# Decentralized Stochastic Optimization with Inherent Privacy Protection

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Abstract—Decentralized stochastic optimization is the basic building block of modern collaborative machine learning, distributed estimation and control, and large-scale sensing. Since involved data usually contain sensitive information like user locations, healthcare records and financial transactions, privacy protection has become an increasingly pressing need in the implementation of decentralized stochastic optimization algorithms. In this paper, we propose a decentralized stochastic gradient descent algorithm which is embedded with inherent privacy protection for every participating agent against other participating agents and external eavesdroppers. This proposed algorithm builds in a dynamics based gradient-obfuscation mechanism to enable privacy protection without compromising optimization accuracy. which is in significant difference from differential-privacy based privacy solutions for decentralized optimization that have to trade optimization accuracy for privacy. The dynamics based privacy approach is encryption-free, and hence avoids incurring heavy communication or computation overhead, which is a common problem with encryption based privacy solutions for decentralized stochastic optimization. Besides rigorously characterizing the convergence performance of the proposed decentralized stochastic gradient descent algorithm under both convex objective functions and non-convex objective functions, we also provide rigorous information-theoretic analysis of its strength of privacy protection. Simulation results for a distributed estimation problem as well as numerical experiments for decentralized learning on a benchmark machine learning dataset confirm the effectiveness of the proposed approach.

#### I. INTRODUCTION

Decentralized optimization and learning has been an area of intensive research due to its wide applications in largescale sensing [1], distributed estimation and control [2], [3], and big data analytics [4]. In recent years, in order to handle the enormous growth in data volumes as well as to address practical data imperfections, decentralized stochastic gradient descent (SGD), in which participating agents use noisy local gradients and peer-to-peer communications to cooperatively solve a network optimization problem, is gaining increased attention. For example, in modern distributed machine learning on massive datasets, multiple training agents cooperatively train a neural network where each participating agent updates a local copy of the neural network model using both information received from neighboring agents and local noisy gradients calculated from one (or a small batch of) data points available to the agent. Computing gradients from only one (or a small batch of) available data point yields a noisy estimation of the exact gradient, but is completely necessary

The work was supported in part by the National Science Foundation under Grants ECCS-1912702 and CCF-2106293.

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because evaluating the precise gradient using all available data points usually is extremely expensive in computation or even simply impractical. Furthermore, with the advent of Internet of things which features the explosive growth of low-cost sensing and communication devices, the data fed to distributed optimization are usually subject to measurement noises [5]. As deterministic optimization approaches will typically falter when involved data are massive or noisy [6], it is mandatory to investigate decentralized stochastic optimization algorithms.

Centralized SGD algorithms date back to the 1950s [7] and decentralized SGD dates back to the 1980s [8]. To date, plenty of results on decentralized SGD have been reported, for both the case with convex objective functions (e.g., [9], [10], [11], [12], [13], [14], [15], [16]) and the case with nonconvex objective functions (e.g., [17], [18], [19], [20], [21], [22]). In such decentralized SGD algorithms, as participating agents only share gradient/model updates and never let raw data leave participants' machines, it was believed that the privacy of participating agents can be protected by the decentralized computing architecture. However, recent studies revealed a completely different picture: not only are properties of the data, such as membership associations, are inferable from gradient/model updates shared in decentralized stochastic optimization [23], [24], but even raw data can be precisely reversely inferred from shared gradients (pixel-wise accurate for images and token-wise matching for texts) [25]. This poses a severe threat to the privacy of participants in decentralized stochastic optimization, because data used in optimization often contain sensitive information such as user positions, healthcare records, or financial transactions.

To protect the privacy of participating agents in distributed stochastic gradient methods, recently several approaches have been proposed. One approach resorts to differential privacy [26] which adds additive noise to shared gradient/model updates (e.g., [27], [28], [29], [30], [31], [32], [33], [34]). [35] and [36] indicate that additive noise used for differential privacy can even facilitate convergence to the global optimum in non-convex optimization. However, recent systematic studies in [25] indicate that a small magnitude of differentialprivacy noise cannot thwart strong privacy attacks and differential privacy based defense can achieve reasonable privacy protection "only when the noise variance is large enough to degrade accuracy [25]." Another commonly used approach to enable privacy in decentralized SGD is to employ secure multi-party computation approaches such as homomorphic encryption [37] or garbled circuit [38], which, however, will incur a runtime overhead of three to four orders of magnitude [39]. Furthermore, homomorphic encryption can only protect against a parameter server [25]. Hardware based approaches such as trusted hardware enclaves have also been discussed in the literature [39], [40], [41]. However, similar to homomorphic encryption based approaches, these approaches cannot be directly used to prevent multiple data providers from inferring each others' data during stochastic optimization. Recently, several results were reported to exploit the structural properties of decentralized optimization to enable privacy. For example, the authors in [42], [43] showed that privacy can be enabled by adding a *constant* uncertain parameter in projection or step sizes. The authors of [44] showed that network structure can be leveraged to construct spatially correlated "structured" noise to cover gradient information. Although these approaches can ensure optimization accuracy, their enabled privacy is restricted: projection based privacy depends on the size of the projection set – a large projection set nullifies privacy protection whereas a small projection set offers strong privacy protection but requires a priori knowledge of the optimal solution; "structured" noise based approaches require each agent to have a certain number of neighbors that do not share information with the adversary. In fact, such a structural constraint is required in most existing accuracy-friendly privacy solutions to decentralized optimization. For example, our results in [45] show that even the partially homomorphic encryption based privacy approaches require that the adversary doesn't have access to a target agent's communications with all of its neighbors.

Inspired by our recent finding that the privacy of participating agents in decentralized consensus computations can be ensured by manipulating inherent consensus dynamics [46], [47], [48], in this paper, we propose to enable intrinsic privacy protection in decentralized stochastic gradient methods by judiciously manipulating the inherent dynamics of inter-agent information fusion and gradient-descent operations. More specifically, we let each participating agent use time-varying and heterogeneous stepsize and an additional stochastic mixing coefficient to obscure its gradients when sharing information with its neighbors. Since the privacy approach will not increase the size or the number of shared messages, it will not incur heavy communication or computation overhead. Furthermore, we rigorously prove that the proposed privacy approach does not affect the convergence to the desired solution, which is in distinct difference from differential privacy based privacy approaches. It is worth noting that although the approaches in [49], [50] use heterogeneous stepsizes, they do not allow the stepsizes to be random and time-varying, which, however, is key to achieving the enabled privacy here. Moreover, the algorithms in [49], [50] have to share two variables (an optimization variable and a gradient-tracking variable) in every iteration whereas our approach only needs to share one variable, which is important since information sharing can create significant communication overhead and even communication bottlenecks in applications like modern deep learning where the dimension of variables scales to hundreds of millions [51].

The main contributions are as follows: 1) We propose an inherently privacy-preserving decentralized SGD algorithm. This algorithm can protect the privacy of all participating agents even when all messages shared by an agent are accessible to an adversary, a scenario which fails existing accuracy-friendly privacy-preserving approaches for decentralized (stochastic) optimization; 2) The inherently privacy-preserving approach

is efficient in communication and computation in that it is encryption-free and does not increase the size or the number of shared messages among participating agents; 3) We prove that the privacy approach does not affect the accuracy of decentralized stochastic optimization in both the convex-objective-function case and the non-convex-objective-function case, which is in distinct difference from difference-privacy based privacy solutions which trade accuracy for privacy; 4) We propose a rigorous information-theoretic privacy analysis approach to evaluate the strength of privacy inherently embedded in the algorithm; 5) We use both simulations for a decentralized estimation problem and numerical experiments for decentralized deep learning on a benchmark dataset to confirm the effectiveness of the proposed results.

The organization of the paper is as follows. Sec. II gives the problem formulation. Sec. III presents the inherently privacy-preserving decentralized SGD algorithm. Sec. IV proves converge of the proposed algorithm when the aggregate objective function is convex. Sec. V analyzes the convergence performance of the inherently privacy-preserving decentralized SGD algorithm when the objective functions are non-convex. Sec. VI proposes an information-theoretic approach to rigorously analyze the strength of enabled privacy. Sec. VII gives simulation results as well as numerical experiments on a benchmark machine learning dataset to confirm the obtained results. Finally Sec. VIII concludes the paper.

**Notation:** We use  $\mathbb{R}$  to denote the set of real numbers and  $\mathbb{R}^d$  the Euclidean space of dimension d. 1 denotes a column vector with all entries equal to 1. A vector is viewed as a column vector, unless otherwise stated. For a vector x,  $x_i$  denotes its ith element.  $A^T$  denotes the transpose of matrix A and  $x^Ty$  denotes the scalar product of two vectors x and y. We use  $\langle \cdot, \cdot \rangle$  to denote inner product and  $\| \cdot \|$  to denote the Euclidean norm for a vector (induced Euclidean norm for a matrix). A square matrix A is said to be column-stochastic when its elements in every column add up to one. A matrix A is said to be doubly-stochastic when both A and  $A^T$  are column-stochastic matrices. We use P(A) to denote the probability of an event A and  $E[x|\mathcal{F}]$  the expectation of a random variable x with  $\mathcal{F}$  denoting the sigma algebra, which will be omitted when clear from the context.

### II. PROBLEM FORMULATION

We consider a network of m agents which solve the following optimization problem cooperatively:

$$\min_{x \in \mathbb{R}^d} F(x) = \frac{1}{m} \sum_{i=1}^m f_i(x), \quad f_i(x) \triangleq \mathbb{E}_{\xi_i \sim \mathcal{D}_i} \left[ \ell_i(x, \xi_i) \right] \quad (1)$$

where x is common to all agents but  $\ell_i : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$  is a local cost function private to agent i.  $\mathcal{D}_i$  is the local distribution of random variable  $\xi_i$  which usually denotes a data sample in machine learning.

The above formulation also covers the empirical risk minimization problem where  $f_i(x)$  will be determined by  $f_i(x) = \frac{1}{n_i} \sum_{j=1}^{n_i} \ell(x, \xi_{i,j})$  with  $n_i$  denoting the number of data samples available to agent i and  $\xi_{i,j}$  representing the randomness associated with the jth sample for agent i.

Because of the randomness in  $\ell_i(x,\xi_i)$ , the gradient that each agent i can obtain is subject to noises. We denote the gradient that agent i has for optimization at iteration k as  $g_i^k(x_i,\xi_i)$ , with  $x_i$  denoting the local copy of the optimization variable on agent i. For the sake of notational simplicity, we will hereafter abbreviate  $g_i^k(x_i,\xi_i)$  as  $g_i^k$ . In this paper, we make the following standard assumption about  $f_i(\cdot)$  and  $g_i^k$ :

**Assumption 1.** *1)* For any  $1 \le i \le m$ ,  $x \in \mathbb{R}^d$ , and  $y \in \mathbb{R}^d$ , we have  $\|\nabla f_i(x) - \nabla f_i(y)\| \le L\|x - y\|$  for some L > 0.

2) All 
$$g_i^k$$
 satisfy  $\mathbb{E}_{\xi_i \sim \mathcal{D}_i} \left[ g_i^k(x_i, \xi_i) \right] = \nabla f_i(x_i)$  and  $\mathbb{E}_{\xi_i \sim \mathcal{D}_i} \left[ \| g_i^k(x_i, \xi_i) - \nabla f_i(x_i) \|^2 \right] \leq \sigma_q^2$  for any  $x \in \mathbb{R}^d$ .

We assume that the network of m agents interact on an undirected graph. The interaction can be described by a matrix W. More specifically, if agents i and j can communicate and interact with each other, then the (i,j)th entry of W, i.e.,  $w_{ij}$ , is positive. Otherwise,  $w_{ij}$  is zero. Because every agent is able to incorporate its own information in its optimization iteration update, we have  $w_{ii}>0$  for all  $1\leq i\leq m$ . The neighbor set  $\mathcal{N}_i$  of agent i is defined as the set of agents  $\{j|w_{ij}>0\}$ . So the neighbor set of agent i always includes itself. To ensure that the network can cooperatively solve (1), we make the following standard assumption about the interaction:

**Assumption 2.** The coupling matrix W is doubly-stochastic, and the spectral radius of the matrix  $W - \frac{\mathbf{1}\mathbf{1}^T}{m}$ , denoted as  $\rho$ , is less than one.

In decentralized stochastic optimization, gradients are directly computed from raw data and hence embed sensitive information of these data. For example, in sensor network based localization, the positions of sensor agents should be kept private in sensitive (hostile) environments [45], [52]. In decentralized-optimization based localization approaches, the gradient is a linear function of sensor locations and disclosing the gradient of an agent amounts to disclosing its position [45]. In machine learning applications, gradients are directly calculated from and embed information of sensitive training data [25]. If disclosed, these gradients can be used by an adversary to reversely recover the raw data used for training. In fact, even when gradients are subject to observation/measurement noise, the DLG attacker in [25] has been shown to be able to recover raw training data from shared gradients, pixel-wise accurate for images and token-wise matching for texts.

We consider two adversaries in decentralized stochastic optimization, honest-but-curious adversaries which are participating agents, and external eavesdroppers [53]:

- Honest-but-curious attacks are attacks in which a participating agent or multiple participating agents (colluding or not) follows all protocol steps correctly but is curious and collects all received intermediate data in an attempt to learn the sensitive information about other participating agents.
- Eavesdropping attacks are attacks in which an external eavesdropper eavesdrops upon all communication channels to intercept exchanged messages so as to learn sensitive information about sending agents.

An honest-but-curious adversary (e.g., agent i) has access to the internal state  $x_i$ , which is unavailable to external eavesdroppers. However, an eavesdropper has access to all shared information in the network, whereas an honest-but-curious agent can only access shared information that is destined to it.

Inspired by recent work on information-theoretic privacy [54], [55], we measure privacy using conditional differential entropy, which characterizes the minimum expected squared error of any adversary's estimator. More specifically, for a private random variable X and adversary's observation Y, we quantify privacy using the conditional differential entropy h(X|Y). From information theory, for any adversary's estimator  $\hat{X}(Y)$ , the following relationship holds [56]:

$$\mathbb{E}\left[\left(X - \hat{X}(Y)\right)^{2}\right] \ge \frac{e^{2h(X|Y)}}{2\pi e} \tag{2}$$

### III. AN INHERENTLY PRIVACY-PRESERVING DECENTRALIZED STOCHASTIC GRADIENT ALGORITHM

Before presenting our privacy-preserving decentralized stochastic gradient descent algorithm, we first discuss why conventional decentralized stochastic gradient descent algorithms leak private information about participating agents.

The most commonly used decentralized stochastic gradient descent algorithm takes the following form [57]:

$$x_i^{k+1} = \sum_{j \in \mathcal{N}_i} w_{ij} x_j^k - \lambda^k g_i^k$$

where  $x_i^k$  denotes the decision variable maintained by agent i at iteration k, and  $\lambda^k$  denotes the stepsize at iteration k. Note that our defined neighbor set  $\mathcal{N}_i$  always includes agent i.

In this conventional decentralized stochastic gradient descent algorithm, every agent i has to share  $x_i^k$  with all its neighbors. Because an adversary, e.g., an external eavesdropper, has access to all shared information, the adversary can calculate the gradient of every agent i based on publicly known W and  $\lambda^k$ . (Note that conventional algorithms need  $w_{ij}$  to be publicly known to ensure that a doubly-stochastic W can be established in a decentralized way.) Therefore, an adversary can infer the gradient of participating agents implementing conventional decentralized stochastic gradient descent algorithms.

Motivated by this observation, we propose to obscure the gradient information using time-varying heterogeneous stepsizes and an additional random mixing coefficient  $b_{ij}^k$ :

$$x_i^{k+1} = \sum_{j \in \mathcal{N}} v_{ij}^k$$
, where  $v_{ij}^k = w_{ij} x_j^k - b_{ij}^k \Lambda_j^k g_j^k$  (3)

where  $\Lambda_j^k \triangleq \mathrm{diag}\{\lambda_{j1}^k, \lambda_{j2}^k, \cdots, \lambda_{jd}^k\} \in \mathbb{R}^d$  is the stepsize matrix. The diagonal elements of matrix  $\Lambda_j^k$  are statistically independent random variables determined by agent j and they have the same mathematical expectation  $\bar{\lambda}_j^k$  and standard deviation  $\sigma_j^k$ . The coefficient  $b_{ij}^k \geq 0$  is also determined by agent j randomly under the constraint  $\sum_{i \in \mathcal{N}_j} b_{ij}^k = 1$ . Note that this constraint can be satisfied locally because agent j

chooses all  $b_{ij}^k$  for its neighbors  $i \in \mathcal{N}_j$  before sending  $v_{ij}^k$  to agent i. It is worth noting that different from gradient-tracking based decentralized optimization approaches (e.g., [13], [58], [59]) that share both gradient information and the optimization variable, our approach only shares one mixed variable, which is significant when the dimension of the optimization variable is high and communication bandwidth is limited.

Note that in our proposed algorithm (3), at iteration k, agent j determines  $\Lambda_j^k$  and  $b_{ij}^k$  randomly, and only sends  $v_{ij}^k$  to its neighbors  $i \in \mathcal{N}_j, i \neq j$ . Because both  $\Lambda_j^k$  and  $b_{ij}^k$  are time-varying and private to agent j, they can obscure the gradient  $g_j^k$  of agent j. It is worth noting that even when the dimension of gradient is high, in, e.g, machine learning applications, the statistically independent random variables in  $\Lambda_j^k$  will be able to obscure individual components of  $g_j^k$  independently. It is also worth noting that even when an adversary has access to all information shared in the network, since  $v_{jj}^k$  is not shared by agent j, the adversary still cannot use the constraint  $\sum_{i \in \mathcal{N}_j} b_{ij}^k = 1$  on  $b_{ij}^k$  to infer gradient information. In fact, in this case, the adversary can obtain  $\sum_{i \in \mathcal{N}_j, i \neq j} v_{ij}^k = (1 - w_{jj}) x_j^k - (1 - b_{jj}^k) \Lambda_j^k g_j^k$ . Since  $b_{jj}^k$  is unknown to the adversary, it can help obfuscate the gradient  $g_i^k$  (so does  $\Lambda_j^k$ ) against the adversary.

Augmenting the decision variables of all agents as

$$x^k = \begin{bmatrix} (x_1^k)^T & (x_2^k)^T & \cdots & (x_m^k)^T \end{bmatrix}^T \in \mathbb{R}^{md \times 1}$$

and representing  $B^k$  as a matrix with the (i, j)th element equal to  $b^k_{ij}$ , we can write the network dynamics of the proposed decentralized stochastic gradient descent algorithm as follows:

$$x^{k+1} = \hat{W}x^k - \hat{B}^k\hat{\Lambda}^k g^k \tag{4}$$

where

$$\hat{W} = W \otimes I_d \in \mathbb{R}^{md \times md}, \quad \hat{B}^k = B^k \otimes I_d \in \mathbb{R}^{md \times md},$$
$$\hat{\Lambda}^k = \operatorname{diag}\{\Lambda_1^k, \Lambda_2^k, \cdots, \Lambda_m^k\} \in \mathbb{R}^{md \times md},$$
$$g^k = \left[ (g_1^k)^T, (g_2^k)^T, \cdots, (g_m^k)^T \right]^T \in \mathbb{R}^{md \times 1}.$$

Here  $\otimes$  denotes Kronecker product and  $I_d$  denotes the identity matrix of dimension d. According to our assumption that all diagonal elements of  $\Lambda^k_i$  have the same mathematical expectation  $\bar{\lambda}^k_i$  and standard deviation  $\sigma^k_i$ , we have  $\mathbb{E}\left[\Lambda^k_i\right] = \bar{\lambda}^k_i I_d$  and  $\operatorname{var}\left[\Lambda^k_i\right] = (\sigma^k_i)^2 I_d$ . Furthermore, since  $b^k_{ij}$  satisfies  $\sum_{i \in \mathcal{N}_i} b^k_{ij} = 1$ ,  $B^k$  is always a column-stochastic matrix.

In the following two sections, we will analyze the convergence of the proposed algorithm (3) under convex objective functions and general non-convex objective functions, respectively.

## IV. CONVERGENCE ANALYSIS IN THE CONVEX OBJECTIVE FUNCTION CASE

In this section, we rigorously prove that when the aggregate objective function  $F(\cdot)$  is convex and the optimal solution set of (1) is nonempty, the proposed algorithm will guarantee that all agents converge to a same optimal solution. Note that only the aggregate objective function  $F(\cdot)$  is assumed to be convex and we do not require all individual objective functions  $f_i(\cdot)$  to be convex.

To analyze the convergence of the proposed algorithm, we first introduce the following lemma:

**Lemma 1.** (Theorem 1 in [34]) Let  $\{\mathbf{v}^k\} \subset \mathbb{R}^d$  and  $\{\mathbf{u}^k\} \subset \mathbb{R}^p$  be random nonnegative vector sequences, and  $\{a^k\}$  and  $\{b^k\}$  be random nonnegative scalar sequences such that

$$\mathbb{E}\left[\mathbf{v}^{k+1}|\mathcal{F}^k\right] \le (V^k + a^k \mathbf{1} \mathbf{1}^T) \mathbf{v}^k + b^k \mathbf{1} - H^k \mathbf{u}^k, \quad \forall k \ge 0$$

holds almost surely, where  $\{V^k\}$  and  $\{H^k\}$  are random sequences of nonnegative matrices and  $\mathbb{E}\left[\mathbf{v}^{k+1}|\mathcal{F}^k\right]$  denotes the conditional expectation given  $\mathbf{v}^\ell, \mathbf{u}^\ell, a^\ell, b^\ell, V^\ell, H^\ell$  for  $\ell = 0, 1, \ldots, k$ . Assume that  $\{a^k\}$  and  $\{b^k\}$  satisfy  $\sum_{k=0}^{\infty} a^k < \infty$  and  $\sum_{k=0}^{\infty} b^k < \infty$  almost surely, and that there exists a (deterministic) vector  $\pi > 0$  such that almost surely

$$\pi^T V^k < \pi^T, \qquad \pi^T H^k > 0 \qquad \forall k > 0$$

Then, we have 1)  $\{\pi^T \mathbf{v}^k\}$  converges almost surely to some random variable  $\pi^T \mathbf{v} \geq 0$ ; 2)  $\{\mathbf{v}^k\}$  is bounded almost surely, and 3)  $\sum_{k=0}^{\infty} \pi^T H^k \mathbf{u}^k < \infty$  holds almost surely.

Based on the above lemma, we first prove the following general convergence results for distributed algorithms for problem (1):

**Theorem 1.** Assume that problem (1) has a solution. Suppose that a distributed algorithm generates sequences  $\{x_i^k\} \subseteq \mathbb{R}^d$  such that almost surely we have for any optimal solution  $\theta^*$  of (1),

$$\begin{bmatrix} \mathbb{E}\left[\|\bar{x}^{k+1} - \theta^*\|^2 | \mathcal{F}^k\right] \\ \mathbb{E}\left[\sum_{i=1}^m \|x_i^{k+1} - \bar{x}^{k+1}\|^2 | \mathcal{F}^k\right] \end{bmatrix}$$

$$\leq \left(\begin{bmatrix} 1 & \frac{1}{m} \\ 0 & \rho \end{bmatrix} + a^k \mathbf{1} \mathbf{1}^T \right) \begin{bmatrix} \|\bar{x}^k - \theta^*\|^2 \\ \sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2 \end{bmatrix}$$

$$+ b^k \mathbf{1} - c^k \begin{bmatrix} F(\bar{x}^k) - F(\theta^*) \\ 0 \end{bmatrix}, \quad \forall k \geq 0$$
(5)

where  $\bar{x}^k = \frac{1}{m} \sum_{i=1}^m x_i^k$ ,  $\mathcal{F}^k = \{x_i^\ell, i \in [m], 0 \le \ell \le k\}$ , the random nonnegative scalar sequences  $\{a^k\}$ ,  $\{b^k\}$  satisfy  $\sum_{k=0}^\infty a^k < \infty$  and  $\sum_{k=0}^\infty b^k < \infty$  almost surely, the deterministic nonnegative sequence  $\{c^k\}$  satisfies  $\sum_{k=0}^\infty c^k = \infty$ , and  $\rho$  satisfies  $0 < \rho < 1$ . Then, we have  $\lim_{k \to \infty} \|x_i^k - \bar{x}^k\| = 0$  almost surely for all agents i, and there is an optimal solution  $\tilde{\theta}^*$  of (1) such that  $\lim_{k \to \infty} \|\bar{x}^k - \tilde{\theta}^*\| = 0$  holds almost surely.

*Proof.* Let  $\theta^*$  be an arbitrary but fixed optimal solution of problem (1). Then, we have  $F(\bar{x}^k) - F(\theta^*) \geq 0$  for all k. Hence, by letting  $\mathbf{v}^k = \left[\|\bar{x}^k - \theta^*\|^2, \sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2\right]^T$ , from relation (5) it follows that almost surely for all  $k \geq 0$ ,

$$\mathbb{E}\left[\mathbf{v}^{k+1}|\mathcal{F}^k\right] \le \left(\begin{bmatrix} 1 & \frac{1}{m} \\ 0 & \rho \end{bmatrix} + a^k \mathbf{1} \mathbf{1}^T\right) \mathbf{v}^k + b^k \mathbf{1} \qquad (6)$$

Consider the vector  $\pi = [1, \frac{1}{1-\rho}]^T$  and note

$$\pi^T \left[ \begin{array}{cc} 1 & \frac{1}{m} \\ 0 & \rho \end{array} \right] \le \pi^T$$

Thus, relation (6) satisfies all the conditions of Lemma 1. By Lemma 1, it follows that the limit  $\lim_{k\to\infty} \pi^T \mathbf{v}^k$  exists almost surely, and that the sequences  $\{\|\bar{x}^k - \theta^*\|^2\}$  and  $\{\sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2\}$  are bounded almost surely. From (6)

we have the following relation almost surely for the second element of  $\mathbf{v}^k$ :

$$\mathbb{E}\left[\sum_{i=1}^{m} \|x_i^{k+1} - \bar{x}^{k+1}\|^2 |\mathcal{F}^k|\right]$$

$$\leq (\rho + a^k) \sum_{i=1}^{m} \|x_i^k - \bar{x}^k\|^2 + \beta^k \quad \forall k \geq 0$$

where

$$\beta^{k} = a^{k} (\|\bar{x}^{k} - \theta^{*}\|^{2} + b^{k})$$

Since  $\sum_{k=0}^{\infty}a^k<\infty$  holds almost surely by our assumption, and the sequences  $\{\|\bar{x}^k-\theta^*\|^2\}$  and  $\{\sum_{i=1}^m\|x_i^k-\bar{x}^k\|^2\}$  are bounded almost surely, it follows that  $\sum_{k=0}^{\infty}\beta^k<\infty$  holds almost surely. Thus, the preceding relation satisfies the conditions of Lemma 2 in the appendix with  $v^k=\sum_{i=1}^m\|x_i^k-\bar{x}^k\|^2$ ,  $\alpha^k=a^k,\ q^k=1-\rho,$  and  $p^k=\beta^k$  due to our assumptions that  $\sum_{k=0}^{\infty}b^k<\infty$  holds almost surely and  $0<\rho<1$ . By Lemma 2 in the appendix it follows that we have the following relations almost surely:

$$\sum_{k=0}^{\infty} \sum_{i=1}^{m} \|x_i^k - \bar{x}^k\|^2 < \infty, \lim_{k \to \infty} \sum_{i=1}^{m} \|x_i^k - \bar{x}^k\|^2 = 0$$
 (7)

It remains to show that  $\|\bar{x}^k - \theta^*\|^2 \to 0$  holds almost surely. For this, we consider relation (5) and focus on the first element of  $\mathbf{v}^k$ , for which we obtain almost surely for all  $k \ge 0$ :

$$\mathbb{E}\left[\|\bar{x}^{k+1} - \theta^*\|^2 |\mathcal{F}^k|\right] \le (1 + a^k) \|\bar{x}^k - \theta^*\|^2 + \left(\frac{1}{m} + a^k\right) \sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2 + b^k - c^k (F(\bar{x}^k) - F(\theta^*))$$

The preceding relation satisfies Lemma 3 in the appendix with  $\phi=F,\ z^*=\theta^*,\ z^k=\bar{x}^k,\ \alpha^k=a^k,\ \eta^k=c^k,\ \text{and}\ \beta^k=(\frac{1}{m}+a^k)\sum_{i=1}^m\|x_i^k-\bar{x}^k\|^2+b^k.$  By our assumptions, the sequences  $\{a^k\}$  and  $\{b^k\}$  are summable almost surely, and  $\sum_{k=0}^\infty c^k=\infty.$  In view of (7), it follows that  $\sum_{k=0}^\infty \beta^k<\infty$  holds almost surely. Hence, all the conditions of Lemma 3 in the appendix are satisfied and, consequently,  $\{\bar{x}^k\}$  converges to some optimal solution almost surely.

Having obtained Theorem 1, we are in position to prove that all agents will converge to a same optimal solution of (1) when the aggregate objective function  $F(\cdot)$  is convex.

**Theorem 2.** Under Assumption 1 and Assumption 2, if the aggregate objective function  $F(\cdot)$  is convex and the optimal solution set of problem (1) is non-empty, then all  $x_i^k$   $(1 \le p \le m)$  converge to a solution of problem (1) if the mathematical expectation and standard deviation of stepsize  $\Lambda_i^k$  (represented as  $\bar{\lambda}_i^k I_d$  and  $\sigma_i^k I_d$ , respectively) satisfy for all i:

1) non-summable but square summable condition:

$$\sum_{k=0}^{\infty} \bar{\lambda}_i^k = +\infty, \quad \sum_{k=0}^{\infty} (\bar{\lambda}_i^k)^2 < \infty, \quad \sum_{k=0}^{\infty} (\sigma_i^k)^2 < \infty$$
 (9)

2) heterogeneity condition:

$$\sum_{k=0}^{\infty} \sum_{i,j \in \{1,2,\cdots,m\}, i \neq j} |\bar{\lambda}_i^k - \bar{\lambda}_j^k| < \infty \tag{10}$$

*Proof.* The basic idea is to apply Theorem 1. We establish needed relationships for  $\mathbb{E}\left[\|\bar{x}^{k+1}-\theta^*\|^2|\mathcal{F}^k\right]$  and  $\mathbb{E}\left[\sum_{i=1}^m\|x_i^{k+1}-\bar{x}^{k+1}\|^2|\mathcal{F}^k\right]$  in Step I and Step II below, respectively.

Part I: We first analyze  $\|\bar{x}^{k+1} - \theta^*\|^2$ .

Using the definition  $\bar{x}^k = \frac{1}{m} \sum_{i=1}^m x_i^k$  and (4), we have

$$\bar{x}^{k+1} = \frac{\mathbf{1}^T \otimes I_d}{m} x^{k+1}$$

$$= \frac{\mathbf{1}^T \otimes I_d}{m} \hat{W} x^k - \frac{\mathbf{1}^T \otimes I_d}{m} \hat{B}^k \hat{\Lambda}^k g^k$$

$$= \bar{x}^k + \frac{\mathbf{1}^T \otimes I_d}{m} \hat{B}^k \hat{\Lambda}^k g^k$$

$$= \bar{x}^k - \frac{1}{m} \sum_{i=1}^m \Lambda_i^k g_i^k$$
(11)

where in the third equality we used the doubly-stochastic property of W, and in the last equality we used the column stochastic property of  $B^k$ . Therefore, for any optimal solution  $\theta^*$ , we have

$$\bar{x}^{k+1} - \theta^* = \bar{x}^k - \theta^* - \frac{1}{m} \sum_{i=1}^m \Lambda_i^k g_i^k$$
 (12)

The preceding relation implies

$$\begin{split} & \left\| \bar{x}^{k+1} - \theta^* \right\|^2 \\ & = \left\| \bar{x}^k - \theta^* \right\|^2 - \frac{2}{m} \sum_{i=1}^m \left\langle \Lambda_i^k g_i^k, \bar{x}^k - \theta^* \right\rangle + \frac{1}{m^2} \left\| \sum_{i=1}^m \Lambda_i^k g_i^k \right\|^2 \\ & = \left\| \bar{x}^k - \theta^* \right\|^2 - \frac{2}{m} \sum_{i=1}^m \left\langle \Lambda_i^k g_i^k, \bar{x}^k - \theta^* \right\rangle + \frac{1}{m^2} \|\Lambda^k g^k\|^2 \\ & \leq \left\| \bar{x}^k - \theta^* \right\|^2 - \frac{2}{m} \sum_{i=1}^m \left\langle \Lambda_i^k g_i^k, \bar{x}^k - \theta^* \right\rangle + \frac{1}{m^2} \|\Lambda^k\|^2 \|g^k\|^2 \\ & \text{where we defined } \Lambda^k = [\Lambda_1^k, \cdots, \Lambda_m^k]^T \in \mathbb{R}^{d \times md}. \end{split}$$

Defining

$$\nabla f(x^k) = \left[ (\nabla f_1(x_1^k))^T, \cdots, (\nabla f_m(x_m^k))^T \right]^T$$
 (14)

we have

$$||g^{k}||^{2} = ||g^{k} - \nabla f(x^{k}) + \nabla f(x^{k})||^{2}$$

$$\leq 2||g^{k} - \nabla f(x^{k})||^{2} + 2||\nabla f(x^{k})||^{2}$$
(15)

Denoting  $x^* = \mathbf{1} \otimes \theta^*$  and then adding and subtracting

 $\nabla f(x^*)$  to  $\nabla f(x^k)$  yield

$$\|\nabla f(x^{k})\|^{2} = \|\nabla f(x^{k}) - \nabla f(x^{*}) + \nabla f(x^{*})\|^{2}$$

$$\leq 2\|\nabla f(x^{k}) - \nabla f(x^{*})\|^{2} + 2\|\nabla f(x^{*})\|^{2}$$

$$\leq 2L^{2}\|x^{k} - x^{*}\|^{2} + 2\|\nabla f(x^{*})\|^{2}$$

$$\leq 2L^{2}\sum_{i=1}^{m}\|x_{i}^{k} - \theta^{*}\|^{2} + 2\|\nabla f(x^{*})\|^{2}$$

$$\leq 2L^{2}\sum_{i=1}^{m}\|x_{i}^{k} - \bar{x}^{k} + \bar{x}^{k} - \theta^{*}\|^{2} + 2\|\nabla f(x^{*})\|^{2}$$

$$\leq 4L^{2}\sum_{i=1}^{m}\|x_{i}^{k} - \bar{x}^{k}\|^{2} + 4mL^{2}\|\bar{x}^{k} - \theta^{*}\|^{2}$$

$$+ 2\|\nabla f(x^{*})\|^{2}$$
(16)

Combining (13), (15), and (16) yields

$$\|\bar{x}^{k+1} - \theta^*\|^2 \le \|\bar{x}^k - \theta^*\|^2 - \frac{2}{m} \sum_{i=1}^m \left\langle \Lambda_i^k g_i^k, \bar{x}^k - \theta^* \right\rangle + \frac{\|\Lambda^k\|^2}{m^2} \times \left( 8L^2 \sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2 + 8mL^2 \|\bar{x}^k - \theta^*\|^2 + 4\|\nabla f(x^*)\|^2 + 2\|g^k - \nabla f(x^k)\|^2 \right)$$

$$(17)$$

Taking the conditional expectation, given  $\mathcal{F}^k = \{x^0,\dots,x^k\}$ , and using the relationship  $\|\Lambda^k\|^2 \leq \|\Lambda^k\|_F^2 = \sum_{j=1}^m \sum_{p=1}^d (\lambda_{jp}^k)^2$ , we obtain

$$\mathbb{E}\left[\left\|\bar{x}^{k+1} - \theta^*\right\|^2 | \mathcal{F}^k\right] \\
\leq \left\|\bar{x}^k - \theta^*\right\|^2 - \frac{2}{m} \sum_{i=1}^m \left\langle \bar{\lambda}_i^k \nabla f_i(x_i^k), \bar{x}^k - \theta^* \right\rangle \\
+ \frac{\sum_{i=1}^m d\left((\bar{\lambda}_i^k)^2 + (\sigma_i^k)^2\right)}{m^2} \times \left(8L^2 \sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2 + 8mL^2 \|\bar{x}^k - \theta^*\|^2 + 4\|\nabla f(x^*)\|^2 + 2m\sigma_g^2\right) \tag{18}$$

Note that  $\sigma_g^2$  is the variance of stochastic gradients from Assumption 1, and we used the fact that the mathematical expectation and standard deviation of  $\Lambda_i^k$  is  $\bar{\lambda}_i^k I_d$  and  $\sigma_i^k I_d$ , respectively.

We next estimate the inner-product term, for which we have

$$\langle \bar{\lambda}_{i}^{k} \nabla f_{i}(x_{i}^{k}), \bar{x}^{k} - \theta^{*} \rangle = \langle \bar{\lambda}_{i}^{k} (\nabla f_{i}(x_{i}^{k}) - \nabla f_{i}(\bar{x}^{k})), \bar{x}^{k} - \theta^{*} \rangle + \langle \bar{\lambda}_{i}^{k} \nabla f_{i}(\bar{x}^{k}), \bar{x}^{k} - \theta^{*} \rangle$$

$$(19)$$

Using the Lipschitz property of  $\nabla f_i$  in Assumption 1, we have

$$\langle \bar{\lambda}_{i}^{k}(\nabla f_{i}(x_{i}^{k}) - \nabla f_{i}(\bar{x}^{k})), \bar{x}^{k} - \theta^{*} \rangle 
\geq -L\bar{\lambda}_{i}^{k} \|x_{i}^{k} - \bar{x}^{k}\| \|\bar{x}^{k} - \theta^{*}\| 
\geq -\frac{1}{2} \|x_{i}^{k} - \bar{x}^{k}\|^{2} - \frac{1}{2} L^{2}(\bar{\lambda}_{i}^{k})^{2} \|\bar{x}^{k} - \theta^{*}\|^{2}$$
(20)

Defining  $\bar{\lambda}^k \triangleq \frac{\sum_{i=1}^m \bar{\lambda}_i^k}{m}$ , we have

$$\langle \bar{\lambda}_{i}^{k} \nabla f_{i}(\bar{x}^{k}), \bar{x}^{k} - \theta^{*} \rangle = \\ \langle (\bar{\lambda}_{i}^{k} - \bar{\lambda}^{k}) \nabla f_{i}(\bar{x}^{k}), \bar{x}^{k} - \theta^{*} \rangle + \langle \bar{\lambda}^{k} \nabla f_{i}(\bar{x}^{k}), \bar{x}^{k} - \theta^{*} \rangle$$
(21)

Further defining a vector  $\bar{\pmb{\lambda}}^k=[\bar{\lambda}_1^k,\cdots,\bar{\lambda}_m^k]^T$  and combining (19)-(21) yield

$$\frac{\sum_{i=1}^{m} \left\langle \bar{\lambda}_{i}^{k} \nabla f_{i}(x_{i}^{k}), \bar{x}^{k} - \theta^{*} \right\rangle}{m} \\
\geq \frac{\sum_{i=1}^{m} (-\|x_{i}^{k} - \bar{x}^{k}\|^{2} - L^{2}(\bar{\lambda}_{i}^{k})^{2}\|\bar{x}^{k} - \theta^{*}\|^{2})}{2m} + \\
\frac{\sum_{i=1}^{m} \left( \left\langle (\bar{\lambda}_{i}^{k} - \bar{\lambda}^{k}) \nabla f_{i}(\bar{x}^{k}), \bar{x}^{k} - \theta^{*} \right\rangle + \left\langle \bar{\lambda}^{k} \nabla f_{i}(\bar{x}^{k}), \bar{x}^{k} - \theta^{*} \right\rangle \right)}{m} \\
= -\frac{\sum_{i=1}^{m} \|x_{i}^{k} - \bar{x}^{k}\|^{2}}{2m} - \frac{L^{2} \|\bar{\lambda}^{k}\|^{2} \|\bar{x}^{k} - \theta^{*}\|^{2}}{2m} + \\
\frac{\sum_{i=1}^{m} \left\langle (\bar{\lambda}_{i}^{k} - \bar{\lambda}^{k}) \nabla f_{i}(\bar{x}^{k}), \bar{x}^{k} - \theta^{*} \right\rangle}{m} + \bar{\lambda}^{k} \left\langle \nabla F(\bar{x}^{k}), \bar{x}^{k} - \theta^{*} \right\rangle} \\
\geq -\frac{\sum_{i=1}^{m} \|x_{i}^{k} - \bar{x}^{k}\|^{2}}{2m} - \frac{L^{2} \|\bar{\lambda}^{k}\|^{2} \|\bar{x}^{k} - \theta^{*}\|^{2}}{2m} + \\
\frac{\sum_{i=1}^{m} \left\langle (\bar{\lambda}_{i}^{k} - \bar{\lambda}^{k}) \nabla f_{i}(\bar{x}^{k}), \bar{x}^{k} - \theta^{*} \right\rangle}{m} + \bar{\lambda}^{k} (F(\bar{x}^{k}) - F(\theta^{*}))$$
(22)

where we used the convexity of  $F(\cdot)$  in the last inequality.

Noting  $\bar{\boldsymbol{\lambda}}^k = [\bar{\lambda}_1^k, \cdots, \bar{\lambda}_m^k]^T$  and  $\bar{\lambda}^k = \frac{\sum_{i=1}^m \bar{\lambda}_i^k}{m}$ , we always have

$$\underline{\sum_{i=1}^{m} \left\langle (\bar{\lambda}_{i}^{k} - \bar{\lambda}^{k}) \nabla f_{i}(\bar{x}^{k}), \bar{x}^{k} - \theta^{*} \right\rangle} \\
= \frac{\left\langle \sum_{i=1}^{m} (\bar{\lambda}_{i}^{k} - \bar{\lambda}^{k}) \nabla f_{i}(\bar{x}^{k}), \bar{x}^{k} - \theta^{*} \right\rangle}{m} \\
\geq -\frac{\|\sum_{i=1}^{m} (\bar{\lambda}_{i}^{k} - \bar{\lambda}^{k}) \nabla f_{i}(\bar{x}^{k}) \| \|\bar{x}^{k} - \theta^{*} \|}{m} \\
= -\frac{\|\left( (\bar{\lambda}^{k} - \bar{\lambda}^{k} \mathbf{1}_{m}) \otimes \mathbf{1}_{d} \right)^{T} \nabla f(\mathbf{1} \otimes \bar{x}^{k}) \| \|\bar{x}^{k} - \theta^{*} \|}{m} \\
\geq -\frac{\sqrt{d}}{m} \|\bar{\lambda}^{k} - \bar{\lambda}^{k} \mathbf{1}_{m} \| \|\nabla f(\mathbf{1} \otimes \bar{x}^{k}) \| \|\bar{x}^{k} - \theta^{*} \|}{2} \\
\geq -\frac{\sqrt{d}}{2m} \|\bar{\lambda}^{k} - \bar{\lambda}^{k} \mathbf{1}_{m} \| (\|\nabla f(\mathbf{1} \otimes \bar{x}^{k}) \|^{2} + \|\bar{x}^{k} - \theta^{*} \|^{2}) \\
(23)$$

where  $\otimes$  is Kronecker product and  $\nabla f$  is defined in (14).

Furthermore,  $\|\nabla f(\mathbf{1}\otimes \bar{x}^k)\|$  can be bounded by

$$\|\nabla f(\mathbf{1} \otimes \bar{x}^{k})\| = \|\nabla f(\mathbf{1} \otimes \bar{x}^{k}) - \nabla f(x^{*}) + \nabla f(x^{*})\|$$

$$\leq \|\nabla f(\mathbf{1} \otimes \bar{x}^{k}) - \nabla f(x^{*})\| + \|\nabla f(x^{*})\|$$

$$\leq L\|\mathbf{1} \otimes \bar{x}^{k} - x^{*}\| + \|\nabla f(x^{*})\|$$

$$= L\sqrt{m} \|\bar{x}^{k} - \theta^{*}\| + \|\nabla f(x^{*})\|$$
(24)

Combining (22), (23), and (24) yields

$$\frac{\sum_{i=1}^{m} \left\langle \bar{\lambda}_{i}^{k} g_{i}^{k}, \bar{x}^{k} - \theta^{*} \right\rangle}{m} \\
\geq -\frac{\sum_{i=1}^{m} \|x_{i}^{k} - \bar{x}^{k}\|^{2}}{2m} - \frac{L^{2} \|\bar{\boldsymbol{\lambda}}^{k}\|^{2} \|\bar{x}^{k} - \theta^{*}\|^{2}}{2m} \\
- \frac{\sqrt{d} (2L^{2}m + 1)}{2m} \|\bar{\boldsymbol{\lambda}}^{k} - \bar{\lambda}^{k} \mathbf{1}_{m} \| \|\bar{x}^{k} - \theta^{*}\|^{2} \\
- \frac{\sqrt{d}}{m} \|\bar{\boldsymbol{\lambda}}^{k} - \bar{\lambda}^{k} \mathbf{1}_{m} \| \|\nabla f(x^{*})\|^{2} + \bar{\lambda}^{k} (F(\bar{x}^{k}) - F(\theta^{*}) \tag{25}$$

Substituting (25) into (18) leads to

$$\mathbb{E}\left[\left\|\bar{x}^{k+1} - \theta^*\right\|^2 | \mathcal{F}^k\right] \\
\leq \left\|\bar{x}^k - \theta^*\right\|^2 + \frac{\sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2}{m} + \frac{L^2 \|\bar{\boldsymbol{\lambda}}^k\|^2 \|\bar{x}^k - \theta^*\|^2}{m} \\
+ \frac{\sqrt{d}(2L^2m+1)}{m} \|\bar{\boldsymbol{\lambda}}^k - \bar{\lambda}^k \mathbf{1}_m\| \|\bar{x}^k - \theta^*\|^2 \\
+ \frac{2\sqrt{d}}{m} \|\bar{\boldsymbol{\lambda}}^k - \bar{\lambda}^k \mathbf{1}_m\| \|\nabla f(x^*)\|^2 - 2\bar{\lambda}^k (F(\bar{x}^k) - F(\theta^*)) \\
+ \frac{\sum_{i=1}^m d\left((\bar{\lambda}_i^k)^2 + (\sigma_i^k)^2\right)}{m^2} \times \left(8L^2 \sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2 \\
+ 8mL^2 \|\bar{x}^k - \theta^*\|^2 + 4\|\nabla f(x^*)\|^2 + 2m\sigma_g^2\right) \tag{26}$$

Part II: Next we analyze  $\sum_{i=1}^m \|x_i^{k+1} - \bar{x}^{k+1}\|^2$ . Using (4) and (11), and defining  $\hat{\Pi} = (I - \frac{\mathbf{1}\mathbf{1}^T}{m}) \otimes I_d$ , we have

$$x^{k+1} - \mathbf{1} \otimes \bar{x}^{k+1} = \hat{W}x^k - \mathbf{1} \otimes \bar{x}^k - \hat{\Pi}\hat{B}^k\hat{\Lambda}^k g^k$$
$$= \hat{W}(x^k - \mathbf{1} \otimes \bar{x}^k) - \hat{\Pi}\hat{B}^k\hat{\Lambda}^k g^k$$

where we used the doubly-stochastic property of W which leads to  $\hat{W}(\mathbf{1} \otimes \bar{x}^k) = \mathbf{1} \otimes \bar{x}^k$ .

Taking norm on both sides and using  $\rho = \|W - \frac{1}{m}\mathbf{1}\mathbf{1}^T\|$  from Assumption 2, we obtain

$$\begin{split} \|x^{k+1} - \mathbf{1} \otimes \bar{x}^{k+1}\| \\ & \leq \rho \|x^k - \mathbf{1} \otimes \bar{x}^k\| + \|\hat{\Pi}\| \|\hat{B}^k\| \|\hat{\Lambda}^k\| \|g^k\| \\ & < \rho \|x^k - \mathbf{1} \otimes \bar{x}^k\| + md\|\hat{\Lambda}^k\| \|g^k\| \end{split}$$

where in the second inequality we used  $\|\Pi\| = 1$  and  $\|\hat{B}^k\| \le \|\hat{B}^k\|_F \le md$  with  $\|\cdot\|_F$  denoting the Frobenius matrix norm.

By taking squares on both sides and using the inequality  $2ab \le \epsilon a^2 + \epsilon^{-1}b^2$  valid for any  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}$ , and  $\epsilon > 0$ , we obtain (noting  $\|x^k - \mathbf{1} \otimes \bar{x}^k\|^2 = \sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2$ ):

$$\sum_{i=1}^{m} \|x_i^{k+1} - \bar{x}^{k+1}\|^2 \le \rho^2 (1+\epsilon) \sum_{i=1}^{m} \|x_i^k - \bar{x}^k\|^2 + m^2 d^2 (1+\epsilon^{-1}) \|\hat{\Lambda}^k\|^2 \|g^k\|^2$$
(27)

By letting  $1 + \epsilon = \frac{1}{\rho}$  and hence  $1 + \epsilon^{-1} = (1 - \rho)^{-1}$ , we arrive at

$$\sum_{i=1}^{m} \|x_i^{k+1} - \bar{x}^{k+1}\|^2 \le \rho \sum_{i=1}^{m} \|x_i^k - \bar{x}^k\|^2 + \frac{m^2 d^2}{1 - \rho} \|\hat{\Lambda}^k\|^2 \|g^k\|^2$$
 (28)

Noting  $\|\hat{\Lambda}^k\| = \max_{1 \leq j \leq m, 1 \leq p \leq d} \lambda_{jp}^k$ , and using the relationship (15) and (16) for  $\|g^k\|$ , we can obtain

$$\sum_{i=1}^{m} \|x_i^{k+1} - \bar{x}^{k+1}\|^2 \\
\leq \rho \sum_{i=1}^{m} \|x_i^k - \bar{x}^k\|^2 + \frac{m^2 d^2 \max_{j,p} (\lambda_{jp}^k)^2}{1 - \rho} \times \\
\left(8L^2 \sum_{i=1}^{m} \|x_i^k - \bar{x}^k\|^2 + 8mL^2 \|\bar{x}^k - \theta^*\|^2 + 4\|\nabla f(x^*)\|^2 + 2\|g^k - \nabla f(x^k)\|^2\right)$$
(29)

Taking the conditional expectation, given  $\mathcal{F}^k = \{x^0, \dots, x^k\}$  yields

$$\mathbb{E}\left[\sum_{i=1}^{m} \|x_i^{k+1} - \bar{x}^{k+1}\|^2\right]$$

$$\leq \rho \sum_{i=1}^{m} \|x_i^k - \bar{x}^k\|^2 + \frac{m^2 d^2 \max_i \left((\bar{\lambda}_i^k)^2 + (\sigma_i^k)^2\right)}{1 - \rho} \times \left(8L^2 \sum_{i=1}^{m} \|x_i^k - \bar{x}^k\|^2 + 8mL^2 \|\bar{x}^k - \theta^*\|^2 + 4\|\nabla f(x^*)\|^2 + 2m\sigma_g^2\right)$$

Combining (26) and (30), we have

$$\mathbb{E}\left[\mathbf{v}^{k+1}|\mathcal{F}^k\right] \le \left(\begin{bmatrix} 1 & \frac{1}{m} \\ 0 & \rho \end{bmatrix} + A^k\right)\mathbf{v}^k + \mathbf{b}^k - 2\bar{\lambda}^k\mathbf{u}^k \tag{31}$$

(30)

where

$$\mathbf{v}^{k} = \left[ \begin{array}{c} \|\bar{x}^{k} - \theta^{*}\|^{2} \\ \sum_{i=1}^{m} \|x_{i}^{k} - \bar{x}^{k}\|^{2} \end{array} \right],$$

$$\mathbf{u}^{k} = \begin{bmatrix} F(\bar{x}^{k}) - F(\theta^{*}) \\ 0 \end{bmatrix}, \quad A^{k} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

with

$$A_{11} = \frac{L^{2} \|\bar{\boldsymbol{\lambda}}^{k}\|^{2}}{m} + \frac{\sqrt{d}(2L^{2}m+1)}{m} \|\bar{\boldsymbol{\lambda}}^{k} - \bar{\boldsymbol{\lambda}}^{k} \mathbf{1}_{m}\|$$

$$+ \frac{8mL^{2} \sum_{i=1}^{m} d\left((\bar{\lambda}_{i}^{k})^{2} + (\sigma_{i}^{k})^{2}\right)}{m^{2}},$$

$$A_{12} = \frac{8L^{2} \sum_{i=1}^{m} d\left((\bar{\lambda}_{i}^{k})^{2} + (\sigma_{i}^{k})^{2}\right)}{m^{2}}$$

$$A_{21} = \frac{8m^{3}L^{2}d^{2} \max_{i} \left((\bar{\lambda}_{i}^{k})^{2} + (\sigma_{i}^{k})^{2}\right)}{1 - \rho}$$

$$A_{22} = \frac{8m^{2}L^{2}d^{2} \max_{i} \left((\bar{\lambda}_{i}^{k})^{2} + (\sigma_{i}^{k})^{2}\right)}{1 - \rho}$$

and 
$$\begin{aligned} \mathbf{b}^k &= \left[\begin{array}{c} b_1^k \\ b_2^k \end{array}\right] \text{ with} \\ b_1^k &= & \frac{2\sqrt{d}}{m} \|\bar{\boldsymbol{\lambda}}^k - \bar{\boldsymbol{\lambda}}^k \mathbf{1}_m \| \|\nabla f(x^*)\|^2 \\ &+ & \frac{\sum_{i=1}^m d\left((\bar{\boldsymbol{\lambda}}_i^k)^2 + (\sigma_i^k)^2\right)}{m^2} (4\|\nabla f(x^*)\|^2 + 2m\sigma_g^2), \\ b_2^k &= & \frac{m^2 d^2 \max\left((\bar{\boldsymbol{\lambda}}_i^k)^2 + (\sigma_i^k)^2\right) (4\|\nabla f(x^*)\|^2 + 2m\sigma_g^2)}{1 - o} \end{aligned}$$

Because  $A^k \leq a^k \mathbf{1} \mathbf{1}^T$  and  $\mathbf{b}^k \leq b^k \mathbf{1}$  hold when  $a^k$  and  $b^k$  are set to  $a^k = \max\{A_{11}, A_{12}, A_{21}, A_{22}\}$  and  $b^k = \max\{b_1^k, b_2^k\}$ , respectively, we can see that (5) in Theorem 1 is satisfied. Further note that under the conditions in the statement, all conditions for  $\{a^k\}$ ,  $\{b^k\}$ , and  $\{c^k\}$  in Theorem 1 are also satisfied. Therefore, we have the claimed results.

**Remark 1.** In implementation, to satisfy the heterogeneity condition in (10), all agents can choose the same expected value for their stepsizes. For example, all agents can set their expected stepsizes to  $\frac{1}{k}$ , i.e.,  $\bar{\lambda}_1^k = \bar{\lambda}_2^k = \cdots = \bar{\lambda}_m^k = \frac{1}{k}$ . Since it is the exact random outcome  $\lambda_i^k$  that is used to obfuscate the gradient, and the exact value  $\lambda_i^k$  is known only to agent i, the privacy of every agent i can still be protected even when  $\bar{\lambda}_i^k$  is publicly known. Moreover, individual agents can also avoid revealing the expected values of their stepsizes. For example, every agent can deviate its expected stepsize from the baseline  $\frac{1}{k}$  in a finite number of iterations. The indices of these iterations are private to individual agents. As long as the deviation in each of these iterations is finite, the heterogeneity condition in (10) will still be satisfied.

Remark 2. Because the evolution of  $x_i^k$  to the optimal solution satisfies the conditions in Theorem 1, we can leverage Theorem 1 to examine the convergence speed. The first relationship in (7) (i.e.,  $\sum_{k=0}^{\infty} \sum_{i=1}^{m} \|x_i^k - \bar{x}^k\|^2 < \infty$ ) implies that  $\sum_{i=1}^{m} \|x_i^k - \bar{x}^k\|^2$  decreases to zero with a rate no slower than  $\mathcal{O}(\frac{1}{k})$ , and hence we have  $x_i^k$  converging to  $\bar{x}^k$  no slower than  $\mathcal{O}(\frac{1}{k^{0.5}})$ . Moreover, given that  $a^k$  and  $b^k$  in (8) are summable (and hence decrease to zero no slower than  $\mathcal{O}(\frac{1}{k})$ ) and  $c^k$  in (8) corresponds to the average expected stepsize  $\bar{\lambda}^k = \frac{\sum_{i=1}^{m} \bar{\lambda}_i^k}{m}$  in our algorithm (which is square summable and hence decreases to zero no slower than  $\mathcal{O}(\frac{1}{k^{0.5}})$ ), we have that  $\bar{x}^k$  converges to an optimal solution with a speed no worse than  $\mathcal{O}(\frac{1}{k^{0.5}})$  [60]. Therefore, the convergence of every  $x_i^k$  to an optimal solution, which is equivalent to the combination of the convergence of  $x_i^k$  to  $\bar{x}^k$  and the convergence of  $\bar{x}^k$  to an optimal solution, should be no slower than  $\mathcal{O}(\frac{1}{k^{0.5}})$ .

# V. CONVERGENCE ANALYSIS IN THE NON-CONVEX OBJECTIVE FUNCTION CASE

In this section, we show that the proposed algorithm will ensure convergence of all agents to a same stationary point when the objective functions are non-convex, as is the case in most machine learning applications.

**Theorem 3.** Under Assumption 1 and Assumption 2, if  $f_i(\cdot)$   $(1 \le i \le m)$  can be non-convex but have bounded gradients, then when the non-summable but square summable condition

in (9) and the heterogeneity condition in (10) for stepsizes are satisfied, we have that the proposed algorithm will guarantee the following results for all  $1 \le p \le m$ :

$$\lim_{k \to \infty} \|x_p^k - \bar{x}^k\| = 0, \quad \text{almost surely}$$
 (32)

$$\lim_{t \to \infty} \frac{\sum_{k=0}^{t} \bar{\lambda}_{p}^{k} \mathbb{E}\left[\left\|\nabla F(\bar{x}^{k})\right\|^{2}\right]}{\sum_{k=0}^{t} \bar{\lambda}_{p}^{k}} = 0$$
 (33)

$$\lim_{t \to \infty} \frac{\sum_{k=0}^{t} \bar{\lambda}_{p}^{k} \mathbb{E}\left[\left\|\frac{\sum_{i=1}^{m} \nabla f_{i}(x_{i}^{k})}{m}\right\|^{2}\right]}{\sum_{k=0}^{t} \bar{\lambda}_{p}^{k}} = 0$$
 (34)

Proof. We first prove (32).

Since the derivation in the proof of Part II of Theorem 2 is independent of the convexity of  $F(\cdot)$ , we still have the relationship in (28) for  $\sum_{i=1}^{m} \|x_i^{k+1} - \bar{x}^{k+1}\|^2$ , which further leads to

$$\sum_{i=1}^{m} \|x_i^{k+1} - \bar{x}^{k+1}\|^2 \le \rho \sum_{i=1}^{m} \|x_i^k - \bar{x}^k\|^2 + \frac{m^2 d^2 \max_{j,p} (\lambda_{jp}^k)^2}{1 - \rho} G^2$$
(35)

where we used  $\|\hat{\Lambda}^k\| = \max_{1 \leq j \leq m, 1 \leq p \leq d} \lambda_{jp}^k$  and that  $\|g^k\|$  is bounded (with the bound represented by G > 0 here).

Taking the conditional expectation, given  $\mathcal{F}^k = \{x^0, \dots, x^k\}$  yields

$$\mathbb{E}\left[\sum_{i=1}^{m} \|x_i^{k+1} - \bar{x}^{k+1}\|^2\right] \le \rho \sum_{i=1}^{m} \|x_i^k - \bar{x}^k\|^2 + \frac{m^2 d^2 \max_i \left((\bar{\lambda}_i^k)^2 + (\sigma_i^k)^2\right)}{1 - \rho} G^2$$
(36)

Since  $\rho < 1$  holds and both  $(\bar{\lambda}_i^k)^2$  and  $(\sigma_i^k)^2$  are summable, we have that  $\sum_{i=1}^m \|x_i^{k+1} - \bar{x}^{k+1}\|^2$  satisfies all conditions in Lemma 2 in the appendix with  $v^k = \sum_{i=1}^m \|x_i^{k+1} - \bar{x}^{k+1}\|^2$ ,  $\alpha^k = 0$ ,  $q^k = 1 - \rho$ , and  $p^k = \frac{m^2 d^2 \max_i ((\bar{\lambda}_i^k)^2 + (\sigma_i^k)^2)}{1 - \rho} G^2$ . Therefore, under the given conditions in the statement, we have  $\sum_{i=1}^m \|x_i^{k+1} - \bar{x}^{k+1}\|^2$  converging to 0 almost surely, and hence the result in (32).

We next proceed to prove (33) and (34).

From the Lipschitz gradient condition in Assumption 1, we have

$$F(y) \le F(x) + \langle \nabla F(x), (y - x) \rangle + \frac{L||y - x||^2}{2}$$

By plugging  $y = \bar{x}^{k+1}$  and  $x = \bar{x}^k$  into the above inequality, we obtain the following relationship based on (11):

$$\mathbb{E}\left[F(\bar{x}^{k+1})\right] \leq \mathbb{E}\left[F(\bar{x}^{k})\right] - \mathbb{E}\left[\left\langle\nabla F(\bar{x}^{k}), \frac{1}{m}\sum_{i=1}^{m}\Lambda_{i}^{k}g_{i}^{k}\right\rangle\right] + \frac{L}{2m^{2}}\mathbb{E}\left[\left\|\sum_{i=1}^{m}\Lambda_{i}^{k}g_{i}^{k}\right\|^{2}\right]$$
(37)

Next we analyze the last two terms on the right hand side of (37). Defining  $\Lambda^k = [\Lambda_1^k, \dots, \Lambda_m^k]^T \in \mathbb{R}^{d \times md}$ , we have

$$\left\| \sum_{i=1}^{m} \Lambda_{i}^{k} g_{i}^{k} \right\|^{2} = \|\Lambda^{k} g^{k}\|^{2} \leq \|\Lambda^{k}\|^{2} \|g^{k}\|^{2}$$

Using the relationship  $\|\Lambda^k\|^2 \leq \|\Lambda^k\|_F^2 = \sum_{j=1}^m \sum_{p=1}^d (\lambda_{jp}^k)^2$  and the independence of  $\Lambda_i^k$  and  $g_i^k$ , we have

$$\frac{L}{2m^2} \mathbb{E}\left[ \left\| \sum_{i=1}^m \Lambda_i^k g_i^k \right\|^2 \right] = \frac{L}{2m^2} \sum_{i=1}^m m dG^2((\bar{\lambda}_i^k)^2 + (\sigma_i^k)^2)$$
(38)

The second last term on the right hand side of (37) can be rewritten as

$$\mathbb{E}\left[\left\langle \nabla F(\bar{x}^k), \frac{1}{m} \sum_{i=1}^m \Lambda_i^k g_i^k \right\rangle \right]$$

$$= \mathbb{E}\left[\left\langle \nabla F(\bar{x}^k), \frac{1}{m} \sum_{i=1}^m \mathbb{E}\left[\Lambda_i^k g_i^k | \mathcal{F}^k\right] \right\rangle \right]$$

$$= \mathbb{E}\left[\left\langle \nabla F(\bar{x}^k), \frac{1}{m} \sum_{i=1}^m \bar{\lambda}_i^k \nabla f_i(x_i^k) \right\rangle \right]$$
(39)

Note that in the first equality, the expectation in the inner product is a conditional expectation taken with respect to  $\mathcal{F}^k$ , the sigma algebra generated by randomness  $\{\Lambda_1^k, \xi_1^k, \cdots, \Lambda_m^k, \xi_m^k\}$  at iteration k. The expectation outside the inner product is the full expectation taken over the sigma algebra generated by randomness in stepsizes and gradients in all iterations up to k.

Given that for any  $1 \le p \le m$ , we have

$$\sum_{i=1}^{m} \bar{\lambda}_{i}^{k} \nabla f_{i}(x_{i}^{k}) = \sum_{i=1}^{m} \bar{\lambda}_{p}^{k} \nabla f_{i}(x_{i}^{k}) + \sum_{i=1}^{m} (\bar{\lambda}_{i}^{k} - \bar{\lambda}_{p}^{k}) \nabla f_{i}(x_{i}^{k})$$

we can further rewrite (39) as

$$\begin{split} & \mathbb{E}\left[\left\langle \nabla F(\bar{x}^k), \frac{1}{m} \sum_{i=1}^m \Lambda_i^k g_i^k \right\rangle \right] \\ & = \mathbb{E}\left[\left\langle \nabla F(\bar{x}^k), \frac{\bar{\lambda}_p^k}{m} \sum_{i=1}^m \nabla f_i(x_i^k) \right\rangle \right] \\ & + \mathbb{E}\left[\left\langle \nabla F(\bar{x}^k), \frac{1}{m} \sum_{i=1}^m (\bar{\lambda}_i^k - \bar{\lambda}_p^k) \nabla f_i(x_i^k) \right\rangle \right] \\ & = \bar{\lambda}_p^k \mathbb{E}\left[\left\langle \nabla F(\bar{x}^k), \frac{1}{m} \sum_{i=1}^m \nabla f_i(x_i^k) \right\rangle \right] \\ & + \mathbb{E}\left[\left\langle \nabla F(\bar{x}^k), \frac{1}{m} \sum_{i=1}^m (\bar{\lambda}_i^k - \bar{\lambda}_p^k) \nabla f_i(x_i^k) \right\rangle \right] \end{split}$$

Applying the relationship  $2\langle X,Y\rangle=\|X\|^2+\|Y\|^2-\|X-Y\|^2$  to the first inner product term on the right hand side of the above inequality leads to

$$\mathbb{E}\left[\left\langle \nabla F(\bar{x}^k), \frac{1}{m} \sum_{i=1}^m \Lambda_i^k g_i^k \right\rangle \right] \\
= \frac{\bar{\lambda}_p^k}{2} \mathbb{E}\left[\left\| \nabla F(\bar{x}^k) \right\|^2 + \left\| \frac{\sum_{i=1}^m \nabla f_i(x_i^k)}{m} \right\|^2 \\
- \left\| \nabla F(\bar{x}^k) - \sum_{i=1}^m \frac{\nabla f_i(x_i^k)}{m} \right\|^2 \right] \\
+ \mathbb{E}\left[\left\langle \nabla F(\bar{x}^k), \frac{1}{m} \sum_{i=1}^m (\bar{\lambda}_i^k - \bar{\lambda}_p^k) \nabla f_i(x_i^k) \right\rangle \right] \\
\geq \frac{\bar{\lambda}_p^k}{2} \mathbb{E}\left[\left\| \nabla F(\bar{x}^k) \right\|^2 + \left\| \frac{\sum_{i=1}^m \nabla f_i(x_i^k)}{m} \right\|^2 \\
- \frac{L^2}{m^2} \sum_{i=1}^m \|\bar{x}^k - x_i^k\|^2 \right] \\
+ \mathbb{E}\left[\left\langle \nabla F(\bar{x}^k), \frac{1}{m} \sum_{i=1}^m (\bar{\lambda}_i^k - \bar{\lambda}_p^k) \nabla f_i(x_i^k) \right\rangle \right]$$

where we used the Lipschitz assumption on the gradient in Assumption 1.

According to the Cauchy-Schwarz inequality, we have  $\langle u, v \rangle \ge -\|u\|\|v\|$ , which leads to the following relationship for the last term in (40):

$$\mathbb{E}\left[\left\langle \nabla F(\bar{x}^k), \frac{1}{m} \sum_{i=1}^m (\bar{\lambda}_i^k - \bar{\lambda}_p^k) \nabla f_i(x_i^k) \right\rangle\right] \\
\geq -\mathbb{E}\left[\left\|\nabla F(\bar{x}^k)\right\| \left\| \frac{1}{m} \sum_{i=1}^m (\bar{\lambda}_i^k - \bar{\lambda}_p^k) \nabla f_i(x_i^k) \right\|\right] \\
\geq -\mathbb{E}\left[\left\|\nabla F(\bar{x}^k)\right\| \frac{1}{m} \sum_{i=1}^m |\bar{\lambda}_i^k - \bar{\lambda}_p^k| \left\|\nabla f_i(x_i^k)\right\|\right] \\
\geq -\mathbb{E}\left[\frac{G^2}{m} \sum_{i=1}^m |\bar{\lambda}_i^k - \bar{\lambda}_p^k|\right] = -\frac{G^2}{m} \sum_{i=1}^m |\bar{\lambda}_i^k - \bar{\lambda}_p^k|$$
(41)

where in the last inequality we used the boundedness of gradients  $\nabla F(\bar{x}^k)$  and  $\nabla f_i(x_i^k)$  by G.

Combining (40) and (41) leads to

$$\mathbb{E}\left[\left\langle \nabla F(\bar{x}^k), \frac{1}{m} \sum_{i=1}^m \Lambda_i^k g_i^k \right\rangle\right] \\
\geq \frac{\bar{\lambda}_p^k}{2} \mathbb{E}\left[\left\|\nabla F(\bar{x}^k)\right\|^2\right] + \frac{\bar{\lambda}_p^k}{2} \mathbb{E}\left[\left\|\frac{\sum_{i=1}^m \nabla f_i(x_i^k)}{m}\right\|^2\right] \\
- \frac{\bar{\lambda}_p^k L^2}{2m^2} \sum_{i=1}^m \mathbb{E}\left[\left\|\bar{x}^k - x_i^k\right\|^2\right] - \frac{G^2}{m} \sum_{i=1}^m |\bar{\lambda}_i^k - \bar{\lambda}_p^k|$$
(42)

Plugging (38) and (42) into (37) leads to

$$\mathbb{E}\left[F(\bar{x}^{k+1})\right] \leq \mathbb{E}\left[F(\bar{x}^{k})\right] \\
-\frac{\bar{\lambda}_{p}^{k}}{2}\mathbb{E}\left[\left\|\nabla F(\bar{x}^{k})\right\|^{2}\right] - \frac{\bar{\lambda}_{p}^{k}}{2}\mathbb{E}\left[\left\|\frac{\sum_{i=1}^{m}\nabla f_{i}(x_{i}^{k})}{m}\right\|^{2}\right] \\
+\frac{\bar{\lambda}_{p}^{k}L^{2}}{2m}\sum_{i=1}^{m}\mathbb{E}\left[\left\|\bar{x}^{k}-x_{i}^{k}\right\|^{2}\right] + \frac{G^{2}}{m}\sum_{i=1}^{m}|\bar{\lambda}_{i}^{k}-\bar{\lambda}_{p}^{k}| \\
+\frac{L\sum_{i=1}^{m}mdG^{2}((\bar{\lambda}_{i}^{k})^{2}+(\sigma_{i}^{k})^{2})}{2m^{2}} \tag{43}$$

or

$$\bar{\lambda}_{p}^{k} \mathbb{E} \left[ \left\| \nabla F(\bar{x}^{k}) \right\|^{2} \right] + \bar{\lambda}_{p}^{k} \mathbb{E} \left[ \left\| \frac{\sum_{i=1}^{m} \nabla f_{i}(x_{i}^{k})}{m} \right\|^{2} \right] \\
\leq 2 \mathbb{E} \left[ F(\bar{x}^{k}) \right] - 2 \mathbb{E} \left[ F(\bar{x}^{k+1}) \right] \\
+ \frac{\bar{\lambda}_{p}^{k} L^{2}}{m} \sum_{i=1}^{m} \mathbb{E} \left[ \left\| \bar{x}^{k} - x_{i}^{k} \right\|^{2} \right] + \frac{2G^{2}}{m} \sum_{i=1}^{m} |\bar{\lambda}_{i}^{k} - \bar{\lambda}_{p}^{k}| \\
+ \frac{L \sum_{i=1}^{m} mdG^{2}((\bar{\lambda}_{i}^{k})^{2} + (\sigma_{i}^{k})^{2})}{2m^{2}} \tag{44}$$

Adding (44) from k = 0 to k = t yields

$$\begin{split} & \sum_{k=0}^{t} \bar{\lambda}_{p}^{k} \mathbb{E}\left[\left\|\nabla F(\bar{x}^{k})\right\|^{2}\right] + \sum_{k=0}^{t} \bar{\lambda}_{p}^{k} \mathbb{E}\left[\left\|\frac{\sum_{i=1}^{m} \nabla f_{i}(x_{i}^{k})}{m}\right\|^{2}\right] \\ & \leq 2\mathbb{E}\left[F(\bar{x}^{0})\right] - 2\mathbb{E}\left[F(\bar{x}^{t+1})\right] \\ & + \sum_{k=0}^{t} \frac{\bar{\lambda}_{p}^{k} L^{2}}{m} \sum_{i=1}^{m} \mathbb{E}\left[\left\|\bar{x}^{k} - x_{i}^{k}\right\|^{2}\right] + \sum_{k=0}^{t} \frac{2G^{2}}{m} \sum_{i=1}^{m} |\bar{\lambda}_{i}^{k} - \bar{\lambda}_{p}^{k}| \\ & + \sum_{k=0}^{t} \frac{L \sum_{i=1}^{m} mdG^{2}((\bar{\lambda}_{i}^{k})^{2} + (\sigma_{i}^{k})^{2})}{m^{2}} \end{split}$$

$$\tag{45}$$

or

$$\frac{\sum_{k=0}^{t} \bar{\lambda}_{p}^{k} \mathbb{E}\left[\left\|\nabla F(\bar{x}^{k})\right\|^{2}\right]}{\sum_{k=0}^{t} \bar{\lambda}_{p}^{k}} + \frac{\sum_{k=0}^{t} \bar{\lambda}_{p}^{k} \mathbb{E}\left[\left\|\frac{\sum_{i=1}^{m} \nabla f_{i}(x_{i}^{k})}{m}\right\|^{2}\right]}{\sum_{k=0}^{t} \bar{\lambda}_{p}^{k}} \\
\leq \frac{2\mathbb{E}\left[F(\bar{x}^{0})\right] - 2\mathbb{E}\left[F(\bar{x}^{t+1})\right]}{\sum_{k=0}^{t} \bar{\lambda}_{p}^{k}} \\
+ \frac{\sum_{k=0}^{t} \frac{\bar{\lambda}_{p}^{k} L^{2}}{m} \sum_{i=1}^{m} \mathbb{E}\left[\left\|\bar{x}^{k} - x_{i}^{k}\right\|^{2}\right]}{\sum_{k=0}^{t} \bar{\lambda}_{p}^{k}} \\
+ \frac{\sum_{k=0}^{t} \frac{2G^{2}}{m} \sum_{i=1}^{m} |\bar{\lambda}_{i}^{k} - \bar{\lambda}_{p}^{k}|}{\sum_{k=0}^{t} \bar{\lambda}_{p}^{k}} \\
+ \frac{\sum_{k=0}^{t} \frac{L}{m} \sum_{i=1}^{m} dG^{2}((\bar{\lambda}_{i}^{k})^{2} + (\sigma_{i}^{k})^{2})}{\sum_{k=0}^{t} \bar{\lambda}_{p}^{k}} \tag{46}$$

Invoking the proven result in (32), we know that the second term on the right hand side of (46) will converge to zero as  $t \to \infty$ . Using the conditions on the expectation and variance of stepsizes, it can be obtained that the third and fourth terms on the right hand side of (46) will also converge to zero as  $t \to \infty$ . Therefore, we have the results in Theorem 3.

Remark 3. Eqn. (46) indicates that the convergence speed of  $\mathbb{E}\left[\left\|\nabla F(\bar{x}^k)\right\|^2\right] \text{ and } \mathbb{E}\left[\left\|\frac{\sum_{i=1}^m \nabla f_i(x_i^k)}{m}\right\|^2\right] \text{ to zero is determined by the convergence speed of } \mathbb{E}\left[\left\|\bar{x}^k - x_i^k\right\|^2\right], |\bar{\lambda}_i^k - \bar{\lambda}_p^k|, \text{ and } (\bar{\lambda}_i^k)^2 + (\sigma_i^k)^2 \text{ to zero. From (36), one obtains that the convergence speed of } \|\bar{x}^k - x_i^k\|^2 \text{ to zero is no slower than } \mathcal{O}(\frac{1}{k}).$  According to the assumption, i.e., all  $|\bar{\lambda}_i^k - \bar{\lambda}_p^k|, (\bar{\lambda}_i^k)^2, \text{ and } (\sigma_i^k)^2 \text{ are summable, one has that they all decrease to zero no slower than } \mathcal{O}(\frac{1}{k}).$  Therefore, we have that  $\mathbb{E}\left[\left\|\nabla F(\bar{x}^k)\right\|^2\right]$  and  $\mathbb{E}\left[\left\|\frac{\sum_{i=1}^m \nabla f_i(x_i^k)}{m}\right\|^2\right]$  converge to zero no slower than  $\mathcal{O}(\frac{1}{k}), \text{ i.e., } \mathbb{E}\left[\left\|\nabla F(\bar{x}^k)\right\|^2\right]$  and  $\mathbb{E}\left[\left\|\frac{\sum_{i=1}^m \nabla f_i(x_i^k)}{m}\right\|^2\right]$  converge to zero no slower than  $\mathcal{O}(\frac{1}{k}), \text{ i.e., } \mathbb{E}\left[\left\|\nabla F(\bar{x}^k)\right\|^2\right]$ 

Although the convergence of weighted sum of gradients in (33) and (34) are the most commonly used form of convergence for stochastic gradient descent under diminishing stepsizes, it only implies that the limit inferior of gradients will be zero [6], i.e.,  $\underline{\lim}_{k\to\infty}\mathbb{E}\left[\left\|\nabla F(\bar{x}^k)\right\|^2\right]=0$  and  $\underline{\lim}_{k\to\infty}\mathbb{E}\left[\left\|\sum_{i=1}^m \nabla f_i(x_i^k)\right\|^2\right]=0$ , which means that the mathematical expectation of gradients cannot stay bounded away from zero. In fact, by adding an additional assumption on the Lipschitz continuity of the Hessian  $H(f_i(\cdot))$  of function  $f_i(\cdot)$ , we can guarantee the convergence of the expectation of gradients to zero, which is stated below:

**Theorem 4.** Under the conditions of Theorem 3, if we further have

$$||H(f_i(x)) - H(f_i(y))|| \le \nu ||x - y||, \quad \forall 1 \le i \le m$$
 (47)

for a  $\nu > 0$  under any x and y, then we can obtain

$$\lim_{k \to \infty} \mathbb{E}\left[\left\|\nabla F(\bar{x}^k)\right\|^2\right] = 0$$

and

$$\lim_{k \to \infty} \mathbb{E}\left[ \left\| \frac{\sum_{i=1}^{m} \nabla f_i(x_i^k)}{m} \right\|^2 \right] = 0$$

*Proof.* The results can be obtained following the same line of derivation of Theorem 5 in [22] and hence we omit the proof here.

Remark 4. The proposed inherently privacy-preserving decentralized SGD algorithm can maintain the accuracy of optimization results while enabling privacy protection. This is in distinct difference from differential-privacy based privacy solutions that have to trade optimization accuracy for privacy.

#### VI. PRIVACY ANALYSIS

In this section, we analyze the inherent privacy embedded in our proposed algorithm against adversaries which try to infer the gradient of participating agents based on received information. Note that both  $b_{ij}^k$  and  $\Lambda_j^k$  are determined by agent j randomly and hence can provide privacy protection for the gradient  $g_j^k$  of agent j. Also note that when the gradient is a vector with dimension d>1,  $\Lambda_j^k$  is a diagonal matrix  $\Lambda_j^k=\mathrm{diag}\{\lambda_{j1}^k,\lambda_{j2}^k,\cdots,\lambda_{jd}^k\}$  with all elements independently chosen by agent j (under the same

mathematical expectation and variance, i.e.,  $\mathbb{E}\Lambda_j^k = \bar{\lambda}_j^k I_d$  and  $\operatorname{var}\left[\Lambda_j^k\right] = (\sigma_j^k)^2 I_d$ ) to obfuscate each element of  $g_j^k$  independently. However,  $b_{ij}^k$  would still be a scalar (although random) when d>1, which limits the privacy it can provide in obfuscating  $g_j^k$ . (For example, the ratio between different entries of  $g_j^k$  will not be covered by  $b_{ij}^k$  if a scalar  $b_{ij}^k$  were used alone to obfuscate  $g_j^k$ .) So here we only consider the privacy enabled by the stepsize  $\Lambda_j^k$ , which is not affected by the dimension of  $g_j^k$ . Note that  $b_{ij}^k$  still contributes to privacy protection since it increases the number of unknown parameters in the system and hence enhances resilience to potential side information attacks. For example, the usage of  $b_{ij}^k$  can avoid an adversary from inferring  $g_j^k$  even after the adversary obtains (for whatever reason) the values of  $\Lambda_j^k$  and  $x_j^k$ .

Because each element of  $g_i^k$  is independently obscured by a corresponding element of  $\Lambda_j^k$ , we only need to consider the scalar case where  $\Lambda_j^k$  becomes a scalar  $\lambda_j^k$  and provides protection for a scalar gradient  $g_j^k$ . This is because when d > 1, if we can quantify the privacy that the *i*th element of  $\Lambda^k_j$  (i.e.,  $\lambda^k_{ji}$ ) enables for the *i*th entry of vector  $g^k_j$  (i.e.,  $g_{ji}^k$ ), we can also quantify the privacy that other entries of  $\Lambda_j^k$ (i.e.,  $\lambda_{jq}^k$  for  $1 \leq q \leq d, q \neq i$ ) provide for other entries of vector  $g_j^k$  (i.e.,  $g_{jq}^k$  for  $1 \leq q \leq d, q \neq i$ ). According to the basic relationship (2) in information theory, if we can quantify the conditional differential entropy  $h(g_i^k|\lambda_i^kg_i^k)$ , then we can obtain an adversary's best estimation accuracy and hence evaluate the privacy enabled by the random stepsize  $\lambda_i^k$ . Clearly, when  $\lambda_i^k$  is deterministic and homogeneous across all agents (and hence publicly known, which is the case in most existing algorithms that do not consider privacy protection), the conditional differential entropy  $h(g_j^k|\lambda_j^kg_j^k)$  will be zero because an adversary can uniquely determine  $g_i^k$  from accessed  $\lambda_i^k g_i^k$ . Next, we quantify the privacy enabled by the random stepsize  $\lambda_i^k$  in our algorithm.

Without loss of generality, we consider the case where  $g_j^k$  is uniformly distributed in the interval  $[-\kappa,\kappa]$  where  $\kappa$  is a positive scalar. Note we assume that this range information of gradients (usually called side information) is available to adversaries in evaluating the protection performance against their inference attacks. The same argument can be easily adapted to cases where  $g_j^k$  has other probability distributions. To obscure the gradient information, we use a random  $\lambda_j^k$  uniformly distributed in  $[0,2\bar{\lambda}_j^k]$ , which can be verified to have expectation  $\mathbb{E}\left[\lambda_j^k\right] = \bar{\lambda}_j^k$  and standard deviation  $\sigma(\lambda_j^k) = \frac{\sqrt{3}\bar{\lambda}_j^k}{3}$ . Note that because  $\lambda_j^k$  can be zero when uniformly distributed in  $[0,2\bar{\lambda}_j^k]$ , an adversary cannot uniquely determine that  $g_j^k$  is zero even if it (for whatever reason) knows that the product  $\lambda_j^k g_j^k$  is zero. For the sake of simplicity, we assume  $2\bar{\lambda}_j^k \leq \kappa$ . Then we can obtain the following result about the adversary's best estimation accuracy:

**Theorem 5.** Under our decentralized stochastic gradient descent algorithm, if an eavesdropping or honest-but-curious adversary tries to estimate the gradient  $g_j^k$  of an agent based on received information, its estimation error will be no less

than 
$$\frac{e^{2\vartheta(\bar{\lambda}_{j}^{k},\kappa)}}{2\pi e}$$
, where 
$$\vartheta(\bar{\lambda}_{i}^{k},\kappa) = \log(4\bar{\lambda}_{i}^{k}\kappa^{2}) - 1 - c(\bar{\lambda}_{i}^{k},\kappa)$$
(48)

with  $c(\bar{\lambda}_i^k, \kappa)$  given by

$$c(\bar{\lambda}_{j}^{k}, \kappa) = -2 \int_{0}^{2\bar{\lambda}_{j}^{k} \kappa} \frac{\log(\frac{2\bar{\lambda}_{j}^{k} \kappa}{x})}{4\bar{\lambda}_{j}^{k} \kappa} \log\left(\frac{\log(\frac{2\bar{\lambda}_{j}^{k} \kappa}{x})}{4\bar{\lambda}_{j}^{k} \kappa}\right) dx \tag{49}$$

*Proof.* To prove that the estimation error is not less than  $\frac{e^{2\vartheta(\bar{\lambda}_j^k,\kappa)}}{2\pi e}$ , we show that the conditional differential entropy  $h(g_j^k|\lambda_j^kg_j^k)$  is no less than  $\vartheta(\lambda_j^k,\kappa)$  in (48). According to information theory,

$$h(g_i^k|\lambda_i^kg_i^k) = h(g_i^k,\lambda_i^kg_i^k) - h(\lambda_i^kg_i^k)$$

where  $h(g_j^k, \lambda_j^k g_j^k)$  denotes the joint differential entropy of  $g_j^k$  and the product variable  $\lambda_j^k g_j^k$ , and  $h(\lambda_j^k g_j^k)$  represents the differential entropy of  $\lambda_j^k g_j^k$ . Next we first calculate the joint differential entropy of  $g_j^k$  and the product variable  $\lambda_j^k g_j^k$ . According to information theory, we have

$$h(g_j^k, \lambda_j^k g_j^k) = -\int_{g_j^k} \int_{\lambda_j^k g_j^k} p(g_j^k, \lambda_j^k g_j^k) \log(p(g_j^k, \lambda_j^k g_j^k)) d\lambda_j^k g_j^k dg_j^k$$
(50)

where  $p(\cdot)$  represents the probability density function.

Making use of the fact that  $\lambda_j^k$  and  $g_j^k$  are stochastically independent random variables with uniform distributions, we have

$$p(g_j^k, \lambda_j^k g_j^k) = p(\lambda_j^k g_j^k | g_j^k) p(g_j^k) = \frac{1}{2\bar{\lambda}_j^k | g_j^k|} \frac{1}{2\kappa} = \frac{1}{4\bar{\lambda}_j^k \kappa | g_j^k|}$$

Substituting this relationship into (50) leads to

$$\begin{split} &h(g_j^k,\lambda_j^kg_j^k)\\ &=-\int_{g_j^k}\int_{\lambda_j^kg_j^k}\frac{1}{4\bar{\lambda}_j^k\kappa|g_j^k|}\log(\frac{1}{4\bar{\lambda}_j^k\kappa|g_j^k|})d\lambda_j^kg_j^kdg_j^k\\ &=-\int_{g_j^k}\int_{\lambda_j^k}\frac{1}{4\bar{\lambda}_j^k\kappa|g_j^k|}g_j^k\log(\frac{1}{4\bar{\lambda}_j^k\kappa|g_j^k|})d\lambda_j^kdg_j^k\\ &=-\int_{g_j^k}2\bar{\lambda}_j^k\frac{1}{4\bar{\lambda}_j^k\kappa|g_j^k|}g_j^k\log(\frac{1}{4\bar{\lambda}_j^k\kappa|g_j^k|})dg_j^k\\ &=-\int_{g_j^k}\frac{g_j^k}{2\kappa|g_j^k|}\log(\frac{1}{4\bar{\lambda}_j^k\kappa|g_j^k|})dg_j^k\\ &=-\int_{-\kappa}^0\frac{1}{2\kappa}\log(\frac{1}{-4\bar{\lambda}_j^k\kappa g_j^k})dg_j^k-\int_0^\kappa\frac{1}{2\kappa}\log(\frac{1}{4\bar{\lambda}_j^k\kappa g_j^k})dg_j^k\\ &=\int_0^\kappa\frac{1}{\kappa}\log(4\bar{\lambda}_j^k\kappa g_j^k)dg_j^k\\ &=\log(4\bar{\lambda}_j^k\kappa^2)-1 \end{split}$$

To calculate the differential entropy of the product variable  $\lambda_j^k g_j^k$ , we first determine its probability distribution  $p(\lambda_j^k g_j^k)$ . It can be seen that  $\lambda_j^k g_j^k$  resides in the interval  $(-2\kappa \bar{\lambda}_j^k, 2\kappa \lambda_j^k)$ . We first determine the probability distribution of  $\lambda_j^k g_j^k$  on the

interval  $(0, 2\kappa \bar{\lambda}_i^k]$ . The cumulative probability of  $\lambda_j^k g_j^k$  on any given interval (0, x] for  $x \leq 2\kappa \bar{\lambda}_i^k$  is given by

$$P(0 \le 2\lambda_j^k g_j^k \le x) = \int_0^\kappa P(0 \le \lambda_j^k g_j^k \le x | g_j^k) p(g_j^k) dg_j^k$$
$$= \int_0^\kappa P(0 \le \lambda_j^k \le \frac{x}{g_j^k}) p(g_j^k) dg_j^k$$
(51)

where  $P(\cdot)$  denotes the cumulative probability.

Because we have  $P(0 \le \lambda_j^k \le \frac{x}{g_j^k}) = 1$  when  $g_j^k \le \frac{x}{2\lambda_j^k}$  and  $P(0 \le \lambda_j^k \le \frac{x}{g_j^k}) = \frac{x}{g_j^k 2\lambda_j^k}$  when  $g_j^k \ge \frac{x}{2\lambda_j^k}$ , we can rewrite (51) as

$$P(0 \le 2\lambda_j^k g_j^k \le x)$$

$$= \int_0^{\frac{x}{2\bar{\lambda}_j^k}} 1p(g_j^k) dg_j^k + \int_{\frac{x}{2\bar{\lambda}_j^k}}^{\kappa} \frac{x}{g_j^k 2\bar{\lambda}_j^k} p(g_j^k) dg_j^k$$

$$= \int_0^{\frac{x}{2\bar{\lambda}_j^k}} \frac{1}{2\kappa} dg_j^k + \int_{\frac{x}{2\bar{\lambda}_j^k}}^{\kappa} \frac{x}{g_j^k 2\bar{\lambda}_j^k} \frac{1}{2\kappa} dg_j^k$$

$$= \frac{x}{4\kappa\bar{\lambda}_j^k} + \frac{x}{4\kappa\bar{\lambda}_j^k} (\log \kappa - \log(\frac{x}{2\bar{\lambda}_j^k}))$$
(52)

Then for  $0 \le \lambda_j^k g_j^k \le 2\lambda_j^k \kappa$ , the probability density function can be obtained as

$$p(\lambda_j^k g_j^k = x) = \frac{d}{dx} P(0 \le 2\lambda_j^k g_j^k \le x) = \frac{\log(\frac{2\bar{\lambda}_j^k \kappa}{x})}{4\bar{\lambda}_j^k \kappa}$$

Similarly, we can get the probability density function of  $\lambda_i^k g_i^k$  in the interval  $-2\lambda_i^k \kappa \leq \lambda_i^k g_i^k \leq 0$  as

$$p(\lambda_j^k g_j^k = x) = \frac{\log(\frac{2\bar{\lambda}_j^k \kappa}{-x})}{4\bar{\lambda}_i^k \kappa}$$

Knowing the probability density function of  $\lambda_j^k g_j^k$  enables us to calculate its differential entropy, which is exactly (49).

Remark 5. According to the convergence results in Theorem 2 - Theorem 4, the amplitude of the expected stepsize  $\bar{\lambda}_i^k$  has to decrease to 0 to guarantee the claimed convergence results. Different from additive-noise based privacy approaches which will lose protection completely when the noise decreases to zero, the stochastic stepsize in our approach obfuscates information in a multiplicative manner and hence still provides protection even when it converges to zero. More specifically, when  $\bar{\lambda}_i^k$  converges to zero, the conditional differential entropy will still be non-zero (due to the uncertainty in gradient  $g_i^k$ ). In this case, the multiplicative relationship between  $\lambda_{j}^{k}$  and  $g_{j}^{k}$ makes an adversary's observation  $\lambda_i^k g_i^k$  be always equal to 0 irrespective of the value of  $g_j^k$ , and hence avoids the adversary from inferring the value of  $g_i^k$ . In fact, using the Matlab numerical integration function "integral", we can obtain that under  $\kappa = 5$ , when  $\bar{\lambda}_i^k$  is set to zero, the expression  $\vartheta(\bar{\lambda}_i^k, \kappa)$  in (48) (a logarithm function minus a constant and a definite integral) is equal to 1.0322, meaning that the conditional differential entropy is no less than 1.0322. Therefore, according to the relationship in (2), by taking exponential and dividing by  $2\pi e$ , we can obtain that the adversary's best expected squared estimation error is no less than 0.4614 after  $\bar{\lambda}_{j}^{k}$  converges to zero.

**Remark 6.** Following the same line of derivation, we can also obtain an adversary's best estimation error when the gradients follow other probability distributions.

Remark 7. It can be verified that under the considered uniform distribution of  $\lambda_j^k$ , a larger  $\bar{\lambda}_j^k$  (and hence a larger variance  $\frac{(\bar{\lambda}_j^k)^2}{3}$  of the stepsize) means a larger conditional differential entropy and hence better privacy protection. But as long as  $\bar{\lambda}_j^k$  satisfies the non-summable but square summable condition in (9) and the summable heterogeneity condition in (10), convergence can always be achieved. Moreover, the proposed algorithm can guarantee the privacy of every participating agent even when an adversary can have access to all shared messages in the network. This is in distinct difference from existing accuracy-friendly privacy-preserving solutions (in, e.g., [42], [43], [44] for decentralized deterministic convex optimization) that have to restrict an adversary from having full access to messages on all channels to enable privacy protection.

Remark 8. It is worth noting that our algorithm can also provide privacy protection for the variable x. This is because the actual variable shared by agent j is  $v_{ij}^k = w_{ij}x_j^k - b_{ij}^k\Lambda_j^kg_j^k$ . Since  $x_j^k$ ,  $b_{ij}^k$ , and  $\Lambda_j^k$  are private to agent j and unknown to adversaries, the value of  $x_j^k$  is not disclosed from shared  $v_{ij}^k$ . However, note that since all states  $x_j^k$  will finally reach consensus and converge to the same value, the final  $x_j^k$  will be disclosed when k tends to infinity. Therefore, our algorithm can avoid the intermediate state value  $x_j^k$  of every agent j from disclosure before the algorithm converges (i.e., before all states  $x_i^k$  reach consensus).

#### VII. NUMERICAL EXPERIMENTS

In this section, we evaluate the performance of our algorithm using numerical experiments. We will consider both the convex-objective-function case and the non-convex-objective-function case.

#### A. Convex case

For the case of convex objective functions, we consider a canonical decentralized estimation problem where a sensor network of m sensors collectively estimate an unknown parameter  $\theta \in \mathbb{R}^d$ , which can be formulated as an empirical risk minimization problem. More specifically, we assume that each sensor i has  $n_i$  noisy measurements of the parameter  $z_{ij} = M_i\theta + w_{ij}$  for  $j = \{1, 2, \cdots, n_i\}$  where  $M_i \in \mathbb{R}^{s \times d}$  is the measurement matrix of agent i and  $w_{ij}$  is measurement noise associated with measurement  $z_{ij}$ . Then the estimation of the parameter  $\theta$  can be solved using empirical risk minimization problem formulated as (1), with each  $f_i(\theta)$  given as

$$f_i(\theta) = \frac{1}{n_i} \sum_{i=1}^{n_i} ||z_{ij} - M_i \theta||^2 + r_i ||\theta||^2$$

where  $r_i$  is a non-negative regularization parameter.

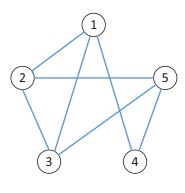


Fig. 1. The interaction topology of the network.

We assume that the network consists of five agents interacting on a graph depicted in Fig. 1. The parameter s was set to three and the parameter d was set to two without loss of generality.  $n_i$  was set to 100 for all i.  $w_{ij}$  were assumed to be uniformly distributed in [0, 1]. To evaluate the performance of our proposed decentralized stochastic gradient descent algorithm, we set the stepsize of agent i as  $\lambda_i^k = \frac{1-\varrho_i^k/k}{k}$  where  $\varrho_i^k$  was randomly chosen by agent i from the interval [0, 1] in each iteration. It can be verified that the stepsize satisfies the summable conditions required in Theorem 2. Every agent i also chose  $b_{ii}^k$  randomly in each iteration under the sumone condition. The evolution of the estimation error averaged over 1,000 runs is illustrated by the solid blue line in Fig. 2. To compare the convergence performance of our algorithm with the conventional decentralized stochastic gradient descent algorithm, we also implemented the decentralized stochastic gradient descent algorithm in [19], whose average convergence trajectory over 1,000 runs is represented by the black dotted line in Fig. 2. The stepsizes of all agents were set to  $\frac{1}{k}$ . Through comparison, it can be seen that our introduced random parameters  $B^k$  and  $\lambda_i^k$  do not slow down the convergence speed. In fact, they significantly increase the optimization speed as well as optimization accuracy compared with the conventional decentralized SGD algorithm.

#### B. Non-convex case

We use the decentralized training of a convolutional neural network (CNN) to evaluate the performance of our proposed decentralized stochastic gradient descent algorithm in nonconvex optimization. More specially, we consider five agents interacting on a topology depicted in Fig. 1. The agents collaboratively train a CNN using the MNIST data set [61], which is a large benchmark database of handwritten digits widely used for training and testing in the field of machine learning [62]. Each agent has a local copy of the CNN. The CNN has 2 convolutional layers with 32 filters with each followed by a max pooling layer, and then two more convolutional layers with 64 filters each followed by another max pooling layer and a dense layer with 512 units. We used Sigmoid activation functions and hence the Lipschitz gradient assumption in

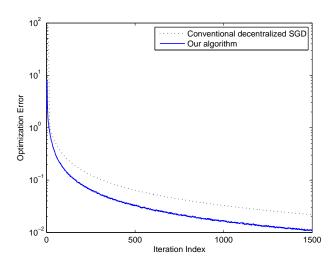


Fig. 2. Comparison of convergence performance between our algorithm and the conventional decentralized SGD in [19].

Theorem 3 is satisfied. Note that ReLU activation functions will lead to non-smooth objective functions and hence non-Lipschitz gradients. Each agent has access to a portion of the MNIST data set, which was further divided into two subsets for training and validation, respectively. We set the stepsize of agent i as  $\Lambda_i^k = \mathrm{diag}\{\frac{1-\varrho_{i1}^k/k}{k}, \frac{1-\varrho_{i2}^k/k}{k}, \cdots, \frac{1-\varrho_{id}^k/k}{k}\}$  where  $\varrho_{ip}^k$   $(1 \le p \le d)$  were statistically independent random variables chosen by agent i from the interval [0, 1] in each iteration. (For the adopted CNN model, the dimension of gradient d is equal to 1,676,266.) It can be verified that the stepsizes satisfy the summable conditions required in Theorem 3. Every agent i also chose  $b_{ji}^k$  randomly in each iteration under the sum-one condition. The evolution of the training and validation accuracies averaged over 1,000 runs are illustrated by the solid and dashed blue lines in Fig. 3. To compare the convergence performance of our algorithm with the conventional decentralized stochastic gradient descent algorithm, we also implemented the decentralized stochastic gradient descent algorithm in [19] to train the same CNN using stepsize  $\frac{1}{k}$ , whose average training and validation accuracies over 1,000 runs are represented by the solid and dashed black lines in Fig. 3. It can be seen that the proposed algorithm has a faster converging speed as well as a slightly better training/validation accuracy.

To show that the proposed algorithm can indeed protect the privacy of participating agents, we also implemented a privacy attacker which tries to infer the raw image of participating agents using received information. The attacker implements the DLG attack model proposed in [25], which is the most powerful inference algorithm reported to date in terms of reconstructing exact raw data from shared gradient/model updates. The attacker was assumed to be able to eavesdrop all messages shared among the agents. Fig. 4 shows that the attacker could effectively recover the original training image from shared model updates in the conventional stochastic gradient descent algorithm [19]. However, under the proposed algorithm, the attacher failed to infer the original training

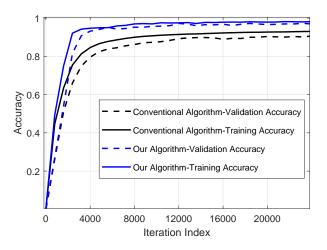
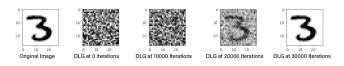


Fig. 3. Comparison of CNN training performance between our algorithm and the conventional decentralized SGD algorithm in [19].

Evolution of DLG attacker inference results under the conventional decentralized SGD



Evolution of DLG attacker inference results under our proposed algorithm

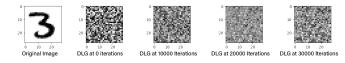


Fig. 4. Comparison of DLG attacher's inference results under the conventional decentralized SGD and our algorithm.

image through information shared in the network. This is also corroborated by the attacker's inference performance measured by the mean-square error (MSE) between the inference result and the original image. More specifically, as illustrated in Fig. 5, the attacker eventually inferred the raw image accurately as its estimation error converged to zero. However, the proposed algorithm successfully thwarted the attacker as attacker's estimation error was always large.

Using the same setup, we also compared our privacy-preserving approach with a differential-privacy based privacy approach that injects additive noise to gradients directly to protect privacy [31], [32], [33]. More specifically, we set  $b_{ij}^k$  and  $\Lambda_i^k$  in (3) as  $b_{ij}^k = \frac{1}{|\mathcal{N}_j|}$  (with  $|\mathcal{N}_j|$  denoting the cardinality of  $\mathcal{N}_j$ ) and  $\Lambda_i^k = \frac{1}{k}I_d$ , respectively, and added d dimensional zero-mean Gaussian noise  $\xi_j$  directly to the gradient  $g_j^k$ . We evaluated the training/validation accuracy and DLG attacker's inference error under different variances  $\sigma_{DP}$  of the differential-privacy noise  $\xi_j$ . The results are summarized in Table I where every entry is the average over 1,000 runs.

In the evaluation, we find that to avoid DLG attackers from obtaining images that are recognizable by human eyes,

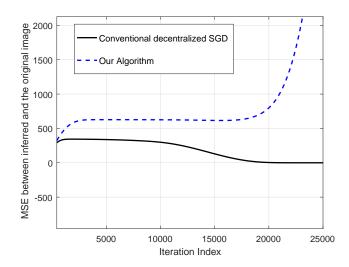


Fig. 5. Comparison of DLG attacher's inference errors under the conventional decentralized SGD and our algorithm.

the DLG inference error should be at least 50. Therefore, according to Table I, to combat DLG inference attacks, we have to use differential-privacy noise on the level of  $10^{-2}$ , which significantly compromises both training and validation accuracies. Given that our privacy-preserving approach can combat DLG attackers without sacrificing training or validation accuracy (see Fig. 3 for training and validation accuracies and Fig. 5 for DLG attacker inference error for our algorithm), the comparison clearly demonstrates the potential of the proposed approach in enabling privacy protection.

### VIII. CONCLUSIONS

By judiciously manipulating inter-agent interaction dynamics, this paper has proposed an inherently privacy-preserving decentralized stochastic gradient descent algorithm. The dynamics based privacy design exploits the inherent robustness of decentralized stochastic optimization to enable privacy while maintaining the accuracy of optimization results, which is in distinct difference from differential-privacy based privacy solutions that trade optimization accuracy for privacy. The inherently private decentralized stochastic gradient descent algorithm is encryption-free, and hence avoids incurring heavy extra communication and computation overhead, an unavoidable problem with existing encryption based privacy solutions for decentralized stochastic optimization. Rigorous theoretical analysis has been provided to confirm the accurate convergence of all decentralized agents to a desired solution, both in the case with convex objective functions and in the case with non-convex objective functions. An informationtheoretic privacy analysis has also been proposed to characterize the strength of privacy protection for each participating agent against honest-but-curious and eavesdropping adversaries. Both simulation results for a convex decentralized estimation problem and numerical experiments for decentralized learning on a benchmark image dataset have confirmed the effectiveness of the proposed algorithm.

	No Noise	$\sigma_{DP} = 10^{-4}$	$\sigma_{DP} = 10^{-3}$	$\sigma_{DP} = 10^{-2}$	$\sigma_{DP} = 10^{-1}$	$\sigma_{DP} = 10^0$
Training Accuracy	0.9877	0.9823	0.7810	0.0999	0.0995	0.0995
Validation Accuracy	0.9879	0.9871	0.7870	0.0993	0.0992	0.0993
Final DLG Error	$3.030\times10^{-7}$	0.1012	16.39	58.48	340.3	1221

#### ACKNOWLEDGEMENT

The authors would like to thanks Ben Liggett for the help in numerical experiments. They would also like to thank the associate editor and anonymous reviewers, whose comments helped improve the paper.

#### APPENDIX

**Lemma 2.** (Lemma 2 in [34]) Let  $\{v^k\}, \{\alpha^k\}$ , and  $\{p^k\}$  be random nonnegative scalar sequences, and  $\{q^k\}$  be a deterministic nonnegative scalar sequence satisfying  $\sum_{k=0}^{\infty} \alpha^k < \infty$  almost surely,  $\sum_{k=0}^{\infty} q^k = \infty$ ,  $\sum_{k=0}^{\infty} p^k < \infty$  almost surely, and the following inequality almost surely:

$$\mathbb{E}\left[v^{k+1}|\mathcal{F}^k\right] \le (1 + \alpha^k - q^k)v^k + p^k, \quad \forall k \ge 0$$

where  $\mathcal{F}^k=\{v^\ell,\alpha^\ell,p^\ell;0\leq \ell\leq k\}$ . Then, we have  $\sum_{k=0}^\infty q^k v^k<\infty$  and  $\lim_{k\to\infty} v^k=0$  almost surely.

**Lemma 3.** (Lemma 3 in [34]) Consider the problem  $\min_{z \in \mathbb{R}^d} \phi(z)$ , where  $\phi : \mathbb{R}^d \to \mathbb{R}$  is a continuous function. Assume that the optimal solution set  $Z^*$  of the problem is nonempty. Let  $\{z^k\}$  be a random sequence such that for any optimal solution  $z^* \in Z^*$ ,

$$\mathbb{E}\left[\|z^{k+1} - z^*\|^2 | \mathcal{F}^k\right] \\ \leq (1 + \alpha^k) \|z^k - z^*\|^2 - \eta^k \left(\phi(z^k) - \phi(z^*)\right) + \beta^k, \ \forall k \geq 0$$

holds almost surely, where  $\mathcal{F}^k = \{z^\ell, \alpha^\ell, \beta^\ell, \ell = 0, 1, \dots, k\}$ ,  $\{\alpha^k\}$  and  $\{\beta^k\}$  are random nonnegative scalar sequences satisfying  $\sum_{k=0}^{\infty} \alpha^k < \infty$  and  $\sum_{k=0}^{\infty} \beta^k < \infty$  almost surely while  $\{\eta^k\}$  is a deterministic nonnegative scalar sequence with  $\sum_{k=0}^{\infty} \eta^k = \infty$ . Then,  $\{z^k\}$  converges almost surely to some solution  $z^* \in Z^*$ .

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