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Wang, Shimin; Zhang, Hongwei; Baldi, Simone; Zhong, Renxin

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Leaderless Consensus of Heterogeneous Multiple Euler–Lagrange Systems With Unknown Disturbance

Shimin Wang, Hongwei Zhang, Simone Baldi, *Senior Member, IEEE*, and Renxin Zhong

***Abstract*—This article studies the leaderless consensus problem of heterogeneous multiple networked Euler–Lagrange systems subject to persistent disturbances with unknown constant biases, amplitudes, initial phases, and frequencies. The main characteristic of this study is that none of the agents has information of a common reference model or of a common reference trajectory. Therefore, the agents must simultaneously and in a distributed way: achieve consensus to a common reference model (group model); achieve consensus to a common reference trajectory; and reject the unknown disturbances. We show that this is possible via a suitable combination of techniques of distributed "observers," internal model principle and adaptive regulation. The proposed design generalizes recent results on group model learning, which have been studied for linear agents over undirected networks. In this article, group model learning is achieved for Euler–Lagrange dynamics over directed networks in the presence of persistent unknown disturbances.**

***Index Terms*—Cooperative control, Euler–Lagrange system, leaderless consensus, multiagent system, output regulation.**

I. Introduction

Euler–Lagrange (EL) systems have found widespread applications in engineering and can model a variety of mechanical systems, such as marine vessels [1], rigid spacecrafts [2], and robot manipulators [3], [4]. Since precise modeling of an EL system is very difficult in practice and disturbances are always entangled with the system movement, control of uncertain EL systems with disturbance rejection has been an important issue in control community [5]–[7]. A recent work [7] solved a global asymptotic tracking control problem of EL systems with disturbance rejection, where the disturbance is a combination of sinusoidal signals with unknown frequencies, amplitudes, and phase angles. However, a similar problem becomes more challenging in a cooperative setting with multiple EL systems since, in addition to rejecting disturbances, the systems should achieve a common behavior with limited information (only using local information from a few neighbors).

Cooperative control of multiple EL systems has been intensively investigated in the past two decades mainly under two formulations, i.e., leader–following consensus (with a single leader or multiple leaders) [8]–[10] and leaderless consensus [11]–[14]. For leader–following consensus, a leader (or a group of leaders) generate a desired trajectory (or a convex hull) that all follower agents should follow. The desired trajectories can be time-varying and the tracking problem will become even more stringent if there exist some external disturbances [15]. In this sense, the tracking control of a single Euler–Lagrange system as in [7] can be viewed as a special case of the leader–following consensus with one leader (i.e., the desired trajectory) and one follower. To tackle the local information challenge, the idea of using a distributed observer [16] or an adaptive distributed observer [9], [17] was proposed for leader–following consensus. The idea is that only part of the follower agents can directly get access to the state and system matrix information of the leader, while the rest of the follower agents should estimate the leader's information using observers.

In many practical scenarios, there is no such leader. For example, when a leaderless swarm of unmanned aerial vehicles (UAVs) performs surveillance missions, individuals need to reach a consensus in altitude and heading angle and must coordinate with each other a commonly agreed trajectory to track [18]. A similar setting has been reported for a group of robotic arms equipped on different mobile robots to cooperatively scan a target area [11]. Most existing works on leaderless consensus of multiple networked EL systems typically allow the common trajectory to be time-invariant [11], [14]. Even when a disturbance is considered, as in [14], it is assumed that the final consensus equilibrium is a constant trajectory. As synchronization of uncertain heterogeneous multiagent systems to more complex trajectories requires either a leader agent generating a desired trajectory, or a common model according to the internal model principle, it is interesting to ask: what can be done without a leader? This problem has not been sufficiently investigated until very recently [19]–[21]. The work [19] gave a first answer for a special class of linear multiagent systems, i.e., heterogeneous oscillator systems. It formulates leaderless consensus as a "virtual" leader–following consensus problem. It shows that there exists a "group model" that has the same structure as the oscillators. Via consensus dynamics, each agent can learn the parameters of the group model without its direct knowledge, and finally synchronize to it. In this sense, synchronization of multiple oscillators to a nonconstant trajectory is achieved. More recently, a similar framework has been proposed in [21] for leaderless consensus of linear time-varying multiagent system, whereas Yan *et al.* [20] proposed a two-step approach, i.e., dynamics synchronization and state synchronization, and provided sufficient conditions for the

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Shimin Wang is with the Department of Chemical Engineering, Queen's University, Kingston, ON K7L 3N6, Canada (e-mail: shimin.wang@queensu.ca).

Hongwei Zhang is with the School of Mechanical Engineering and Automation, Harbin Institute of Technology, Shenzhen 518055, China (e-mail: hwzhang@hit.edu.cn).

Simone Baldi is with the School of Mathematics, Southeast University, Nanjing 211189, China, and also with Delft Center for Systems and Control, Delft University of Technology, 2628 CD Delft, The Netherlands (e-mail: s.baldi@tudelft.nl).

Renxin Zhong is with the Guangdong Key Laboratory of Intelligent Transportation Systems, School of Intelligent Systems Engineering, Sun Yat-Sen University, Guangzhou 510275, China (e-mail: zhrenxin@mail.sysu.edu.cn).

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efficacy of this two-step design. However, the authors in [19]–[21] only consider linear dynamics or undirected communication graphs.

Motivated by these recent achievements, this article aims to solve a leaderless consensus problem of uncertain heterogeneous EL systems with unknown disturbances over directed graphs. The disturbance is a compound sinusoidal signal with unknown magnitudes, frequencies, and phase angles. Each agent aims to achieve consensus to a complex time-varying trajectory, cooperatively contributed by the whole group of agents. This include the constant consensus equilibrium [14] as a special case. Therefore, the agents must simultaneously and in a distributed way: achieve consensus to a common group system matrix; achieve consensus to a common reference trajectory; and reject the unknown harmonic disturbances. Inspired by both [6], [7], [9], and [19], we show that this is possible via a suitable combination of consensus dynamics, internal model principle, and adaptive regulation. More specifically, we propose an "observer" for each agent, whose task is to "observe" the state and system matrix of an autonomous system, which is not prespecified but arising from the inherent properties and the initial states of the agents. We put the term "observe" in quotes since this autonomous system does not exist *a priori*. In other words, it is an imaginary one, and is generated through the collaboration of all observers of the group of agents. The contribution and novelties of our approach are summarized as follows.

1) In place of considering linear dynamics and undirected graphs, we solve a leaderless consensus problem of uncertain heterogeneous EL systems with unknown disturbances over directed graphs. This requires to develop new technical results (Lemmas 2–4 in this work) not reported in the literature.
2) Based on the consensus stage, we design a cooperative controller for each EL system to synchronize to the observer while rejecting in an adaptive way the external unknown disturbances.
3) Instead of a bounded tracking signal as in the single Euler–Lagrange system case [7], we only require that the derivative of the final consensus state is bounded without imposing bounds on the cooperatively agreed trajectory.

The rest of this article is organized as follows. The problem is formulated in Section II. In Section III, distributed "observers" are designed for all agents, which collaboratively generate an autonomous system, which is not prespecified but arising from the inherent properties and the initial states of the agents. The main result is presented in Section IV, followed by a numerical example in Section V. Finally, Section VI concludes this article.

Notation: Notation $\|\cdot\|$ is the Euclidean norm. The set of (positive) real numbers are denoted by $(\mathbb{R}_+)$ $\mathbb{R}$. For $X_i \in \mathbb{R}^{n_i\times m}$, $i=1,\ldots,N$, let $\mathrm{col}(X_1,\ldots,X_N)=\left[X_1^T,\ldots,X_N^T\right]^T$, and $\mathbb{1}_N=\mathrm{col}(1,\ldots,1)\in\mathbb{R}^N$. For $X_i\in\mathbb{R}^{m\times n_i}$, $i=1,\ldots,N$, let $\mathrm{row}(X_1,\ldots,X_N)=[X_1,\ldots,X_N]$. For any matrix $X\in\mathbb{R}^{m\times n}$, let $\mathrm{vec}(X)=\mathrm{col}(X_1,\ldots,X_n)$, where $X_i\in\mathbb{R}^m$ denotes the ith column of X. Finally, $\otimes$ denotes the Kronecker product, and $\circ$ denotes the Tracy–Singh product.

II. Problem Formulation

Consider N agents represented by the following Euler–Lagrange dynamics:

$$\mathcal{M}_i(q_i)\ddot{q}_i+\mathcal{C}_i(q_i,\dot{q}_i)\dot{q}_i+G_i(q_i)=\tau_i+d_i \tag{1}$$

where for each agent i, $q_i\in\mathbb{R}^n$ is the vector of generalized coordinates, $\mathcal{M}_i(q_i)\in\mathbb{R}^{n\times n}$ is the symmetric positive definite inertia matrix, $\mathcal{C}_i(q_i,\dot{q}_i)\dot{q}_i\in\mathbb{R}^n$ is the vector of Coriolis and centripetal forces, $G_i(q_i)\in\mathbb{R}^n$ is the vector of gravitational force, $\tau_i\in\mathbb{R}^n$ is the control torque, and $d_i=\mathrm{col}(d_{i1},\ldots,d_{in})\in\mathbb{R}^n$ is the external disturbance, taking the form

$$d_{is}(t)=\psi_{is,0}+\sum_{k=1}^{n_{is}}\psi_{is,k}\sin(\sigma_{is,k}t+\phi_{is,k})$$
$$i=1,\ldots,N,\ s=1,\ldots,n \tag{2}$$

where $\psi_{is,0},\phi_{is,k}\in\mathbb{R}$, $\psi_{is,k},\sigma_{is,k}\in\mathbb{R}_+$ are constant biases, initial phases, amplitudes, and frequencies. Biases, initial phases, amplitudes, and frequencies can all be arbitrary and unknown. In line with most Euler–Lagrange literature [3], let the dynamics (1) satisfy the following properties.

1) The inertia matrix $\mathcal{M}_i(q_i)$ is symmetric and uniformly positive definite such that $k_{\underline{m}}I\le\mathcal{M}_i(q_i)\le k_{\overline{m}}I$ for some positive scalars $k_{\underline{m}}$ and $k_{\overline{m}}$. Also, $\|\mathcal{C}_i(q_i,\dot{q}_i)\|\le k_c\|\dot{q}_i\|$, and $\|G_i(q_i)\|\le k_g$ for some positive scalars k_c and k_g.
2) For all $x,y\in\mathbb{R}^n$, $\mathcal{M}_i(q_i)x+\mathcal{C}_i(q_i,\dot{q}_i)y+G_i(q_i)=Y_i(q_i,\dot{q}_i,x,y)\Theta_i$, where $Y_i(q_i,\dot{q}_i,x,y)\in\mathbb{R}^{n\times q}$ is a known regression matrix and $\Theta_i\in\mathbb{R}^q$ is a constant vector consisting of the uncertain parameters of (1).
3) $\dot{\mathcal{M}}_i(q_i)-2\mathcal{C}_i(q_i,\dot{q}_i)$ is skew symmetric, $\forall q_i,\dot{q}_i\in\mathbb{R}^n$.

Let the agents (1) interact according to a static directed graph $\mathcal{G}=\{\mathcal{V},\mathcal{E},\mathcal{A}\}$ where the vertex set is $\mathcal{V}=\{1,2,\ldots,N\}$, and the edge set is $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$. We use $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$ to denote the adjacency matrix of graph $\mathcal{G}$, where $a_{ij}>0$ if $(j,i)\in\mathcal{E}$, and $a_{ij}=0$ otherwise. Let $\mathcal{L}\in\mathbb{R}^{N\times N}$ be the Laplacian matrix of graph $\mathcal{G}$, and $\mathcal{N}_i=\{j|(j,i)\in\mathcal{E}\}$ be the neighbor set of agent i. For more details on graph theory, readers are referred to [22]. The following property holds for the Laplacian matrix $\mathcal{L}$.

Lemma 1 (see [23]): If the communication graph $\mathcal{G}$ contains a spanning tree, then 0 is a simple eigenvalue of the Laplacian matrix $\mathcal{L}$, and all the other $N-1$ eigenvalues have positive real parts.

Problem 1 (Leaderless Consensus Problem): Consider the networked Euler–Lagrange systems (1) with communication graph $\mathcal{G}$. Find a distributed control law such that, for any external disturbance with arbitrary $\psi_{is,0}$, $\psi_{is,k}$, $\phi_{is,k}$, and $\sigma_{is,k}$ as in (2), and arbitrary initial conditions $q_i(0)$ and $\dot{q}_i(0)$, the trajectories $q_i(t)$ and $\dot{q}_i(t)$ exist and are bounded for all $t\ge 0$, and the following consensus results are achieved:

$$\lim_{t\to\infty}(q_i(t)-q_j(t))=0,\lim_{t\to\infty}(\dot{q}_i(t)-\dot{q}_j(t))=0\quad\forall i,j.$$

To solve Problem 1, we need the following assumption, standard for directed static communication graphs [23].

Assumption 1: The communication graph $\mathcal{G}$ contains a spanning tree.

Remark 1: Under Assumption 1, for the Laplacian matrix $\mathcal{L}\in\mathbb{R}^{N\times N}$ of the communication graph $\mathcal{G}$, there exists a nonsingular matrix $U\in\mathbb{R}^{N\times N}$ such that $U^{-1}\mathcal{L}U=J_{\mathcal{L}}$, where $J_{\mathcal{L}}$ is the Jordan canonical form of $\mathcal{L}$. In the following, let us denote λ_1 as the nonzero minimum real part among the eigenvalues of $\mathcal{L}$.

III. Distributed Observer and Dynamic Compensator

In this section, a distributed observer is designed for each agent so that all these observers will achieve consensus to an autonomous system determined by the inherent properties and the initial states of the agents. Additionally, an internal model based dynamic compensator is designed to deal with the uncertain disturbances.

A. Design of a Distributed Observer

We propose a distributed observer for each agent as follows:

$$\dot{S}_i=\mu_1\sum_{j\in\mathcal{N}_i}a_{ij}(S_j-S_i) \tag{3a}$$

$$\dot{\eta}_i = S_i\eta_i + \mu_2 \sum_{j\in\mathcal{N}_i} a_{ij}(\eta_j - \eta_i) \tag{3b}$$

where $S_i \in \mathbb{R}^{n\times n}$ and $\eta_i \in \mathbb{R}^n$ are the estimated system matrix and state of the autonomous system, respectively. The main difference between (3) and other adaptive distributed observers in the literature, e.g., [9], [24] is that the adaptive distributed observers in [9] and [24] require an explicit leader agent, generating an *a priori* reference trajectory for the network, while (3) requires no leader agent and all agents works cooperatively to construct an autonomous system.

In the following development, we shall show how to construct an autonomous system by the proposed observer (3). To this purpose, a technical lemma is needed.

Lemma 2: Consider the system

$$\dot{x} = F(t)x \tag{4}$$

where $x \in \mathbb{R}^n$, and $F(\cdot) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is bounded and piecewise continuous for all $t \geq 0$. If $F(t)$ vanishes exponentially, then x converges to a bounded vector.

Proof: Since $F(t)$ vanishes exponentially, there exist positive constants α and λ, such that $\|F(t)\| \leq \alpha e^{-\lambda t}$. Let $V = x^T x$. The time derivative of V along system (4) is

$$\begin{aligned}\dot{V} &= x^T\left(F(t) + F^T(t)\right)x\\ &\leq 2\alpha e^{-\lambda t}V.\end{aligned}$$

Then, $\forall t \geq 0$

$$\begin{aligned}V(t) &\leq e^{\int_0^t 2\alpha e^{-\lambda\tau}d\tau}V(0)\\ &\leq e^{\frac{2\alpha}{\lambda}}\|x(0)\|^2\end{aligned}$$

which implies that $\|x(t)\|$ is bounded for all $x(0)$ and $t \geq 0$. Hence, for system (4), $F(t)x$ will converge to zero exponentially at the rate of λ. Clearly, there exists an $x^* \in \mathbb{R}^n$ such that $\lim_{t\to\infty} x(t) = x^*$ exponentially at the rate of λ.

Remark 2: A related result is reported in [24, Lemma 1]. However, [24, Lemma 1] considers the system $\dot{x} = F_0x + F(t)x$, where matrix F_0 needs to be Hurwitz, proving that x converges to zero. Clearly, system (4) in the proposed Lemma 2 cannot be covered by [24], due to the absence of the Hurwitz matrix F_0.

Now we are ready to show the consensus of dynamics (3).

Lemma 3: Consider dynamics (3a). Under Assumption 1, for any positive μ_1 and any initial $S_i(0)$, the matrix signals $S_i(t)$ will achieve consensus exponentially, for $i = 1, \ldots, N$.

Proof: For notational conciseness, define $\bar{S} = \mathrm{col}(S_1, \ldots, S_N)$. Then, we can rewrite dynamics (3a) in a compact way

$$\dot{\bar{S}} = -\mu_1(\mathcal{L}\otimes I_n)\bar{S}. \tag{5}$$

By Remark 1, let $\Phi = (U^{-1}\otimes I_n)\bar{S} \in \mathbb{R}^{Nn\times n}$. Then, (5) can be rewritten as

$$\dot{\Phi} = -\mu_1(J_{\mathcal{L}}\otimes I_n)\Phi \tag{6}$$

where $J_{\mathcal{L}}$ is the Jordan canonical form of $\mathcal{L}$. Since the graph $\mathcal{G}$ contains a spanning tree, we have, from Lemma 1, that 0 is a simple eigenvalue of $J_{\mathcal{L}}$, and all other $N-1$ eigenvalues have positive real parts. For convenience, let us rearrange

$$J_{\mathcal{L}} = \text{block diag}\,(0, J_{N-1})$$

where $J_{N-1} \in \mathbb{R}^{(N-1)\times(N-1)}$ consists of the last $(N-1)$ rows and the last $(N-1)$ columns of the matrix $J_{\mathcal{L}}$. Let $\Phi = \mathrm{col}(\Phi_1, \Psi)$ and $\Psi = \mathrm{col}(\Phi_2 \ldots, \Phi_N)$, where $\Phi_i \in \mathbb{R}^{n\times n}$ for $i = 1, \ldots, N$. Then, system (6) can be rewritten as

$$\dot{\Phi}_1 = 0I_n \tag{7a}$$

$$\dot{\Psi} = -\mu_1(J_{N-1}\otimes I_n)\Psi. \tag{7b}$$

From (7b), and the properties of J_{N-1}, we obtain $\lim_{t\to\infty}\Psi(t) = 0$ exponentially with decay rate $\mu_1\lambda_1$, which implies

$$\lim_{t\to\infty}\Phi(t) = \mathrm{col}\left(\Phi_1(0), 0_{(N-1)n\times n}\right)$$

exponentially. Thus,

$$\lim_{t\to\infty}\bar{S}(t) = (U\otimes I_n)\,\mathrm{col}\left(\Phi_1(0), 0_{(N-1)n\times n}\right)$$

exponentially. Let $\mathbb{1}_N$ be the eigenvector associated to the 0 simple eigenvalue of $\mathcal{L}$. Then, arrange U so that its first column is $\mathbb{1}_N$. Thus, for any positive μ_1 and any initial $S_i(0) \in \mathbb{R}^{n\times n}$, $\lim_{t\to\infty}\bar{S}(t) = (\mathbb{1}_N \otimes \Phi_1(0))$ exponentially, i.e., $\lim_{t\to\infty} S_i(t) = \Phi_1(0), \forall i$ with decay rate $\mu_1\lambda_1$. ■

Remark 3: After denoting the first row of U^{-1} as $u^T = \mathrm{col}(u_1, \ldots, u_N)$, the following equality holds:

$$\begin{aligned}\Phi(0) &= \left(U^{-1}\otimes I_n\right)\bar{S}(0)\\ &= \mathrm{col}\left(\Phi_1(0), \Phi_2(0), \ldots, \Phi_N(0)\right).\end{aligned}$$

Thus, $\Phi_1(0) = \sum_{i=1}^N u_iS_i(0)$. Denote $S^* = \Phi_1(0)$, which can be treated as the system dynamics of the autonomous system determined by the initial conditions of each agent and communication network.

Next, we show that dynamics (3b) achieve consensus to the state of the autonomous system constructed by all the agents through communication network.

Lemma 4: Consider dynamics (3b) with an arbitrary $\eta_i(0)$. Under Assumption 1, for sufficiently large μ_1 and μ_2, the signals $\eta_i(t)$ achieve consensus exponentially, for $i = 1, \ldots, N$.

Proof: For notational conciseness, let $\eta = \mathrm{col}(\eta_1, \ldots, \eta_N)$ and $\hat{S}_d = \text{block diag}(S_1, \ldots, S_N)$, Then, we can put (3b) into the following compact form:

$$\dot{\eta} = \left[\hat{S}_d - \mu_2(\mathcal{L}\otimes I_n)\right]\eta. \tag{8}$$

Perform the following transformation:

$$\hat{\eta} = P(t)\eta \tag{9}$$

where $P(t) = e^{Qt}$ and $Q = \mu_2(\mathcal{L}\otimes I_n) - (I_N\otimes S^*)$. The time derivative of $\hat{\eta}$ along the trajectory (8) is

$$\begin{aligned}\dot{\hat{\eta}} &= P(t)\left[\hat{S}_d(t) - (I_N\otimes S^*)\right]P^{-1}(t)\hat{\eta}\\ &= e^{Qt}\left[\hat{S}_d(t) - (I_N\otimes S^*)\right]e^{-Qt}\hat{\eta}\\ &= F(t)\hat{\eta}.\end{aligned} \tag{10}$$

We know from Lemma 3 that $\lim_{t\to\infty} S_i(t) = S^*$ exponentially with decay rate $\mu_1\lambda_1$. Note that $\|e^{Qt}\|$ and $\|e^{-Qt}\|$ are upper bounded by $e^{(\mu_2\|\mathcal{L}\| + \|S^*\|)t}$. Therefore, we have $\lim_{t\to\infty} F(t) = 0$ exponentially for

$$\mu_1 \geq 2(\mu_2\|\mathcal{L}\| + \|S^*\|)/\lambda_1.$$

Then, by Lemma 2, for any initial states $\hat{\eta}(0) \in \mathbb{R}^{Nn}$, $\hat{\eta}(t)$ converges to a bounded vector $\hat{\eta}^* = \mathrm{col}(\hat{\eta}_1^*, \ldots, \hat{\eta}_N^*)$, $\hat{\eta}_i^* \in \mathbb{R}^n$. Since graph $\mathcal{G}$ contains a spanning tree, for any positive μ_2 and any initial $\hat{\eta}(0)$, we have from Lemma 1 that

$$\lim_{t\to\infty} e^{-\mu_2(\mathcal{L}\otimes I_n)t}\hat{\eta}(t) = \lim_{t\to\infty} e^{-\mu_2(\mathcal{L}\otimes I_n)t} \lim_{t\to\infty}\hat{\eta}(t) = \mathbb{1}_N \otimes \chi^* \tag{11}$$

where $\chi^* = \sum_{i=1}^N u_i \hat{\eta}_i^*$ and u_i is defined in Remark 3. Let

$$\eta_0(t) = \mathbb{1}_N \otimes \left(e^{S^* t}\chi^*\right). \tag{12}$$

According to (9), we have

$$\eta(t) = e^{-Qt}\hat{\eta}(t) = e^{(I_N\otimes S^*)t} e^{-\mu_2(\mathcal{L}\otimes I_n)t}\hat{\eta}(t).$$

Since $\|e^{(I_N\otimes S^*)t}\| \le e^{\|S\|t}$

$$\|\eta(t)-\eta_0(t)\| = \left\|e^{(I_N\otimes S^*)t}\left[e^{-\mu_2(\mathcal{L}\otimes I_n)t}\hat{\eta}(t) - \mathbb{1}_N\otimes\chi^*\right]\right\| \le e^{\|S^*\|t}\left\|e^{-\mu_2(\mathcal{L}\otimes I_n)t}\hat{\eta}(t) - \mathbb{1}_N\otimes\chi^*\right\|.$$

Considering (11), the exponentially decay rate is $\mu_2\lambda_1$. Then, we have

$$\|\eta(t)-\eta_0(t)\| \le e^{\|S^*\|t}e^{-\mu_2\lambda_1 t} = e^{-(\mu_2\lambda_1 - \|S^*\|)t}.$$

Hence, for $i, j \in \mathcal{N}$ and $\mu_2 > \frac{\|S^*\|}{\lambda_1}$

$$\lim_{t\to\infty}(\eta_i(t) - \eta_j(t)) = 0 \tag{13}$$

exponentially. This further implies that

$$\lim_{t\to\infty}(\eta_i(t) - e^{S^* t}\chi^*) = 0$$

exponentially for all i.

Remark 4: Note that the convergence analysis of Lemma 4 does not require the consensus state to be bounded, whereas the convergence analysis in some recent works such as [9] relies on the condition that the state of the leader is bounded. The idea of constructing an autonomous system in a distributed way was proposed in [19] for agents in the form of heterogeneous oscillators over undirected graphs. More specifically, in [19], the matrix S_i takes the following form:

$$S_i = \begin{bmatrix} 0 & 1 \\ -\beta_i & 0 \end{bmatrix} \tag{14}$$

together with the following distributed dynamics:

$$\dot{\beta}_i = \sum_{j\in\mathcal{N}_i} a_{ij}(\beta_j - \beta_i)$$

where $\beta_i \in \mathbb{R}$. Lemma 4 extends this result to directed graphs and more general S_i. In the following, an internal model design is discussed to handle the unknown disturbances.

B. Design of a Dynamic Compensator

A so-called internal model approach can be adopted to reject the disturbances $d_i(t)$. For compactness, let $\sigma_{is} = \mathrm{col}(\sigma_{is,1},\ldots,\sigma_{is,n_{is}})$ and $\sigma_i = \mathrm{col}(\sigma_{i1},\ldots,\sigma_{in})$, $i = 1,\ldots,N$ and $s = 1,\ldots,n$. According to [2], [6], [7], [25]–[28], we know that for each $i = 1,\ldots,N$ and $s = 1,\ldots,n$, there exist positive integers r_{is} and real numbers $c_{is,1},\ldots,c_{is,r_{is}}$, which may depend on σ_{is}, such that

$$d_{is}^{(r_{is})} = c_{is,1}d_{is} + c_{is,2}\dot{d}_{is} + \cdots + c_{is,r_{is}} d_{is}^{(r_{is}-1)}.$$

Let $T_{is}^{\sigma_{is}}$ be a nonsingular matrix of dimension r_{is}, and

$$\vartheta_{is} = \mathrm{col}\left(d_{is}, \dot{d}_{is}, d_{is}^{(2)},\ldots,d_{is}^{(r_{is}-1)}\right).$$

Then, we have

$$\dot{\vartheta}_{is} = \Phi_{is}^{\sigma_{is}}\vartheta_{is}, d_{is} = \Psi_{is}\vartheta_{is}$$

where

$$\Phi_{is}^{\sigma_{is}} = \left[\begin{array}{c|c} 0 & I_{r_{is}-1} \\ \hline c_{is,1} & c_{is,2},\ldots,c_{is,r_{is}} \end{array}\right],\ \Psi_{is} = \mathrm{row}(1, 0_{r_{is}-1}).$$

Let $M_{is} \in \mathbb{R}^{r_{is}\times r_{is}}$ be Hurwitz, $N_{is} \in \mathbb{R}^{r_{is}}$, and (M_{is}, N_{is}) be controllable. Then, there exists a nonsingular matrix $T_{is}^{\sigma_{is}}$ satisfying the Sylvester equation

$$T_{is}^{\sigma_{is}}\Phi_{is}^{\sigma_{is}} - M_{is}T_{is}^{\sigma_{is}} = N_{is}\Psi_{is}. \tag{15}$$

Let $\theta_{is}(t) = -T_{is}^{\sigma_{is}}\vartheta_{is}(t)$, $\theta_i = \mathrm{col}(\theta_{i1},\ldots,\theta_{in})$, $\Psi_i = \text{block diag}(\Psi_{i1},\ldots,\Psi_{in})$, $M_i = \text{block diag}(M_{i1},\ldots,M_{in})$, $T_i^{\sigma_i} = \text{block diag}(T_{i1}^{\sigma_{i1}},\ldots,T_{in}^{\sigma_{in}})$, and $N_i = \text{block diag}(N_{i1},\ldots,N_{in})$. Then, we have

$$d_i = -\Psi_i\left(T_i^{\sigma_i}\right)^{-1}\theta_i.$$

The dynamic compensator is designed as

$$\dot{\xi}_i = M_i\xi_i + N_i\tau_i \tag{16}$$

where $\xi_i \in \mathbb{R}^{n_i}$ with $n_i = \sum_{s=1}^n r_{is}$. The following section concerns the design of the distributed control τ_i.

IV. Main Results

To propose a distributed control law for the EL agents, we assume that $\dot{\eta}_0 = \mathbb{1}_N \otimes (S^* e^{S^* t}\chi^*)$ in (12) is bounded for all $t \ge 0$, which implies that $\dot{\eta}_i$ is bounded for all $t \ge 0$, for $i = 1,\ldots,N$. Let

$$\dot{q}_{ri} = S_i\eta_i - \alpha(q_i - \eta_i) \tag{17a}$$

$$s_i = \dot{q}_i - \dot{q}_{ri} \tag{17b}$$

where $\alpha > 0$ and η_i and S_i are generated by (3). Then,

$$\ddot{q}_{ri} = \dot{S}_i\eta_i + S_i\dot{\eta}_i - \alpha(\dot{q}_i - \dot{\eta}_i) \tag{18a}$$

$$\dot{s}_i = \ddot{q}_i - \ddot{q}_{ri}. \tag{18b}$$

By Property 2, there exists a known matrix $Y_i(q_i,\dot{q}_i,\dot{q}_{ri},\ddot{q}_{ri})$ and an unknown constant vector Θ_i such that

$$Y_i(q_i,\dot{q}_i,\dot{q}_{ri},\ddot{q}_{ri})\Theta_i = \mathcal{M}_i(q_i)\ddot{q}_{ri} + G_i(q_i) + \mathcal{C}_i(q_i,\dot{q}_i)\dot{q}_{ri}. \tag{19}$$

Next, substituting $Y_i(q_i,\dot{q}_i,\dot{q}_{ri},\ddot{q}_{ri})\Theta_i$ into system (1) gives

$$\mathcal{M}_i(q_i)(\ddot{q}_i - \ddot{q}_{ri}) + \mathcal{C}_i(q_i,\dot{q}_i)(\dot{q}_i - \dot{q}_{ri}) + Y_i(q_i,\dot{q}_i,\dot{q}_{ri},\ddot{q}_{ri})\Theta_i = \tau_i + d_i. \tag{20}$$

Then, from (17b) and (20), we have

$$\mathcal{M}_i(q_i)\dot{s}_i = \tau_i - \mathcal{C}_i(q_i,\dot{q}_i)s_i - Y_i(q_i,\dot{q}_i,\dot{q}_{ri},\ddot{q}_{ri})\Theta_i + d_i. \tag{21}$$

Consider the augmented system composed of (16) and (21), and the following coordinate transformation:

$$\bar{\xi}_i = \xi_i - \theta_i \tag{22a}$$

$$\tilde{\tau}_i = \tau_i - A_i\xi_i \tag{22b}$$

$$d_i = -B_i\theta_i \tag{22c}$$

where $A_i = \Psi_i(T_i^0)^{-1}$ and $B_i = \Psi_i(T_i^{\sigma_i})^{-1}$ with T_i^0 being a nonsingular matrix, Ψ_i and $T_i^{\sigma_i}$ given in (15). We have

$$\dot{\bar{\xi}}_i = [M_i + N_iA_i]\bar{\xi}_i + N_i\breve{u} + N_iE_i^{\sigma_i}\theta_i,$$

$$\mathcal{M}_i(q_i)\dot{s}_i = \tilde{\tau}_i - \mathcal{C}_i(q_i,\dot{q}_i)s_i + A_i\bar{\xi}_i$$

$$- Y_i\left(q_i, \dot{q}_i, \dot{q}_{ri}, \ddot{q}_{ri}\right)\Theta_i + E_i^{\sigma_i}\theta_i$$

with $E_i^{\sigma_i} = A_i - B_i$. Then, a further transformation

$$\tilde{\xi}_i = \bar{\xi}_i - N_i\mathcal{M}_i\left(q_i\right)s_i$$

gives

$$\begin{aligned}\dot{\tilde{\xi}}_i &= M_i\tilde{\xi}_i + P_i(q_i, \dot{q}_i, s_i)\Theta_i\\ \mathcal{M}_i\left(q_i\right)\dot{s}_i &= \tilde{\tau}_i - \mathcal{C}_i\left(q_i, \dot{q}_i\right)s_i + A_i\tilde{\xi}_i + Q_i(q_i, \dot{q}_i, s_i)\Theta_i\\ &\quad + E_i^{\sigma_i}\xi_i - E_i^{\sigma_i}\left[\tilde{\xi}_i + N_i\mathcal{M}_i\left(q_i\right)s_i\right]\end{aligned}$$

where $\xi_i \in \mathbb{R}^{n_i}$, $s_i \in \mathbb{R}^n$, and

$$\begin{aligned}P_i(q_i, \dot{q}_i, s_i)\Theta_i &= M_iN_i\mathcal{M}_i\left(q_i\right)s_i + N_i\mathcal{C}_i\left(q_i, \dot{q}_i\right)s_i\\ &\quad - N_i\dot{\mathcal{M}}_i\left(q_i\right)s_i + N_iY_i\left(q_i, \dot{q}_i, \dot{q}_{ri}, \ddot{q}_{ri}\right)\Theta_i\\ Q_i(q_i, \dot{q}_i, s_i)\Theta_i &= A_iN_i\mathcal{M}_i\left(q_i\right)s_i\\ &\quad - Y_i\left(q_i, \dot{q}_i, \dot{q}_{ri}, \ddot{q}_{ri}\right)\Theta_i \end{aligned} \tag{23}$$

with $P_i(q_i, \dot{q}_i, s_i)$ and $Q_i(q_i, \dot{q}_i, s_i)$ being known regression matrices. Let $\zeta_i \in R^{n_i\times p}$ be produced by an auxiliary system

$$\dot{\zeta}_i = M_i\zeta_i + P_i(q_i, \dot{q}_i, s_i). \tag{24}$$

Let $\hat{\xi}_i = \tilde{\xi}_i - \zeta_i\Theta_i$. A straightforward computation shows

$$\begin{aligned}\dot{\hat{\xi}}_i &= M_i\tilde{\xi}_i + P_i(q_i, \dot{q}_i, s_i)\Theta_i\\ &\quad - \left[M_i\zeta_i + P_i(q_i, \dot{q}_i, s_i)\right]\Theta_i\\ &= M_i\hat{\xi}_i \end{aligned} \tag{25a}$$

$$\begin{aligned}\mathcal{M}_i\left(q_i\right)\dot{s}_i &= \tilde{\tau}_i - \mathcal{C}_i\left(q_i, \dot{q}_i\right)s_i + B_i\hat{\xi}_i\\ &\quad + \left[A_i\zeta_i + Q_i(q_i, \dot{q}_i, s_i)\right]\Theta_i\\ &\quad + E_i^{\sigma_i}\left[\xi_i - N_i\mathcal{M}_i\left(q_i\right)s_i\right] - E_i^{\sigma_i}\zeta_i\Theta_i. \end{aligned} \tag{25b}$$

Since M_i is Hurwitz in (25a), we only need to concentrate on the second equation of (25). To handle the uncertain term in (25b) [i.e., the last two lines of (25b)] with adaptive control technique, we note that the uncertainty in the matrix $E_i^{\sigma_i}$ can be linearly parameterized for some integer $l \geq 1$ as follows:

$$\begin{aligned}E_i^{\sigma_i} &= \sum_{j=1}^{l} E_{ij}\varrho_{ij}\\ &= E_i\left[\varrho_i \otimes I_{n_i}\right]\end{aligned}$$

where $E_i = \mathrm{row}(E_{i1}, \ldots, E_{il})$, $\varrho_i = \mathrm{col}(\varrho_{i1}, \ldots, \varrho_{il})$, $E_{ij} \in \mathbb{R}^{n\times n_i}$ is a constant matrix and $\varrho_{ij} \in \mathbb{R}$ is a smooth function of σ_i. As a result

$$E_i^{\sigma_i}\zeta_i\Theta_i = \left[E_i \circ \zeta_i\right]\left[\varrho_i \otimes \Theta_i\right]$$

where $E_i \circ \zeta_i = \mathrm{row}(E_{i1}\zeta_i, \ldots, E_{il}\zeta_i)$. Besides

$$\begin{aligned}E_i^{\sigma_i}\xi_i &= \left[E_i \circ \xi_i\right]\varrho_i\\ E_i^{\sigma_i}N_iM_i\left(q_i\right)s_i &= E_i\left[\varrho_i \otimes I_{n_i}\right]N_iL_i\left(q_i, s_i\right)\Theta_i\\ &= E_i \circ \left[N_iL_i\left(q_i, s_i\right)\right]\left[\varrho_i \otimes \Theta_i\right]\end{aligned}$$

where $L_i(q_i, s_i)\Theta_i = \mathcal{M}_i(q_i)s_i$, and $L_i(q_i, s_i)$ is a known regression matrix. Now, system (25) can be further written in the following linearly parameterized form:

$$\dot{\hat{\xi}}_i = M_i\hat{\xi}_i \tag{26a}$$

$$\begin{aligned}\mathcal{M}_i\left(q_i\right)\dot{s}_i &= \tilde{\tau}_i - \mathcal{C}_i\left(q_i, \dot{q}_i\right)s_i\\ &\quad + B_i\hat{\xi}_i + \rho_i(q_i, \dot{q}_i, s_i, \zeta_i)\omega_i \end{aligned} \tag{26b}$$

where $\omega_i = \mathrm{col}(\Theta_i, \varrho_i \otimes \Theta_i, \varrho_i)$ is a constant vector consisting of the uncertain parameters of (1) and (2), and $\rho_i(q_i, \dot{q}_i, s_i, \zeta_i)$ is a known regression matrix with

$$\begin{aligned}\rho_i(q_i, \dot{q}_i, s_i, \zeta_i) &= \mathrm{col}(\rho_{i1}(q_i, \dot{q}_i, s_i, \zeta_i), \rho_{i2}(q_i, s_i, \zeta_i), \rho_{i3}(\xi_i))\\ &= \begin{bmatrix} A_i\zeta_i + Q_i(q_i, \dot{q}_i, s_i)\\ E_i \circ \left[\zeta_i + N_iL_i\left(q_i, s_i\right)\right]\\ E_i \circ \xi_i \end{bmatrix}. \end{aligned} \tag{27}$$

The last step for solving the regulation problem of system (26) is to introduce the control law as follows:

$$\tilde{\tau}_i = -K_is_i - \rho_i(q_i, \dot{q}_i, s_i, \zeta_i)\hat{\omega}_i \tag{28a}$$

$$\dot{\hat{\omega}}_i = \Lambda_i^{-1}\rho_i^T(q_i, \dot{q}_i, s_i, \zeta_i)s_i \tag{28b}$$

where s_i is calculated from (17b), ζ_i is generated by (24), the vector $\hat{\omega}_i$ is used to estimate ω_i, K_i is a positive definite matrix, and Λ_i a positive definite diagonal matrix representing the estimator update rate. Now we are in a position to present our main result.

Theorem 1: Consider system (1) over a communication graph satisfying Assumption 1. Problem 1 is solvable by the control law consisting of (3), (16), (24), and (28).

Proof: Substituting (28) into (26) gives

$$\dot{\hat{\xi}}_i = M_i\hat{\xi}_i \tag{29a}$$

$$\begin{aligned}\mathcal{M}_i\left(q_i\right)\dot{s}_i &= -\mathcal{C}_i\left(q_i, \dot{q}_i\right)s_i - K_is_i\\ &\quad - \rho_i(q_i, \dot{q}_i, s_i, \zeta_i)\tilde{\omega}_i + B_i\hat{\xi}_i \end{aligned} \tag{29b}$$

$$\dot{\tilde{\omega}}_i = \Lambda_i^{-1}\rho_i^T(q_i, \dot{q}_i, s_i, \zeta_i)s_i \tag{29c}$$

where $\tilde{\omega}_i = \hat{\omega}_i - \omega_i$. Let $\mathcal{Q}_i$ be the symmetric positive definite matrix satisfying

$$\mathcal{Q}_iM_i + M_i^T\mathcal{Q}_i = -I$$

and pick a real number $\epsilon \geq \frac{\|B_i\|^2}{\lambda_{\min}(K_i)}$, where $\|B_i\| = \max_{\|x\|=1}\|B_ix\|$. Pick the following Lyapunov function candidate:

$$V_i = \epsilon\hat{\xi}_i^T\mathcal{Q}\hat{\xi}_i + \frac{1}{2}\left[s_i^T\mathcal{M}_i\left(q_i\right)s_i + \tilde{\omega}_i^T\Lambda_i\tilde{\omega}_i\right].$$

The time derivative of V along the trajectory (29) is

$$\begin{aligned}\dot{V}_i = &-s_i^TK_is_i + \frac{1}{2}s_i^T\left[\dot{\mathcal{M}}_i\left(q_i\right) - 2\mathcal{C}_i\left(q_i, \dot{q}_i\right)\right]s_i + s_i^TB_i\hat{\xi}_i\\ &- s_i^T\rho_i(q_i, \dot{q}_i, s_i, \zeta_i)\tilde{\omega}_i + \tilde{\omega}_i^T\rho_i^T(q_i, \dot{q}_i, s_i, \zeta_i)s_i - \epsilon\|\hat{\xi}_i\|^2.\end{aligned}$$

Since $\dot{\mathcal{M}}_i(q_i) - 2\mathcal{C}_i(q_i, \dot{q}_i)$ is skew symmetric, we have

$$\begin{aligned}\dot{V}_i &= -s_i^TK_is_i + s_i^TB_i\hat{\xi}_i - \epsilon\|\hat{\xi}_i\|^2\\ &\leq -s_i^TK_is_i + \frac{1}{2\epsilon}\|s_i^TB_i\|^2 + \frac{\epsilon}{2}\|\hat{\xi}_i\|^2 - \epsilon\|\hat{\xi}_i\|^2\\ &\leq -\frac{\epsilon}{2}\|\hat{\xi}_i\|^2 - \frac{1}{2}s_i^TK_is_i\\ &= -a\left(\hat{\xi}_i, s_i\right). \end{aligned} \tag{30}$$

Thus, s_i, $\hat{\xi}_i$, and $\tilde{\omega}_i$ are bounded. From (3) and (17), we have

$$\dot{q}_i - \dot{\eta}_i + \alpha\left(q_i - \eta_i\right) = s_i - \mu_2\sum_{j\in\mathcal{N}_i}a_{ij}(\eta_j - \eta_i)$$

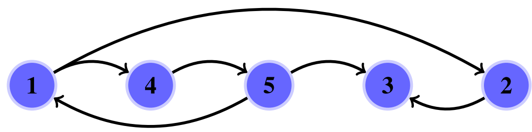

Fig. 1. Communication graph $\bar{\mathcal{G}}$.

which can be further rewritten as

$$\dot{e}_i + \alpha e_i = s_i - \mu_2 \sum_{j\in\mathcal{N}_i} a_{ij}(\eta_j - \eta_i). \tag{31}$$

This can be viewed as a stable first-order differential equation in e_i with $s_i - \mu_2 \sum_{j\in\mathcal{N}_i} a_{ij}(\eta_j - \eta_i)$ as the input. Since this input is bounded for all $t \geq 0$, we conclude that both $e_i = q_i - \eta_i$ and $\dot{e}_i = \dot{q}_i - \dot{\eta}_i$ are bounded for all $t \geq 0$, which further implies $\dot{q}_i$ is bounded for all $t \geq 0$ because of $\dot{\eta}_i$ is bounded for all $t \geq 0$.

By Property 1, we obtain that $\mathcal{M}_i(q_i)$, $\mathcal{C}_i(q_i, \dot{q}_i)$, and $G_i(q_i)$ are all bounded for all $t \geq 0$. It is noted that

$$\lim_{t\to\infty} \dot{S}_i(t)\eta_i(t) = 0 \text{ and } \lim_{t\to\infty} \left[S_i(t)\dot{\eta}_i(t) - (S^*)^2 e^{S^* t}\chi^*\right] = 0$$

from Lemmas 3 and 4, where χ^* and S^* are defined in (11) and Remark 3, respectively. Hence, $\dot{q}_{ri}$ and $\ddot{q}_{ri}$ are bounded from (17) for all $t \geq 0$. By (29b), we have $Y_i(q_i, \dot{q}_i, \dot{q}_{ri}, \ddot{q}_{ri})$ is bounded. Noted that, $P_i(q_i, \dot{q}_i, s_i)$ and $Q_i(q_i, \dot{q}_i, s_i)$ are bounded for all $t \geq 0$ from (23). Thus, ζ_i is also bounded for all $t \geq 0$ from a stable differential (24) with a bounded input $P_i(q_i, \dot{q}_i, s_i)$. As a result, ξ_i is bounded for all $t \geq 0$ from (22a) and the fact that θ_i is bounded for all $t \geq 0$. Then, $\rho(q_i, \dot{q}_i, s_i, \zeta_i)$ is bounded for all $t \geq 0$ from (27). Hence, we have $\dot{s}(t)$ is bounded for all $t \geq 0$ from (29b).

By integrating both sides of (30), we can show that

$$\int_0^t a\left(\hat{\xi}_i(\tau), s_i(\tau)\right) d\tau \leq V(0) - V(t) \leq V(0).$$

Thus, $\lim_{t\to\infty} \int_0^t a(\hat{\xi}_i(\tau), s_i(\tau)) d\tau$ exists and is finite. Therefore

$$\dot{a}\left(\hat{\xi}_i(t), s_i(t)\right) = \frac{\partial a}{\partial \hat{\xi}_i}\dot{\hat{\xi}}_i + \frac{\partial a}{\partial s_i}\dot{s}_i$$

is bounded for all $t \geq 0$, and hence, $a(\hat{\xi}_i(t), s_i(t))$ is uniformly continuous in t. Applying Barbalat's lemma, we have $\lim_{t\to\infty} a(\hat{\xi}_i(t), s_i(t)) = 0$, thus, $\lim_{t\to\infty} s_i(t) = 0$.

Since the input in (31) is bounded for all $t \geq 0$ and tends to zero as $t \to \infty$, we conclude that both $e_i = q_i - \eta_i$ and $\dot{e}_i = \dot{q}_i - \dot{\eta}_i$ are bounded for all $t \geq 0$ and will decay to zero. Together with (13), the proof is completed.

Remark 5: For the single Euler–Lagrange system as in [7], the tracking signal is bounded. In our multiple Euler–Lagrange setting we only require that the derivative of the final consensus state is bounded without imposing bounds on the cooperatively agreed trajectory.

V. Numerical Example

Consider a group of 5 EL agents with the communication network described in Fig. 1. Let each EL agent represent a two-link robotic arm, whose dynamics is described by (1), with generalized coordinates $q_i = \mathrm{col}(\theta_{i1}, \theta_{i2})$

$$\mathcal{M}_i(q_i) = \begin{bmatrix} a_{i1} + a_{i2} + 2a_{i3}\cos\theta_{i2} & a_{i2} + a_{i3}\cos\theta_{i2} \\ a_{i2} + a_{i3}\cos\theta_{i2} & a_{i2} \end{bmatrix}$$

$$\mathcal{C}_i(q_i, \dot{q}_i) = \begin{bmatrix} -a_{i3}(\sin\theta_{i2})\dot{\theta}_{i2} & -a_{i3}\sin\theta_{i2}\left(\dot{\theta}_{i1} + \dot{\theta}_{i2}\right) \\ a_{i3}\sin\theta_{i2}\dot{\theta}_{i1} & 0 \end{bmatrix}$$

$$G_i(q_i) = \begin{bmatrix} a_{i4}g\cos\theta_{i1} + a_{i5}g\cos(\theta_{i1} + \theta_{i2}) \\ a_{i5}g\cos(\theta_{i1} + \theta_{i2}) \end{bmatrix}$$

and $\Theta_i = \mathrm{col}(a_{i1}, a_{i2}, a_{i3}, a_{i4}, a_{i5})$. This dynamics is adopted from [3, Example 3.2-2] with some simplified modification of notations. The physical interpretation of each parameter can be found in [3]. We consider the disturbance

$$d_{ik} = \psi_{ik}\sin(\sigma_{ik}t + \phi_{ik}), k = 1, 2.$$

According to the internal model approach, we can select

$$\Phi_{ik} = \begin{bmatrix} 0 & 1 \\ -\sigma_{ik}^2 & 0 \end{bmatrix}, \Psi_{ik} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Choosing

$$M_{ik} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}, N_{ik} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

gives

$$\begin{aligned} T_{ik}^{\sigma_{ik}} &= \begin{bmatrix} 3 - \sigma_{ik}^2 & -2 \\ 2\sigma_{ik}^2 & 3 - \sigma_{ik}^2 \end{bmatrix} \frac{1}{(3 - \sigma_{ik}^2)^2 + 4\sigma_{ik}^2} \\ \Psi_{ik}\left(T_{ik}^{\sigma_{ik}}\right)^{-1} &= \begin{bmatrix} 3 - \sigma_{ik}^2 & 2 \end{bmatrix} \\ T_{ik}^0 &= \begin{bmatrix} 3 & -2 \\ 0 & 3 \end{bmatrix}\frac{1}{9}. \end{aligned}$$

Let $\sigma_i = \mathrm{col}(\sigma_{i1}, \sigma_{i2})$, $\psi_i = \mathrm{col}(\psi_{i1}, \psi_{i2})$, $\phi_i = \mathrm{col}(\phi_{i1}, \phi_{i2})$, $M_i = \text{block diag}(M_{i1}, M_{i2})$, $N_i = \text{block diag}(N_{i1}, N_{i2})$, $T_i = \text{block diag}(T_{i1}, T_{i2})$, and $\Psi_i = \text{block diag}(\Psi_{i1}, \Psi_{i2})$. For the nominal value $\sigma_i = 0$, we have

$$\begin{aligned} E_i^{\sigma_i} &= \Psi_i\left(T_i^0\right)^{-1} - \Psi_i\left(T_i^{\sigma_i}\right)^{-1} = \begin{bmatrix} \sigma_{i1}^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{i2}^2 & 0 \end{bmatrix} \\ &= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \left[\begin{bmatrix} \varrho_{i1} \\ \varrho_{i2} \end{bmatrix} \otimes I_4\right] \\ &= \begin{bmatrix} E_{i1} & E_{i2} \end{bmatrix}[\varrho_i \otimes I_4] \\ &= E_i[\varrho_i \otimes I_4] \end{aligned}$$

where $\varrho_i = \mathrm{col}(\varrho_{i1}, \varrho_{i2}) = \mathrm{col}(\sigma_{i1}^2, \sigma_{i2}^2)$. Then, we have $\omega_i = \mathrm{col}(\Theta_i, \varrho_i \otimes \Theta_i, \varrho_i)$. Next, the terms $\rho_i(q_i, \dot{q}_i, s_i, \zeta_i)$ and $P_i(q_i, \dot{q}_i, s_i)$ can be obtained from the following equations:

$$\rho_i(q_i, \dot{q}_i, s_i, \zeta_i) = \begin{bmatrix} A_i\zeta_i + Q_i(q_i, \dot{q}_i, s_i) \\ E_i \circ [\zeta_i + NL_i(q_i, s_i)] \\ E_i \circ \xi_i \end{bmatrix}$$

$$\begin{aligned} P_i(q_i, \dot{q}_i, s_i)\Theta_i &= M_iN_i\mathcal{M}_i(q_i)s_i + N_iC_i(q_i, \dot{q}_i)s_i \\ &\quad - N_i\dot{\mathcal{M}}_i(q_i)s_i + N_iY_i(q_i, \dot{q}_i, \dot{q}_{ri}, \ddot{q}_{ri})\Theta_i \end{aligned}$$

with

$$\begin{aligned} Q_i(q_i, \dot{q}_i, s_i)\Theta_i &= A_iN_i\mathcal{M}_i(q_i)s_i - Y_i(q_i, \dot{q}_i, \dot{q}_{ri}, \ddot{q}_{ri})\Theta_i \\ L_i(q_i, s_i)\Theta_i &= \mathcal{M}_i(q_i)s_i. \end{aligned}$$

Now, we are ready to construct the control law as follows:

$$\begin{aligned} \tau_i &= -K_is_i - \rho_i(q_i, \dot{q}_i, s_i, \zeta_i)\hat{\omega}_i + A_i\xi_i \\ \dot{\xi}_i &= M_i\xi_i + N_i\tau_i \\ \dot{\hat{\omega}}_i &= \Lambda^{-1}\rho_i^T(q_i, \dot{q}_i, s_i, \zeta_i)s_i \\ \dot{\zeta}_i &= M_i\zeta_i + P_i(q_i, \dot{q}_i, s_i) \end{aligned}$$

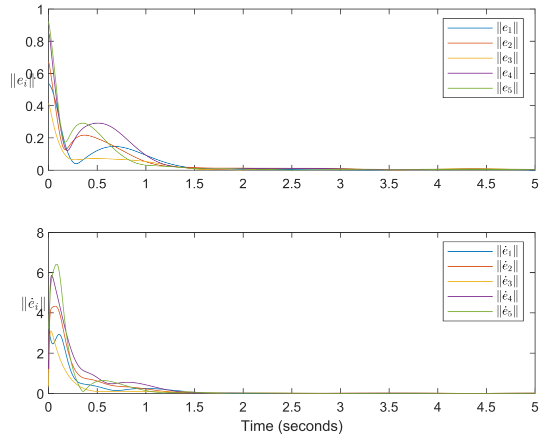

Fig. 2. Trajectories of $\|e_i\|$ and $\|\dot{e}_i\|$, for $i = 1, \ldots, 5$.

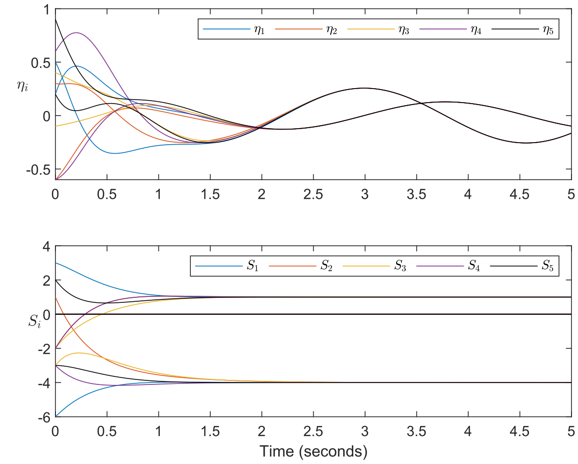

Fig. 3. Trajectories of η_i and S_i, for $i = 1, \ldots, 5$.

$$\dot{S}_i = \mu_1 \sum_{j=1}^{5} a_{ij}(S_j - S_i)$$

$$\dot{\eta}_i = S_i \eta_i + \mu_2 \sum_{j=1}^{5} a_{ij}(\eta_j - \eta_i).$$

Select the following parameters: $\mu_1 = \mu_2 = 2$, $K_i = 40I_2$, $\alpha = 6$, $\Lambda_i = 0.15I_{17}$. The actual values of Θ_i, ψ_i, σ_i, and ϕ_{ik} are given as

$$\Theta_1 = \text{col}(0.64, 1.10, 0.08, 0.64, 0.32), \psi_1 = \text{col}(6, 8)$$
$$\Theta_2 = \text{col}(0.76, 1.17, 0.14, 0.93, 0.44), \psi_2 = \text{col}(-1, -2)$$
$$\Theta_3 = \text{col}(0.91, 1.26, 0.22, 1.27, 0.58), \psi_3 = \text{col}(-2, -5)$$
$$\Theta_4 = \text{col}(1.10, 1.36, 0.32, 1.67, 0.73), \psi_4 = \text{col}(3, 5)$$
$$\Theta_5 = \text{col}(1.21, 1.16, 0.12, 1.45, 1.03), \psi_5 = \text{col}(-3, -2.5)$$

$\sigma_i = \text{col}(0.1, 0.2)$ and $\phi_{ik} = 0$. The simulation is conducted with the following initial conditions: $q_i = 0$, $\hat{\Theta}_i = 0$, $\zeta_i = 0$, $\hat{\omega}_i = 0$, $\xi_i = 0$, $\forall i$, and

$$S_1(0) = \begin{bmatrix} 0 & 3 \\ -6 & 0 \end{bmatrix}, S_2(0) = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$$
$$S_3(0) = \begin{bmatrix} 0 & -2 \\ -3 & 0 \end{bmatrix}, S_4(0) = \begin{bmatrix} 0 & -2 \\ -3 & 0 \end{bmatrix}$$
$$S_5(0) = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}, \eta_1(0) = \text{col}(0.2, 0.5)$$
$$\eta_2(0) = \text{col}(-0.6, 0.3), \eta_3(0) = \text{col}(-0.1, 0.4)$$
$$\eta_4(0) = \text{col}(-0.6, 0.6), \eta_5(0) = \text{col}(0.9, 0.2).$$

The errors in Fig. 2 show that consensus of both q_i and $\dot{q}_i$ is achieved among all the five agents. The trajectories of η_i and S_i in Fig. 3 show that all five agents converge to an autonomous system arising from the communication network, the inherent properties and the initial states of the agents.

VI. Conclusion

This article proposed a novel design for leaderless consensus and disturbance rejection problem of multiple Euler–Lagrange agents. In this setting, all agents must converge to a common behavior while being affected by persistent disturbances with unknown biases, amplitudes, initial phases, and frequencies. The main feature of the proposed design is that none of the agents has information of a common reference model or of a common reference trajectory. Rather, all agents collaborate with each other through a communication network to achieve a common reference trajectory, and simultaneously reject persistent disturbances. The analysis shows that the generalized coordinates and velocities of the multiple Euler–Lagrange systems converge to common time-varying states in a distributed way.

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