by the event-triggered protocol (23). Moreover, the closed-loop system does not exhibit the Zeno behavior.

*Proof:* In light of Lemma 5, to achieve the robust tracking performance  $||T_{re}||_{\infty} < \frac{1}{\gamma}$  for all  $\begin{bmatrix} \bar{\Delta}^{a} \\ \bar{\Delta}^{b} \end{bmatrix} \in \begin{bmatrix} \Delta_{1} \\ \Delta_{2} \end{bmatrix}$  and  $\left\| \begin{bmatrix} \bar{\Delta}^{a} \\ \bar{\Delta}^{b} \end{bmatrix} \right\|_{\infty} \leq \gamma$ , it is enough to let  $\left( \begin{bmatrix} G_{11} & G_{12} & G_{13} \end{bmatrix} \right)_{-1}$ 

$$\mu_{\tilde{\boldsymbol{\Delta}}} \left( \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \right) < \frac{1}{\gamma},$$

where

$$ilde{\mathbf{\Delta}} = egin{bmatrix} \mathbf{\Delta}_1 & & \ & \mathbf{\Delta}_2 & \ & & \mathbf{\Delta}_f \end{bmatrix}$$

It can be derived by some simple calculations that

$$\begin{split} \|G_{11}(j\omega)\|^{2} &= \frac{\omega^{2}\theta^{2}\beta^{2}\lambda_{N}^{2}}{(\omega^{2}+\theta^{2})(\omega^{2}+\beta^{2}\lambda_{N}^{2})}, \\ \|G_{12}(j\omega)\| &= \sqrt{\frac{\beta^{2}\lambda_{N}^{2}\omega^{2}}{\omega^{2}+\beta^{2}\lambda_{N}^{2}}}, \\ \|G_{21}(j\omega)\| &= \sqrt{\frac{\omega^{2}\theta^{2}}{(\omega^{2}+\theta^{2})(\omega^{2}+\beta^{2}\lambda_{2}^{2})}}, \\ \|G_{22}(j\omega)\|^{2} &= \frac{\omega^{2}}{\omega^{2}+\beta^{2}\lambda_{2}^{2}}, \\ \|G_{13}(j\omega)\| &= \sqrt{\frac{\theta^{2}\beta^{2}\lambda_{N}^{2}\omega^{2}}{(\omega^{2}+\theta^{2})(\omega^{2}+\beta^{2}\lambda_{N}^{2})}} \leq \frac{\theta\beta\lambda_{N}}{\theta+\beta\lambda_{N}} \\ \|G_{31}(j\omega)\| &= \sqrt{\frac{\theta^{2}\beta^{2}\lambda_{N}^{2}}{(\omega^{2}+\theta^{2})(\omega^{2}+\beta^{2}\lambda_{2}^{2})}} \leq 1, \\ \|G_{23}(j\omega)\| &= \sqrt{\frac{\theta^{2}\lambda_{N}^{2}}{(\omega^{2}+\theta^{2})(\omega^{2}+\beta^{2}\lambda_{2}^{2})}} \leq 1, \\ \|G_{32}(j\omega)\| &= \sqrt{\frac{\beta^{2}\lambda_{N}^{2}}{(\omega^{2}+\beta^{2}\lambda_{N}^{2}}} \leq 1, \\ \|G_{33}(j\omega)\|^{2} &= \max_{i=2,\cdots,N} \frac{\omega^{4}+(\theta+\beta\lambda_{i})^{2}\omega^{2}}{(\omega^{2}+\theta^{2})(\omega^{2}+\beta^{2}\lambda_{i}^{2})} \\ &\leq (\frac{\theta}{\beta\lambda_{2}}+1)^{2}. \end{split}$$

In light of Lemma 4, we can derive that  $||T_{re}||_{\infty} < \frac{1}{\gamma}$ , if (29) holds. The Zeno behavior can be similarly excluded as in Theorem 2 and is omitted here for brevity.

*Remark 12:* Not surprisingly, we can easily see from (29) that a necessary condition of the robust performance specification is  $\frac{\theta\beta\lambda_N}{\theta+\beta\lambda_N} + 1 < \frac{1}{\gamma}$ . This coincides with the fact that the robust performance specification requires that the robust consensus specification holds when there are no reference signals. Moreover, it can also be observed from (29) that to ensure that the event-triggered DAC algorithm has a better



Fig. 4. The network graph.

robustness and tracking performance under additive uncertainties, the proportional gain  $\theta$  should be chosen to be as small as possible. Moreover, the parameter  $\alpha$  in the event-triggered mechanism should be relatively small. This is because when  $\alpha$  increases, the gain of the operator will be larger, which will lead to larger sampling errors and thus a worse control performance.

### **VI. SIMULATION RESULTS**

In this section, simulation examples will be provided to validate the effectiveness of the theoretical results.

Consider a multi-agent system with six agents whose communication graph is shown in Fig. 4. The Laplacian matrix of the graph is

$$L = \begin{bmatrix} 3.5 & -1 & -2 & 0 & -0.5 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 \\ -2 & 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ -0.5 & 0 & -1 & -1 & 3.5 & -1 \\ 0 & -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

We randomly generate transfer matrices representing the additive dynamic uncertainties from  $\Delta_1^a$  to  $\Delta_6^a$  as follows:

$$\begin{split} \Delta_1^a &= \begin{bmatrix} -55.4 & 140.7 & | & -8.24 \\ -155.7 & -71.41 & | & -1.28 \\ \hline 3.3989 & -5.4689 & | & 0.1022 \end{bmatrix}, \\ \Delta_2^a &= \begin{bmatrix} -55.4 & 140.7 & | & -5.1520 \\ -163.7 & -39.56 & | & -9.2230 \\ \hline 2.03 & -2.14 & | & 0.2760 \end{bmatrix}, \\ \Delta_3^a &= \begin{bmatrix} -0.3 & | & -0.28 \\ \hline 0.5454 & | & 0.0460 \end{bmatrix}, \\ \Delta_4^a &= \begin{bmatrix} -5.4 & 14.7 & | & -1.24 \\ -15.7 & -1.41 & | & -0.28 \\ \hline 0.1150 & -2.4610 & | & 0.0920 \end{bmatrix}, \\ \Delta_5^a &= \begin{bmatrix} -55.4 & 140.7 & | & -3.0150 \\ -155.7 & -71.41 & | & -3.1122 \\ \hline 0.33 & -2.14 & | & 0.4133 \end{bmatrix}, \end{split}$$



Fig. 5. The evolution of the state x when  $\beta = 0.2$  under additive dynamic uncertainties.



Fig. 6. The evolution of the state x when  $\beta = 0.1$  under additive dynamic uncertainties.

$$\Delta_6^a = \begin{bmatrix} -44.4 & 140.7 & -57.4 & -0.24 \\ -19.7 & -18.41 & -6.32 & -1.28 \\ 45.70 & 29 & -130.84 & 1.16 \\ \hline 4.4563 & -10.2542 & 4.2646 & 0.2875 \end{bmatrix}$$

Note that  $\Delta_i^a \in \mathcal{RH}_\infty \ \forall i = 1, \cdots, 6 \text{ and } \|\Delta_i^a\|_\infty \leq$  $0.4654 = \eta$ . We then choose  $\alpha = 0.02$ ,  $\mu = 0.1$  and  $\nu = 5$ in the triggering function (11). It is easy to calculate that  $\gamma = \max\{\eta, \sqrt{2\alpha\lambda_6}\} = 0.4680$ . We select the controller gain  $\beta$  to be 0.2 such that the robust consensus condition in Theorem 4 is satisfied. Theorem 4 states that the network reaches consensus asymptotically under the event-triggered algorithm (3). To illustrate this, we depict the evolution of the state x in Fig. 5. Denote the consensus error  $\zeta = Mx$ . The time instant after which  $\|\zeta\| < 0.1$  is denoted as  $t_{min}$ . When  $\beta = 0.2, t_{min} = 14.7304$ . We then decrease the parameter  $\beta$  to 0.1 and let the other parameters remain unchanged. The evolution of the state x with respect to the time t is depicted in Fig. 6. When  $\beta = 0.1$ ,  $t_{min} = 26.6796$ , which is evidently larger than that when  $\beta = 0.2$ . Clearly, the convergence speed slows down with the decrease of  $\beta$ . On the other hand, if we increase the parameter  $\beta$  for too much, then in virtue of Theorem 4, the robust consensus cannot be guaranteed. For example, if we increase  $\beta$  to 1.2. Then the state x cannot reach consensus anymore. To show that the closed-loop system does not exhibit the Zeno behavior, we draw the triggering instants



Fig. 7. Triggering instants of the six agents when  $\beta = 0.1$  under additive dynamic uncertainties.



Fig. 8. The evolution of the state x under network topology uncertainties.

of the six agents when  $\beta = 0.2$  in Fig. 7.

We next move on to test the robustness against network topology uncertainties and verify the result in Theorem 5. We consider the communication topology as in Fig.4. For illustration, we randomly generate a dynamical perturbation for each edge within a  $H_{\infty}$  norm bound  $\delta = 0.1315$ , namely,  $\Delta_i^c, i = 1, \cdots, 7$ . To satisfy the condition provided in Theorem 5, we choose  $\beta = 0.08$ ,  $\alpha = 0.002$ ,  $\mu = 0.1$  and  $\nu = 5$ . It is easy to verify that  $\gamma = \max{\{\delta, \sqrt{2\alpha\lambda_6}\}} = 0.1480$  and  $\gamma < \sqrt{\frac{\lambda_2^2}{\lambda_2^2 + \lambda_6^2}}$ . The evolution of the states of the six agents is depicted in Fig.8. It is shown that the six agents reach state consensus. Furthermore, the triggering instants of the six agents are drawn in Fig.9.

To test the robust performance of the event-triggered DAC algorithm (23) in the presence of additive dynamic uncertainties, we choose  $\alpha = 0.02$ ,  $\mu = 0.1$ , and  $\nu = 5$  in the triggering function (24). The perturbations  $\Delta_i^a$  are the same as in the first case (the case when agents are subject to additive dynamic uncertainties). To satisfy the robust performance specification in Theorem 8, we can choose  $\theta = 0.25$  and  $\beta = 1.2$  in the DAC algorithm (23). Supposing that the reference signals whose average the agents need to track are  $r = [6.1 \sin(0.02t); 19.1 \cos(0.02t); 4.8 \cos(0.07t + 8); 2.2 \sin(0.06t); 1.9 \cos(0.041t)e^{-0.09t}; 2.5 \sin(0.05t)]^T$ , we can calculate the evolution of the average tracking error e.



Fig. 9. Triggering instants of the six agents under network topology uncertainties.



Fig. 10. The evolution of the DAC tracking errors e.



Fig. 11. Triggering instants of the agents from 194s to 200s.

By depicting the evolution of e, Fig. 10 illustrates that the event-triggered DAC algorithm has a pre-specified tracking performance. The triggering instants of the six agents from 194s to 200s are depicted in Fig. 11.

## **VII. CONCLUSION**

In this paper, a novel operator-theoretic approach has been put forward to study the robustness of the event-triggered consensus algorithms against frequency-domain uncertainties. By treating the event-triggered sampling mechanism as a negative feedback loop and the sampling errors resulted by event triggering as the images of the finite-gain  $\mathcal{L}_2$  stable operators, a frequency-domain analysis framework has been established. The developed approach effectively builds a bridge between the time-domain triggering mechanism and the frequency-domain uncertainties.

There are many potential extensions to this paper. Perhaps the most direct one is to consider the case where the agent dynamics are higher-order integrators or even general linear systems. It should be noted that the introduction of the system matrix A will definitely make the problem much more challenging. This is for sure an important problem we will consider in the future.

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