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Linear Quadratic Control of Positive Systems: A Projection-Based Approach

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***Abstract*— This technical note investigates the linear quadratic regulator (LQR) design for continuous-time positive linear systems. Based on positive systems theory and Lyapunov theory, the solvability and optimality of the positivity-preserving LQR problem are analyzed through the lens of optimization, and two projection theorems are derived for single-input and multi-input positive systems, respectively, which paves the way for developing a projected gradient descent (PGD) algorithm. The proposed results fill the literature gap by considering both the optimality and positivity in LQR design, and numerical simulations are provided to verify the effectiveness of the results.**

***Index Terms*— Linear quadratic regulator, nonfragile control, positive linear systems, projected gradient descent.**

I. Introduction

Positive linear systems have attracted much attention in recent years for its broad applications in biochemical engineering [1], mobile robots [2], and fault detection [3], to name just a few. The research on such type of systems can be traced back to David G. Luenberger who, for the first time, introduced the concept of positive systems in a fundamental book [4]. For many electrical or mechanical systems, the descriptor variables are often intrinsically nonnegative or positive, otherwise the controller or observer design will lose its physical attributes [5]. For example, the power level of transmitters in a communication system should always be nonnegative [6]. Meanwhile, positive systems theory has also seen applications in stochastic processes such as Markov processes [7] since probabilities are naturally nonnegative values. The emerging development of nonnegative matrices [8] and co-positive programming [9] is providing more advanced mathematical tools for the analysis and design of positive system, which identifies its uniqueness than usual systems. The research on positive linear systems mainly focuses on, positive controllability and controller design [10], [11], positive observability and observer design [12]–[14], and positive realization [15]. Moreover, positive systems theory is also used in cooperative control [16], time-delay systems [17] and epidemic spreading dynamics [18].

In recent years, the performance of positive systems has received much attention. In [19], the extension of *Bounded Real Lemma* and H_∞ performance of positive linear systems were investigated. In [20], the finite-horizon linear quadratic optimal control was analyzed using the maximum principle. In [21], the H_2 state-feedback controller design of positive linear systems was studied using linear matrix inequalities (LMIs). Moreover, as pointed out by [21], the answer to infinite-horizon LQR for positive systems is still missing. The H_2 control of positive systems was further investigated in [22] where a monotonically nondecreasing sequence of lower bounds for the optimal H_2 performance is constructed using semidefinite programmings (SDPs). In another recent paper [9], the performance of positive systems is characterized by not-self-dual cones and analyzed using co-positive programming but the numerical method is still missing. Due to the nonconvexity inherited by the positive systems [21], the progress on this challenging topic is still very limited and the problem is far from being solved. This motivates us to propose a new projection-based method in this paper, which will shed new light on solving these problems.

The nonfragility of controllers is an important topic in control systems theory [23]–[25]. Due to the inaccuracies or uncertainties in the implementation of a designed controller, there may exist variations over the gain matrix, which can also be regarded as structural uncertainty of actuators [25]. The designed controller may fail easily without considering such type of uncertainty. Thus a section of this paper is devoted to discussing the extension of the proposed optimization method to tackle gain variations. It turns out that the nonfragility of the controller can be reduced to variation over constraints in the proposed optimization framework thus easily solved.

The contributions of this note are multi-fold. First, the infinite-horizon LQR design is investigated through the lens of optimization which is a completely new attempt in positive systems theory. Second, a systematic optimization-based framework is proposed for linear quadratic control of positive LTI systems which can be readily extended to analyze other types of positive systems, such as positive delay systems and positive switched systems. The derivations may also be utilized to study robust control or filtering of positive systems. Third, through using convexification and positive systems theory, two projection theorems are provided and a tractable PGD

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algorithm is developed for numerical computation.

Compared with the LMI-based methods in [21] and [22] which investigated H_2 performance of positive linear systems, this paper proposes a new projection-based approach for the LQR design of positive systems where a local optimal solution is obtained. The proposed approach also turns out to be flexible to gain variations and is extended to design nonfragile optimal controllers. The method in [22] characterized the upper and lower bounds for the optimal H_2 performance while the local optimality is not guaranteed. Moreover, to illustrate the utility of our proposed optimization method versus the LMI-based method in [21], a numerical comparison is included in Section V in which the proposed PGD algorithm shows a better LQR performance than the algorithm in [21].

The remainder of this paper is organized as follows. In Section II, some useful results in positive systems theory are introduced and the problem to be solved by this paper is formulated. In Section III, a systematic optimization-based framework is proposed and some analyses are provided. Two projection theorems are, respectively, derived for single-input and multi-input positive systems, and the corresponding PGD algorithm is developed. In Section IV, an extension of the proposed optimization method to design nonfragile optimal controllers is discussed. In Section V, simulations are given to verify the effectiveness of the proposed results. In Section VI, the whole paper is summarized and concluded, where the potential direction for future research is also discussed.

Notations: The notations used throughout this paper are standard. The absolute value of matrix X is defined as $|X|$ such that $[|X|]_{ij} = |[X]_{ij}|$. The notation $\|v\|_p$ means the L_p norm of vector v. The L_p projection of vector $\hat{v}$ on set S is defined as $v = \arg\min_{v\in\bar{S}} \|\hat{v} - v\|_p$. The notation $\mathrm{vec}(X)$ denotes the vectorization of matrix X. To avoid matrix norms, we use vectorization for matrix projection. The notation $X \succ 0$ (or $X \succeq 0$) means matrix X is positive definite (or semidefinite). The notation $X > 0$ (or $X \geq 0$) means all elements of matrix X are positive (or nonnegative).

II. Preliminaries

In this section, some backgrounds and useful results on positive linear systems are introduced and the problem to be investigated by this paper is formulated.

A. Positive Systems Theory

Consider a continuous-time positive linear system:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where initial state $x(0) \geq 0$, $A \in \mathbb{R}^{n\times n}$ is Metzler and $B \in \mathbb{R}^{n\times m}$ is nonnegative. We assume that (A, B) is stabilizable. Without loss of generality, we assume that the initial value $x(0)$ is random (as the deterministic one is a special case) with $\mathbb{E}[x(0)x(0)^T] = \Omega \succeq 0$.

Lemma 1: [26] For Metzler matrices M_1, M_2, if $M_1 \leq M_2$, $\alpha(M_1) \leq \alpha(M_2)$ where $\alpha(\cdot)$ is the spectral abscissa.

Lemma 2: [27] Metzler matrix A is Hurwitz if and only if there exists a diagonal matrix $P \succ 0$ such that

$$A^{\mathrm{T}}P + PA \prec 0 \quad \text{or} \quad PA^{\mathrm{T}} + AP \prec 0.$$

Lemma 3: [28] Metzler matrix A is Hurwitz if and only if there exists a vector $\omega > 0$ such that $\omega^{\mathrm{T}}A < 0$ or $A\omega < 0$.

B. Problem Formulation

We consider the positive linear system in (1) with a static state-feedback controller:

$$u(t) = Kx(t) \tag{2}$$

The infinite-horizon LQR performance index of the control system is defined as

$$\mathcal{J}(K) = \mathbb{E}[\int_0^\infty x(t)^{\mathrm{T}}Qx(t) + u(t)^{\mathrm{T}}Ru(t)\ dt] \tag{3}$$

where $Q \succeq 0$ and $R \succ 0$ are, respectively, the state cost and input cost matrices, and the expectation is taken with respect to the initial values. More specifically, $x(t)^{\mathrm{T}}Qx(t)$ measures the state deviation cost at time t, and $u(t)^{\mathrm{T}}Ru(t)$ measures the input authority cost at time t. Throughout this paper, we restrict our attention to static state feedback controller for convenience which may be suboptimal due to its time-invariance. The problem to be solved is defined as follows.

Problem LQCPS (Linear Quadratic Control of Positive Systems): Consider the positive linear system in (1), design a controller gain K in (2) such that the LQR performance index $\mathcal{J}(K)$ is minimized, and meanwhile the positivity of system (1) is preserved, i.e., $x(t) \geq 0$ for $t \geq 0$.

Remark 1: The main difficulty of the above problem is to simultaneously preserve the stability and positivity as well as the optimality of the controller gain matrix. It could be very complicated to tackle the above problem from an LMI-based perspective due to the conflict between positivity constraints and algebraic Riccati equations. Hence in this paper, we will analyze **Problem LQCPS** through the lens of optimization and propose a projection-based approach to solve it using Lyapunov theory and positive systems theory.

III. Main Results

In this section, we first analyze the solvability of **Problem LQCPS** and then transform it into an optimization-based framework. To preserve the positivity and stability of the positive linear system in (1), two projection theorems are derived and a PGD algorithm is developed.

Proposition 1: **Problem LQCPS** is solvable if and only if there exists a diagonal matrix $L \succ 0$ and U such that all the following conditions are satisfied,

1) $AL + BU$ is Metzler.
2) $AL + LA^{\mathrm{T}} + BU + U^{\mathrm{T}}B^{\mathrm{T}} \prec 0$.

Proof. Since diagonal matrix $L \succ 0$, then $L^{-1} \geq 0$ and $(AL + BU)L^{-1} = A + BUL^{-1}$ is a Metzler matrix. Taking $K = UL^{-1}$, we simply have that $A + BK$ is Metzler thus the positivity of system (1) is preserved. Meanwhile pre- and post-multiply condition 2) by matrix $L^{-1} \succ 0$, we obtain that $L^{-1}A + A^{\mathrm{T}}L^{-1} + L^{-1}BUL^{-1} + L^{-1}U^{\mathrm{T}}B^{\mathrm{T}}L^{-1} = L^{-1}(A + BK) + (A + BK)^{\mathrm{T}}L^{-1} \prec 0$. By *Lemma 1*, we have that (1) is Hurwitz stable. Therefore **Problem LQCPS** is solvable. On the other hand, if **Problem LQCPS** is solved by some gain matrix K, then $A + BK$ is a Metzler matrix and system

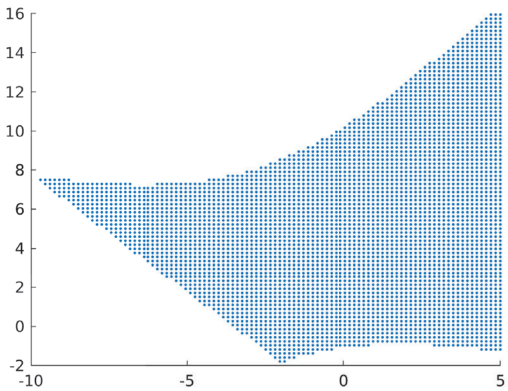

Fig. 1: An arbitrary cut plane of the set $\mathcal{L}$.

(1) is asymptotically stable. Since $A+BK$ is simultaneously Metzler and Hurwitz, by *Lemma 2*, there exists a diagonal matrix $L \succ 0$ such that $(A+BK)L + L(A+BK)^{\mathrm{T}} \prec 0$. Taking $U = KL$, it is easy to see that both conditions 1) and 2) hold. The proof is completed. $\square$

To solve **Problem LQCPS**, we have to optimize the LQR performance subject to the positivity and stability constraints, which can be well described as a constrained optimization problem. Moreover, we have the following results.

Theorem 1: **Problem LQCPS** is equivalent to the following constrained optimization:

$$\min_K \ f := \mathrm{tr}(F\Omega)$$
$$\begin{cases} A+BK \text{ is Hurwitz} \\ A+BK \text{ is Metzler} \\ (A+BK)F + F(A+BK)^{\mathrm{T}} + K^{\mathrm{T}}RK + Q = 0 \end{cases} \tag{4}$$

Proof. **Problem LQCPS** is feasible if and only if there exists a gain matrix K such that $A+BK$ is simultaneously Hurwitz and Metzler. By Bellman's lemma [29], we have that the LQR performance $\mathcal{J}(K) = \mathbb{E}[x(0)^{\mathrm{T}}Fx(0)]$ where matrix $F \succeq 0$ satisfies the algebraic equation

$$(A+BK)^{\mathrm{T}}F + F(A+BK) = -(Q + K^{\mathrm{T}}RK). \tag{5}$$

Since $A+BK$ is Hurwitz, that is, system (1) is asymptotically stable, then Eqn (5) has a unique solution $F \succeq 0$. Further notice the fact that $\mathcal{J}(K) = \mathbb{E}[x(0)^{\mathrm{T}}Fx(0)] = \mathrm{tr}(F\mathbb{E}[x(0)x(0)^{\mathrm{T}}]) = \mathrm{tr}(F\Omega)$. Hence **Problem LQCPS** is equivalent to optimization (4). The proof is completed. $\square$

Based on the above formulation, we define the feasible region for gain matrix K as

$$\mathcal{L} := \{K \mid A+BK \text{ is both Hurwitz and Metzler}\}$$

and the following example shows that $\mathcal{L}$ is in general not a convex set. Consider the positive linear system in (1) with the following system matrices:

$$A = \begin{bmatrix} -1.00 & 1.20 \\ 2.00 & 1.00 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1.00 & 0.50 \\ 0.30 & 1.00 \end{bmatrix}.$$

An arbitrary cut plane of the feasible region $\mathcal{L}$ is shown in Fig. 1, which turns out to be nonconvex. Thus the feasible region for gain matrix K is in general not a convex set.

Corollary 1: Given that **Problem LQCPS** is solvable, the feasible region $\mathcal{L}$ is a connected set.

Proof. To show the connectedness, first notice that set $\mathcal{L}$ is equal to set $\hat{\mathcal{L}} = \{K = UL^{-1} \mid \exists \epsilon, U, L \succ 0,\ AL + BU + \epsilon I \geq 0,\ AL + LA^{\mathrm{T}} + BU + U^{\mathrm{T}}B^{\mathrm{T}} \prec 0\}$. Notice that $\hat{\mathcal{L}}$ is connected since set $\{(L,U) \mid \exists \epsilon,\ AL + BU + \epsilon I \geq 0,\ AL + LA^{\mathrm{T}} + BU + U^{\mathrm{T}}B^{\mathrm{T}} \prec 0\}$ is convex, thus connected, and $\mathcal{F} : (U,L) \mapsto K = UL^{-1}$ is a continuous mapping. Hence $\mathcal{L} = \hat{\mathcal{L}}$ is a connected set. The proof is completed. $\square$

Corollary 2: The derivative of function f in K is

$$\frac{\partial f}{\partial K} = 2(RK + B^{\mathrm{T}}F)H \tag{6}$$

where H is the unique solution of

$$(A+BK)H + H(A+BK)^{\mathrm{T}} + \Omega = 0. \tag{7}$$

Proof. Notice that the incremental equation $(A+BK)^{\mathrm{T}}\partial F + \partial K^{\mathrm{T}}B^{\mathrm{T}}F + \partial F(A+BK) + FB\partial K + \partial K^{\mathrm{T}}RK + K^{\mathrm{T}}R\partial K = 0$ is equivalent to $(A+BK)^{\mathrm{T}}\partial F + \partial F(A+BK) + (FB + K^{\mathrm{T}}R)\partial K + \partial K^{\mathrm{T}}(B^{\mathrm{T}}F + RK) = 0$. Since $(A+BK)$ is Hurwitz, then the Lyapunov equation $(A+BK)H + H(A+BK)^{\mathrm{T}} + \Omega = 0$ has a unique solution $H \succeq 0$. Premultiply the incremental equation by matrix H, we obtain that $H(A+BK)^{\mathrm{T}}\partial F + H\partial F(A+BK) + H(FB + K^{\mathrm{T}}R)\partial K + H\partial K^{\mathrm{T}}(B^{\mathrm{T}}F + RK) = 0$. Take the trace and reorganize the equation, we further have that $-\mathrm{tr}(\partial F\Omega) + 2\mathrm{tr}(H(B^{\mathrm{T}}F + RK)^{\mathrm{T}}\partial K) = -\partial f + \langle 2(RK + B^{\mathrm{T}}F)H, \partial K\rangle = 0$, which implies that Eqn (6) holds. The proof is completed. $\square$

Notice that (4) is a constrained optimization with nonconvex and nonsmooth feasible region. To tackle these constraints, we further derive the following *projection theorems* for single-input and multi-input positive linear systems.

Theorem 2: If system (1) is single-input, for any matrix $\hat{K}$, its L_p projection on the feasible region $\mathcal{L}$ is given by K, which can be obtained by the following convex programming:

$$\min_{K,\eta} \ \|\mathrm{vec}(K) - \mathrm{vec}(\hat{K})\|_p$$
$$\begin{cases} \eta > 0 \\ \eta^{\mathrm{T}}B = 1 \\ A + BK + \epsilon I \geq 0 \\ \eta^{\mathrm{T}}A + K < 0 \end{cases} \tag{8}$$

Proof. Notice that there exists a scalar ϵ such that $A+BK+\epsilon I \geq 0$ if and only if $A+BK$ is a Metzler matrix. Substituting $\eta^{\mathrm{T}}B = 1$ into inequality $\eta^{\mathrm{T}}A + K < 0$, we obtain that

$$\eta^{\mathrm{T}}(A+BK) = \eta^{\mathrm{T}}A + \eta^{\mathrm{T}}BK < 0, \tag{9}$$

By *Lemma 3*, since $A+BK$ is Metzler, we have that $A+BK$ is a Hurwitz matrix. On the other hand, if matrix $A+BK$ is simultaneously Metzler and Hurwitz, then there must exist a positive vector $\hat{\eta} > 0$ such that

$$\hat{\eta}^{\mathrm{T}}A + \hat{\eta}^{\mathrm{T}}BK = \hat{\eta}^{\mathrm{T}}(A+BK) < 0. \tag{10}$$

Taking $\eta = \hat{\eta}/\hat{\eta}^{\mathrm{T}}B$, we can obtain that $\eta^{\mathrm{T}}B = 1$, meanwhile multiply Eqn (10) by scalar $1/\hat{\eta}^{\mathrm{T}}B > 0$, we have that

$$\eta^{\mathrm{T}}A + \eta^{\mathrm{T}}BK = \eta^{\mathrm{T}}A + K < 0.$$

The convexity of optimization (8) follows from the fact that its constraints are all linear and its objective function is convex for any L_p norm. The proof is completed. □

Remark 2: Notice that, the feasible region for gain matrix K is in general not a convex set as shown in Fig. 1, thus finding the projection of $\hat{K}$ on nonconvex set $\mathcal{L}$ is usually an NP-hard problem. However, for single-input positive linear systems, the above theorem shows that its L_p projection can always be obtained by convex programming with linear constraints. Based on positive systems theory and Lyapunov theory, we further derive the following projection theorem for general multi-input positive linear systems.

Theorem 3: For any matrix $\hat{K}$, its L_∞ projection on the feasible region $\mathcal{L}$ is given by $K = UL^{-1}$ which can be obtained by the following biconvex programming:

$$\min_{L,U,\epsilon,\psi} \quad \psi$$
$$\begin{cases} U - \hat{K}L \leq \psi \mathbf{1}_{m\times n} L \\ -\psi \mathbf{1}_{m\times n} L \leq U - \hat{K}L \\ L \text{ is diagonal and } L \succ 0 \\ AL + BU + \epsilon I \geq 0 \\ AL + LA^{\mathrm{T}} + BU + U^{\mathrm{T}} B^{\mathrm{T}} \prec 0 \end{cases} \tag{11}$$

Proof. Notice that there exists scalar ϵ such that $AL + BU + \epsilon I \geq 0$ if and only if $AL+BU$ is a Metzler matrix. Since diagonal matrix $L \succ 0$, thus $L > 0$, then $AL+BU$ is Metzler if and only if $A+BK$ is Metzler. By congruence transformation with L^{-1}, it is easy to see that $AL+LA^{\mathrm{T}}+BU+U^{\mathrm{T}}B^{\mathrm{T}} \prec 0$ if and only if $L^{-1}(A+BK) + (A+BK)^{\mathrm{T}} L^{-1} \prec 0$, that is, $A+BK$ is Hurwitz. Denote the L_∞ distance between K and $\hat{K}$ as $\phi := \|\mathrm{vec}(K) - \mathrm{vec}(\hat{K})\|_\infty \geq 0$. Notice that $|K - \hat{K}| \leq \phi \mathbf{1}_{m\times n}$, thus $-\phi \mathbf{1}_{m\times n} \leq K - \hat{K} \leq \mathbf{1}_{m\times n}\phi$. Moreover, $\phi = \arg\min_\phi \{\phi \in \mathbb{R} \mid -\phi \mathbf{1}_{m\times n} \leq K - \hat{K} \leq \mathbf{1}_{m\times n}\phi\}$. As $K = UL^{-1}$ where $L \succ 0$ is a diagonal matrix, equivalently we have $-\phi \mathbf{1}_{m\times n} L \leq U - \hat{K}L \leq \phi \mathbf{1}_{m\times n} L$. It is easy to see that $\phi = \psi^*$ where ψ^* is the optimal value of optimization (11). The proof is completed. □

Corollary 3: The biconvex programming in *Theorem 3* can always be solved by the bisection method.

Proof. Notice that, if the value of ψ is fixed, optimization (11) becomes a feasibility problem as follows,

$$\text{Find } L, U$$
$$\begin{cases} U - \hat{K}L \leq \psi \mathbf{1}_{m\times n} L \\ -\psi \mathbf{1}_{m\times n} L \leq U - \hat{K}L \\ L \text{ is diagonal and } L \succ 0 \\ AL + BU + \omega I \geq 0 \\ AL + LA^{\mathrm{T}} + BU + U^{\mathrm{T}} B^{\mathrm{T}} \prec 0 \end{cases} \tag{12}$$

which turns out to be linear, thus convex. Notice that, if (12) is feasible for some ψ, then it is also feasible for any $\psi^{'} > \psi$. On the other hand, if (12) is infeasible for some ψ, then it is also infeasible for any $\psi^{'} < \psi$. Hence optimization (11) can be solved by bisection method. The proof is completed. □

Remark 3: Due to the nonconvexity and nonsmooothness of the feasible region, the L_p projection for multi-input cases is very complicated. Thus here we relax solving the original L_p projection problem to solving an L_∞ projection problem, which is further equivalently transformed into the biconvex programming in *Theorem 3* which can be completely solved by the bisection method in *Corollary 3*.

Based on the above derivations, a PGD algorithm is developed in Algorithm 1 to solve **Problem LQCPS**.

Algorithm 1 LQR Design for Positive Systems

- Initialize $k = 1$, $\xi > 0$, $\tau \in (0, \frac{1}{2})$, $\theta \in (0, 1)$, $\lambda > 0$.
- Initialize the gain $K^{(1)} = UL^{-1}$ through *Proposition 1*.

repeat
  - Update step size $\lambda^{(k)} = \lambda$.
  - Determine $f^{(k)}$ and $F^{(k)}$ by solving $(A + BK^{(k)})F + F(A + BK^{(k)})^{\mathrm{T}} + K^{(k)\mathrm{T}} RK^{(k)} + Q = 0$.

  repeat
    - Determine $\hat{K}^{(k)}$ by gradient descent $\hat{K}^{(k)} = K^{(k)} + \lambda^{(k)} \mathcal{D}(K^{(k)})$ where $\mathcal{D}(K^{(k)})$ denotes the descent direction that can be determined by *Corollary 2*.
    - Determine the projection of $\hat{K}^{(k)}$ on the feasible region as $\tilde{K}^{(k)}$ through *Theorem 2* or *Corollary 3*.
    - Determine $\tilde{f}^{(k)}$ and $\tilde{F}^{(k)}$ by solving $(A+B\tilde{K}^{(k)})F + F(A + B\tilde{K}^{(k)})^{\mathrm{T}} + \tilde{K}^{(k)\mathrm{T}} R\tilde{K}^{(k)} + Q = 0$.
    - Update step size $\lambda^{(k)} = \theta\lambda^{(k)}$.

  until $\tilde{f}^{(k)} < f^{(k)} - \tau\lambda^{(k)} |\langle \mathcal{D}(K^{(k)}), \tilde{K}^{(k)} - K^{(k)} \rangle|$;
  - Update $K^{(k+1)} = \tilde{K}^{(k)}$, $f^{(k+1)} = \tilde{f}^{(k)}$, $k = k + 1$.

until $|f^{(k)} - f^{(k-1)}|/|f^{(k)}| < \xi$;
return $K^* = K^{(k)}$ and $f^* = f^{(k)}$.

Lemma 4: Algorithm 1 converges to a local optimal point.

Proof. Through backtracking linesearch, the above algorithm will generate a nonincreasing sequence $\{K^{(k)}, k \geq 1\}$ where $f(K^{(k+1)}) \leq f(K^{(k)})$ for any $k \geq 1$. Due to the boundedness of function $f(\cdot)$ over feasible domain, it is guaranteed that Algorithm 1 converges to some K^*. If K^* is an interior point, its local optimality follows easily from that $\nabla f(K^*) = 0$. Otherwise K^* exists over the Metzler constraint. Assume K^* is not local optimal, that is, there is a neighboring point K' sufficiently close to K^* such that $f(K') < f(K^*)$. By the smoothness of function $f(\cdot)$, we have that $K' - K^*$ is a descent direction thus $\langle \mathcal{D}(K^*), K' - K^* \rangle > 0$. Denote $\bar{K}$ as the projection of $K^* + \lambda\mathcal{D}(K^*)$ for a sufficiently small $\lambda > 0$, then we have $\langle \mathcal{D}(K^*), \bar{K} - K^* \rangle \geq \langle \mathcal{D}(K^*), K' - K^* \rangle > 0$. By the convexity of the Metzler constraint, we have that $\bar{K} - K^*$ is a feasible descent direction thus $f(\bar{K}) < f(K^*)$. This contradicts with the convergence of Algorithm 1 to K^*. Thus K^* is local optimal. The proof is completed. □

Remark 4: Notice the fact that strict inequalities are not executable in numerical solvers. To avoid singular cases, we can replace '$\prec 0$' (or '$\succ 0$') and '< 0' (or '> 0') by '$\preceq -\delta$' (or '$\succeq \delta$') and '$\leq -\delta$' (or '$\geq \delta$'), respectively, where δ denotes a sufficiently small positive scalar or vector.

IV. FURTHER DISCUSSIONS

This section discusses **Problem LQCPS** with nonfragile controllers. On one hand, the controller designed in this way is more robust in practical implementations. On the other hand, the following analysis may serve as an example for more extensions and variations, which also illustrates the flexibility and compatibility of the proposed optimization method.

The nonfragility of controllers is an important topic in control engineering [23]–[25]. The variation over controller gains is caused by inaccuracies or uncertainties in the implementation of a designed controller.

Here we consider interval gain variations, that is, the practical control input is in the form of

$$\tilde{u}(t) = (K + \Delta)x(t) = u(t) + \Delta x(t) \tag{13}$$

with the bounded gain variation as follows

$$-\underline{\Delta} \leq \Delta \leq \overline{\Delta} \tag{14}$$

where $\underline{\Delta}$ and $\overline{\Delta}$ are non-negative matrices with compatible dimensions. Since matrix Δ is usually inaccessible and narrow-range during the dynamic process, we use the so-called nominal LQR performance [30]–[32] for optimization. In other words, we optimize the LQR performance in (3) with nominal controller (2) and meanwhile guarantee the positivity and stability of the system with practical controller (13).

Proposition 2: Matrix $A + B(K + \Delta)$ is simultaneously Hurwitz and Metzler if and only if $A + B(K - \underline{\Delta})$ is Metzler and $A + B(K + \overline{\Delta})$ is Hurwitz.

Proof. The necessity is obvious and it suffices to show the sufficiency. If matrix $A + B(K - \underline{\Delta})$ is Metzler, as matrix B is non-negative, then $A + B(K + \Delta) \geq A + B(K - \underline{\Delta})$ is Metzler for any Δ in (14). If $A + B(K + \overline{\Delta})$ is Hurwitz, then we have $\alpha(A + B(K + \Delta)) \leq \alpha(A + B(K + \overline{\Delta})) < 0$ thus $A + B(K + \Delta)$ is Hurwitz for any Δ in (14), which follows from *Lemma 1*. The proof is completed. □

Remark 5: Based on the above proposition, we can solve the nonfragile version of **Problem LQCPS** by simply changing the constraints in optimization (4) correspondingly. Then the projection theorems and PGD algorithm proposed in Section III follows immediately in a similar way.

V. Simulations

In this section, two numerical examples are given to verify the effectiveness of Algorithm 1 for single-input and multi-input positive linear systems, respectively. Throughout this section, we simply set that $\Omega = I$.

A. Single-Input Positive Systems

Consider a positive linear system in (1) where

$$A = \begin{bmatrix} -3.00 & 1.00 & 5.00 \\ 3.00 & -1.00 & 3.00 \\ 2.00 & 1.00 & -8.00 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1.00 \\ 0.50 \\ 1.50 \end{bmatrix}$$

with

$$Q = \begin{bmatrix} 1.00 & 0.20 & 0.10 \\ 0.20 & 1.00 & 0.20 \\ 0.10 & 0.20 & 1.00 \end{bmatrix} \quad \text{and} \quad R = 2.00.$$

The gain variations are $\overline{\Delta} = \begin{bmatrix} 0.02 & 0.01 & 0.01 \end{bmatrix}$ and $\underline{\Delta} = \begin{bmatrix} 0.01 & 0.02 & 0.01 \end{bmatrix}$. We first computed the optimal controller gain without considering the positivity and nonfragility, that is, $K^\dagger = \begin{bmatrix} -1.1801 & -1.0390 & -1.0134 \end{bmatrix}$ with $f^\dagger = 2.3877$. It is easy to verify that $A + BK^\dagger$ is Hurwitz but not Metzler.

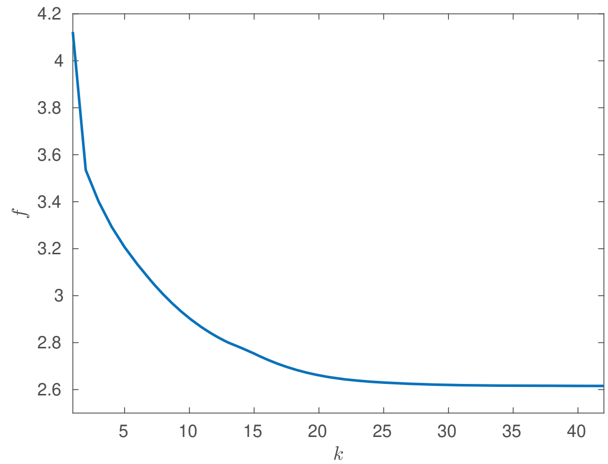

Fig. 2: Convergence process of Algorithm 1.

We implemente Algorithm 1 with the convex programming in *Theorem 2*. The gain matrix was initialized using *Proposition 1* as $K^{(1)} = \begin{bmatrix} -0.0595 & -0.5738 & -4.7821 \end{bmatrix}$ with $f^{(1)} = 4.1229$. After 30 iterations, it converged and we obtained that $K^* = \begin{bmatrix} -1.1211 & -0.6466 & -1.6713 \end{bmatrix}$ with $f^* = 2.6152$. Notice the fact that

$$A + B(K^* - \underline{\Delta}) = \begin{bmatrix} -4.1311 & 0.3333 & 3.3187 \\ 2.4345 & -1.3333 & 2.1593 \\ 0.3034 & 0.0000 & -10.5220 \end{bmatrix}$$

is a Metzler matrix and

$$A + B(K^* + \overline{\Delta}) = \begin{bmatrix} -4.1011 & 0.3633 & 3.3387 \\ 2.4495 & -1.3183 & 2.1693 \\ 0.3484 & 0.0450 & -10.4920 \end{bmatrix}$$

has eigenvalues $\{-0.9875, -4.2523, -10.6716\}$, thus is Hurwitz. Hence K^* is a feasible point to the optimization in (4). The convergence process is shown in Fig. 2.

Denote $\mathcal{M} := \{K \mid A + B(K - \underline{\Delta}) \text{ is Metzler}\}$ and $\mathcal{H} := \{K \mid A + B(K + \overline{\Delta}) \text{ is Hurwitz}\}$. The cross-section of the feasible region and optimal points is depicted in Fig. 3, where $\mathcal{M} \setminus \mathcal{H}$ is denoted by red dots, $\mathcal{H} \setminus \mathcal{M}$ is denoted by green dots, and $\mathcal{M} \cap \mathcal{H}$ is denoted by blue dots.

To verify the local optimality of K^*, we randomly generated 10 feasible neighboring points and obtained their performances as {2.6165, 2.6154, 2.6170, 2.6161, 2.6182, 2.6175, 2.6180, 2.6177, 2.6168, 2.6174}, which are all larger than f^*.

B. Multi-Input Positive Systems

Consider a positive linear system in (1) where

$$A = \begin{bmatrix} -1.00 & 2.00 \\ 2.00 & 2.00 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1.00 & 0.50 \\ 0.30 & 1.00 \end{bmatrix}$$

with

$$Q = \begin{bmatrix} 1.00 & 0.10 \\ 0.10 & 1.00 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 2.00 & 0.20 \\ 0.20 & 1.00 \end{bmatrix}.$$

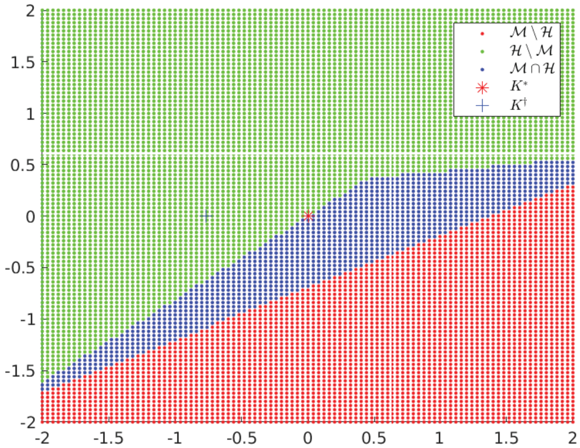

Fig. 3: An arbitrary cut plane passing through optimal points with K^* shifted to the origin of the plot.

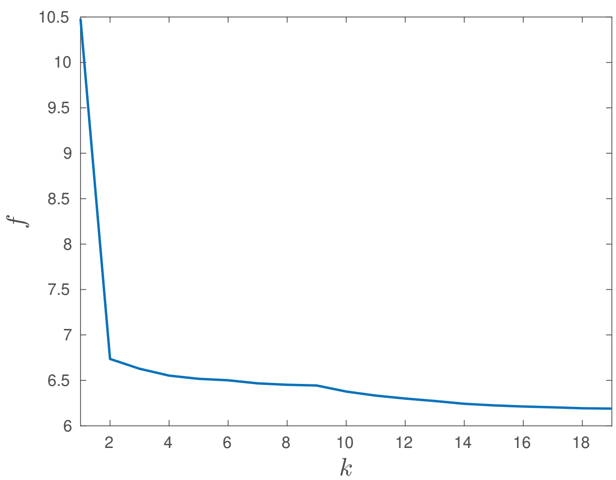

Fig. 4: Convergence process of Algorithm 1.

The gain variations are

$$\overline{\Delta} = \begin{bmatrix} 0.10 & 0.05 \\ 0.10 & 0.10 \end{bmatrix} \quad \text{and} \quad \underline{\Delta} = \begin{bmatrix} 0.10 & 0.10 \\ 0.05 & 0.10 \end{bmatrix}.$$

We first computed the optimal controller gain without considering the positivity and nonfragility, that is,

$$K^\dagger = \begin{bmatrix} -0.5789 & -0.9758 \\ -2.1327 & -4.3662 \end{bmatrix}$$

with $f^\dagger = 4.7750$. It is easy to verify that $A + BK^\dagger$ is Hurwitz but not Metzler. Then we implement Algorithm 1 with the bisection method in *Corollary 3*. The gain matrix was initialized using *Proposition 1* as

$$K^{(1)} = \begin{bmatrix} -1.3715 & -0.0935 \\ -0.2366 & -2.8586 \end{bmatrix}$$

with $f^{(1)} = 10.4818$. After 18 iterations, Algorithm 1 converged, and finally we obtained that

$$K^* = \begin{bmatrix} -0.8682 & -0.2090 \\ -1.0601 & -3.2820 \end{bmatrix}$$

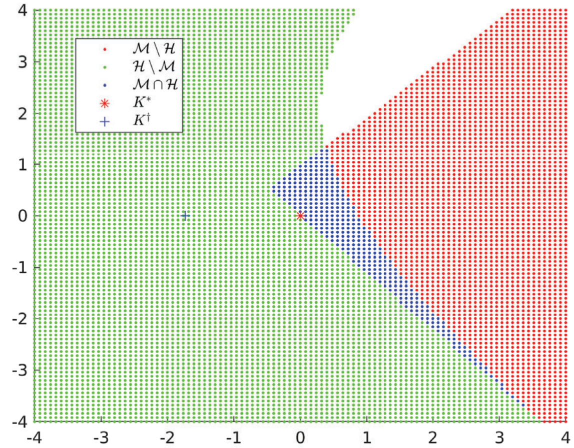

Fig. 5: An arbitrary cut plane passing through optimal points with K^* shifted to the origin of the plot.

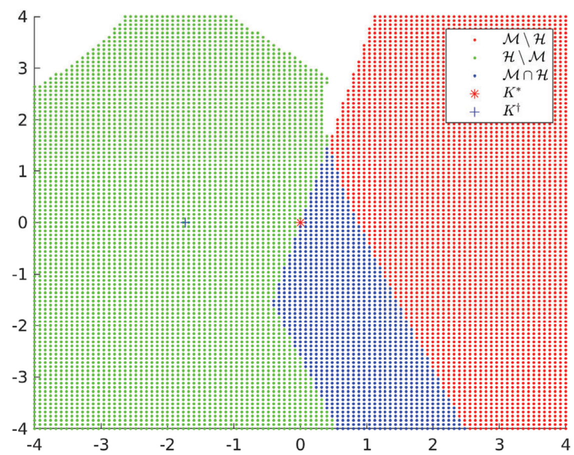

Fig. 6: An arbitrary cut plane passing through optimal points with K^* shifted to the origin of the plot.

with $f^* = 6.1893$. Notice the fact that

$$A + B(K^* - \underline{\Delta}) = \begin{bmatrix} -2.5233 & 0.0000 \\ 0.5995 & -1.4747 \end{bmatrix}$$

is a Metzler matrix and

$$A + B(K^* + \overline{\Delta}) = \begin{bmatrix} -2.2483 & 0.2500 \\ 0.8095 & -1.2297 \end{bmatrix}$$

has eigenvalues $\{-2.4185, -1.0595\}$, thus is Hurwitz. Hence K^* is a feasible point to the optimization in (4). The convergence process is shown in Fig. 4.

Two cross-sections of the feasible region and optimal points are depicted in Fig. 5 and Fig. 6.

To verify the local optimality of K^*, we randomly generated 10 feasible neighboring points and obtained their performances as $\{6.1990, 6.2171, 6.2293, 6.2421, 6.2630, 6.2712, 6.2823, 6.2903, 6.2965, 6.3123\}$, which are all larger than f^*.

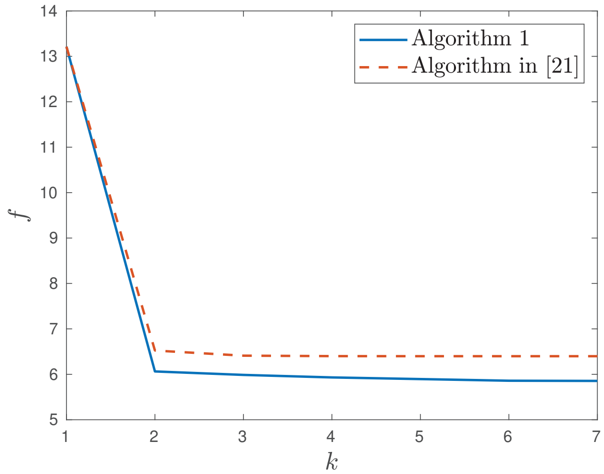

Fig. 7: Convergence process of the proposed PGD algorithm versus the LMI-based algorithm in [21].

C. Numerical Comparisons

Then we compare our algorithm with the method proposed in [21] through simulating *Example B* without gain variations since the latter did not include uncertainty in the analysis.

The gain matrix is initialized using *Proposition 1* as

$$K_0 = \begin{bmatrix} -0.6822 & -0.2574 \\ -0.4965 & -2.6758 \end{bmatrix}.$$

We implement Algorithm 1 and the LMI-based algorithm in [21]. Our algorithm converged after 6 iterations and returned

$$K^* = \begin{bmatrix} -0.7382 & -0.3338 \\ -1.1251 & -3.3325 \end{bmatrix}$$

with the LQR performance $f^* = 5.8545$. The LMI-based algorithm converged after 7 iterations and returned

$$K^* = \begin{bmatrix} -1.6447 & -0.1159 \\ -0.1159 & -3.7682 \end{bmatrix}$$

with the LQR performance $f^* = 6.3985$. Their convergence processes are depicted in Fig. 7.

TABLE I: Simulation results for different initial values.

Algorithm	Algorithm 1	Algorithm in [21]
Avg Iteration	6.47	7.81
Avg Value	5.8341	6.3988
Best Value	5.8031	6.3985
Worst Value	5.8496	6.4004

To evaluate the algorithm's performance for different initial conditions, we use *Proposition 1* to generate 100 initial values then implement Algorithm 1 and the algorithm in [21]. The simulation results are shown in Table 1.

VI. Conclusion

In this paper, we studied the design of linear quadratic regulator for positive linear systems subject to gain variations. Based on Lyapunov theory and positive systems theory, a systematic optimization framework is proposed for positivity-preserving controller design, and several analyses on solvability and feasibility were provided. The feasible region for gain matrices is in general nonconvex and nonsmooth. However, using convexification and positive systems theory, two projection theorems were derived for single-input and multi-input positive systems, respectively, and a tractable PGD algorithm was developed for computation. Finally, two numerical examples were provided to verify the effectiveness of the results. The analyses and approach proposed in this paper can also be potentially extended to investigate other types of positive systems, such as positive switched systems and positive delay systems, which are left to future research.

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