

A deterministic least squares approach for simultaneous input and state estimation

Grigorios Gakis and Malcolm C. Smith

Abstract—This paper considers a deterministic estimation problem to find the input and state of a linear dynamical system which minimise a weighted integral squared error between the resulting output and the measured output. A completion of squares approach is used to find the unique optimum in terms of the solution of a Riccati differential equation. The optimal estimate is obtained from a two-stage procedure that is reminiscent of the Kalman filter. The first stage is an end-of-interval estimator for the finite horizon which may be solved in real time as the horizon length increases. The second stage computes the unique optimum over a fixed horizon by a backwards integration over the horizon. A related tracking problem is solved in an analogous manner. Making use of the solution to both the estimation and tracking problems a constrained estimation problem is solved which shows that the Riccati equation solution has a least squares interpretation that is analogous to the meaning of the covariance matrix in stochastic filtering. The paper shows that the estimation and tracking problems considered here include the Kalman filter and the linear quadratic regulator as special cases. The infinite horizon case is also considered for both the estimation and tracking problems. Stability and convergence conditions are provided and the optimal solutions are shown to take the form of left inverses of the original system.

Index Terms—Continuous time systems, Kalman filter, deterministic optimisation, least squares, input and state estimation, trajectory tracking, Riccati differential equation.

I. INTRODUCTION

OUR goal in this paper is to pose and solve a filtering/estimation problem for the simultaneous estimation of inputs and states in a continuous time linear finite dimensional dynamical system. A model of the physical system is assumed to be available. The output of the dynamical system is the vector of *all* variables that are measured (e.g. by means of sensors). The filter should make use only of these measured outputs for the estimation. The filter should produce the best estimate of the system variables treating the exogenous inputs and states on an equal footing. The meaning of “best” is to minimise a weighted integral squared error between output and measured output. The problem set-up is illustrated in Fig. 1. The filtered signals $w_1(t), x_1(t)$ provide best estimates of w and x at a given time instant t based on measurements up to that time, while the estimates $\hat{w}(t), \hat{x}(t)$ provide the best estimates over a time interval $[0, T]$.

The estimation problem of Fig. 1 has a strong motivation in engineering applications. For example, the motion of an automobile is determined by the external forces acting on

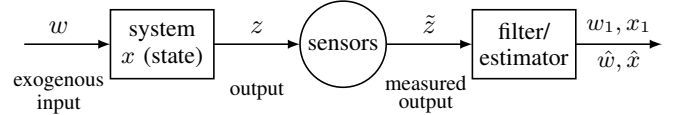


Fig. 1. Estimation problem for a dynamical system with state x , exogenous input w and output z which is measured.

it (principally tyre and aerodynamic forces). These are very difficult to measure directly but may be estimated together with the system state from an appropriate set of sensors. Similar considerations apply to other types of vehicles, e.g. aerial or nautical vehicles or vessels. In the field of structural dynamics an example would be the estimation of forces applied to a structure together with the resulting vibrational displacements in the structure. In power systems an example would be the estimation of the external excitation to an electrical machine together with the machine states. The standard Kalman filter is often used in such examples, however such an approach imposes a Gaussian assumption on the exogenous input which may be unjustified.

Our solution to the problem of Fig. 1 is based on the method of completion of squares and builds on the work of Willems [1] which gave a deterministic derivation of the standard Kalman filter. A key step is the construction of a dynamical system derived from the system model together with a matrix $P_1(t)$ which is the solution of a Riccati differential equation. This allows the cost functional to be written in a form that allows the unique optimal solution to be determined. The dynamical system used in the construction turns out to be an “end-of-interval estimator” which generates signals $w_1(t), x_1(t)$ by integration forwards in time on the interval $[0, T]$ (Lemma 3). The completion of squares points to the construction of a further dynamical system which is solved backwards in time on the interval $[0, T]$ to generate the unique optimum (Theorem 1). The structure of the solution is reminiscent of the standard (causal) Kalman filter and the non-causal process of smoothing, which involves backwards integration and is not computable in real time, though the solution to the present problem has a more general structure through placing the estimation of state and exogenous input on an equal footing, and without prior assumption on the nature of the exogenous input.

An important further contribution of the paper is to provide a deterministic interpretation of the matrix $P_1(t)$ (Section VI). To do this we pose and solve a constrained optimisation problem which requires the state to pass through a prescribed

G. Gakis and M. C. Smith are with the Department of Engineering, University of Cambridge, UK, e-mail: gg402@cam.ac.uk and mcs@eng.cam.ac.uk. The first author would like to acknowledge McLaren Automotive Ltd for a CASE studentship awarded for doctoral research.

point at a given time. The optimal cost for the new problem is increased by a term which is the norm squared error between the state and the optimal state at the given time weighted by the inverse of a matrix Lyapunov differential equation solution $P_2(t)$ (Theorem 4). The latter coincides with $P_1(t)$ if the prescribed time for the state constraint is at the end of the interval, i.e. $t = T$. This allows the interpretation that if $P_1(t)$ is small, the measurements suggest strongly that $x_1(t)$ is an accurate estimate of the state at time t given measurements up to that time, while conversely if its inverse $P_1(t)^{-1}$ is near singular then there exist trajectories with $x(t)$ far from $x_1(t)$ that fit the measurements \tilde{z} up to time t almost as well. There is a similar interpretation for $P_2(t)$ valid for any time t within the fixed horizon length. We note that the simpler version of the question of providing a deterministic interpretation of $P_1(t)$ and $P_2(t)$ for the standard Kalman filter has not been addressed in the literature so far. Our result achieves this as a special case, namely, the result provides a deterministic interpretation of the filtered and smoothed state covariances in the stochastic formulation which is analogous to the meaning of covariance in that context (see Section VII-A).

In order to solve the constrained optimisation problem (Section VI), a tracking problem is introduced that has a close relation to the estimation problem and has independent interest (Section V). In one respect the estimation and tracking problems are identical, namely it is desired to minimise the weighted integral squared error between the output of the dynamical system and another signal that is given, i.e. a set of sensor measurements or a desired trajectory. In the estimation problem we seek the input and state that is the best explanation of the observed trajectory given the measurements made, while in the tracking problem we seek the input (and state) that gives the closest output trajectory of the system to the desired one. In another respect the two problems differ in that there is a quadratic penalty in the cost criterion on the initial state (estimation problem) or on the final state (tracking problem). This results in the two problems having solutions which are dual to each other: in the tracking problem the first stage of the solution solves a Riccati differential equation backwards in time after which the optimal control and state trajectory are found by integrating forwards in time, which is the opposite way round to the estimation problem. This duality is a generalisation of the well-known duality between the solutions of the Kalman filter and the linear quadratic regulator.

In Section VII we show how the results of the paper specialise to some standard problems. Section VII-A considers the standard finite horizon Kalman filter and shows how Theorems 1 and 2 reduce to the well-known results for that case. In particular the end-of-interval estimator of Lemma 3 coincides with the minimum variance estimator with $P_1(t)$ equal to the covariance. Theorem 2 reduces to the smoothed estimate in Kalman filtering with $P_2(t)$ equal to the smoothed covariance. Section VII-B shows how the corresponding results are deduced for the Kalman filter with direct feedthrough of input to measured output. Section VII-C shows that Theorem 3 reduces to the linear quadratic (LQ) tracking problem as a special case, which further specialises

to the linear quadratic regulator.

The paper considers the infinite horizon case for the estimation problem in Section VIII. It is shown under mild conditions that the limiting form of the end-of-interval estimator can be written as a linear system solved forwards in time with the system matrices determined via the solution of an algebraic Riccati equation (Theorem 8). We show that the end-of-interval estimator of the input is a stable left inverse of the original system (Theorem 9). We also show that the unique solution of the estimation problem has a limiting form which includes a second stage of processing via an anti-stable system, equivalently a system that is stable in the backwards time direction. We show that the series connection of the end-of-interval estimator (with judiciously chosen output) and this anti-causal “smoother” is also a left inverse of the original system (Theorem 10).

Section IX considers the infinite horizon case for the tracking problem. On a finite horizon the tracking problem solution has a natural two-stage form where the first stage involves a backwards-in-time integration and the second stage has an integration forwards in time. This form is maintained in the infinite horizon limit with, under mild conditions, the first stage being an anti-stable system (equivalently a stable anti-causal system) and the second stage being a stable system. The analysis takes care to show that the optimal control converges for any fixed time t to the finite-horizon solutions in the limit as the horizon length tends to infinity (Theorem 11 and Theorem 13). Moreover, we show that the first stage system is an anti-stable left inverse of the original system (Theorem 12), and that the series connection of the first stage’s computation of a modified output and the second stage system is also a left inverse of the original system (Theorem 14).

II. LITERATURE REVIEW

The close connection between the deterministic concept of estimation by least squares and the minimisation of a mean square error in a statistical sense has long been appreciated (see [2], [3], [4], [5]). Kalman followed earlier work of Wiener and Kolmogorov in taking a statistical view of signal estimation, though his approach specified a linear state-space model for the system and estimation of the state rather than estimation of a signal from a noisy measurement with assumptions on the spectral characteristics. The resulting filter was initially introduced for discrete time systems [6] and then for continuous time systems first by analogy with optimal control [7] and later as a limit of the discrete time equations [8]. The discrete (continuous) time filter has the form of a recursive algorithm (differential equation) which allows for easy implementation in real time applications and a lot of its success can be attributed to this property. Very important is Kalman’s work [9] in deriving conditions for stability in the time invariant case. A (deterministic) least squares formulation and derivation of the continuous time Kalman filter was given more recently by Willems [1]. As noted in [1] the possibility of such a derivation had been “system theory folklore” ever since the first appearance of Kalman’s work and he provides a number of earlier references in this direction. The work by

Willems inspired the analogous study [10] for discrete time systems that are controllable and observable.

An active direction of research that emerged following Kalman's initial contribution focused on estimation in systems with unknown inputs. Noise-free systems were considered first and observers were designed for systems with a full column rank first Markov parameter and zero feedthrough matrix [11], [12], [13], [14]. This work was extended to systems with a non-zero feedthrough matrix in [15] while the first Markov parameter rank condition was relaxed in [16]. Estimation of unknown inputs and states in a stochastic setting was considered in [17] for discrete time systems, and extended in [18], [19], [20] and [21] for systems without feedthrough. Systems with a non-zero feedthrough matrix were treated in [22] using a technique developed in [15] for noise-free systems. In [23] an additional least-squares procedure was used for input estimation to derive a filter for systems with a full column rank feedthrough matrix. More recently [24], [25], [26] inspired by [23] and [18] describe a procedure to derive filters which relax the above matrix conditions. In [27], [28] filters have been derived for linear discrete time systems in the zero informational limit for the process noise as a method to treat the estimation problem for unknown inputs. In [27] there is no direct feedthrough of the process noise to the measurements and there is an assumption that the first Markov parameter has full column rank. In [29] there is further analysis with reference to stability, convergence and system inversion. In [28] the case where there is a direct feedthrough matrix of full column rank is treated. We mention that [27], [28] and [29] contrast with the present work in considering discrete time stochastic systems, whereas here we deal with the continuous time case and present a first principles approach which is purely deterministic.

The problem formulation and assumptions in this paper are in part motivated by examples in mechanical systems where a direct feedthrough of unknown inputs to measured outputs frequently occurs due to the use of accelerometers. We will make the assumption that the direct feedthrough matrix of the system has full column rank, though it is possible that this could be relaxed at the expense of making the filter equations more involved. Even so, it should be emphasised that the assumption is mild enough to include the standard Kalman filter, the Kalman filter with direct feedthrough of process noise to measurements and the linear quadratic tracking problem as special cases, as we demonstrate in Section VII.

III. PRELIMINARIES

A. Notation

A real scalar, a real m dimensional vector and a real $m \times l$ dimensional matrix are denoted by \mathbb{R} , \mathbb{R}^m and $\mathbb{R}^{m \times l}$ respectively. A square symmetric matrix $\Theta = \Theta^T \in \mathbb{R}^{m \times m}$ is positive (semi-positive or negative) definite and is denoted by $\Theta > 0$ ($\Theta \geq 0$ or $\Theta < 0$) if $\theta^T \Theta \theta > 0$ ($\theta^T \Theta \theta \geq 0$ or $\theta^T \Theta \theta < 0$) for all $0 \neq \theta \in \mathbb{R}^m$ and we denote by $\|\theta\|_{\Theta^{-1}}^2$ the norm on \mathbb{R}^m defined by $\|\theta\|_{\Theta^{-1}}^2 := \theta^T \Theta^{-1} \theta$ if $\Theta > 0$. Furthermore, $\mathcal{L}_{2,e}^m$ denotes the space of vector signals of dimension m whose Lebesgue integrated squared 2-norm exists on any finite interval. Let \mathcal{L}_{∞}^m denote the space of

Lebesgue integral m dimensional vector functions of bounded ∞ -norm (see Appendix for a precise definition).

B. The Riccati equation

Consider the non-linear matrix differential equation:

$$\dot{P} = AP + PA^T - PC^T R^{-1} CP + BQB^T \quad (1)$$

in the unknown matrix function $P : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ where $A, B, C, R > 0, Q > 0$ are fixed real known matrices. The equation is known as a *Riccati differential equation* (RDE) and is central to estimation problems (Kalman filter). Assuming $P(0) > 0$, then the RDE has a unique positive definite solution $P(t) > 0$ for all $t \geq 0$ (see [30, p. 165]). In steady-state the RDE (1) is given by the *algebraic Riccati equation* (ARE):

$$AP + PA^T - PC^T R^{-1} CP + BQB^T = 0 \quad (2)$$

in the unknown matrix $P \in \mathbb{R}^{n \times n}$.

Lemma 1: The ARE (2) has a unique solution P^∞ that is stabilizing, i.e. $A - P^\infty C^T R^{-1} C$ is Hurwitz, if and only if (C, A) is detectable and (A, B) has no uncontrollable modes on the imaginary axis. If these conditions hold $P^\infty \geq 0$. Furthermore, P^∞ is nonsingular if and only if (A, B) has no stable uncontrollable modes.

Proof: See [31, Theorem 13.7], [32, p. 985], [33] or [34]. \square

Lemma 2: Let the conditions of Lemma 1 hold such that P^∞ is the unique positive semi-definite and stabilising solution of the ARE (2). Then the unique positive definite solution $P(t) > 0$ for all $t \geq 0$ of the RDE (1) with the initial condition $P(0) > 0$ has a limit as $t \rightarrow \infty$ which is given by P^∞ .

Proof: See [32], [35], [36] and [37, Theorem 3.7] noting that the null space of $P(0)$ is empty by assumption. \square

Similarly the differential equation:

$$\dot{S} = FS + SF^T + SH^T R^{-1} HS - GQG^T \quad (3)$$

in the unknown matrix function $S : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ where $F, G, H, R > 0, Q > 0$ are fixed real known matrices is an RDE central to control problems (linear-quadratic-regulator). Assuming $S(0) > 0$, then the RDE (3) has a unique positive definite solution $S(t) > 0$ for all $t \leq 0$. This follows from the positive time case by considering the transformation $t \rightarrow -t$. In steady-state the RDE (3) is given by the *algebraic Riccati equation* (ARE):

$$FS + SF^T + SH^T R^{-1} HS - GQG^T = 0 \quad (4)$$

in the unknown matrix $S \in \mathbb{R}^{n \times n}$.

Remark 1: Lemmas 1 and 2 carry over easily to (3) and (4). Namely the ARE (4) has a unique solution S^∞ that is “anti-stabilizing”, i.e. $-F - S^\infty H^T R^{-1} H$ is Hurwitz, if and only if $(H, -F)$ is detectable and (F, G) has no uncontrollable modes on the imaginary axis. If these conditions hold $S^\infty \geq 0$. Furthermore, S^∞ is nonsingular if and only if (F, G) has no unstable uncontrollable modes. The unique positive definite solution $S(t) > 0$ for $t \leq 0$ of the RDE (3) with the terminal condition $S(0) > 0$ has a limit as $t \rightarrow -\infty$ which is given by S^∞ .

IV. ESTIMATION PROBLEM

Consider the linear, finite-dimensional, continuous time system with the state space description:

$$\dot{x} = Ax + Bw, \quad (5)$$

$$z = Cx + Dw \quad (6)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$ and $D \in \mathbb{R}^{p \times m}$ (full column rank) are fixed known matrices¹ and $w \in \mathcal{L}_{2,e}^m$, $x \in \mathcal{L}_{2,e}^n$ and $z \in \mathcal{L}_{2,e}^p$ are input, state and output related through this linear system. We consider the problem to estimate w and x , which is the same as estimating w and $x(0)$ since x is generated by (5), from the measurement of the signal z and a prior estimate of the initial state $x(0)$. We assume that the state x and driving input w are not measured directly, other than (indirectly) through the measurement of z (i.e. all measurements of the system are made through the output vector z). Each element of z may correspond to an individual sensor or multiple entries of z may be generated by a single device. To pose our problem precisely we will denote by $\tilde{z} \in \mathcal{L}_{2,e}^p$ the actual measured output signal in an experiment (see Fig. 1). We introduce the performance index:

$$C_T(w, x(0)) = \int_0^T \|\tilde{z}(t) - z(t)\|_{R^{-1}}^2 dt + \|x(0) - \gamma\|_{\Gamma^{-1}}^2 \quad (7)$$

where $0 < R \in \mathbb{R}^{p \times p}$, $0 < \Gamma \in \mathbb{R}^{n \times n}$, $\gamma \in \mathbb{R}^n$, $0 < T \in \mathbb{R}$ are specified. The vector γ represents the estimate of the initial state available a-priori and Γ the accuracy/confidence of this estimate. The problem we wish to solve is:

$$\inf_{w, x(0)} C_T(w, x(0)) \quad (8)$$

subject to (5) and (6). In particular we wish to compute the optimal w and $x(0)$ which we will denote by \hat{w} and $\hat{x}(0)$.

A key step in our solution of (8) is a ‘‘completion of squares’’ construction for the performance index (7) which is given in the following lemma.

Lemma 3: Consider the system:

$$\dot{x}_1 = (A_1 - K_1 C_1)x_1 + (B_1 + K_1)\tilde{z}, \quad (9)$$

$$\dot{P}_1 = A_1 P_1 + P_1 A_1^T - K_1 R K_1^T + B_1 R B_1^T, \quad (10)$$

$$K_1 = P_1 C_1^T R^{-1}, \quad (11)$$

$$w_1 = D^\dagger(\tilde{z} - z_1), \quad (12)$$

$$z_1 = Cx_1 \quad (13)$$

with the initial conditions $P_1(0) = \Gamma$ and $x_1(0) = \gamma$, where we have defined:

$$A_1 = A - B_1 C, \quad (14)$$

$$B_1 = B D^\dagger, \quad (15)$$

$$C_1 = (I - \Pi)C, \quad (16)$$

$$\Pi = D D^\dagger, \quad (17)$$

$$D^\dagger = (D^T R^{-1} D)^{-1} D^T R^{-1}. \quad (18)$$

¹The assumption that the system matrices are constant is for notational convenience. We note that all finite horizon results in Sections IV-VII are valid if the system matrices A , B , C , D are time-varying, with no change required in the proofs.

Then the RDE (10) has a unique positive definite solution $P_1(t) > 0$ for all $t \in [0, T]$. Furthermore, the performance index defined in (7) is given by:

$$C_T(w, x(0)) = \int_0^T \|(I - \Pi)(\tilde{z}(t) - z_1(t))\|_{R^{-1}}^2 dt + \int_0^T \|\Pi(RB_1^T P_1(t)^{-1}(x(t) - x_1(t)) + \tilde{z}(t) - z(t))\|_{R^{-1}}^2 dt + \|x(T) - x_1(T)\|_{P_1(T)^{-1}}^2 \quad (19)$$

(w_1 is defined here for convenience and will be first used in Theorem 2).

Proof: The parallel projection Π satisfies $\Pi^2 = \Pi$ and:

$$(I - \Pi)^T R^{-1} D = 0. \quad (20)$$

Hence the following identities hold:

$$K_1 D = 0, \quad (21)$$

$$K_1 C_1 = K_1 C. \quad (22)$$

From (6) $w = D^\dagger(z - Cx)$. Substituting into (5) gives:

$$\dot{x} = A_1 x + B_1 z = (A_1 - K_1 C_1)x + (B_1 + K_1)z \quad (23)$$

using (21) and (22). From (23) and (9) we obtain:

$$\dot{x} - \dot{x}_1 = (A_1 - K_1 C_1)(x - x_1) - (B_1 + K_1)(\tilde{z} - z). \quad (24)$$

We note that $(I - \Pi)(z - z_1) = C_1(x - x_1)$ from which it follows that:

$$C_1(x - x_1) + (\tilde{z} - z) = (I - \Pi)(\tilde{z} - z_1) + \Pi(\tilde{z} - z). \quad (25)$$

Hence from (20) and (25):

$$\|C_1(x - x_1) + (\tilde{z} - z)\|_{R^{-1}}^2 = \|(I - \Pi)(\tilde{z} - z_1)\|_{R^{-1}}^2 + \|\Pi(\tilde{z} - z)\|_{R^{-1}}^2. \quad (26)$$

Using (10), (24) and (26) we verify the calculation:

$$\begin{aligned} \frac{d}{dt} (\|x - x_1\|_{P_1^{-1}}^2) &= \frac{d}{dt} ((x - x_1)^T P_1^{-1} (x - x_1)) \\ &= 2(x - x_1)^T P_1^{-1} (\dot{x} - \dot{x}_1) \\ &\quad - (x - x_1)^T P_1^{-1} \dot{P}_1 P_1^{-1} (x - x_1) \\ &= 2(x - x_1)^T P_1^{-1} ((A_1 - K_1 C_1)(x - x_1) \\ &\quad - (B_1 + K_1)(\tilde{z} - z)) \\ &\quad - (x - x_1)^T (P_1^{-1} A_1 + A_1^T P_1^{-1} - C_1^T R^{-1} C_1 \\ &\quad + P_1^{-1} B_1 R B_1^T P_1^{-1}) (x - x_1) \\ &= -2(x - x_1)^T (P_1^{-1} B_1 + C_1^T R^{-1}) (\tilde{z} - z) \\ &\quad - (x - x_1)^T (P_1^{-1} B_1 R B_1^T P_1^{-1} + C_1^T R^{-1} C_1) (x - x_1) \\ &= -\|R B_1^T P_1^{-1} (x - x_1) + \Pi(\tilde{z} - z)\|_{R^{-1}}^2 \\ &\quad + \|\Pi(\tilde{z} - z)\|_{R^{-1}}^2 \\ &\quad - \|C_1(x - x_1) + (\tilde{z} - z)\|_{R^{-1}}^2 \\ &\quad + \|(\tilde{z} - z)\|_{R^{-1}}^2 \\ &= -\|R B_1^T P_1^{-1} (x - x_1) + \Pi(\tilde{z} - z)\|_{R^{-1}}^2 \\ &\quad - \|(I - \Pi)(\tilde{z} - z_1)\|_{R^{-1}}^2 + \|(\tilde{z} - z)\|_{R^{-1}}^2 \end{aligned} \quad (27)$$

where in the penultimate step we have noted that $B_1 \Pi = B_1$. Integrating (27) in the interval $[0, T]$ gives the required expression on noting that $R B_1^T = \Pi R B_1^T$. \square

Theorem 1: The optimisation problem in (8) has a unique solution \hat{w} , $\hat{x}(0)$ which is obtained as follows: first integrate (9)–(11) forwards in time in the interval $0 \leq t \leq T$ with initial conditions $P_1(0) = \Gamma$ and $x_1(0) = \gamma$; then integrate:

$$\dot{\hat{x}} = A_2 \hat{x} + B_2 \tilde{z}_2 \quad (28)$$

backwards in time with terminal condition:

$$\hat{x}(T) = x_1(T) \quad (29)$$

to compute $\hat{x}(0)$ (and indeed \hat{x}); and lastly set:

$$\hat{w} = D^\dagger(\tilde{z}_2 - C_2 \hat{x}) \quad (30)$$

where:

$$A_2 = A - B_2 C_2, \quad (31)$$

$$B_2 = B_1, \quad (32)$$

$$C_2 = C - R B_1^T P_1^{-1}, \quad (33)$$

$$\tilde{z}_2 = \tilde{z} - R B_1^T P_1^{-1} x_1. \quad (34)$$

Furthermore, the minimum of the performance index (7) is:

$$\begin{aligned} \inf_{w, x(0)} C_T(w, x(0)) &= \int_0^T \|(I - \Pi)(\tilde{z}(t) - z_1(t))\|_{R^{-1}}^2 dt \\ &= \int_0^T \|\tilde{z}(t) - \hat{z}(t)\|_{R^{-1}}^2 dt + \|\hat{x}(0) - \gamma\|_{\Gamma^{-1}}^2 \end{aligned} \quad (35)$$

where we have denoted the optimal output by $\hat{z} = C\hat{x} + D\hat{w}$. *Proof:* We note by an application of [30, Theorem 1, p. 40] that (28) may be integrated on the interval $[0, T]$ to yield \hat{x} . Next we verify that (5) driven by $w = \hat{w}$ from the initial state $x(0) = \hat{x}(0)$ generates the state trajectory $x = \hat{x}$, i.e. $\dot{\hat{x}} = A\hat{x} + B\hat{w}$. This is easily seen by substituting for \hat{w} to obtain (28). We now claim that the following lower bound:

$$C_T(w, x(0)) \geq \int_0^T \|(I - \Pi)(\tilde{z}(t) - z_1(t))\|_{R^{-1}}^2 dt$$

holds for all w and $x(0)$. To see this note that all terms in (19) are non-negative and the first term is independent of w and $x(0)$. We proceed to claim that for $x(0) = \hat{x}(0)$ and $w = \hat{w}$ the last two terms in (19) are zero. To see that the second term is zero we substitute $w = \hat{w}$ from (30) and $x = \hat{x}$ into (19) with z defined in (6), i.e. $z = \hat{z} = C\hat{x} + D\hat{w}$. The third term is zero with $x = \hat{x}$ from (29). We therefore conclude that $w = \hat{w}$, $x(0) = \hat{x}(0)$ achieve a minimum of the performance index with the minimum given by (35). The second line in (35) is given by substitution of the optimal solution into (7).

It remains to show that this solution is unique, which we will now establish by contradiction. Let:

$$x = \hat{x} + \delta x, \quad (36)$$

$$w = \hat{w} + \delta w \quad (37)$$

be another solution that satisfies (5) with $\delta x, \delta w$ not identically zero in the interval $[0, T]$. Substituting (36) and (37) into (5) gives:

$$\dot{\delta x} = A\delta x + B\delta w \quad (38)$$

by noting that \hat{x}, \hat{w} also satisfy (5) by construction. The difference in the output is given by:

$$\delta z = C\delta x + D\delta w. \quad (39)$$

We now substitute (36) and (37) into the performance index (19) which gives:

$$\begin{aligned} C_T(w, x(0)) &= \int_0^T \|(I - \Pi)(\tilde{z}(t) - z_1(t))\|_{R^{-1}}^2 dt \\ &+ \int_0^T \|\Pi(RB_1^T P_1(t)^{-1} \delta x(t) - \delta z(t))\|_{R^{-1}}^2 dt \\ &+ \|\delta x(T)\|_{P_1(T)^{-1}}^2 \end{aligned} \quad (40)$$

using the fact that with $x = \hat{x}$ and $w = \hat{w}$ the integrand in the second term of (19) is identically zero in the interval $[0, T]$. Under the assumption that the trajectory in (36), (37) is a solution to the optimisation problem, the last two terms in (40) have to be zero, which gives:

$$D^\dagger(RB_1^T P_1^{-1} \delta x - \delta z) = 0 \quad (41)$$

on the interval $[0, T]$ since D is full column rank, and:

$$\delta x(T) = 0 \quad (42)$$

since $P_1(T) > 0$. Substituting δz from (39) into (41) gives:

$$\delta w = -D^\dagger C_2 \delta x \quad (43)$$

using (33). Substituting δw from (43) into (38) gives:

$$\dot{\delta x} = A_2 \delta x \quad (44)$$

using (31). Solving (44) backwards in time with the terminal condition (42) gives $\delta x = 0$ identically in the interval $[0, T]$, and from (43) we have $\delta w = 0$ identically in the same interval, which results in a contradiction. \square

We now turn our attention to the filtered estimates (i.e. end-of-interval estimates) of the state and input, namely $\hat{x}(T)$ and $\hat{w}(T)$. In real time applications, the horizon T is itself a variable. It would appear at first glance that a new optimisation problem needs to be solved at every T to compute end-of-interval estimates. The following result shows that this is not the case.

Theorem 2: Let $x_1(t)$, $w_1(t)$ be defined by (9)–(12) with $P_1(0) = \Gamma$ and $x_1(0) = \gamma$, and $\hat{x}(t)$, $\hat{w}(t)$ as in Theorem 1, for a fixed time interval $[0, T]$. Then:

$$\hat{x}(T) = x_1(T), \quad (45)$$

$$\hat{w}(T) = w_1(T). \quad (46)$$

Proof: The result follows from (12), (29), (30), (33) and (34). \square

Theorem 2 shows that integrating (9)–(11) as the measurements \tilde{z} become available is sufficient to recover the end-of-interval estimates without computing $x(0)$ or w . Hence the filter of Lemma 3 can be used in a receding horizon manner as the optimal filter in real time. We further note that this filter is non-anticipating, meaning that the end-of-interval estimates $x_1(T)$ and $w_1(T)$ do not depend on future measurements, i.e. $\tilde{z}(t)$ for $t > T$. This property is required of any filter to be applied in a real time application.

V. TRACKING PROBLEM

In this section we will consider a related tracking problem. Assume the state q and input u satisfy:

$$\dot{q} = Fq + Gu, \quad (47)$$

$$y = Hq + Ju \quad (48)$$

where $F \in \mathbb{R}^{n \times n}$, $G \in \mathbb{R}^{n \times m}$, $H \in \mathbb{R}^{p \times n}$ and $J \in \mathbb{R}^{p \times m}$ (full column rank) are fixed known matrices and $u \in \mathcal{L}_{2,e}^m$, $q \in \mathcal{L}_{2,e}^n$ and $y \in \mathcal{L}_{2,e}^p$ are input, state and output related through this linear system. We wish to find an input u such that the output y best tracks a desired signal $\tilde{y} \in \mathcal{L}_{2,e}^p$ over a finite horizon T together with a penalty on the deviation of the terminal state from a desired state ξ for a given but arbitrary initial state η . More precisely, we introduce the performance index:

$$W_T(u) = \int_0^T \|\tilde{y}(t) - y(t)\|_{R^{-1}}^2 dt + \|q(T) - \xi\|_{\Xi^{-1}}^2 \quad (49)$$

where $0 < R \in \mathbb{R}^{p \times p}$, $\xi \in \mathbb{R}^n$, $0 < \Xi \in \mathbb{R}^{n \times n}$ and propose the problem:

$$\inf_u W_T(u) \quad (50)$$

subject to (47), (48) and $q(0) = \eta$. We denote the optimal solution to (50) by \hat{u} . We first give a completion of squares result similar to Lemma 3 which we then use to solve (50) in Theorem 3.

Lemma 4: Consider the system:

$$\dot{q}_1 = (F_1 + M_1 H_1)q_1 + (G_1 - M_1)\tilde{y}, \quad (51)$$

$$\dot{S}_1 = F_1 S_1 + S_1 F_1^T + M_1 R M_1^T - G_1 R G_1^T, \quad (52)$$

$$M_1 = S_1 H_1^T R^{-1}, \quad (53)$$

$$u_1 = J^\dagger(\tilde{y} - y_1), \quad (54)$$

$$y_1 = H q_1 \quad (55)$$

with the terminal conditions $S_1(T) = \Xi$ and $q_1(T) = \xi$, where we have defined the matrices:

$$F_1 = F - G_1 H, \quad (56)$$

$$G_1 = G J^\dagger, \quad (57)$$

$$H_1 = (I - \Lambda)H, \quad (58)$$

$$\Lambda = J J^\dagger, \quad (59)$$

$$J^\dagger = (J^T R^{-1} J)^{-1} J^T R^{-1}. \quad (60)$$

Then the RDE (52) has a unique positive definite solution $S_1(t) > 0$ for all $t \in [0, T]$. Furthermore, the performance index in (49) is given by:

$$\begin{aligned} W_T(u) = & \int_0^T \|(I - \Lambda)(\tilde{y}(t) - y_1(t))\|_{R^{-1}}^2 dt \\ & + \int_0^T \|\Lambda(R G_1^T S_1(t)^{-1}(q(t) - q_1(t)) - \tilde{y}(t) + y(t))\|_{R^{-1}}^2 dt \\ & + \|\eta - q_1(0)\|_{S_1(0)^{-1}}^2. \end{aligned} \quad (61)$$

Proof: We sketch the outline of two proofs. A direct proof is a completion of squares argument analogous to Lemma 3. It differs from Lemma 3 only in the signs of some terms. An indirect proof is to recognise that Lemma 4 is the time reversed

Lemma 3. The transformations $\frac{d}{dt} \rightarrow -\frac{d}{dt}$, $A \rightarrow -F$, $B \rightarrow -G$, $C \rightarrow H$, $D \rightarrow J$, $P_1 \rightarrow S_1$, $x_1 \rightarrow q_1$, $z_1 \rightarrow y_1$, $\tilde{z} \rightarrow \tilde{y}$ and consequential correspondences $A_1 \rightarrow -F_1$ etc. suffice to give the result. Furthermore, $S_1(t) > 0$ for all $t \in [0, T]$ is guaranteed by the reversed time Lemma 2 (see Remark 1). \square

Theorem 3: The optimisation problem in (50) has a unique solution \hat{u} which is obtained as follows: first integrate (51)–(53) backwards in time with terminal conditions $S_1(T) = \Xi$ and $q_1(T) = \xi$; then integrate:

$$\dot{\hat{q}} = F\hat{q} + G\hat{u} \quad (62)$$

forwards in time with \hat{u} given by the feedback law:

$$\hat{u} = J^\dagger(\tilde{y}_2 - H_2 \hat{q}) \quad (63)$$

and initial condition $\hat{q}(0) = \eta$, where we have defined:

$$H_2 = H + R G_1^T S_1^{-1}, \quad (64)$$

$$\tilde{y}_2 = \tilde{y} + R G_1^T S_1^{-1} q_1. \quad (65)$$

Furthermore, the minimum of the performance index (49) is:

$$\begin{aligned} \inf_u W_T(u) = & \int_0^T \|(I - \Lambda)(\tilde{y}(t) - y_1(t))\|_{R^{-1}}^2 dt \\ & + \|\eta - q_1(0)\|_{S_1(0)^{-1}}^2. \end{aligned} \quad (66)$$

Proof: We note that (62) with \hat{u} given by (63) and $\hat{q}(0) = \eta$ may be integrated [30, Theorem 1, p. 40] on the interval $[0, T]$ to yield \hat{q} , while \hat{u} can be computed by substituting \hat{q} into (63). We now claim that the following lower bound:

$$\begin{aligned} W_T(u) \geq & \int_0^T \|(I - \Lambda)(\tilde{y}(t) - y_1(t))\|_{R^{-1}}^2 dt \\ & + \|\eta - q_1(0)\|_{S_1(0)^{-1}}^2 \end{aligned}$$

holds for all u . To see this note that all terms in (61) are non-negative and the first and third terms are independent of u . We proceed to claim that for $u = \hat{u}$ the second term in (61) is zero. To see this we substitute $u = \hat{u}$ and $q = \hat{q}$ into (61) with y defined in (48), i.e. $y = \hat{y} = H\hat{q} + J\hat{u}$, and noting that $\Lambda^2 = \Lambda$. We therefore conclude that $u = \hat{u}$ is a solution to the optimisation problem and the minimum of the performance index (49) is given by (66).

It remains to show that this solution is unique, which we will now establish by contradiction. Let:

$$q = \hat{q} + \delta q \quad (67)$$

$$u = \hat{u} + \delta u \quad (68)$$

be another solution that satisfies (47) with δq , δu not identically zero in the interval $[0, T]$. Substituting (67) and (68) into (47) gives:

$$\dot{\delta q} = F\delta q + G\delta u \quad (69)$$

using (62). The difference in the output is given by:

$$\delta y = H\delta q + J\delta u. \quad (70)$$

We now substitute (69) and (70) into the performance index (61) which gives:

$$W_T(u) = \int_0^T \|(I - \Lambda)(\tilde{y}(t) - y_1(t))\|_{R^{-1}}^2 dt + \int_0^T \|\Lambda(RG_1^T S_1(t)^{-1} \delta q(t) + \delta y(t))\|_{R^{-1}}^2 dt + \|\eta - q_1(0)\|_{S_1(0)^{-1}}^2 \quad (71)$$

using the fact that with $q = \hat{q}$ and $u = \hat{u}$ the integrand in the second term of (61) is identically zero in the interval $[0, T]$. Under the assumption that the trajectory in (67), (68) is a solution to the optimisation problem, the second term in (71) has to be zero, which gives:

$$J^\dagger(RG_1^T S_1^{-1} \delta q + \delta y) = 0 \quad (72)$$

on the interval $[0, T]$ since J is full column rank. Substituting δy from (70) into (72) gives:

$$\delta u = -J^\dagger H_2 \delta q \quad (73)$$

using (64). Substituting δu from (73) into (69) gives:

$$\dot{\delta q} = (F - G_1 H_2) \delta q. \quad (74)$$

Solving (74) forwards in time with the initial constraint $\delta q(0) = 0$, which follows since $q(0) = \hat{q}(0) = \eta$, gives $\delta q = 0$ identically in the interval $[0, T]$. Using (73) we have $\delta u = 0$ identically in the same interval, which results in a contradiction. \square

VI. CONSTRAINED ESTIMATION PROBLEM

We now turn our attention to the constrained optimisation problem:

$$\inf_{w, x(0)} C_T(w, x(0)) \text{ subject to } x(\tau) = \zeta \quad (75)$$

for $\zeta \in \mathbb{R}^n$ and $0 \leq \tau \leq T$ where (5) and (6) hold. Here $C_T(w, x(0))$ is defined as in (7) with the same meaning for \tilde{z} , γ and Γ . Again we wish to compute the optimal w and $x(0)$ which we will denote by \hat{w} and $\hat{x}(0)$. A solution of this optimisation problem will show how the optimal cost increases compared to the unconstrained value when we demand that the state passes through a prescribed point at a given time. This will give an indication in a least squares sense of how “likely” it is that the state passes through the optimum point for the unconstrained problem. For example, if there is a sharp rise in the cost when the state is required to pass through a different point, then we may have more confidence in the value of the unconstrained optimum state at that time. We will first give Lemma 5 before deriving the solution to the optimisation problem (75) in Theorem 4.

Lemma 5: Let x_1 , P_1 , z_1 , A_2 , B_2 , C_2 , \tilde{z}_2 be defined as in Lemma 3 and Theorem 1 and consider the system:

$$\dot{x}_2 = A_2 x_2 + B_2 \tilde{z}_2, \quad (76)$$

$$\dot{P}_2 = A_2 P_2 + P_2 A_2^T - B_2 R B_2^T, \quad (77)$$

$$w_2 = D^\dagger(\tilde{z}_2 - C_2 x_2) \quad (78)$$

with the terminal conditions $P_2(T) = P_1(T)$, $x_2(T) = x_1(T)$. (Note that x_2 , w_2 are the optimal trajectories of the optimisation problem given by (8), namely $x_2 = \hat{x}$, $w_2 = \hat{w}$ as defined in Theorem 1.) Then $C_T(w, x(0))$ is equivalently given by:

$$C_T(w, x(0)) = \int_0^T \|(I - \Pi)(\tilde{z}(t) - z_1(t))\|_{R^{-1}}^2 dt + \int_0^\tau \|\Pi(\tilde{z}_2(t) - C_2 x(t) - D w(t))\|_{R^{-1}}^2 dt + \int_\tau^T \|\Pi(\tilde{z}_3(t) - C_3 x(t) - D w(t))\|_{R^{-1}}^2 dt + \|x(\tau) - x_2(\tau)\|_{P_2^{-1}(\tau)}^2 \quad (79)$$

where C_3 and \tilde{z}_3 are given by:

$$C_3 = C_2 + R B_2^T P_2^{-1}, \quad (80)$$

$$\tilde{z}_3 = \tilde{z}_2 + R B_2^T P_2^{-1} x_2. \quad (81)$$

Proof: Using (6), (33) and (34) we obtain:

$$\Pi(R B_1^T P_1^{-1}(x - x_1) + \tilde{z} - z) = \Pi(\tilde{z}_2 - C_2 x - D w). \quad (82)$$

Substituting (82) into (19) gives:

$$C_T(w, x(0)) = \int_0^T \|(I - \Pi)(\tilde{z}(t) - z_1(t))\|_{R^{-1}}^2 dt + \int_0^\tau \|\Pi(\tilde{z}_2(t) - C_2 x(t) - D w(t))\|_{R^{-1}}^2 dt + \int_\tau^T \|\Pi(\tilde{z}_2(t) - C_2 x(t) - D w(t))\|_{R^{-1}}^2 dt + \|x(T) - x_1(T)\|_{P_1^{-1}(T)}^2. \quad (83)$$

We next note that Lemma 4 remains true for time varying matrices. The proof will apply Lemma 4 to the last two terms of (83) in the interval $[\tau, T]$ rather than $[0, T]$. First we make the following notational substitutions: $q \rightarrow x$, $u \rightarrow w$, $\tilde{y} \rightarrow \Pi \tilde{z}_2$, $F \rightarrow A$, $G \rightarrow B$, $H \rightarrow \Pi C_2$, $J \rightarrow D$ and setting $\xi = x_1(T)$, $\Xi = P_1(T)$. Making these replacements in (56)–(60) and then (53) gives $F_1 = A_2$, $G_1 = B_2$, $H_1 = 0$, $\Lambda = \Pi$, $J^\dagger = D^\dagger$ and $M_1 = 0$. Substituting these into (51), (52) and (54) with the notational substitutions $q_1 \rightarrow x_2$, $S_1 \rightarrow P_2$ and $u_1 \rightarrow w_2$ gives (76)–(78). We next note that the expression in (49) with the notational substitutions and the lower limit replaced by τ is the same as the last two terms in (83). Therefore, using Lemma 4, we can replace these terms by the expression in (61), which gives (79) using (80) and (81) and the fact that the first integral on the right hand side of (61) is zero since $(I - \Pi)\Pi = 0$. \square

Theorem 4: The optimisation problem (75) has a unique solution \hat{w} , $\hat{x}(0)$ which is obtained as follows: first integrate (9)–(11) forwards in time in the interval $0 \leq t \leq T$ with initial conditions $P_1(0) = \Gamma$ and $x_1(0) = \gamma$ which gives x_1 and P_1 in the interval $0 \leq t \leq T$; then integrate (76)–(78) backwards in time in the interval $\tau \leq t \leq T$ with terminal conditions $P_2(T) = P_1(T)$ and $x_2(T) = x_1(T)$ which gives x_2 and P_2 in the interval $\tau \leq t \leq T$; then integrate:

$$\dot{\hat{x}} = A \hat{x} + B \hat{w} \quad (84)$$

backwards in time in the interval $0 \leq t \leq \tau$ with the feedback law: $\hat{w} = D^\dagger(\tilde{z}_2 - C_2\hat{x})$; and the terminal condition $x(\tau) = \zeta$ to find \hat{x} , \hat{w} in the interval $0 \leq t \leq \tau$; then integrate (84) forwards in time in the interval $\tau \leq t \leq T$ with the feedback law: $\hat{w} = D^\dagger(\tilde{z}_3 - C_3\hat{x})$ and the initial condition $x(\tau) = \zeta$ to find \hat{x} , \hat{w} in the interval $\tau \leq t \leq T$. The minimum of the performance index is:

$$\int_0^T \|(I - \Pi)(\tilde{z}(t) - z_1(t))\|_{R^{-1}}^2 dt + \|\zeta - x_2(\tau)\|_{P_2^{-1}(\tau)}^2. \quad (85)$$

Proof: We proceed similarly to the proof of Theorem 1. The first and fourth terms of the performance index in (79) are independent of $x(0)$ and w , subject to the constraint $x(\tau) = \zeta$. The two integrands in (79) are identically zero in their respective intervals for $x = \hat{x}$ and $w = \hat{w}$, which can be verified by substitution. Uniqueness is proven similarly to Theorem 1 for the intervals $0 \leq t \leq \tau$ and $\tau \leq t \leq T$ separately. \square

The solution to the constrained optimisation problem (75) given in Theorem 4 introduced the vector and matrix variables x_2 , w_2 and P_2 . We recall that x_2 and w_2 coincide with the state and input trajectories on the interval $[0, T]$ that minimise the performance criterion (7) as shown in Theorem 1. We may now provide an interpretation of the matrix variable P_2 . In Theorem 4 it is shown that the unique minimum of (75) takes the same form as the first expression on the right hand side of (35) but with an additional quadratic term which is zero when $\zeta = x_2(\tau)$, in which case we recover the results of Theorem 1. Consider now the eigenvector-eigenvalue decomposition of $P_2(\tau)$. Components of $\zeta - x_2(\tau)$ in those eigenvector directions of $P_2(\tau)$ which have small eigenvalues (i.e. large eigenvalues of $P_2^{-1}(\tau)$) give a large contribution to the second term in (85). Hence the measurements provide high confidence that the state $x(\tau)$ should be close to $x_2(\tau)$ in those directions. Fig. 2 illustrates the interpretation of $P_2(\tau)$ in the 2-D case. The figure shows an ellipse with centre $x_2(\tau)$ whose axes are aligned with the eigenvectors of $P_2(\tau)$ and lengths given by the corresponding eigenvalue square roots. All points on the ellipse increase the minimum performance index (85) by 1.

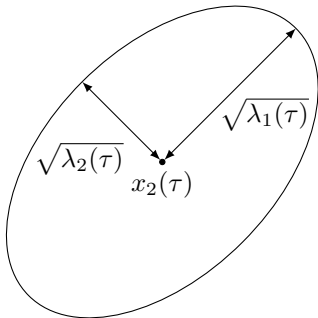


Fig. 2. An ellipse with semi-axes of length given by the eigenvalue square roots of $P_2(\tau)$, $\sqrt{\lambda_1(\tau)}$ and $\sqrt{\lambda_2(\tau)}$, and aligned with the corresponding eigenvectors.

VII. SPECIAL CASES

A. Standard Kalman filter

We now show how the continuous time Kalman filter can be derived as a special case of the filter in Theorems 1 and 2. We therefore consider a system described by:

$$\dot{x} = Ax + Bw \quad (86)$$

$$z = Cx \quad (87)$$

with noisy measurement \tilde{z} of z . Note that we assume as standard that sensor measurements of the state are not directly affected by the process noise w . In the standard Kalman filter the process noise w can be interpreted as a small magnitude disturbance to the system. Hence we need to incorporate a weighted 2-norm constraint on w in the performance index (7). In particular, we consider the following optimisation problem:

$$\inf_{w, x(0)} \left(\int_0^T \|\tilde{z}(t) - z(t)\|_{R^{-1}}^2 dt + \int_0^T \|w\|_{Q^{-1}}^2 dt + \|x(0) - \gamma\|_{\Gamma^{-1}}^2 \right) \quad (88)$$

where γ is given and R , Q and Γ are given positive-definite matrices of appropriate dimension. To translate this into the framework of this paper we introduce a virtual measurement of w which is equal to zero. More precisely, we consider an augmented system with (real and virtual) outputs given by:

$$z_a = \begin{bmatrix} C \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} w \quad (89)$$

and for which we have the measurement:

$$\tilde{z}_a = \begin{bmatrix} \tilde{z} \\ 0 \end{bmatrix}. \quad (90)$$

We define an augmented block diagonal weighting matrix R_a given by:

$$R_a = \begin{bmatrix} R & 0 \\ 0 & Q \end{bmatrix}. \quad (91)$$

The following result is obtained by applying Lemma 3 and Theorem 1 to this augmented system.

Theorem 5: Consider the system:

$$\dot{x}_1 = Ax_1 + K(\tilde{z} - Cx_1), \quad (92)$$

$$\dot{P}_1 = AP_1 + P_1A^T - K RK^T + BQB^T, \quad (93)$$

$$K = P_1 C^T R^{-1} \quad (94)$$

for $P_1(0) = \Gamma$ and $x_1(0) = \gamma$. The optimisation problem (88) where z is defined by (86)–(87) has a unique solution \hat{w} , $\hat{x}(0)$ where \hat{w} is defined by the feedback law:

$$\hat{w} = QB^T P_1^{-1}(\hat{x} - x_1) \quad (95)$$

and $\hat{x}(t)$ is obtained by solving $\dot{\hat{x}} = A\hat{x} + B\hat{w}$ backwards on the interval $[0, T]$ with terminal condition $\hat{x}(T) = x_1(T)$. Furthermore, the optimal cost (88) is given by:

$$\int_0^T \|\tilde{z} - Cx_1\|_{R^{-1}}^2 dt. \quad (96)$$

Proof: Replacing (6) by (89), \tilde{z} and R in (7) by (90) and (91), and applying Lemma 3 gives equations (92)–(94) after some simplification. Equations (95) and (96) are obtained by substituting from (89), (90) and (91) into (30) and (35) respectively. \square

The filter (92)–(94) is an end-of-interval estimator (cf. Theorem 2) in the sense that $\hat{x}(T) = x_1(T)$, $\hat{w}(T) = w_1(T)$ and takes the form of the standard Kalman filter with gain K . The above result reduces to that given in [1] with $R = I$ and $Q = I$.

It is interesting to note that by substituting for \hat{w} from (95) we obtain an equation for the optimal state estimate:

$$\dot{\hat{x}} = A\hat{x} + BQB^T P_1^{-1}(\hat{x} - x_1)$$

where $\hat{x}(T) = x_1(T)$ that coincides with the standard form for the smoothed estimate in Kalman filtering (see [38, eqn. 34(a)]). Similarly by specialising (77) to the present case we have the equation:

$$\dot{P}_2 = (A + BQB^T P_1^{-1})P_2 + P_2(A + BQB^T P_1^{-1})^T - BQB^T$$

where $P_2(T) = P_1(T)$, which is the corresponding form for the smoothed covariance (see [38, eqn. 34(b)]). We note that the interpretations of $P_1(t)$ and $P_2(t)$ derived in Section VI for the more general problem now apply immediately to the present case, and hence provide an analogous deterministic interpretation of the state covariances in standard Kalman filtering.

B. Kalman filter with input feedthrough

We consider the extension of the standard Kalman filter to the case where there is direct feedthrough of the input to the measurements. In particular we consider a system described by:

$$\dot{x} = Ax + Bw, \quad (97)$$

$$z = Cx + Dw \quad (98)$$

with noisy measurement \tilde{z} of z . As in section VII-A we incorporate a weighted 2-norm constraint on w in the performance index (7) and consider the optimisation problem (88) with z given by (98) rather than (87). To solve this we proceed similarly and consider an augmented output given by:

$$z_a = \begin{bmatrix} C \\ 0 \end{bmatrix} x + \begin{bmatrix} D \\ I \end{bmatrix} w \quad (99)$$

and for which we have the measurement \tilde{z}_a as in (90) and we define an augmented weighting matrix as in (91).

Theorem 6: Consider the system:

$$\dot{x}_1 = Ax_1 + Bw_1 + K_x(\tilde{z} - Cx_1 - Dw_1), \quad (100)$$

$$\dot{P}_1 = AP_1 + P_1A^T - K_xRK_x^T + (B - K_xD)P_w(B - K_xD)^T, \quad (101)$$

$$w_1 = K_w(\tilde{z} - Cx_1), \quad (102)$$

$$K_w = QD^T(DQD^T + R)^{-1}, \quad (103)$$

$$K_x = P_1C^TR^{-1}, \quad (104)$$

$$P_w = (I - K_wD)Q \quad (105)$$

with $P_1(0) = \Gamma$ and $x_1(0) = \gamma$. The optimisation problem (88) where z is given by (97)–(98) has a unique solution \hat{w} , $\hat{x}(0)$ where \hat{w} is defined by the feedback law:

$$\hat{w} = K_w(\tilde{z} - C\hat{x}) + P_wB^TP_1^{-1}(\hat{x} - x_1) \quad (106)$$

and $\hat{x}(t)$ is obtained by solving $\dot{\hat{x}} = A\hat{x} + B\hat{w}$ backwards on the interval $[0, T]$ with terminal condition $\hat{x}(T) = x_1(T)$. Furthermore, the optimal cost (88) is:

$$\int_0^T \|\tilde{z} - Cx_1\|_{(DQD^T + R)^{-1}}^2 dt. \quad (107)$$

Proof: We will apply Lemma 3 replacing (6) by (99), \tilde{z} and R in (7) by (90) and (91). Substituting into (18) gives:

$$D^\dagger = [K_w \quad I - K_wD] \quad (108)$$

where we have used the definition (103) and the matrix inversion identities (3.1) and (3.2) of Section 6.3 in [39]. Substituting into (14), (15), (16) and (12) using (108) gives:

$$A_1 = A - BK_wC, \quad (109)$$

$$B_1 = B[K_w \quad I - K_wD], \quad (110)$$

$$C_1 = \begin{bmatrix} I - DK_w \\ -K_w \end{bmatrix} C \quad (111)$$

and (102) respectively. Noting the symmetry $DK_wR = (DK_wR)^T$ we find after substituting into (11) using (111) and the definition (104) that:

$$K_1 = K_x[I - DK_w \quad -RK_w^TQ^{-1}]. \quad (112)$$

We now verify:

$$\begin{aligned} & \begin{bmatrix} I - DK_w \\ -K_w \end{bmatrix}^T \begin{bmatrix} R & 0 \\ 0 & Q \end{bmatrix}^{-1} \begin{bmatrix} I - DK_w \\ -K_w \end{bmatrix} \\ &= R^{-1}(I - DK_w) \\ &+ K_w^TD^TR^{-1}(-I + DK_w + R(DQD^T + R)^{-1}) \\ &= R^{-1}(I - DK_w) \quad (113) \\ &= (DQD^T + R)^{-1} \quad (114) \\ &= R^{-1}(R - DP_wD^T)R^{-1}. \quad (115) \end{aligned}$$

From (11) using (111), (113) and (104) we obtain:

$$K_1C_1 = K_x(I - DK_w)C. \quad (116)$$

Similarly from (11) using (111), (115) and (104) we obtain:

$$K_1R_aK_1^T = K_xRK_x^T - (K_xD)P_w(K_xD)^T. \quad (117)$$

Substituting into (9) using (109), (116), (110), (112) and (102) gives (100). Using (103), (104) and (105) we obtain:

$$K_wCP_1 = P_w(K_xD)^T. \quad (118)$$

By substituting from (108) we obtain:

$$D^\dagger R_a D^{\dagger T} = P_w. \quad (119)$$

Substituting into (10) using (109), (117), (118) and (119) gives (101). Equation (106) follows from (30) using (108) and (119). Equation (107) follows from the first expression in (35) using (108) and (114). \square

We again note that by substituting for \hat{w} from (106) we obtain the differential equation for the optimal state estimate:

$$\dot{\hat{x}} = A\hat{x} + BK_w(\tilde{z} - C\hat{x}) + BP_wB^TP_1^{-1}(\hat{x} - x_1) \quad (120)$$

where $\hat{x}(T) = x_1(T)$. Similarly (77) specialises to:

$$\begin{aligned} \dot{P}_2 &= (A - BK_wC + BP_wB^TP_1^{-1})P_2 \\ &+ P_2(A - BK_wC + BP_wB^TP_1^{-1})^T - BP_wB^T. \end{aligned} \quad (121)$$

Finally, we point out that the augmented input feedthrough matrix in (99) is full column rank for all D . Furthermore, the standard Kalman filter can be recovered trivially by setting $D = 0$ in (100)–(105), (120) and (121).

C. Standard LQ tracking on a finite time horizon

We now show how the standard linear-quadratic (LQ) tracking solution on a finite time horizon can be derived as a special case of the tracking problem in Theorem 3. In the standard LQ tracking problem we wish to find a low energy input such that the output tracks a desired output trajectory $\tilde{y}(t)$. More precisely, we consider the optimisation problem:

$$\inf_u \left(\int_0^T \|\tilde{y}(t) - Hq(t)\|_{R^{-1}}^2 + \|u(t)\|_{Q^{-1}}^2 dt + \|q(T)\|_{\Xi^{-1}}^2 \right), \quad (122)$$

for given H and positive-definite matrices R , Q and Ξ , where the state q and input u satisfy $\dot{q} = Fq + Gu$, and the initial state $q(0) = \eta$ is known. To put this into the form required to apply Lemma 4 and Theorem 3 we introduce the augmented output, desired output and weight matrix given by:

$$y_a = \begin{bmatrix} H \\ 0 \end{bmatrix} q + \begin{bmatrix} 0 \\ I \end{bmatrix} u, \quad (123)$$

$$\tilde{y}_a = \begin{bmatrix} \tilde{y} \\ 0 \end{bmatrix}, \quad (124)$$

$$R_a = \begin{bmatrix} R & 0 \\ 0 & Q \end{bmatrix}. \quad (125)$$

Theorem 7: Consider the system:

$$\dot{q}_1 = Fq_1 - K(\tilde{y} - Hq_1), \quad (126)$$

$$\dot{S}_1 = FS_1 + S_1F^T + S_1H^TR^{-1}HS_1^T - GQG^T, \quad (127)$$

$$K = S_1H^TR^{-1} \quad (128)$$

with the terminal condition $S_1(T) = \Xi$ and $q_1(T) = \xi$. The optimisation problem (122) has a unique solution \hat{u} given by the feedback law:

$$\hat{u} = -QG^TS_1^{-1}(\hat{q} - q_1) \quad (129)$$

and $\hat{q}(t)$ is obtained by solving $\dot{\hat{q}} = F\hat{q} + G\hat{u}$ forwards in the interval $[0, T]$ with the initial condition $\hat{q}(0) = \eta$. Furthermore, the optimal cost (122) is:

$$\int_0^T \|\tilde{y} - Hq_1\|_{R^{-1}}^2 dt + \|\eta - q_1(0)\|_{S_1(0)^{-1}}^2. \quad (130)$$

Proof: Replacing (48) by (123), \tilde{y} and R in (49) by (124)–(125) and applying Lemma 4 gives equations (126)–(128) after some simplification. Equations (129) and (130) are obtained by

substituting from (123)–(125) into (63) and (66) respectively. \square

The above result is the standard LQ tracking solution [40, Section 8.3] and the dual to the Kalman filtering and smoothing solution derived in Section VII-A. In the standard LQ regulator problem we wish to find the low energy input u that brings the state q to the origin. This is a specialisation of the LQ tracking problem which corresponds to setting $\tilde{y} = 0$ and $\xi = 0$. In this case $q_1(t) = 0$ for all t and hence the feedback law (129) is given by $\hat{u} = -QG^TS_1^{-1}\hat{q}$ and the minimum cost is given by $\|\eta\|_{S_1(0)^{-1}}^2$. This is recognised as the classical LQ regulator result on a finite time horizon.

VIII. STEADY STATE FILTER

A. Stability

We first consider the convergence properties of the filter of Lemma 3 (end of interval estimator) as $T \rightarrow \infty$. In order to express convergence conditions directly in terms of A, B, C, D we first need to establish the following two Lemmas.

Lemma 6: Let D have full column rank. $s_0 \in \mathbb{C}$ is an uncontrollable mode of (A_1, B_1) if and only if it is an uncontrollable mode of (A, B) .

Proof: If s_0 is an uncontrollable mode of (A, B) then there exists $0 \neq x \in \mathbb{C}^n$ such that $x^*A = x^*s_0$ and $x^*B = 0$. Hence $x^*(A - BD^\dagger C) = x^*s_0$ and $x^*BD^\dagger = 0$. The converse follows since D^\dagger has full row rank. \square

Lemma 7: Let D have full column rank. $s_0 \in \mathbb{C}$ is an unobservable mode of (C_1, A_1) if and only if it is an invariant zero of the system (5)–(6), i.e. $\begin{bmatrix} A - s_0I & B \\ C & D \end{bmatrix}$ does not have full column rank.

Proof: The proof is a more general result to [31, Lemma 13.9] to all system modes. \square

Theorem 8: Suppose (A, B) has no uncontrollable mode $s_0 \in \mathbb{C}$ with $\text{Re}(s_0) = 0$, the system (5)–(6) has no invariant zero $s_0 \in \mathbb{C}$ with $\text{Re}(s_0) \geq 0$ and $\tilde{z}(t) \in \mathcal{L}_\infty^p[0, \infty)$. Then the ARE:

$$A_1P_1^\infty + P_1^\infty A_1^T - K_1^\infty R K_1^{\infty T} + B_1 R B_1^T = 0 \quad (131)$$

where $K_1^\infty = P_1^\infty C_1^T R^{-1}$ has a unique solution P_1^∞ such that $A^\infty = A_1 - K_1^\infty C_1$ is Hurwitz. Furthermore, $P_1^\infty \geq 0$ and $P_1(t) \rightarrow P_1^\infty$ as $t \rightarrow \infty$ where $P_1(t)$ is given by (10) with the initial condition $P_1(0) = \Gamma$. Consider the system:

$$\dot{x}_1^\infty = A^\infty x_1^\infty + B^\infty \tilde{z}, \quad (132)$$

$$w_1^\infty = D^\dagger(\tilde{z} - Cx_1^\infty) \quad (133)$$

with any initial condition $x_1^\infty(0) \in \mathbb{R}^n$ where $B^\infty = B_1 + K_1^\infty$. Then:

- 1) $x_1(t) - x_1^\infty(t) \rightarrow 0$ as $t \rightarrow \infty$;
- 2) $w_1(t) - w_1^\infty(t) \rightarrow 0$ as $t \rightarrow \infty$

where $x_1(t)$ and $w_1(t)$ are given by (9) and (12) with the initial condition $x_1(0) = \gamma$.

Proof: The claims in regard to (131) follow directly by applying Lemmas 1 and 2 to the RDE (10) and expressing the convergence conditions in terms of A, B, C, D using Lemmas 6 and 7. The convergence results 1) and 2) follow from Lemma 11 (Appendix). \square

We remark that the system (132)–(133) is the “limiting form” of the end-of-interval estimator of Lemma 3 in which $P_1(t)$ is replaced by P_1^∞ . We do not assert any convergence property other than 1) and 2) in Theorem 8.

B. The steady state filter as a stable left inverse

We adopt the notation introduced in [31, Ch. 3] and denote the transfer function of (5)–(6) by:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = C(sI - A)^{-1}B + D. \quad (134)$$

We consider the transfer function of the steady state filter (132)–(133):

$$\left[\begin{array}{c|c} A^\infty & B^\infty \\ \hline -D^\dagger C & D^\dagger \end{array} \right]. \quad (135)$$

Theorem 9: Suppose (A, B) has no uncontrollable mode $s_0 \in \mathbb{C}$ with $\text{Re}(s_0) = 0$ and the system (5)–(6) has no invariant zero $s_0 \in \mathbb{C}$ with $\text{Re}(s_0) \geq 0$. Then (135) is a stable left inverse of (134).

Proof: To see this we consider the cascade connection of (135) with (134) and verify the calculation:

$$\begin{aligned} & \left[\begin{array}{c|c} A^\infty & B^\infty \\ \hline -D^\dagger C & D^\dagger \end{array} \right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \\ &= \left[\begin{array}{cc|c} A^\infty & B^\infty C & B^\infty D \\ 0 & A & B \\ \hline -D^\dagger C & D^\dagger C & D^\dagger D \end{array} \right] \\ &= \left[\begin{array}{cc|c} A^\infty & B^\infty C & B \\ 0 & A & B \\ \hline -D^\dagger C & D^\dagger C & I \end{array} \right] \end{aligned}$$

where we have used the transfer matrix product operation in [31, Sec. 3.6] and noting that $K_1^\infty D = 0$ (cf. (21)). The product is equivalently given by:

$$\left[\begin{array}{cc|c} A^\infty & 0 & 0 \\ 0 & A & B \\ \hline -D^\dagger C & 0 & I \end{array} \right] = I$$

using the similarity transformation:

$$\begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$$

after some simplification and noting that $K_1^\infty C_1 = K_1^\infty C$ (cf. (22)). We recall from Theorem 8 that A^∞ is Hurwitz. \square

C. The infinite time smoother as a left inverse

Lemma 8: Suppose (A, B) has no uncontrollable mode $s_0 \in \mathbb{C}$ with $\text{Re}(s_0) \leq 0$ and the system (5)–(6) has no invariant zero $s_0 \in \mathbb{C}$ with $\text{Re}(s_0) \geq 0$. Then $P_1^\infty > 0$. Furthermore, let:

$$A_2^\infty = A - B_1 C_2^\infty, \quad (136)$$

$$C_2^\infty = C - R B_1^T (P_1^\infty)^{-1}. \quad (137)$$

Then $-A_2^\infty$ is Hurwitz.

Proof: $P_1^\infty > 0$ follows by applying Lemma 1 to the ARE (131) using Lemmas 6 and 7 to express the convergence conditions in terms of A, B, C, D . Now note that:

$$A_2^\infty = A_1 + B_1 R B_1^T (P_1^\infty)^{-1}$$

by substituting (137) into (136) and then using (14). We may then verify that:

$$A_2^\infty P_1^\infty + P_1^\infty (A^\infty)^T = 0$$

by substituting for A^∞ , A_2^∞ and using (131). Hence $(A^\infty)^T$ and $-A_2^\infty$ are similar which means that $-A_2^\infty$ is Hurwitz. \square

We now assume that the conditions of Lemma 8 hold and consider the cascade connection of two systems. The first system has input \tilde{z} and output \tilde{z}_2^∞ . It is given by:

$$\dot{x}_1^\infty = A^\infty x_1^\infty + B^\infty \tilde{z}, \quad (138)$$

$$\tilde{z}_2^\infty = \tilde{z} - R B_1^T (P_1^\infty)^{-1} x_1^\infty \quad (139)$$

and has the transfer function:

$$\left[\begin{array}{c|c} A^\infty & B^\infty \\ \hline -R B_1^T (P_1^\infty)^{-1} & I \end{array} \right]. \quad (140)$$

(Note that (138) coincides with (132).) The second system is driven by the output of the first and has output \hat{w}^∞ . It is given by:

$$\begin{aligned} \dot{\hat{x}}^\infty &= A_2^\infty \hat{x}^\infty + B_2 \tilde{z}_2^\infty, \\ \hat{w}^\infty &= D^\dagger (\tilde{z}_2^\infty - C_2^\infty \hat{x}^\infty) \end{aligned}$$

and has the transfer function:

$$\left[\begin{array}{c|c} A_2^\infty & B_2 \\ \hline -D^\dagger C_2^\infty & D^\dagger \end{array} \right]. \quad (141)$$

This cascade connection is the “limiting form” of the construction for the optimal estimator of Theorem 1 and is shown next to be a left inverse of the original system. We do not assert any formal convergence property for this cascade connection.

Theorem 10: Suppose (A, B) has no uncontrollable mode $s_0 \in \mathbb{C}$ with $\text{Re}(s_0) \leq 0$ and the system (5)–(6) has no invariant zero $s_0 \in \mathbb{C}$ with $\text{Re}(s_0) \geq 0$. Then the cascade connection of (140) with (141):

$$\left[\begin{array}{c|c} A_2^\infty & B_2 \\ \hline -D^\dagger C_2^\infty & D^\dagger \end{array} \right] \left[\begin{array}{c|c} A^\infty & B^\infty \\ \hline -R B_1^T (P_1^\infty)^{-1} & I \end{array} \right] \quad (142)$$

is a left inverse of the system (5)–(6).

Proof: To see this consider the cascade connection of (142) with (134) which is given by:

$$\left[\begin{array}{ccc|c} A_2^\infty & -B_1 R B_1^T (P_1^\infty)^{-1} & B_2 C & B \\ 0 & A^\infty & B^\infty C & B \\ 0 & 0 & A & B \\ \hline -D^\dagger C_2^\infty & -D^\dagger R B_1^T (P_1^\infty)^{-1} & D^\dagger C & I \end{array} \right]$$

where we have used the transfer matrix product operation and noting that $B_2 D = B$ and $B^\infty D = B$ (cf. proof of Theorem 9). The product is equivalently given by:

$$\left[\begin{array}{ccc|c} A_2^\infty & -B_1 R B_1^T (P_1^\infty)^{-1} & 0 & 0 \\ 0 & A^\infty & 0 & 0 \\ 0 & 0 & A & B \\ \hline -D^\dagger C_2^\infty & -D^\dagger R B_1^T (P_1^\infty)^{-1} & 0 & I \end{array} \right] = I$$

using the similarity transformation:

$$\begin{bmatrix} \mathbf{I} & 0 & -\mathbf{I} \\ 0 & \mathbf{I} & -\mathbf{I} \\ 0 & 0 & \mathbf{I} \end{bmatrix}$$

after some simplification. \square

Remark 2: The conditions of Theorem 10 can be written in several alternative ways, e.g. (C_1, A_1) detectable and $(-A_1, B_1)$ stabilizable, or equivalently $(-A, B)$ stabilizable. We can interpret this as a forwards in time detectability condition for the first stage of the inversion and a backwards in time stabilisability condition for the second stage.

Remark 3: It is interesting to note that there is a natural state transformation given by $x_2 = (P_1^\infty)^{-1}x_1$ which leads to the alternative state space representation of the transfer function in (140):

$$\left[\begin{array}{c|c} (-A_2^\infty)^T & (P_1^\infty)^{-1}B^\infty \\ \hline -RB_1^T & \mathbf{I} \end{array} \right]. \quad (143)$$

Writing the state-space form of the cascade connection of (142) with the right hand factor replaced by (143) and then making use of the similarity transformation:

$$\left[\begin{array}{c|c} \mathbf{I} & 0 \\ \hline -(P_1^\infty)^{-1} & \mathbf{I} \end{array} \right]$$

we obtain the following state space realisation for the left inverse of Theorem 10:

$$\left[\begin{array}{cc|c} A_1 & -B_1RB_1^T & B_1 \\ -C_1^TRC_1 & -A_1^T & C_1^TR^{-1} \\ \hline -D^\dagger C & -D^\dagger RB_1^T & D^\dagger \end{array} \right]. \quad (144)$$

It is notable that this state-space realisation does not depend explicitly on the solution P_1^∞ of the Riccati equation (131) but has the Hamiltonian for the Riccati equation for $-(P_1^\infty)^{-1}$ as its A -matrix. We will encounter a similar form for the infinite horizon tracking problem in Remark 7.

D. Remarks on the optimal steady-state solutions as a left inverse

Theorems 9 and 10 show that the filtering (end-of-interval estimate) and smoothing (whole interval estimate) solutions of the estimation problem of Section IV take the form of left inverses of the system. In the former case the left inverse is stable under certain conditions, while in the latter case the left inverse has poles split between the left and right half-planes. The stable left inverse of (135) can be written as

$$\left[\begin{array}{c|c} A_1 - K_1^\infty C_1 & B_1 + K_1^\infty \\ \hline -D^\dagger C & D^\dagger \end{array} \right] \quad (145)$$

and it is interesting to note that this is a different left inverse to one that is widely known in the literature (e.g. [41], [31]) which only coincides with (145) when $K_1^\infty = 0$. It is of further interest to note that (145) is a left inverse of the the system for any matrix K_1^∞ which satisfies the condition $K_1^\infty D = 0$. (The proof of Theorem 9 continues to be valid with just this assumption.) We will encounter another left inverse of the form (145) in the next section.

We now turn to the left inverse of Theorem 10. After applying a similarity transformation (142) takes the form

$$\left[\begin{array}{cc|c} A_2^\infty & K_1^\infty C_1 & -K_1^\infty \\ 0 & A_1 - K_1^\infty C_1 & B_1 + K_1^\infty \\ \hline -D^\dagger C_2^\infty & -D^\dagger C & D^\dagger \end{array} \right]. \quad (146)$$

Generally, when $p > m$, (146) is different to (145). However, if $p = m$, we can easily show that $K_1^\infty = 0$ and all the unstable modes associated with A_2^∞ in (146) cancel out. It is then seen that (146) coincides with (145) which is also the same as the left inverse in [41]. The equality of all three left inverses in the square case is expected since the left inverse is unique in this case.

The above observations show that the steady-state solution of the smoothing problem has the identical solution to the end-of-interval estimation problem for the special case when $p = m$. Since the latter is a stable, causal filter, it appears that there is no extra benefit from the anti-causal processing that is characteristic of the smoothed solution in this special case of $p = m$. It is interesting to ask if the same conclusion holds for the finite horizon problem. In fact it is possible to deduce from Theorem 1 that

$$\frac{d}{dt}(\hat{x} - x_1) = B_1RB_1^TP_1^{-1}(\hat{x} - x_1)$$

when $p = m$ which means that \hat{x} and x_1 are equal identically on the whole interval $[0, T]$ and not just at time T . Thus, indeed, the smoothed solution is identical with the end-of-interval estimate for the special case of $p = m$.

IX. INFINITE HORIZON TRACKING

A. An anti-stable left inverse

We begin by considering the convergence of the construction of Lemma 4 as the horizon length increases. We show that $q_1(t)$ and $u_1(t)$ converge for any fixed t to the state and input of an anti-stable time invariant system solved backwards in time.

Theorem 11: Suppose (F, G) has no uncontrollable mode $s_0 \in \mathbb{C}$ with $\text{Re}(s_0) = 0$, the system (47)–(48) has no invariant zero $s_0 \in \mathbb{C}$ with $\text{Re}(s_0) \leq 0$ and $\tilde{y}(t) \in \mathcal{L}_\infty^p[0, \infty)$. Then the ARE:

$$F_1S_1^\infty + S_1^\infty F_1^T + M_1^\infty R M_1^{\infty T} - G_1 R G_1^T = 0 \quad (147)$$

where $M_1^\infty = S_1^\infty H_1^T R^{-1}$ has a unique solution S_1^∞ such that $-F^\infty$ is Hurwitz where $F^\infty = F_1 + M_1^\infty H_1$. Furthermore, $S_1^\infty \geq 0$ and $S_1(t, T) \rightarrow S_1^\infty$ as $T \rightarrow \infty$ for any fixed $t \geq 0$ where $S_1(t, T)$ equals $S_1(t)$ in (52) with the terminal condition $S_1(T) = \Xi$. Consider the system:

$$\dot{q}_1^\infty = F^\infty q_1^\infty + G^\infty \tilde{y}, \quad (148)$$

$$u_1^\infty = J^\dagger(\tilde{y} - H q_1^\infty) \quad (149)$$

with any terminal condition $q_1^\infty(T) \in \mathbb{R}^n$ where $G^\infty = G_1 - M_1^\infty$. Then:

- 1) $q_1(t, T) - q_1^\infty(t, T) \rightarrow 0$ as $T \rightarrow \infty$ for any fixed $t \geq 0$;
 - 2) $u_1(t, T) - u_1^\infty(t, T) \rightarrow 0$ as $T \rightarrow \infty$ for any fixed $t \geq 0$
- where $q_1(t, T)$ equals $q_1(t)$ in (51) with the terminal condition $q_1(T) = \xi$ and $u_1(t, T)$ equals $u_1(t)$ in (54).

Proof: The claims in regard to (147) follow directly by applying Remark 1 to the RDE (52) and expressing the convergence conditions in terms of F, G, H, J using Lemmas 6 and 7 with the appropriate notational substitutions. The convergence results 1) and 2) follow from the time reversed Lemma 11 (Appendix). \square

Remark 4: It is interesting to note the contrasting form of 1) and 2) in Theorems 8 and 11. At first sight this is unexpected since the estimation and tracking problems are dual to each other. The difference arises since the infinite horizon limit of the time window $[0, T]$ is taken to be $[0, \infty)$ in both cases, namely the left hand limit is fixed at the origin while the right hand limit tends to ∞ , which is not symmetric since the two problems are dual by time reversal.

We now consider the transfer function of the system (148)–(149):

$$\left[\begin{array}{c|c} F^\infty & G^\infty \\ \hline -J^\dagger H & J^\dagger \end{array} \right]. \quad (150)$$

Theorem 12: Suppose (F, G) has no uncontrollable mode $s_0 \in \mathbb{C}$ with $\text{Re}(s_0) = 0$ and the system (47)–(48) has no invariant zero $s_0 \in \mathbb{C}$ with $\text{Re}(s_0) \leq 0$. Then (150) is an anti-stable left inverse of the system (47)–(48).

Proof: To see this we consider the cascade connection of (150) with the transfer function of the system (47)–(48) and then perform the transfer matrix product and similarity transformation similarly to the proof of Theorem 9. \square

Remark 5: After the notational substitutions $F \leftrightarrow A$ etc (150) reduces to (145) with $K_1^\infty = Q_1^\infty C_1^T R^{-1}$ and Q_1^∞ equal to the *anti-stabilizing solution* of the Riccati equation (131). As before $K_1^\infty D = 0$. It is thus interesting to note that the steady-state form of the end-of-interval estimator and the steady-state form of the construction of Lemma 4 reduce to the same type of left inverse (145) of the system but with K_1^∞ determined from a stabilizing (resp. anti-stabilizing) solution of the same Riccati equation.

B. The infinite horizon controller

We now consider the convergence properties of the unique solution \hat{u} of the tracking problem (50), which is given in Theorem 3, in the infinite time horizon limit, i.e. as $T \rightarrow \infty$.

Theorem 13: Suppose (F, G) has no uncontrollable mode $s_0 \in \mathbb{C}$ with $\text{Re}(s_0) \geq 0$, i.e. it is stabilizable, the system (47)–(48) has no invariant zero $s_0 \in \mathbb{C}$ with $\text{Re}(s_0) = 0$ and $\tilde{y}(t) \in \mathcal{L}_\infty^p[0, \infty)$. Then the ARE:

$$S_2^\infty F_1 + F_1^T S_2^\infty - S_2^\infty G_1 R G_1^T S_2^\infty + H_1^T R^{-1} H_1 = 0 \quad (151)$$

has a unique solution S_2^∞ that is stabilising, i.e. F_2^∞ in (154) is Hurwitz, and $S_2^\infty \geq 0$. Consider the system:

$$\dot{\hat{q}}^\infty(t) = F \hat{q}^\infty(t) + G \hat{u}^\infty(t), \quad (152)$$

$$\hat{u}^\infty(t) = J^\dagger (\tilde{y}_2^\infty(t) - H_2^\infty \hat{q}^\infty(t)) \quad (153)$$

with the initial condition $\hat{q}^\infty(0) = \eta$ where:

$$F_2^\infty = F - G_1 H_2^\infty, \quad (154)$$

$$G_2^\infty = H_1^T R^{-1} - S_2^\infty G_1, \quad (155)$$

$$H_2^\infty = H + R G_1^T S_2^\infty, \quad (156)$$

$$\tilde{y}_2^\infty(t) = \tilde{y}(t) + R G_1^T q_2^\infty(t), \quad (157)$$

$$q_2^\infty(t) = \int_t^\infty e^{F_2^\infty(\tau-t)} G_2^\infty \tilde{y}(\tau) d\tau. \quad (158)$$

Then the unique optimal control input $\hat{u}(t, T)$ of the tracking problem (50) with the initial condition $q(0) = \eta$ (i.e. \hat{u} as defined in (63)) converges as $T \rightarrow \infty$, i.e. $\lim_{T \rightarrow \infty} \hat{u}(t, T)$ exists for any fixed $t \geq 0$, and the limit is given by $\hat{u}^\infty(t)$.

Proof: Let S_1, q_1 be defined by (51), (52) with the terminal conditions $S_1(T) = \Xi, q_1(T) = \xi$. We introduce the variables: $S_2 = S_1^{-1}, q_2 = S_1^{-1} q_1$. Hence S_2, q_2 are generated by solving:

$$\dot{S}_2 = -S_2 F_1 - F_1^T S_2 + S_2 G_1 R G_1^T S_2 - H_1^T R^{-1} H_1, \quad (159)$$

$$\dot{q}_2 = \left(-F_1^T + S_2 G_1 R G_1^T \right) q_2 + \left(S_2 G_1 - H_1^T R^{-1} \right) \tilde{y} \quad (160)$$

backwards in time with the given terminal conditions:

$$S_2(T) = S_1^{-1}(T) = \Xi^{-1},$$

$$q_2(T) = S_1^{-1}(T) q_1(T) = \Xi^{-1} \xi.$$

We now apply Remark 1 to the ARE (151) and the RDE (159). The conditions of the theorem are obtained in terms of F, G, H, J using Lemmas 6 and 7 with the appropriate notational substitutions. Furthermore F_2^∞ is Hurwitz and:

$$\lim_{T \rightarrow \infty} S_1^{-1}(t, T) = \lim_{T \rightarrow \infty} S_2(t, T) = S_2^\infty \quad (161)$$

for all $\Xi > 0$ and for any fixed $t \geq 0$. We introduce the anti-stable and time-invariant equation:

$$\dot{q}_2^\infty = -F_2^\infty q_2^\infty - G_2^\infty \tilde{y}. \quad (162)$$

We now consider the time reversed equations (160) and (162) for a given T (i.e. solved forwards in time). These equations take the form of (176) and (177) in Lemma 11 (Appendix) on the interval $[0, T]$, where we note that $u(t)$ depends on T , but with $\sup_{0 \leq t \leq T} |u(t)|_\infty \leq \|\tilde{y}(t)\|_\infty$ for any T . Now choose any $\epsilon > 0$ and find the T_0 guaranteed by Lemma 11. Then the time reversed solutions are within ϵ in norm for $T_0 \leq t \leq T$ providing $T_0 \leq T$. Hence, for any $T > T_0$, $|q_2(t) - q_2^\infty(t)|_\infty < \epsilon$ for $0 \leq t \leq T - T_0$. It follows that:

$$\lim_{T \rightarrow \infty} q_2(t, T) = q_2^\infty(t) \quad (163)$$

for any fixed $t \geq 0$, where $q_2^\infty(t)$ is given by the convolution form (158). We now return to compute the limit of the unique optimal input $\hat{u}(t, T)$ given in Theorem 3, i.e. $\lim_{T \rightarrow \infty} \hat{u}(t, T)$. Taking the limit in (64), (65) and substituting from (161), (163) gives:

$$\lim_{T \rightarrow \infty} H_2(t, T) = H_2^\infty,$$

$$\lim_{T \rightarrow \infty} \tilde{y}_2(t, T) = \tilde{y}_2^\infty(t)$$

where $\tilde{y}_2^\infty(t)$ and H_2^∞ are given by (154)–(158). Rewriting (62) for the infinite horizon with a notational substitution and taking the limit in (63) gives the feedback law (153). \square

Remark 6: Evaluating q_2^∞ for all finite t using (158) is costly even if it is possible. It can be approximated for any finite t_0 as accurately as required by:

$$q_2^\lambda(t_0) = \int_{t_0}^{t_0+\lambda} e^{F_2^\infty T(\tau-t_0)} G_2^\infty \tilde{y}(\tau) d\tau \quad (164)$$

for a sufficiently large $\lambda > 0$ since F_2^∞ is Hurwitz. Integrating (162) forwards in time for $t > t_0$ with the initial condition $q_2^\lambda(t_0)$ obtained from (164) gives $q_2^\infty(t)$ approximately for t near t_0 but errors amplify since (162) is anti-stable. A practical compromise is to evaluate (164) at regular intervals and to integrate (162) within those intervals.

C. The steady state controller as an unstable left inverse

Theorem 14: Suppose (F, G) has no uncontrollable mode $s_0 \in \mathbb{C}$ with $\text{Re}(s_0) \geq 0$, i.e. it is stabilizable, and the system (47)–(48) has no invariant zero $s_0 \in \mathbb{C}$ with $\text{Re}(s_0) = 0$. Then the infinite horizon controller is given by the cascade connection of an anti-stable system with input \tilde{y} and output \tilde{y}_2^∞ and transfer function:

$$\left[\begin{array}{c|c} (-F_2^\infty)^T & -G_2^\infty \\ \hline RG_1^T & I \end{array} \right]$$

followed by a stable system with input \tilde{y}_2^∞ and output \hat{u} and transfer function:

$$\left[\begin{array}{c|c} F_2^\infty & G_1 \\ \hline -J^\dagger H_2^\infty & J^\dagger \end{array} \right].$$

Furthermore, their cascade connection, given by:

$$\left[\begin{array}{c|c} F_2^\infty & G_1 \\ \hline -J^\dagger H_2^\infty & J^\dagger \end{array} \right] \left[\begin{array}{c|c} (-F_2^\infty)^T & -G_2^\infty \\ \hline RG_1^T & I \end{array} \right] \quad (165)$$

is a left inverse of the system (47)–(48).

Proof: Apply the transfer matrix product operation and a system similarity transformation similarly to the proof of Theorem 10. Note that F_2^∞ is Hurwitz from Theorem 13. \square

Remark 7: Theorem 14 can also be seen by noting that (165) is in fact identical with the left inverse of Theorem 10 after the appropriate notational substitutions $A \leftrightarrow F$, etc. This can be seen by forming the state-space of the cascade connection of (165) and then making use of the similarity transformation:

$$\begin{bmatrix} I & 0 \\ S_2^\infty & -I \end{bmatrix}$$

to obtain the realisation:

$$\left[\begin{array}{cc|c} F_1 & -G_1 R G_1^T & G_1 \\ -H_1^T R H_1 & -F_1^T & H_1^T R^{-1} \\ \hline -J^\dagger H & -J^\dagger R G_1^T & J^\dagger \end{array} \right]$$

which has the same form as (144). In this case the Hamiltonian corresponds exactly to the Riccati equation (151) for S_2^∞ .

X. CONCLUSIONS

The paper has proposed a framework for estimation in which the output of the dynamical system comprises *all* variables that are measured, and the variables to be estimated comprise, equally, system states and exogenous inputs. The unique optimum solution on a finite horizon takes a two-stage form in which the first stage provides an end-of-interval estimator which can be solved in real time as the horizon length increases. The estimation problem is general enough to include the Kalman filter and, for the dual tracking problem, the linear quadratic tracking problem as special cases.

A further contribution of this paper has been to provide an interpretation of the solution of the Riccati differential equation which is analogous to the meaning of the covariance matrix in stochastic filtering. The solution of a matrix Lyapunov differential equation $P_2(t)$ is shown to have an analogous interpretation to the smoothed covariance in the stochastic case. This has been achieved by considering the least-squares estimation problem with an additional constraint that the state passes through a prescribed point at a given time in the fixed horizon.

The paper has considered the natural time invariant limiting forms of the estimation and tracking problems. Conditions were given for the convergence of the finite horizon solutions to these limits. Stability of the end-of-interval estimator on the infinite horizon requires a minimum phase condition (i.e. that there are no invariant zeros of the system in the closed right half plane) as well as the absence of uncontrollable modes on the imaginary axis. The time invariant systems were shown to be stable or anti-stable left inverses of the original system under appropriate conditions.

XI. APPENDIX

We define the vector norm and induced matrix norm:

$$|x(t)|_\infty = \max_j |x_j(t)|,$$

$$|G(t)|_\infty = \max_i \sum_j |G_{ij}(t)|$$

where $x(t) \in \mathbb{R}^n$ and $G(t) \in \mathbb{R}^{n \times n}$ and the signal norms:

$$\|x(t)\|_\infty = \sup_{t \geq 0} |x(t)|_\infty, \quad (166)$$

$$\|G(t)\|_\infty = \sup_{t \geq 0} |G(t)|_\infty \quad (167)$$

for signals which belong to the corresponding Lebesgue space $\mathcal{L}_\infty^n[0, \infty)$ or $\mathcal{L}_\infty^{n \times n}[0, \infty)$. (Strictly we should take the *essential supremum* in (166) and (167) though this will always coincide with the supremum for signals encountered here.) We further define the norm:

$$\|G(t)\|_1 = \max_i \sum_j \int_0^\infty |G_{ij}(\tau)| d\tau.$$

Lemma 9: Suppose $x(t) \in \mathbb{R}^n$ satisfies:

$$\dot{x}(t) = A(t)x(t) + f(t) \quad (168)$$

where $A(t)$ is continuously time varying and $\lim_{t \rightarrow \infty} A(t) = A$ with A Hurwitz, $f(t) \in \mathcal{L}_\infty^n[0, \infty)$ and $x(0) \in \mathbb{R}^n$. Then $x(t)$ is uniformly bounded, i.e. $\|x(t)\|_\infty < \infty$.

Proof: First we set $M_1 = \|e^{At}\|_1$ noting that $M_1 < \infty$ since A is Hurwitz. Next we choose $\delta > 0$ such that $\delta M_1 < 1$. Since $A(t) \rightarrow A$ we can find t_0 such that $|A(t) - A|_\infty < \delta$ for all $t > t_0$. We next consider the free and forced solution of (168) on the interval $[0, t_0]$. We define:

$$M_2 = \left\{ \sup_{0 \leq t \leq t_0} |x(t)|_\infty : \dot{x}(t) = A(t)x(t), x(0) = x_0 \right\}$$

where $M_2 = M_2(x_0)$ and:

$$M_3 = \left\{ \sup_{0 \leq t \leq t_0} |x(t)|_\infty : \dot{x}(t) = A(t)x(t) + f(t), x(0) = 0, \|f(t)\|_\infty \leq 1 \text{ for } t \in [0, t_0] \right\}.$$

We note that $M_2 < \infty$ and $M_3 < \infty$ follows from [30, Theorem 1, p. 40]. Hence:

$$|x(t)|_\infty < M_2 + M_3\|f(t)\|_\infty \text{ for all } t \in [0, t_0]. \quad (169)$$

We next define: $M_4 = \|e^{At}\|_\infty$. We can see that $M_4 < \infty$ as follows. Let $A = TJT^{-1}$ be a Jordan decomposition and let $\bar{\lambda}$ be the largest real part among the eigenvalues of A . Then:

$$|e^{At}|_\infty \leq |T|_\infty |T^{-1}|_\infty e^{-\bar{\lambda}t} \left(1 + t + \dots + \frac{t^{n-1}}{(n-1)!} \right) \quad (170)$$

which is uniformly bounded since A is Hurwitz and thus $\bar{\lambda} < 0$. We now consider the solution of (168) for $t \geq t_0$. We can write:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}u(\tau)d\tau$$

where we have defined: $u(t) = (A(t) - A)x(t) + f(t)$. Then:

$$\begin{aligned} |x(t)|_\infty &= \left| e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}u(\tau)d\tau \right|_\infty \\ &\leq |e^{A(t-t_0)}x(t_0)|_\infty + \left| \int_{t_0}^t e^{A(t-\tau)}f(\tau)d\tau \right|_\infty \\ &\quad + \left| \int_{t_0}^t e^{A(t-\tau)}(A(t) - A)x(\tau)d\tau \right|_\infty \\ &\leq M_4(M_2 + M_3\|f(t)\|_\infty) \\ &\quad + M_1(\|f(t)\|_\infty + \delta \sup_{t_0 \leq \tau \leq t} |x(\tau)|_\infty). \end{aligned} \quad (171)$$

Combining (171) with (169) we obtain:

$$\begin{aligned} |x(t)|_\infty &\leq \max\{1, M_4\}(M_2 + M_3\|f(t)\|_\infty) \\ &\quad + M_1(\|f(t)\|_\infty + \delta \sup_{0 \leq t \leq t_1} |x(t)|_\infty) \end{aligned} \quad (172)$$

for all $t \in [0, t_1]$ and any t_1 . Since this is true for all t we can replace the LHS of (172) by $\sup_{0 \leq t \leq t_1} |x(t)|_\infty$. Therefore:

$$\begin{aligned} \sup_{0 \leq t \leq t_1} |x(t)|_\infty &\leq \frac{1}{1 - \delta M_1} \\ &\left(\max\{1, M_4\}(M_2 + M_3\|f(t)\|_\infty) + M_1\|f(t)\|_\infty \right) \end{aligned} \quad (173)$$

Since this is true for all t_1 the RHS is an upper bound for $\|x(t)\|_\infty$ which completes the proof. \square

Lemma 10: Suppose $x(t) \in \mathbb{R}^n$ satisfies:

$$\dot{x}(t) = Ax(t) + f(t) \quad (174)$$

where $A \in \mathbb{R}^{n \times n}$ is Hurwitz, $f(t) \in \mathcal{L}_\infty^n[0, \infty)$, $\lim_{t \rightarrow \infty} f(t) = 0$ and $x(0) \in \mathbb{R}^n$. Then $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof: We consider the solution to (174):

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}f(\tau)d\tau. \quad (175)$$

Since A is Hurwitz it follows that $\lim_{t \rightarrow \infty} e^{At} = 0$ and thus without loss of generality we set $x(0) = 0$ in (175). We set $M = \int_0^\infty |e^{At}|_\infty dt$, where $M < \infty$ using (170). Choose any $\epsilon > 0$. We first set $\delta = \epsilon/2M$. Since $\lim_{t \rightarrow \infty} f(t) = 0$ we can find t_0 such that $|f(t)|_\infty < \delta$ for all $t > t_0$. Then for $t > t_0$:

$$\begin{aligned} |x(t)|_\infty &= \left| \int_0^t e^{A(t-\tau)}f(\tau)d\tau \right|_\infty \\ &\leq \int_0^{t_0} |e^{A(t-\tau)}|_\infty |f(\tau)|_\infty d\tau \\ &\quad + \int_{t_0}^t |e^{A(t-\tau)}|_\infty |f(\tau)|_\infty d\tau \\ &\leq \|f(t)\|_\infty \int_{t-t_0}^t |e^{A\tau}|_\infty d\tau + \delta \int_0^{t-t_0} |e^{A\tau}|_\infty d\tau \\ &< \|f(t)\|_\infty \left(\sup_{t-t_0 \leq \tau \leq t} |e^{A\tau}|_\infty \right) t_0 + \delta M. \end{aligned}$$

We note using (170) that $\lim_{t \rightarrow \infty} |e^{At}|_\infty = 0$ and thus there exists $t_1 > t_0$ such that:

$$\sup_{t-t_0 \leq \tau \leq t} |e^{A\tau}|_\infty \leq \frac{\epsilon}{2\|f(t)\|_\infty t_0}$$

for all $t > t_1$. It follows that $|x(t)|_\infty < \epsilon$ for all $t > t_1$. \square

Lemma 11: Suppose $x(t), x_1(t) \in \mathbb{R}^n$ satisfy:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad (176)$$

$$\dot{x}_1(t) = Ax_1(t) + Bu(t) \quad (177)$$

where $A(t), B(t)$ are continuously time varying, $\lim_{t \rightarrow \infty} A(t) = A$ with A Hurwitz, $\lim_{t \rightarrow \infty} B(t) = B$, $u(t) \in \mathcal{L}_\infty^m[0, \infty)$ and $x(0), x_1(0) \in \mathbb{R}^n$. Then $\lim_{t \rightarrow \infty} (x(t) - x_1(t)) = 0$. More precisely, given any $\epsilon > 0$, $\exists T_0$ such that $|x(t) - x_1(t)|_\infty < \epsilon$ for all $t > T_0$, where T_0 depends on $\|u(t)\|_\infty$ but not $u(t)$ itself.

Proof: First note from Lemma 9 that $\|x(t)\|_\infty$ is finite. Moreover it can be seen from the proof of Lemma 9 that $\|x(t)\|_\infty$ has an upper bound which depends on $\|B(t)\|_\infty \|u(t)\|_\infty$ but otherwise does not depend on $u(t)$ (see (173)). Now write:

$$\begin{aligned} \dot{x}(t) - \dot{x}_1(t) &= A(x(t) - x_1(t)) + (A(t) - A)x(t) \\ &\quad + (B(t) - B)u(t). \end{aligned}$$

The conclusion follows from Lemma 10 by noting that the choice of t_0 and t_1 can be made independent of the choice of $u(t)$ for a given bound on $\|u(t)\|_\infty$. \square

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Grigorios Gakis received degrees in mechanical and nuclear engineering from the Universities of Bristol and Cambridge, England. He is currently a PhD student in control engineering at the University of Cambridge. His research interests are in the areas of estimation and control.



Malcolm C. Smith (M'90 SM'00 F'02) received degrees in mathematics and control engineering from Cambridge University, England. He was subsequently a Research Fellow at the German Aerospace Centre, Oberpfaffenhofen, Germany, a Visiting Assistant Professor and Research Fellow with the Department of Electrical Engineering at McGill University, Montreal, Canada, and an Assistant Professor with the Department of Electrical Engineering, Ohio State University, Columbus. In 1990, he joined the Engineering Department, University of Cambridge,

where he is currently a Professor and Head of the Control Group. He is a Fellow of Gonville and Caius College. His research interests are in the areas of robust control, nonlinear systems, electrical and mechanical networks, and automotive applications. Dr. Smith received the 1992 and 1999 George Axelby Best Paper Awards, in the IEEE Transactions on Automatic Control, both times for joint work with T.T. Georgiou. He is a winner of the 2020 IEEE Control Systems Technology Award (with W. Houlst and P. Brezas). He is a Fellow of the Royal Academy of Engineering.