# Distributed Optimization Over Dependent Random Networks 

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#### Abstract

We study the averaging-based distributed optimization solvers over random networks. We show a general result on the convergence of such schemes using weight-matrices that are row-stochastic almost surely and column-stochastic in expectation for a broad class of dependent weight-matrix sequences. In addition to implying many of the previously known results on this domain, our work shows the robustness of distributed optimization results to link-failure. Also, it provides a new tool for synthesizing distributed optimization algorithms. To prove our main theorem, we establish new results on the rate of convergence analysis of averaging dynamics over (dependent) random networks. These secondary results, along with the required martingale-type results to establish them, might be of interest to a broader research endeavors in distributed computation over random networks.


## 1 Introduction

Distributed optimization has received increasing attention in recent years due to its applications in distributed control of robotic networks [3], study of opinion dynamics in social networks [10], distributed estimation and signal processing [4, 22, 28, and power networks [6] 8].

In distributed optimization, we are often interested in finding an optimizer of a decomposable function $F(z)=\sum_{i=1}^{n} f_{i}(z)$ such that $f_{i}(\cdot)$ s are distributed through a network of $n$ agents, i.e., agent $i$ only knows $f_{i}(\cdot)$, and we are seeking to solve this problem without sharing the local objective function $f_{i}(z)$. Therefore, the goal is to find distributed dynamics over time-varying networks that, asymptotically, all the nodes agree on an optimizer of $F(\cdot)$.

The most well-know algorithm that achieves this is, what we refer to as, the averaging-based distributed optimization solver where each node maintains an estimate of an optimal point, and at each time step, each node computes the average of the estimates of its neighbors and performs (sub-)gradient descent on its local objective function [19]. However, in order for such an algorithm to converge, the corresponding weight matrices should be doubly stochastic. While making a row-stochastic or a column-stochastic matrix is easy, this is not the case for doubly stochastic matrices. Therefore, it is often assumed that such a doubly stochastic matrix sequence is given.

A solution to overcome this challenge is to establish more complicated distributed algorithms that effectively reconstruct the average-state distributively. The first algorithm in this category was proposed in [32], which is called subgradient-push (or push-sum), and later was extended for time-varying networks [18]. In this scheme, the weight matrices are assumed to be column-stochastic, and through the use of auxiliary state variables the approximate average state is reconstructed. Another scheme in this category that works with row-stochastic matrices, but does not need the column-stochastic assumption, is proposed in [15, 33]. However, to use this scheme, every node needs to be assigned and know its unique label. Assigning those labels distributively is also another challenge in this respect. In addition, both these schemes invoke division operation which results in theoretical challenges in establishing their stability in random networks [24, 25].

Another solution to address this challenge is to use gossip 12 and broadcast gossip [17] algorithms over random networks. The weight matrices of gossip algorithms are row-stochastic and in-expectation column-stochastic. This fact was generalized in [16], where it is proven that it is sufficient to have rowstochastic weight matrices, that are column-stochastic in-expectation. In all the above works on distributed optimization over random networks, all weight matrices are assumed to be independent and identically

[^0]distributed (i.i.d.). In [14], the broader class of random networks, which is Markovian networks was studied in distributed optimization; however, weight matrices were assumed to be doubly stochastic almost surely. Also, our work is closely related to the existing works on distributed averaging on random networks [11 29-31.

In this paper, we study distributed optimization over random networks, where the randomness is not only time-varying but also, possibly, dependent on the past. Under the standard assumptions on the local objective functions and step-size sequences for the gradient descent algorithm, we show that the averagingbased distributed optimization solver at each node converges to a global optimizer almost surely if the weight matrices are row-stochastic almost surely, column-stochastic in-expectation, and satisfy certain connectivity assumptions.

The paper is organized as follows: we conclude this section by introducing mathematical notations that will be used subsequently. In Section2, we formulate the problem of interest and state the main result of this work and discuss some of its immediate consequences. To prove the main result, first we study the behavior of the distributed averaging dynamics over random networks in Section 3. Then, in Section 4, we extent this analysis to the dynamics with arbitrary control inputs. Finally, the main result is proved in Section 5 We conclude this work in Section 6 .

Notation and Basic Terminology: The following notation will be used throughout the paper. We let $[n] \triangleq\{1, \ldots, n\}$. We denote the space of real numbers by $\mathbb{R}$ and natural (positive integer) numbers by $\mathbb{N}$. We denote the space of $n$-dimensional real-valued vectors by $\mathbb{R}^{n}$. In this paper, all vectors are assumed to be column vectors. The transpose of a vector $x \in \mathbb{R}^{n}$ is denoted by $x^{T}$. For a vector $x \in \mathbb{R}^{n}, x_{i}$ represents the $i$ th coordinate of $x$. We denote the all-one vector in $\mathbb{R}^{n}$ by $e^{n}=[1,1, \ldots, 1]^{T}$. We drop the superscript $n$ in $e^{n}$ whenever the dimension of the space is understandable from the context. A non-negative matrix $A$ is a row-stochastic (column-stochastic) matrix if $A e=e\left(e^{T} A=e^{T}\right)$.

Let $(\Omega, \mathcal{F}, \operatorname{Pr})$ be a probability space and let $\{W(t)\}$ be a chain of random matrices, i.e., for all $t \geq 0$ and $i, j \in[n], w_{i j}(t): \Omega \rightarrow \mathbb{R}$ is a Borel-measurable function. For random vectors (variables) $\mathbf{x}(1), \ldots, \mathbf{x}(t)$, we denote the sigma-algebra generated by these random variables by $\sigma(\mathbf{x}(0), \ldots, \mathbf{x}(t))$. We say that $\{\mathcal{F}(t)\}$ is a filtration for $(\Omega, \mathcal{F})$ if $\mathcal{F}(0) \subseteq \mathcal{F}(1) \subseteq \cdots \subseteq \mathcal{F}$. Further, we say that a random process $\{V(t)\}$ (of random variables, vectors, or matrices) is adapted to $\{\mathcal{F}(t)\}$ if $V(t)$ is measurable with respect to $\mathcal{F}(t)$.

Throughout this paper we mainly deal with directed graphs. A directed graph $\mathcal{G}=([n], \mathcal{E})$ (on $n$ vertices) is defined by a vertex set (identified by) $[n]$ and an edge set $\mathcal{E} \subset[n] \times[n]$. A graph $\mathcal{G}=([n], \mathcal{E})$ has a spanning directed rooted tree if it has a vertex $r \in[n]$ as $a$ root such that there exists a (directed) path from $r$ to every other vertex in the graph. For a matrix $A=\left[a_{i j}\right]_{n \times n}$, the associated directed graph with parameter $\gamma>0$ is the graph $\mathcal{G}^{\gamma}(A)=\left([n], \mathcal{E}^{\gamma}(A)\right)$ with the edge set $\mathcal{E}^{\gamma}(A)=\left\{(j, i) \mid i, j \in[n], a_{i j}>\gamma\right\}$. Later, we fix the value $0<\gamma<1$ throughout the paper and hence, unless otherwise stated, for notational convenience, we use $\mathcal{G}(A)$ and $\mathcal{E}(A)$ instead of $\mathcal{G}^{\gamma}(A)$ and $\mathcal{E}^{\gamma}(A)$.

The function $f: \mathbb{R}^{m} \rightarrow \mathbb{R}$ is convex if for all $x, y \in \mathbb{R}^{m}$ and all $\theta \in[0,1]$,

$$
f(\theta x+(1-\theta) y) \leq \theta f(x)+(1-\theta) f(y) .
$$

We say that $g \in \mathbb{R}^{m}$ is a subgradient of the function $f(\cdot)$ at $\hat{x}$ if for all $x \in \mathbb{R}^{m}, f(x)-f(\hat{x}) \geq\langle g, x-\hat{x}\rangle$, where $\left\langle u_{1}, u_{2}\right\rangle=u_{1}^{T} u_{2}$ is the standard inner product in $\mathbb{R}^{m}$. The set of all subgradients of $f(\cdot)$ at $x$ is denoted by $\nabla f(x)$. For a convex function $f(\cdot), \nabla f(x)$ is not empty for all $x \in \mathbb{R}^{m}$ (see e.g., Theorem 3.1.15 in [21]). Finally, for convenience and due to the frequent use of $\ell_{\infty}$ norm in the study of averaging dynamics, we use $\|\cdot\|$ to denote the $\ell_{\infty}$ norm $\|x\| \triangleq \max _{i \in[m]}\left|x_{i}\right|$.

## 2 Problem Formulation and Main Result

In this section, we discuss the main problem and the main result of this work. The proof of the result is provided in the subsequent sections.

### 2.1 General Framework

Consider a communication network with $n$ nodes or agents such that node $i$ has the cost function $f_{i}: \mathbb{R}^{m} \rightarrow \mathbb{R}$. Let $F(z) \triangleq \sum_{i=1}^{n} f_{i}(z)$. The goal of this paper is to solve

$$
\begin{equation*}
\arg \min _{z \in \mathbb{R}^{m}} F(z) \tag{1}
\end{equation*}
$$

distributively with the following assumption on the objective function.
Assumption 1 (Assumption on the Objective Function) We assume that:
(a) All $f_{i}(z)$ are convex functions over $\mathbb{R}^{m}$.
(b) The optimizer set $\mathcal{Z} \triangleq \arg \min _{z \in \mathbb{R}^{m}} F(z)$ is non-empty.
(c) The subgradients $f_{i}(z)$ s are uniformly upper bounded, i.e., for all $g \in \nabla f_{i}(z),\|g\| \leq L_{i}$ for all $z \in \mathbb{R}^{m}$ and all $i \in[n]$. We let $L \triangleq \sum_{i=1}^{n} L_{i}$.

In this paper, we are dealing with the dynamics of the $n$ agents estimates of an optimizer $z^{*} \in \mathcal{Z}$ which we denote them by $\mathbf{x}_{i}(t)$ for all $i \in[n]$. Therefore, we view $\mathbf{x}(t)$ as a vector of $n$ elements in the vector space $\mathbb{R}^{m}$.

A distributed solution of (1) was first proposed in 19 using the following deterministic dynamics

$$
\mathbf{x}_{i}(t+1)=\sum_{j=1}^{n} w_{i j}(t+1) \mathbf{x}_{j}(t)-\alpha(t) \mathbf{g}_{i}(t)
$$

for $t \geq t_{0}$ for an initial time $t_{0} \in \mathbb{N}$, initial conditions $\mathbf{x}_{i}\left(t_{0}\right) \in \mathbb{R}^{m}$ for all $i \in[n]$, where $\mathbf{g}_{i}(t) \in \mathbb{R}^{m}$ is a subgradient of $f_{i}(z)$ at $z=\mathbf{x}_{i}(t)$ for $i \in[n]$, and $\{\alpha(t)\}$ is a step-size sequence (in [19] the constant step-sizes variation of this dynamics was studied). We simply refer to this dynamics as the averaging-based distributed optimization solver. We can compactly write the above dynamics as

$$
\begin{equation*}
\mathbf{x}(t+1)=W(t+1) \mathbf{x}(t)-\alpha(t) \mathbf{g}(t) \tag{2}
\end{equation*}
$$

where, $\mathbf{g}(t)=\left[\mathbf{g}_{1}(t), \ldots, \mathbf{g}_{n}(t)\right]^{T}$ is the vector of the sub-gradient vectors and matrix multiplication should be understood over the vector-field $\mathbb{R}^{m}$, i.e.,

$$
[W(t+1) \mathbf{x}(t)]_{i} \triangleq \sum_{j=1}^{n} w_{i j}(t+1) \mathbf{x}_{j}(t)
$$

In distributed optimization, the goal is to find distributed dynamics $\mathbf{x}_{i}(t)$ s such that $\lim _{t \rightarrow \infty} \mathbf{x}_{i}(t)=z$ where $z \in \mathcal{Z}$ for all $i \in[n]$.

### 2.2 Our Contribution

In the paper, we consider the random variation of (2), i.e., when $\{W(t)\}$ is a chain of random matrices. This random variation was first studied in [13] where to ensure the convergence, it was assumed that this sequence is doubly stochastic almost surely and i.i.d.. This was generalized to random networks that is Markovian in [14]. The dynamics (2) with i.i.d. weight matrices that are row-stochastic almost surely and columnstochastic in-expectation was studied in [16]. A special case of [16] is the asynchronous gossip algorithm that were introduced in (17). In this work, we provide an overarching framework for the study of (2) with random weight matrices that are row-stochastic almost surely and column-stochastic in-expectation, that are not even independent in general. More precisely, we have following assumptions.

Assumption 2 (Stochastic Assumption) We assume that the weight matrix sequence $\{W(t)\}$, adapted to a filtration $\{\mathcal{F}(t)\}$, satisfies
(a) For all $t \geq t_{0}, W(t)$ is row-stochastic almost surely.
(b) For every $t>t_{0}, \mathbb{E}[W(t) \mid \mathcal{F}(t-1)]$ is column-stochastic (and hence, doubly stochastic) almost surely.

Similar to other works in this domain, our goal is to ensure that $\lim _{t \rightarrow \infty} \mathbf{x}_{i}(t)=z$ almost surely for some optimal $z \in \mathcal{Z}$ for all $i \in[n]$. To reach such a consensus value, we need to ensure enough flow of information between the agents, i.e., the associated graph sequence of $\{W(t)\}$ satisfies some form of connectivity over time. More precisely, we assume the following connectivity conditions.

Assumption 3 (Conditional B-Connectivity Assumption) We assume that for all $t \geq t_{0}$
(a) Every node in $\mathcal{G}(W(t))$ has a self-loop, almost surely.
(b) There exists an integer $B>0$ such that the random graph $\mathcal{G}_{B}(t)=\left([n], \mathcal{E}_{B}(t)\right)$ where

$$
\mathcal{E}_{B}(t)=\bigcup_{\tau=t B+1}^{(t+1) B} \mathcal{E}(\mathbb{E}[W(\tau) \mid \mathcal{F}(t B)])
$$

has a spanning rooted tree almost surely.
Note that Assumption (3) (b) is satisfied if the random graph with vertex set $[n]$ and the edge set $\bigcup_{\tau=t B+1}^{(t+1) B} \mathcal{E}(\mathbb{E}[W(\tau) \mid \mathcal{F}(\tau-1)])$ has a spanning rooted tree almost surely.

Finally, we assume the following standard condition on the step-size sequence $\{\alpha(t)\}$.
Assumption 4 (Assumption on Step-size) For the step-size sequence $\{\alpha(t)\}$, we assume that $0<$ $\alpha(t) \leq K t^{-\beta}$ for some $K, \beta>0$ and all $t \geq t_{0}, \lim _{t \rightarrow \infty} \frac{\alpha(t)}{\alpha(t+1)}=1$, and

$$
\begin{equation*}
\sum_{t=t_{0}}^{\infty} \alpha(t)=\infty \quad \text { and } \quad \sum_{t=t_{0}}^{\infty} \alpha^{2}(t)<\infty \tag{3}
\end{equation*}
$$

The main result of this paper is the following theorem.
Theorem 1 Under the Assumptions 14 on the model and the dynamics (2), $\lim _{t \rightarrow \infty} \mathbf{x}_{i}(t)=z^{*}$ almost surely for all $i \in[n]$ and all initial conditions $\mathbf{x}_{i}\left(t_{0}\right) \in \mathbb{R}^{m}$, where $z^{*}$ is a random vector that is supported on the optimal set $\mathcal{Z}$.

Before continuing with the technical details of the proof, let us first discuss some of the higher-level implications of this result:

1. Gossip-based sequential solvers: Gossip algorithms, which were originally studied in [1, 2, have been used in solving distributed optimization problems [12, 17. In gossip algorithms, at each round, a node randomly wakes up and shares its value with all or some of its neighbors. However, it is possible to leverage Theorem 1 to synthesize algorithms that do not require choosing a node independently and uniformly at random or use other coordination methods to update information at every round. An example of such a scheme is as follows:

Example 1 Consider a connected undirected networ ${ }^{11} \mathcal{G}=([n], E)$. Consider a token that is handed sequentially in the network and initially it is handed to an arbitrary agent $\ell(0) \in[n]$ in the network. If at time $t \geq 0$, agent $\ell(t) \in[n]$ is in the possession of the token, it chooses one of its neighbors $s(t+1) \in[n]$ randomly and by flipping a coin, i.e., with probability $\frac{1}{2}$ shares its information to $s(t+1)$ and passes the token and with probability $\frac{1}{2}$ keeps the token and asks for information from $s(t+1)$. It means

$$
\ell(t+1)= \begin{cases}\ell(t), & \text { with probability } \frac{1}{2} \\ s(t+1), & \text { with probability } \frac{1}{2}\end{cases}
$$

Finally, the agent $\ell(t+1)$, who has the token at time $t+1$ and is receiving the information, does

$$
\mathbf{x}_{\ell(t+1)}(t+1)=\frac{1}{2}\left(\mathbf{x}_{s(t+1)}(t)+\mathbf{x}_{\ell(t)}(t)\right)-\alpha(t) \mathbf{g}_{\ell(t+1)}(t)
$$

For the other agents $i \neq \ell(t+1)$, we set

$$
\mathbf{x}_{i}(t+1)=\mathbf{x}_{i}(t)-\alpha(t) \mathbf{g}_{i}(t)
$$

[^1]Let $\mathcal{F}(t)=\sigma(\mathbf{x}(0), \ldots, \mathbf{x}(t), \ell(t))$, and the weight matrix $W(t)=\left[w_{i j}(t)\right]$ be

$$
w_{i j}(t)= \begin{cases}\frac{1}{2}, & i=j=\ell(t) \\ \frac{1}{2}, & i=\ell(t), j \in\{s(t), \ell(t-1)\} \backslash\{\ell(t)\} \\ 1, & i=j \neq \ell(t) \\ 0, & \text { otherwise }\end{cases}
$$

which is the weight matrix of this scheme. Note that $\mathbb{E}[W(t) \mid \mathcal{F}(t-1)]=V(\ell(t-1))$ where $V(h)=\left[v_{i j}(h)\right]$ with

$$
v_{i j}(h)= \begin{cases}\frac{3}{4}, & i=j=h \\ \frac{1}{4 \delta_{i}}, & i=h,(i, j) \in E \\ \frac{1}{4 \delta_{i}}, & j=h,(i, j) \in E \\ 1, & i=j \neq h \\ 0, & \text { otherwise }\end{cases}
$$

where $\delta_{i}$ is the degree of the node $i$. Note that the matrix $\mathbb{E}[W(t) \mid \mathcal{F}(t-1)]$ is doubly stochastic, satisfies Assumption 3 -( (a), and only depends on $\ell(t-1)$. Now, we need to check whether $\{W(t)\}$ satisfies Assumption (3-(b). We have

$$
\begin{aligned}
\mathbb{E}[W(t+n) \mid \mathcal{F}(t)] & =\mathbb{E}[\mathbb{E}[W(t+n) \mid \mathcal{F}(t+n-1)] \mid \mathcal{F}(t)] \\
& =\mathbb{E}[V(\ell(t+n-1)) \mid \mathcal{F}(t)] \\
& =\mathbb{E}\left[\sum_{i=1}^{n} V(\ell(t+n-1)) 1_{\{\ell(t+n-1)=i\}} \mid \mathcal{F}(t)\right] \\
& =\sum_{i=1}^{n} \mathbb{E}\left[V(i) 1_{\{\ell(t+n-1)=i\}} \mid \mathcal{F}(t)\right] \\
& =\sum_{i=1}^{n} V(i) \mathbb{E}\left[1_{\{\ell(t+n-1)=i\}} \mid \mathcal{F}(t)\right]
\end{aligned}
$$

If the network is connected, starting from any vertex, after $n-1$ steps, the probability of reaching any other vertex is at least $(2 \Delta)^{-(n-1)}>0$, where $\Delta \triangleq \max _{i \in[n]} \delta_{i}$. Therefore, we have $\mathbb{E}\left[1_{\{\ell(t+n-1)=i\}} \mid \mathcal{F}(t)\right]>0$ for all $i \in[n]$ and $t$, and hence, Assumption (3) (b) is satisfied with $B=n$.
2. Robustness to link-failure: Our result shows that (2) is robust to random link-failures. Note that the results such as [13] will not imply the robustness of the algorithms to link failure as it assumes that the resulting weight matrices remain doubly stochastic. To show the robustness of averaging-based solvers, suppose that we have a deterministic doubly stochastic sequence $\{A(t)\}$, and suppose that each link at any time $t$ fails with some probability $p(t)>0$. More precisely, let $B(t)$ be a failure matrix where $b_{i j}(t)=0$ if a failure on link $(i, j)$ occurs at time $t$ and otherwise $b_{i j}(t)=1$ and we have

$$
\begin{equation*}
\mathbb{E}\left[b_{i j}(t) \mid \mathcal{F}(t-1)\right]=1-p(t) \tag{4}
\end{equation*}
$$

for $i, j \in[n]$. For example, if $B(t)$ is independent and identically distributed, i.e.,

$$
b_{i j}(t)= \begin{cases}0, & \text { with probability } p \\ 1, & \text { with probability } 1-p\end{cases}
$$

then $B(t)$ satisfies (4). Define $W(t)=\left[w_{i j}(t)\right]$ as follows

$$
w_{i j}(t) \triangleq \begin{cases}a_{i j}(t) b_{i j}(t), & i \neq j \\ 1-\sum_{j \neq i} a_{i j}(t) b_{i j(t)}, & i=j\end{cases}
$$

Note that $W(t)$ is row-stochastic, and since $A(t)$ is column-stochastic, $\mathbb{E}[W(t) \mid \mathcal{F}(t-1)]$ is column-stochastic. Thus, Theorem 11, using $W(t)$, translates to a theorem on robustness of the distributed dynamics (2): as long as the connectivity conditions of Theorem 1 holds, the dynamics will reach a minimizer of the distributed problem almost surely. For example, if the link failure probability satisfies $p(t) \leq \bar{p}$ for all $t$ and some $\bar{p}<1$, our result implies that the result of Proposition 4 in $[20$ (for unconstrained case) would still hold under the above link-failure model. It is worth mentioning that if $\{A(t)\}$ is time-varying, then $\mathbb{E}[W(t)]$ would be time-varying and hence, the previous results on distributed optimization using i.i.d. row-stochastic weight matrices that are column-stochastic in-expectation [16] would not imply such a robustness result.

## 3 Autonomous Averaging Dynamics

To prove Theorem 1 we need to study the time-varying distributed averaging dynamics with a particular control input (gradient-like dynamics). To do this, first we study the autonomous averaging dynamics (i.e., without any input) and then, we use the established results to study the controlled dynamics.

For this, consider the time-varying distributed averaging dynamics

$$
\begin{equation*}
\mathbf{x}(t+1)=W(t+1) \mathbf{x}(t) \tag{5}
\end{equation*}
$$

where $\{W(t)\}$ satisfying Assumption 3. Defining transition matrix

$$
\Phi(t, \tau) \triangleq W(t) \cdots W(\tau+1)
$$

and $\Phi(\tau, \tau)=I$, we have $\mathbf{x}(t)=\Phi(t, \tau) \mathbf{x}(\tau)$. Note that since $W(t)$ s are row-stochastic matrices (a.s.) and the set of row-stochastic matrices is a semi-group (with respect to multiplication), the transition matrices $\Phi(t, \tau)$ are all row-stochastic matrices (a.s.).

We say that a chain $\{W(t)\}$ achieves consensus for the initial time $t_{0} \in \mathbb{N}$ if for all $i, \lim _{t \rightarrow \infty}\left\|\mathbf{x}_{i}(t)-\tilde{x}\right\|=0$ almost surely, for all choices of initial condition $\mathbf{x}\left(t_{0}\right) \in\left(\mathbb{R}^{m}\right)^{n}$ in (5) and some random vector $\tilde{x}=\tilde{x}_{\mathbf{x}\left(t_{0}\right)}$. It can be shown that an equivalent condition for consensus (for time $t_{0}$ ) that $\lim _{t \rightarrow \infty} \Phi\left(t, t_{0}\right)=e \pi^{T}\left(t_{0}\right)$ for a random stochastic vector $\pi\left(t_{0}, \omega\right) \in \mathbb{R}^{n}$, almost surely where $\omega \in \Omega$ is a sample point.

For a matrix $A=\left[a_{i j}\right]$, let

$$
\operatorname{diam}(A)=\max _{i, j \in[n]} \frac{1}{2} \sum_{\ell=1}^{n}\left|a_{i \ell}-a_{j \ell}\right|,
$$

and the mixing parameter

$$
\Lambda(A)=\min _{i, j \in[n]} \sum_{\ell=1}^{n} \min \left\{a_{i \ell}, a_{j \ell}\right\}
$$

Note that for a row-stochastic matrix $A, \operatorname{diam}(A) \in[0,1]$. For a vector $\mathbf{x}=\left[\mathbf{x}_{i}\right]$ where $\mathbf{x}_{i} \in \mathbb{R}^{m}$ for all $i$, let

$$
\mathrm{d}(\mathbf{x})=\max _{i, j \in[n]}\left\|\mathbf{x}_{i}-\mathbf{x}_{j}\right\|
$$

Note that $\mathrm{d}(\mathbf{x}) \leq 2 \max _{i \in[n]}\left\|\mathbf{x}_{i}\right\|$. Also, if we have consensus, then $\lim _{t \rightarrow \infty} \mathrm{~d}(\mathbf{x}(t))=0$ and $\lim _{t \rightarrow \infty} \operatorname{diam}\left(\Phi\left(t, t_{0}\right)\right)=0$ and in fact, the reverse implications are true [5], i.e., a chain achieves consensus (for time $t_{0}$ ) if and only if $\lim _{t \rightarrow \infty} \mathrm{~d}(\mathbf{x}(t))=0$ for all $\mathbf{x}\left(t_{0}\right) \in\left(\mathbb{R}^{m}\right)^{n}$ or $\lim _{t \rightarrow \infty} \operatorname{diam}\left(\Phi\left(t, t_{0}\right)\right)=0$.

The following results relating the above quantities are useful for our future discussions.
Lemma $1([\mathbf{9}, \mathbf{2 7}])$ For $n \times n$ row-stochastic matrices $A, B$, we have

$$
\operatorname{diam}(A B) \leq(1-\Lambda(A)) \operatorname{diam}(B)
$$

Lemma 2 For any $n \times n$ row-stochastic matrices $A, B$, we have
(a) $\mathrm{d}(A \mathbf{x}) \leq \operatorname{diam}(A) \mathrm{d}(\mathbf{x})$ for all $\mathbf{x} \in\left(\mathbb{R}^{m}\right)^{n}$,
(b) $\mathrm{d}(\mathbf{x}+\mathbf{y}) \leq \mathrm{d}(\mathbf{x})+\mathrm{d}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in\left(\mathbb{R}^{m}\right)^{n}$,
(c) $\operatorname{diam}(A)=1-\Lambda(A)$,
(d) $\operatorname{diam}(A B) \leq \operatorname{diam}(A) \operatorname{diam}(B)$, and
(e) $\left\|\mathbf{x}_{i}-\sum_{j=1}^{n} \pi_{j} \mathbf{x}_{j}\right\| \leq \mathrm{d}(\mathbf{x})$ for all $i \in[n], \mathbf{x} \in\left(\mathbb{R}^{m}\right)^{n}$, and any stochastic vector $\pi \in[0,1]^{n}$ (i.e., $\sum_{i=1}^{n} \pi_{i}=1$ ).

Proof: The proof is provided in Appendix.
The main goal of this section is to obtain an exponentially decreasing upper bound (in terms of $t_{1}-\tau_{1}$ and $\left.t_{2}-\tau_{2}\right)$ on $\mathbb{E}\left[\operatorname{diam}\left(\Phi\left(t_{2}, \tau_{2}\right)\right) \operatorname{diam}\left(\Phi\left(t_{1}, \tau_{1}\right)\right) \mid \mathcal{F}\left(\tau_{1}\right)\right]$.

Using this result and the connectivity assumption 3 we can show that the transition matrices $\Phi(t, s)$ become mixing in-expectation for large enough $t-s$.

Lemma 3 Under Assumption圆 (Connectivity), there exists a parameter $\theta>0$ such that for every $s \geq t_{0}$, we have almost surely

$$
\mathbb{E}\left[\Lambda\left(\Phi\left(\left(n^{2}+s\right) B, s B\right)\right) \mid \mathcal{F}(s B)\right] \geq \theta .
$$

Proof: Fix $s \geq 0$. Let $\mathbb{T}$ be the set of all collection of edges $E$ such that the graph $([n], E)$ has a spanning rooted tree, and for $k \in\left[n^{2}\right]$,

$$
\mathcal{E}_{B}(k) \triangleq \bigcup_{\tau=(s+k-1) B+1}^{(s+k) B} \mathcal{E}(\mathbb{E}[W(\tau) \mid \mathcal{F}((s+k-1) B)]) .
$$

For notational simplicity, denote $\mathcal{F}(s B)$ by $\mathcal{F}$ and $\mathcal{F}((s+k) B)$ by $\mathcal{F}_{k}$ for $k \in\left[n^{2}\right]$. Let $V=\left\{\omega \mid \forall k \mathcal{E}_{B}(k) \in\right.$ $\mathbb{T}\}$. From Assumption 3, we have $P(V)=1$. For $\omega \in V$ and $k \geq 1$, define the random graph ( $[n], \mathcal{T}_{k}$ ) on $n$ vertices by

$$
\mathcal{T}_{k}= \begin{cases}\mathcal{T}_{k-1}, & \text { if } \mathcal{T}_{k-1} \in \mathbb{T} \\ \mathcal{T}_{k-1} \cup\left\{u_{k}\right\}, & \text { if } \mathcal{T}_{k-1} \notin \mathbb{T}\end{cases}
$$

with $\mathcal{T}_{0}=\emptyset$, where

$$
\begin{equation*}
u_{k} \in \mathcal{E}_{B}(k) \cap \overline{\mathcal{T}}_{k-1}, \tag{6}
\end{equation*}
$$

and $\overline{\mathcal{T}}_{k}$ is the edge-set of the complement graph of $\left([n], \mathcal{T}_{k}\right)$. Note that since $\mathcal{E}_{B}(k)$ has a spanning rooted tree, if $\mathcal{T}_{k-1} \notin \mathbb{T}$, then $\mathcal{E}_{B}(k)$ should contain an edge that does not belong to $\mathcal{T}_{k-1}$, which we identify it as $u_{k}$ in (6). Hence, $\mathcal{T}_{k}$ is well-defined. Since there are at most $n(n-1)$ potential edges in a graph on $n$ vertices, $\mathcal{T}_{n^{2}}$ has a spanning rooted tree for $\omega \in V$.

For $k \in\left[n^{2}\right]$, let

$$
\mathcal{D}_{B}(k) \triangleq \bigcup_{\tau=(s+k-1) B+1}^{(s+k) B} \mathcal{E}^{\nu}(W(\tau)),
$$

for some fixed $0<\nu<\gamma$, and

$$
\mathcal{H}(k) \triangleq \cup_{\tau=1}^{k} \mathcal{D}_{B}(\tau) .
$$

Consider the sequences of events $\left\{U_{k}\right\}$ defined by $U_{k} \triangleq\left\{\omega \in V \mid \mathcal{T}_{k} \subset \mathcal{H}(k)\right\}$, for $k \geq 1$, and $U_{0}=V$. Note that if $\mathcal{T}_{k-1} \in \mathbb{T}$, then $\mathcal{T}_{k-1} \subset \mathcal{H}(k-1)$ implies $\mathcal{T}_{k} \subset \mathcal{H}(k)$, and if $\mathcal{T}_{k-1} \notin \mathbb{T}$, then $\mathcal{T}_{k-1} \subset \mathcal{H}(k-1)$ and $u_{k} \in \mathcal{D}_{B}(k)$ imply $\mathcal{T}_{k} \subset \mathcal{H}(k)$. Hence, for $k \geq 1$

$$
\begin{equation*}
1_{\left\{U_{k}\right\}} \geq 1_{\left\{U_{k-1}\right\}} 1_{\left\{\mathcal{T}_{k-1} \notin \mathbb{T}\right\}} 1_{\left\{u_{k} \in \mathcal{D}_{B}(k)\right\}}+1_{\left\{U_{k-1}\right\}} 1_{\left\{\mathcal{T}_{k-1} \in \mathbb{T}\right\}} . \tag{7}
\end{equation*}
$$

On the other hand, from Tower rule, we have

$$
\begin{align*}
& \mathbb{E}\left[1_{\left\{U_{k-1}\right\}} 1_{\left\{\mathcal{T}_{k-1} \notin \mathbb{T}\right\}} 1_{\left\{u_{k} \in \mathcal{D}_{B}(k)\right\}} \mid \mathcal{F}\right]  \tag{8}\\
&=\mathbb{E}\left[1_{\left\{U_{k-1}\right\}} 1_{\left\{\mathcal{T}_{k-1} \notin \mathbb{T}\right\}} \mathbb{E}\left[1_{\left\{u_{k} \in \mathcal{D}_{B}(k)\right\}} \mid \mathcal{F}_{k-1}\right] \mid \mathcal{F}\right] .
\end{align*}
$$

Let $u_{k}(\omega)=\left(j_{k}(\omega), i_{k}(\omega)\right)$. Since $u_{k} \in \mathcal{E}_{B}(k)$, there exists $(s+k-1) B<\tau_{k} \leq(s+k) B$ such that

$$
u_{k} \in \mathcal{E}\left(\mathbb{E}\left[W\left(\tau_{k}\right) \mid \mathcal{F}_{k-1}\right]\right),
$$

and, we have

$$
\mathbb{E}\left[(1-\nu) 1_{\left\{1-w_{i_{k} j_{k}}\left(\tau_{k}\right) \geq 1-\nu\right\}} \mid \mathcal{F}_{k-1}\right] \leq \mathbb{E}\left[1-w_{i_{k} j_{k}}\left(\tau_{k}\right) \mid \mathcal{F}_{k-1}\right] \leq 1-\gamma .
$$

Therefore,

$$
\begin{aligned}
\mathbb{E}\left[1_{\left\{u_{k} \in \mathcal{D}_{B}(k)\right\}} \mid \mathcal{F}_{k-1}\right] & \geq \mathbb{E}\left[1_{\left\{u_{k} \in \mathcal{E}\right.}\left(W\left(\tau_{k}\right)\right)\right\} \\
& =\mathbb{E}\left[1_{\left\{w_{i_{k} j_{k}}\left(\tau_{k}\right)>\nu\right\}} \mid \mathcal{F}_{k-1}\right] \\
& =1-\mathbb{E}\left[1_{\left\{w_{i_{k} j_{k}}\left(\tau_{k}\right) \leq \nu\right\}} \mid \mathcal{F}_{k-1}\right] \\
& =1-\mathbb{E}\left[1_{\left\{1-w_{i_{k} j_{k}}\left(\tau_{k}\right) \geq 1-\nu\right\}} \mid \mathcal{F}_{k-1}\right] \\
& \geq 1-\frac{1-\gamma}{1-\nu} \triangleq p>0,
\end{aligned}
$$

which holds as $\nu<\gamma$. This inequality and (8) imply that

$$
\mathbb{E}\left[1_{\left\{U_{k-1}\right\}} 1_{\left\{\mathcal{T}_{k-1} \notin \mathbb{T}\right\}} 1_{\left\{u_{k} \in \mathcal{D}_{B}(k)\right\}} \mid \mathcal{F}\right] \geq p \mathbb{E}\left[1_{\left\{U_{k-1}\right\}} 1_{\left\{\mathcal{T}_{k-1} \notin \mathbb{T}\right\}} \mid \mathcal{F}\right] .
$$

Therefore, (7) implies

$$
\begin{aligned}
\mathbb{E}\left[1_{\left\{U_{k}\right\}} \mid \mathcal{F}\right] & \geq p \mathbb{E}\left[1_{\left\{U_{k-1}\right\}} 1_{\left\{\mathcal{T}_{k-1} \notin \mathbb{T}\right\}} \mid \mathcal{F}\right]+\mathbb{E}\left[1_{\left\{U_{k-1}\right\}} 1_{\left\{\mathcal{T}_{k-1} \in \mathbb{T}\right\}} \mid \mathcal{F}\right] \\
& \geq p\left(\mathbb{E}\left[1_{\left\{U_{k-1}\right\}} 1_{\left\{\mathcal{T}_{k-1} \notin \mathbb{T}\right\}} \mid \mathcal{F}\right]+\mathbb{E}\left[1_{\left\{U_{k-1}\right\}} 1_{\left\{\mathcal{T}_{k-1} \in \mathbb{T}\right\}} \mid \mathcal{F}\right]\right) \\
& =p \mathbb{E}\left[1_{\left\{U_{k-1}\right\}} \mid \mathcal{F}\right],
\end{aligned}
$$

and hence $\mathbb{E}\left[1_{\left\{U_{k}\right\}} \mid \mathcal{F}\right] \geq p^{k}$. Finally, since $\mathcal{T}_{n^{2}}$ has a spanning rooted tree, from Lemma 1 in [13], we have

$$
\Lambda\left(W\left(n^{2} B, \omega\right) \cdots W(n(n-1) B+n-1, \omega) \cdots W(1, \omega)\right) \geq \nu^{n^{2} B},
$$

for $\omega \in U_{n^{2}}$. Therefore, we have

$$
\mathbb{E}\left[\Lambda\left(\Phi\left(\left(n^{2}+s\right) B, s B\right)\right) \mid \mathcal{F}(s B)\right] \geq \nu^{n^{2} B} \mathbb{E}\left[1_{\left\{U_{n^{2}}\right\}} \mid \mathcal{F}\right] \geq \nu^{n^{2} B} p^{n^{2}} \triangleq \theta>0,
$$

which completes the proof.
Finally, we need the following result, which is proved in Appendix, to prove the main result of this section.
Lemma 4 For a random process $\{Y(k)\}$, adapted to a filtration $\{\mathcal{F}(k)\}$, let

$$
\mathbb{E}[Y(k) \mid \mathcal{F}(k-1)] \leq a(k)
$$

for $K_{1} \leq k \leq K_{2}$ almost surely, where $K_{1} \leq K_{2}$ are arbitrary positive integers and a(k)s are (deterministic) scalars. Then, we have almost surely

$$
\mathbb{E}\left[\prod_{k=K_{1}}^{K_{2}} Y(k) \mid \mathcal{F}\left(K_{1}-1\right)\right] \leq \prod_{k=K_{1}}^{K_{2}} a(k) .
$$

Now, we are ready to prove the main result for the convergence rate of the autonomous random averaging dynamics.

Lemma 5 Under Assumption 3 (Connectivity), there exist $0<C$ and $0 \leq \lambda<1$ such that for every $t_{0} \leq \tau_{1} \leq t_{1}$ and $t_{0} \leq \tau_{2} \leq t_{2}$ with $\tau_{1} \leq \tau_{2}$, we have almost surely

$$
\mathbb{E}\left[\operatorname{diam}\left(\Phi\left(t_{2}, \tau_{2}\right)\right) \operatorname{diam}\left(\Phi\left(t_{1}, \tau_{1}\right)\right) \mid \mathcal{F}\left(\tau_{1}\right)\right] \leq C \lambda^{t_{1}-\tau_{1}} \lambda^{t_{2}-\tau_{2}}
$$

Proof: First, we prove

$$
\begin{equation*}
\mathbb{E}[\operatorname{diam}(\Phi(t, \tau)) \mid \mathcal{F}(\tau)] \leq \tilde{C} \tilde{\lambda}^{t-\tau} \tag{9}
\end{equation*}
$$

for some $0<\tilde{C}$ and $0 \leq \tilde{\lambda}<1$. Let $s \triangleq\left\lceil\frac{\tau}{B}\right\rceil$ and $K \triangleq\left\lfloor\frac{t-s B}{n^{2} B}\right\rfloor$. We have

$$
\begin{aligned}
\mathbb{E}[\operatorname{diam}(\Phi(t, \tau)) \mid \mathcal{F}(\tau)] & =\mathbb{E}\left[\operatorname{diam}\left(\Phi\left(t, s B+K n^{2} B\right)\left[\prod_{k=1}^{K} \Phi\left(s B+k n^{2} B, s B+(k-1) n^{2} B\right)\right] \Phi(s B, \tau)\right) \mid \mathcal{F}(\tau)\right] \\
& \stackrel{(a)}{\leq} \mathbb{E}\left[\prod_{k=1}^{K}\left(1-\Lambda\left(\Phi\left(s B+k n^{2} B, s B+(k-1) n^{2} B\right)\right) \mid \mathcal{F}(\tau)\right]\right. \\
& \stackrel{(b)}{\leq}(1-\theta)^{K} \\
& \leq \tilde{C}(1-\theta)^{\frac{t-\tau}{n^{2} B}}
\end{aligned}
$$

where $\tilde{C}=(1-\theta)^{-1-\frac{1}{n^{2}}}$ and (a) follows from Lemma 1, Lemma 2 (d), and $\operatorname{diam}(\Phi(.,)) \leq$.1 , and (b) follows from Lemma 3 and 4. Since $\theta>0$, we have $\tilde{\lambda} \triangleq(1-\theta)^{\frac{1}{n^{2} B}}<1$.

To prove the main statement, we consider two cases:
(i) intervals $\left(\tau_{1}, t_{1}\right]$ and $\left(\tau_{2}, t_{2}\right]$ do not have an intersection, and
(ii) $\left(\tau_{1}, t_{1}\right]$ and $\left(\tau_{2}, t_{2}\right]$ intersect.

For case (i), since the two intervals do not overlap, we have $t_{1} \leq \tau_{2}$, and hence

$$
\begin{aligned}
\mathbb{E}\left[\operatorname{diam}\left(\Phi\left(t_{2}, \tau_{2}\right)\right) \operatorname{diam}\left(\Phi\left(t_{1}, \tau_{1}\right)\right) \mid \mathcal{F}\left(\tau_{1}\right)\right] & =\mathbb{E}\left[\mathbb{E}\left[\operatorname{diam}\left(\Phi\left(t_{2}, \tau_{2}\right)\right) \mid \mathcal{F}\left(\tau_{2}\right)\right] \operatorname{diam}\left(\Phi\left(t_{1}, \tau_{1}\right)\right) \mid \mathcal{F}\left(\tau_{1}\right)\right] \\
& \leq \tilde{C} \tilde{\lambda}^{t_{1}-\tau_{1}} \tilde{C} \tilde{\lambda}^{t_{2}-\tau_{2}}
\end{aligned}
$$

which follows from (9). For case (ii), let's write the union of the intervals $\left(\tau_{1}, t_{1}\right]$ and $\left(\tau_{2}, t_{2}\right]$ as disjoint union of three intervals:

$$
\left(\tau_{1}, t_{1}\right] \cup\left(\tau_{2}, t_{2}\right]=\left(s_{1}, s_{2}\right] \cup\left(s_{2}, s_{3}\right] \cup\left(s_{3}, s_{4}\right]
$$

for $s_{1} \leq s_{2} \leq s_{3}$ where $\left(s_{2}, s_{3}\right] \triangleq\left(\tau_{1}, t_{1}\right] \cap\left(\tau_{2}, t_{2}\right],\left(s_{1}, s_{2}\right] \cup\left(s_{3}, s_{4}\right] \triangleq\left(\tau_{1}, t_{1}\right] \triangle\left(\tau_{2}, t_{2}\right]$. Using this, it can be verified that

$$
\begin{aligned}
\mathbb{E}\left[\operatorname{diam}\left(\Phi\left(t_{2}, \tau_{2}\right)\right) \operatorname{diam}\left(\Phi\left(t_{1}, \tau_{1}\right)\right) \mid \mathcal{F}\left(\tau_{1}\right)\right] & \stackrel{(a)}{\leq} \mathbb{E}\left[\operatorname{diam}\left(\Phi\left(s_{4}, s_{3}\right)\right) \operatorname{diam}^{2}\left(\Phi\left(s_{3}, s_{2}\right)\right) \operatorname{diam}\left(\Phi\left(s_{2}, s_{1}\right)\right) \mid \mathcal{F}\left(\tau_{1}\right)\right] \\
& \stackrel{(b)}{\leq} \mathbb{E}\left[\operatorname{diam}\left(\Phi\left(s_{4}, s_{3}\right)\right) \operatorname{diam}\left(\Phi\left(s_{3}, s_{2}\right)\right) \operatorname{diam}\left(\Phi\left(s_{2}, s_{1}\right)\right) \mid \mathcal{F}\left(\tau_{1}\right)\right] \\
& \stackrel{(c)}{\leq} \tilde{C} \tilde{\lambda}^{s_{2}-s_{1}} \tilde{C} \tilde{\lambda}^{s_{3}-s_{2}} \tilde{C} \tilde{\lambda}^{s_{4}-s_{3}} \\
& =\tilde{C} \tilde{\lambda}^{s_{2}-s_{1}} 2 \tilde{C} \sqrt{\tilde{\lambda}}^{2\left(s_{3}-s_{2}\right)} \tilde{C} \tilde{\lambda}^{s_{4}-s_{3}} \\
& \leq \tilde{C}^{3} \sqrt{\tilde{\lambda}}^{t_{1}-\tau_{1}} \sqrt{\tilde{\lambda}}^{t_{2}-\tau_{2}}
\end{aligned}
$$

where ( $a$ ) follows from Lemma $2(\mathrm{~d}),(b)$ follows $\operatorname{from} \operatorname{diam}(A) \leq 1$ for all row-stochastic matrices $A$, and (c) follows from (9) and Lemma 4. Letting $C \triangleq \max \left\{\tilde{C}^{2}, \tilde{C}^{3}\right\}$ and $\lambda \triangleq \sqrt{\tilde{\lambda}}$, we arrive at the conclusion.

## 4 Averaging Dynamics with Gradient-Flow Like Feedback

In this section, we study the controlled linear time-varying dynamics

$$
\begin{equation*}
\mathbf{x}(t+1)=W(t+1) \mathbf{x}(t)+\mathbf{u}(t) \tag{10}
\end{equation*}
$$

Note that the feedback $\mathbf{u}(t)=-\alpha(t) \mathbf{g}(t)$ leads to the dynamics (2). The goal of this section is to establish bounds on the convergence-rate of $\mathrm{d}(\mathbf{x})$ (to zero) in-expectation and almost surely for a class of regularized input $\mathbf{u}(t)$.

We start with the following two lemmas.
Lemma 6 For dynamics (10) and every $t_{0} \leq \tau \leq t$, we have

$$
\begin{equation*}
\mathrm{d}(\mathbf{x}(t)) \leq \operatorname{diam}(\Phi(t, \tau)) \mathrm{d}(\mathbf{x}(\tau))+\sum_{s=\tau}^{t-1} \operatorname{diam}(\Phi(t, s+1)) \mathrm{d}(\mathbf{u}(s)) \tag{11}
\end{equation*}
$$

Proof: Note that the general solution for the dynamics (10) is given by

$$
\begin{equation*}
\mathbf{x}(t)=\Phi(t, \tau) \mathbf{x}(\tau)+\sum_{s=\tau}^{t-1} \Phi(t, s+1) \mathbf{u}(s) \tag{12}
\end{equation*}
$$

Therefore, using the sub-linearity property of $\mathrm{d}(\cdot)$ (Lemma 2 (b)), we have

$$
\begin{aligned}
\mathrm{d}(\mathbf{x}(t)) & \leq \mathrm{d}(\Phi(t, \tau) \mathbf{x}(\tau))+\sum_{s=\tau}^{t-1} \mathrm{~d}(\Phi(t, s+1) \mathbf{u}(s)) \\
& \leq \operatorname{diam}(\Phi(t, \tau)) \mathrm{d}(\mathbf{x}(\tau))+\sum_{s=\tau}^{t-1} \operatorname{diam}(\Phi(t, s+1)) \mathrm{d}(\mathbf{u}(s))
\end{aligned}
$$

where the last inequality follows from Lemma 2 (a).

Lemma 7 Let $\{\beta(t)\}$ be a positive (scalar) sequence such that $\lim _{t \rightarrow \infty} \frac{\beta(t)}{\beta(t+1)}=1$. Then for any $\theta \in[0,1)$, there exists some $M>0$ such that

$$
\sum_{s=\tau}^{t-1} \beta(s) \theta^{t-s} \leq M \beta(t)
$$

for all $t \geq \tau \geq t_{0}$.
Proof: The proof is provided in Appendix.
To prove the main theorem, we need to study how fast $\mathbb{E}[\mathrm{d}(\mathbf{x}(t))]$ and $\mathbb{E}\left[\mathrm{d}^{2}(\mathbf{x}(t))\right]$ approach to zero when the diameter of the control input $\mathrm{d}(\mathbf{u}(t))$ goes to zero. Since $\mathbb{E}\left[\mathrm{d}^{2}(\mathbf{x}(t))\right] \geq \mathbb{E}^{2}[\mathrm{~d}(\mathbf{x}(t))]$, it suffice to study convergence rate of $\mathbb{E}\left[\mathrm{d}^{2}(\mathbf{x}(t))\right]$.

Lemma 8 Under Assumptions 圆, 3, and 4, if almost surely $\mathrm{d}(\mathbf{u}(t))<q \alpha(t)$ for some $q>0$, then we have,

$$
\frac{\mathbb{E}\left[\mathrm{d}^{2}(\mathbf{x}(t))\right]}{\alpha^{2}(t)} \leq \hat{M}
$$

for some $\hat{M}>0$ and all $t \geq t_{0}$.

Proof: Taking the square of both sides of (11), for $t>\tau \geq t_{0}$, we have

$$
\begin{aligned}
& \mathrm{d}^{2}(\mathbf{x}(t)) \leq \operatorname{diam}^{2}(\Phi(t, \tau)) \mathrm{d}^{2}(\mathbf{x}(\tau))+2 \operatorname{diam}(\Phi(t, \tau)) \mathrm{d}(\mathbf{x}(\tau)) \sum_{s=\tau}^{t-1} \operatorname{diam}(\Phi(t, s+1)) \mathrm{d}(\mathbf{u}(s)) \\
&+\sum_{s=\tau}^{t-1} \sum_{\ell=\tau}^{t-1} \operatorname{diam}(\Phi(t, s+1)) \mathrm{d}(\mathbf{u}(s)) \operatorname{diam}(\Phi(t, \ell+1)) \mathrm{d}(\mathbf{u}(\ell))
\end{aligned}
$$

Taking the expectation of both sides of the above inequality, and using $\mathrm{d}(\mathbf{u}(t))<q \alpha(t)$ almost surely, we have

$$
\begin{aligned}
& \mathbb{E}\left[\mathrm{d}^{2}(\mathbf{x}(t))\right] \leq \mathbb{E}\left[\operatorname{diam}^{2}(\Phi(t, \tau)) \mathrm{d}^{2}(\mathbf{x}(\tau))\right]+2 \sum_{s=\tau}^{t-1} \mathbb{E}[\operatorname{diam}(\Phi(t, \tau)) \mathrm{d}(\mathbf{x}(\tau)) \operatorname{diam}(\Phi(t, s+1)) \mathrm{d}(\mathbf{u}(s))] \\
&+\sum_{s=\tau}^{t-1} \sum_{\ell=\tau}^{t-1} \mathbb{E}[\operatorname{diam}(\Phi(t, s+1)) \mathrm{d}(\mathbf{u}(s)) \operatorname{diam}(\Phi(t, \ell+1)) \mathrm{d}(\mathbf{u}(\ell))] \\
& \leq \mathbb{E}\left[\mathbb{E}\left[\operatorname{diam}^{2}(\Phi(t, \tau)) \mid \mathcal{F}(\tau)\right] \mathrm{d}^{2}(\mathbf{x}(\tau))\right]+2 \sum_{s=\tau}^{t-1} \mathbb{E}[\mathbb{E}[\operatorname{diam}(\Phi(t, \tau)) \operatorname{diam}(\Phi(t, s+1)) \mid \mathcal{F}(\tau)] \mathrm{d}(\mathbf{x}(\tau))] \alpha(s) q \\
&+\sum_{s=\tau}^{t-1} \sum_{\ell=\tau}^{t-1} \mathbb{E}[\operatorname{diam}(\Phi(t, s+1)) \operatorname{diam}(\Phi(t, \ell+1))] \alpha(s) \alpha(\ell) q^{2}
\end{aligned}
$$

Therefore, from Lemma 5, we have

$$
\begin{aligned}
\mathbb{E}\left[\mathrm{d}^{2}(\mathbf{x}(t))\right] & \leq C \lambda^{2(t-\tau)} \mathbb{E}\left[\mathrm{d}^{2}(\mathbf{x}(\tau))\right]+\frac{2 C q}{\lambda} \lambda^{t-\tau} \mathbb{E}[\mathrm{d}(\mathbf{x}(\tau))] \sum_{s=\tau}^{t-1} \lambda^{t-s} \alpha(s)+\frac{C q^{2}}{\lambda^{2}} \sum_{s=\tau}^{t-1} \sum_{\ell=\tau}^{t-1} \alpha(s) \alpha(\ell) \lambda^{t-s} \lambda^{t-\ell} \\
& \leq C \lambda^{2(t-\tau)} \mathbb{E}\left[\mathrm{d}^{2}(\mathbf{x}(\tau))\right]+\frac{2 C q M}{\lambda} \lambda^{t-\tau} \mathbb{E}[\mathrm{d}(\mathbf{x}(\tau))] \alpha(t)+\frac{C q^{2} M^{2}}{\lambda^{2}} \alpha^{2}(t)
\end{aligned}
$$

where the last inequality follows from Lemma 7 and the fact that

$$
\sum_{s=\tau}^{t-1} \sum_{\ell=\tau}^{t-1} \alpha(s) \alpha(\ell) \lambda^{t-s} \lambda^{t-\ell}=\left(\sum_{s=\tau}^{t-1} \alpha(s) \lambda^{t-s}\right)^{2}
$$

Dividing both sides of the above inequality by $\alpha^{2}(t)$ and noting $\frac{\alpha(\tau)}{\alpha(t)} \lambda^{t-\tau}=\prod_{\kappa=\tau}^{t-1} \frac{\alpha(\kappa)}{\alpha(\kappa+1)} \lambda$, we have

$$
\frac{\mathbb{E}\left[\mathrm{d}^{2}(\mathbf{x}(t))\right]}{\alpha^{2}(t)} \leq C \frac{\mathbb{E}\left[\mathrm{~d}^{2}(\mathbf{x}(\tau))\right]}{\alpha^{2}(\tau)}\left(\prod_{\kappa=\tau}^{t-1} \frac{\alpha(\kappa)}{\alpha(\kappa+1)} \lambda\right)^{2}+2 \frac{C q M}{\lambda} \frac{\mathbb{E}[\mathrm{~d}(\mathbf{x}(\tau))]}{\alpha(\tau)}\left(\prod_{\kappa=\tau}^{t-1} \frac{\alpha(\kappa)}{\alpha(\kappa+1)} \lambda\right)+\frac{C q^{2} M^{2}}{\lambda^{2}}
$$

Since $\lim _{\tau \rightarrow \infty} \frac{\alpha(\tau)}{\alpha(\tau+1)}=1$, for any $\hat{\lambda} \in(\lambda, 1)$, there exists $\hat{\tau}$ such that for $\tau \geq \hat{\tau}$, we have $\frac{\mu(\tau)}{\mu(\tau+1)} \lambda \leq \hat{\lambda}$. Therefore,

$$
\frac{\mathbb{E}\left[\mathrm{d}^{2}(\mathbf{x}(t))\right]}{\alpha^{2}(t)} \leq C \frac{\mathbb{E}\left[\mathrm{~d}^{2}(\mathbf{x}(\tau))\right]}{\alpha^{2}(\tau)} \hat{\lambda}^{2(t-\tau)}+\frac{2 C q M}{\lambda} \frac{\mathbb{E}[\mathrm{~d}(\mathbf{x}(\tau))]}{\alpha(\tau)} \hat{\lambda}^{t-\tau}+\frac{C q^{2} M^{2}}{\lambda^{2}}
$$

Taking the limit of the above inequality, we get

$$
\begin{aligned}
\limsup _{t \rightarrow \infty} \frac{\mathbb{E}\left[\mathrm{~d}^{2}(\mathbf{x}(t))\right]}{\alpha^{2}(t)} & \leq \lim _{t \rightarrow \infty} C \frac{\mathbb{E}\left[\mathrm{~d}^{2}(\mathbf{x}(\tau))\right]}{\alpha^{2}(\tau)} \hat{\lambda}^{2(t-\tau)}+\lim _{t \rightarrow \infty} \frac{2 C q M}{\lambda} \frac{\mathbb{E}[\mathrm{~d}(\mathbf{x}(\tau))]}{\alpha(\tau)} \hat{\lambda}^{t-\tau}+\frac{C q^{2} M^{2}}{\lambda^{2}} \\
& =\frac{C q^{2} M^{2}}{\lambda^{2}}
\end{aligned}
$$

As a result, there exists an $\hat{M}>0$ such that $\frac{\mathbb{E}\left[\mathrm{d}^{2}(\mathbf{x}(t))\right]}{\alpha^{2}(t)} \leq \hat{M}$.

To prove the main theorem, we also need to show that $\mathrm{d}(\mathbf{x}(t))$ converges to zero almost surely (as will be proved in Lemma 11). To do so, using the previous results, we will show (in Lemma 11) that

$$
\mathbb{E}[\mathrm{d}(\mathbf{x}(t)) \mid \mathcal{F}(\tau)] \leq \mathbb{E}[\operatorname{diam}(\Phi(t, \tau)) \mid \mathcal{F}(\tau)] \mathrm{d}(\mathbf{x}(\tau))+K \tau^{-\gamma}
$$

for some $K, \gamma>0$. However, since $\sum_{\tau=t_{0}}^{\infty} \tau^{-\gamma}$ is not necessarily summable, we cannot use the standard Siegmund-Robbins Theorem [26] to argue $\mathrm{d}(\mathbf{x}(t)) \rightarrow 0$ based on this inequality. We will use the facts that $\mathbb{E}[\operatorname{diam}(\Phi(t, \tau)) \mid \mathcal{F}(\tau)]<1$ if $t-\tau$ is large enough, and $\operatorname{diam}(\Phi(t, \tau)) \leq 1$ for all $t \geq \tau \geq t$. This leads us to prove a martingale-type result in Lemma 10. To prove this lemma, we first show the following.

Lemma 9 Consider a non-negative random process $a(t)$ such that

$$
\mathbb{E}[a(t+1) \mid \mathcal{F}(t)] \leq \tilde{\lambda}
$$

for some $\tilde{\lambda}<1$ and all $t \geq 0$. For $\lambda$ satisfying $\tilde{\lambda}<\lambda<1$, define the sequence of stopping-times $\left\{t_{s}\right\}_{s \geq 0}$ by

$$
t_{s} \triangleq \inf \left\{t>t_{s-1} \mid a(t) \leq \lambda\right\}
$$

with $t_{0}=0$. Then, $\lim _{s \rightarrow \infty}\left(t_{s+1}-t_{s}\right) t_{s}^{-\beta}=0$ almost surely for all $\beta>0$.
Proof: Let us define the martingale $S(t)$ by

$$
S(t)=S(t-1)+\left(1_{\{a(t)>\lambda\}}-\mathbb{E}\left[1_{\{a(t)>\lambda\}} \mid \mathcal{F}(t-1)\right]\right),
$$

where $S(0)=0$. Noting $|S(t+1)-S(t)| \leq 1$, from Azuma's inequality, we have

$$
\begin{equation*}
P(S(t+\sigma)-S(t)>\sigma \rho) \leq \exp \left(-\frac{\sigma^{2} \rho^{2}}{2 \sigma}\right) \tag{13}
\end{equation*}
$$

for all $\sigma \in \mathbb{N}$ and $\rho \in(0,1)$. For $\theta>0$, let the sequences of events

$$
A_{\theta}(t) \triangleq\left\{\omega \mid S\left(t+\left\lfloor\theta t^{\beta}\right\rfloor\right)-S(t)>\left\lfloor\theta t^{\beta}\right\rfloor \rho\right\} .
$$

From (13), we have

$$
P\left(A_{\theta}(t)\right) \leq \exp \left(-\frac{1}{2}\left(\theta t^{\beta}-1\right) \rho^{2}\right)
$$

implying $\sum_{t=1}^{\infty} P\left(A_{\theta}(t)\right)<\infty$ as $\exp \left(-t^{\beta}\right) \leq \frac{M}{t^{2}}$ for sufficiently large $M$ (depending on $\beta$ ). Therefore, the Borel-Cantelli Theorem implies that $P\left(\left\{A_{\theta}(t)\right.\right.$ i.o. $\left.\}\right)=0$ for all $\theta>0$.

For $\theta>0$, let the sequences of events

$$
B_{\theta}(t) \triangleq\left\{\omega \left\lvert\, \frac{\hat{t}-t}{t^{\beta}}>\theta\right. \text { where } \hat{t}=\inf \{\tau>t \mid a(\tau) \leq \lambda\}\right\}
$$

We show that $B_{\theta}(t) \subset A_{\theta}(t)$ for all $t, \theta$. Fix a constant $\rho \in(0,1)$ such that $1-\frac{\tilde{\lambda}}{\lambda}>\rho$. Since $\mathbb{E}[a(\tau) \mid$ $\mathcal{F}(\tau-1)] \leq \tilde{\lambda}$, we have

$$
\begin{aligned}
\mathbb{E}\left[\lambda 1_{\{a(\tau) \geq \lambda\}} \mid \mathcal{F}(\tau-1)\right] & \leq \mathbb{E}[a(\tau) \mid \mathcal{F}(\tau-1)] \\
& \leq \tilde{\lambda}<\lambda(1-\rho)
\end{aligned}
$$

and hence,

$$
\begin{equation*}
\mathbb{E}\left[1_{\{a(\tau) \geq \lambda\}} \mid \mathcal{F}(\tau-1)\right]<1-\rho . \tag{14}
\end{equation*}
$$

Let $\sigma(t) \triangleq\left\lfloor\theta t^{\beta}\right\rfloor$. If $\hat{t}-t>\theta t^{\beta}$, then

$$
\begin{aligned}
S(t+\sigma(t))-S(t) & =\sigma(t)-\sum_{\tau=t+1}^{t+\sigma(t)} \mathbb{E}\left[1_{\{a(\tau)>\lambda\}} \mid \mathcal{F}(\tau-1)\right] \\
& >\sigma(t)-\sigma(t)(1-\rho)=\sigma(t) \rho
\end{aligned}
$$

which follows from (14). Therefore, we have $B_{\theta}(t) \subset A_{\theta}(t)$, and hence, $P\left(\left\{B_{\theta}(t)\right.\right.$ i.o. $\left.\}\right)=0$ for all $\theta>0$.
Finally, by contradiction, we show that $\lim _{s \rightarrow \infty}\left(t_{s+1}-t_{s}\right) t_{s}^{-\beta}=0$. Since, if $\lim _{s \rightarrow \infty}\left(t_{s+1}-\right.$ $\left.t_{s}\right) t_{s}^{-\beta} \neq 0$ almost surely, then $\lim \sup _{s \rightarrow \infty}\left(t_{s+1}-t_{s}\right) t_{s}^{-\beta}>0$ almost surely, and hence, $P\left(\limsup _{s \rightarrow \infty}\left(t_{s+1}-t_{s}\right) t_{s}^{-\beta}>\epsilon\right)>0$ for some $\epsilon>0$. Therefore, $P\left(\left\{B_{\epsilon}(t)\right.\right.$ i.o. $\left.\}\right)>0$, which is a contradiction.

Lemma 10 Suppose that $\{D(t)\}$ is a non-negative random (scalar) process such that

$$
\begin{equation*}
D(t+1) \leq a(t+1) D(t)+b(t), \quad \text { almost surely } \tag{15}
\end{equation*}
$$

where $\{b(t)\}$ is a deterministic sequence and $\{a(t)\}$ is an adapted process (to $\{\mathcal{F}(t)\})$, such that $a(t) \in[0,1]$ and

$$
\mathbb{E}[a(t+1) \mid \mathcal{F}(t)] \leq \tilde{\lambda}
$$

almost surely for some $\tilde{\lambda}<1$ and all $t \geq 0$. Then, if

$$
0 \leq b(t) \leq K t^{-\tilde{\beta}}
$$

for some $K, \tilde{\beta}>0$, we have $\lim _{t \rightarrow \infty} D(t) t^{\beta}=0$, almost surely, for all $\beta<\tilde{\beta}$.
Proof: Let $t_{s} \triangleq \inf \left\{t>t_{s-1} \mid a(t) \leq \lambda\right\}$ and $t_{0}=0$ for some $\tilde{\lambda}<\lambda<1$, and $c(s) \triangleq \sum_{\tau=t_{s}+1}^{t_{s+1}-1} b(\tau)$. Also, define

$$
A \triangleq\left\{\omega \left\lvert\, \lim _{s \rightarrow \infty} \frac{t_{s+1}-t_{s}}{t_{s}^{\min \{\tilde{\beta}-\beta, 1\}}}=0\right.\right\}
$$

Note that Lemma 9 implies $P(A)=1$. On the other hand, using (15), we have

$$
\begin{aligned}
D\left(t_{s+1}\right) & \leq D\left(t_{s}\right) \prod_{\ell=t_{s}+1}^{t_{s+1}} a(\ell)+\sum_{\tau=t_{s}}^{t_{s+1}-1} b(\tau) \prod_{\ell=\tau+2}^{t_{s+1}} a(\ell) \\
& \leq D\left(t_{s}\right) \lambda+c(s)
\end{aligned}
$$

where the last inequality follows from $a(t) \in[0,1]$ and $a\left(t_{s+1}\right) \leq \lambda$. Letting $R(t)=D(t) t^{\beta}$, we have

$$
R\left(t_{s+1}\right) \leq\left(\frac{t_{s+1}}{t_{s}}\right)^{\beta} R\left(t_{s}\right) \lambda+c(s) t_{s+1}^{\beta}
$$

Note that, for $\omega \in A$, we have

$$
\begin{equation*}
\lim _{s \rightarrow \infty} \frac{t_{s+1}}{t_{s}}=\lim _{s \rightarrow \infty} 1+\frac{t_{s+1}-t_{s}}{t_{s}}=1 \tag{16}
\end{equation*}
$$

As a result, for any $\hat{\lambda} \in(\lambda, 1)$, there exists $\hat{s}$ such that for $s \geq \hat{s}$, we have $\left(\frac{t_{s+1}}{t_{s}}\right)^{\beta} \lambda \leq \hat{\lambda}$, and hence

$$
R\left(t_{s+1}\right) \leq R\left(t_{s}\right) \hat{\lambda}+c(s) t_{s+1}^{\beta}
$$

Therefore,

$$
R\left(t_{s}\right) \leq \hat{\lambda}^{s-\hat{s}} R\left(t_{\hat{s}}\right)+\sum_{\tau=\hat{s}}^{s-1} c(\tau) t_{\tau+1}^{\beta} \hat{\lambda}^{s-\tau-1}
$$

Taking the limits of the both sides, we have

$$
\begin{aligned}
\limsup _{s \rightarrow \infty} R\left(t_{s}\right) & \leq \limsup _{s \rightarrow \infty} \hat{\lambda}^{s-\hat{s}} R\left(t_{\hat{s}}\right)+\sum_{\tau=\hat{s}}^{s-1} c(\tau) t_{\tau+1}^{\beta} \hat{\lambda}^{s-\tau-1} \\
& =\lim _{s \rightarrow \infty} c(s) t_{s+1}^{\beta}
\end{aligned}
$$

which is implied by Lemma 3.1-(a) in [23]. For $\omega \in A$, we have

$$
\begin{align*}
\lim _{s \rightarrow \infty} c(s) t_{s+1}^{\beta} & \stackrel{(a)}{\leq} \lim _{s \rightarrow \infty} \frac{K\left(t_{s+1}-t_{s}\right)}{t_{s}^{\tilde{\beta}}} t_{s+1}^{\beta} \\
& =\lim _{s \rightarrow \infty} \frac{K\left(t_{s+1}-t_{s}\right)}{t_{s}^{\tilde{\beta}-\beta}} \frac{t_{s+1}^{\beta}}{t_{s}^{\beta}} \\
& \stackrel{(b)}{=} \lim _{s \rightarrow \infty} \frac{K\left(t_{s+1}-t_{s}\right)}{t_{s}^{\tilde{\beta}-\beta}}=0 \tag{17}
\end{align*}
$$

where ( $a$ ) follows from $b(t) \leq K t^{-\tilde{\beta}}$, and (b) follows from (16). Therefore, $\lim _{s \rightarrow \infty} R\left(t_{s}\right)=0$. Now for any $t>$ 0 with $t_{s} \leq t<t_{s+1}$, let $\sigma(t)=s$. By the definition of $R(t)$, we have $R(t) \leq\left(\frac{t}{\sigma(t)}\right)^{\beta} R\left(t_{\sigma(t)}\right)+c(\sigma(t)) t_{\sigma(t)+1}^{\beta}$. Therefore, $\lim _{s \rightarrow \infty} R\left(t_{s}\right)=0$ and Inequality (17) imply

$$
\limsup _{t \rightarrow \infty} R(t) \leq \lim _{t \rightarrow \infty}\left(\frac{t}{\sigma(t)}\right)^{\beta} R\left(t_{\sigma(t)}\right)+c(\sigma(t)) t_{\sigma(t)+1}^{\beta}=0
$$

which is the desired conclusion as $R(t)=D(t) t^{\beta}$.
Finally, we are ready to show the almost sure convergence $\lim _{t \rightarrow \infty} \mathrm{~d}(\mathbf{x}(t))=0$ (and more) under our connectivity assumption and a regularity condition on the input $\mathbf{u}(t)$ for the controlled averaging dynamics (10).

Lemma 11 Suppose that $\{W(t)\}$ satisfies Assumption 3. Then, if $\mathrm{d}(\mathbf{u}(t))<q t^{-\tilde{\beta}}$ almost surely for some $q \geq 0$, we have $\lim _{t \rightarrow \infty} \mathrm{~d}(\mathbf{x}(t)) t^{\beta}=0$, almost surely, for $\beta<\tilde{\beta}$.

Proof: From inequality (11), we have

$$
\begin{align*}
\mathrm{d}(\mathbf{x}(k)) & \leq \operatorname{diam}(\Phi(k, \tau)) \mathrm{d}(\mathbf{x}(\tau))+\sum_{s=\tau}^{k-1} \operatorname{diam}(\Phi(\tau, s+1)) \mathrm{d}(\mathbf{u}(s)) \\
& \stackrel{(a)}{\leq} \operatorname{diam}(k, \tau)) \mathrm{d}(\mathbf{x}(\tau))+\sum_{s=\tau}^{k-1} q s^{-\tilde{\beta}} \\
& \leq \operatorname{diam}(\Phi(k, \tau)) \mathrm{d}(\mathbf{x}(\tau))+(k-\tau) q \tau^{-\tilde{\beta}} \tag{18}
\end{align*}
$$

where $(a)$ follows from $\operatorname{diam}(\Phi(.,)) \leq$.1 . Let $C>0$ and $\lambda \in[0,1)$ be the constants satisfying the statement of Lemma 5. Since $\lambda<1$, for $T=\left\lceil\left|\frac{\log C}{\log \lambda}\right|\right\rceil+1$, we have $\tilde{\lambda} \triangleq C \lambda^{T}<1$. Then, Lemma 5 implies that

$$
\begin{equation*}
\mathbb{E}[\operatorname{diam}(\Phi(T(t+1), T t)) \mid \mathcal{F}(T t)] \leq C \lambda^{T}=\tilde{\lambda}<1 \tag{19}
\end{equation*}
$$

Let $D(t) \triangleq \mathrm{d}(\mathbf{x}(T t))$. From inequality (18), for $\tau=T t$ and $k=T(t+1)$, we have

$$
D(t+1) \leq \operatorname{diam}(\Phi(T(t+1), T t)) D(t)+T^{1-\tilde{\beta}} q t^{-\tilde{\beta}}
$$

Taking conditional expectation of both sides of the above inequality given $\mathcal{F}(T t)$, we have

$$
\mathbb{E}[D(t+1) \mid \mathcal{F}(T t)] \leq \mathbb{E}[\operatorname{diam}(\Phi(T(t+1), T t)) \mid \mathcal{F}(T t)] D(t)+T^{1-\tilde{\beta}} q t^{-\tilde{\beta}}
$$

By letting $a(t+1) \triangleq \operatorname{diam}(\Phi(T(t+1), T t))$ and $b(t) \triangleq T q t^{-\tilde{\beta}}$, we are in the setting of Lemma 10. Therefore, by Inequality (19) and $\operatorname{diam}(\Phi(T(t+1), T t)) \leq 1$, the conditions of Lemma 10 hold, and hence,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} D(t) t^{\beta}=0 \tag{20}
\end{equation*}
$$

almost surely.

On the other hand, letting $\tau=T\left\lfloor\frac{k}{T}\right\rfloor$ in (18) we have

$$
\mathrm{d}(\mathbf{x}(k)) \leq D\left(\left\lfloor\frac{k}{T}\right\rfloor\right)+T^{1-\tilde{\beta}} q\left\lfloor\frac{k}{T}\right\rfloor^{-\tilde{\beta}}
$$

Therefore,

$$
\begin{aligned}
\lim _{k \rightarrow \infty} \mathrm{~d}(\mathbf{x}(k)) k^{\beta} & \leq \lim _{k \rightarrow \infty} D\left(\left\lfloor\frac{k}{T}\right\rfloor\right) k^{\beta}+T^{1-\tilde{\beta}} q\left\lfloor\frac{k}{T}\right\rfloor^{-\tilde{\beta}} k^{\beta} \\
& \leq \lim _{k \rightarrow \infty} D\left(\left\lfloor\frac{k}{T}\right\rfloor\right)\left(T\left(\left\lfloor\frac{k}{T}\right\rfloor+1\right)\right)^{\beta}+T q(k-T)^{-\tilde{\beta}} k^{\beta} \\
& =0
\end{aligned}
$$

where the last equality follows from (20) and $\tilde{\beta}>\beta$.

## 5 Convergence Analysis of the Main Dynamics

Finally, in this section, we will study the main dynamics (2), i.e., the dynamics (10) with the feedback policy $\mathbf{u}_{i}(t)=-\alpha(t) \mathbf{g}_{i}(t)$ where $\mathbf{g}_{i}(t) \in \nabla f_{i}\left(\mathbf{x}_{i}(t)\right)$. Throughout this section, we let $\overline{\mathbf{x}} \triangleq \frac{1}{n} e^{T} \mathbf{x}$ for a vector $\mathbf{x} \in\left(\mathbb{R}^{m}\right)^{n}$,

First, we prove an inequality (Lemma 13) which plays a key role in the proof of Theorem 1 and to do so, we make use of the following result which is proven as a part of the proof of Lemma 8 (Equation (27)) in 18.

Lemma 12 ( [18]) Under Assumption 1, for all $v \in \mathbb{R}^{m}$, we have

$$
n\langle\overline{\mathbf{g}}(t), \overline{\mathbf{x}}(t)-v\rangle \geq F(\overline{\mathbf{x}}(t))-F(v)-2 \sum_{i=1}^{n} L_{i}\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\|
$$

Lemma 13 For the dynamics (2), under Assumption 11, for all $v \in \mathbb{R}^{m}$, we have

$$
\begin{aligned}
\mathbb{E}\left[\|\overline{\mathbf{x}}(t+1)-v\|^{2} \mid \mathcal{F}(t)\right] \leq \| \overline{\mathbf{x}}(t)- & v\left\|^{2}+\alpha^{2}(t) \frac{L^{2}}{n^{2}}+\sum_{i=1}^{n}\right\| \mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t) \|^{2} \\
& -\frac{2 \alpha(t)}{n}(F(\overline{\mathbf{x}}(t))-F(v))+\frac{4 \alpha(t)}{n} \sum_{i=1}^{n} L_{i}\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\| .
\end{aligned}
$$

Proof: Multiplying $\frac{1}{n} e^{T}$ from left to both sides of (2), we have

$$
\begin{aligned}
\overline{\mathbf{x}}(t+1) & =\bar{W}(t+1) \mathbf{x}(t)-\alpha(t) \overline{\mathbf{g}}(t) \\
& =\overline{\mathbf{x}}(t)-\alpha(t) \overline{\mathbf{g}}(t)+\bar{W}(t+1) \mathbf{x}(t)-\overline{\mathbf{x}}(t)
\end{aligned}
$$

where $\bar{W}(t) \triangleq \frac{1}{n} e^{T} W(t)$. Therefore, we can write

$$
\begin{aligned}
\|\overline{\mathbf{x}}(t+1)-v\|^{2}= & \|\overline{\mathbf{x}}(t)-v-\alpha(t) \overline{\mathbf{g}}(t)+\bar{W}(t+1) \mathbf{x}(t)-\overline{\mathbf{x}}(t)\|^{2} \\
= & \|\overline{\mathbf{x}}(t)-v\|^{2}+\|\alpha(t) \overline{\mathbf{g}}(t)\|^{2}+\|\bar{W}(t+1) \mathbf{x}(t)-\overline{\mathbf{x}}(t)\|^{2}-2 \alpha(t)\langle\overline{\mathbf{g}}(t), \bar{W}(t+1) \mathbf{x}(t)-v\rangle \\
& +2\langle\overline{\mathbf{x}}(t)-v, \bar{W}(t+1) \mathbf{x}(t)-\overline{\mathbf{x}}(t)\rangle
\end{aligned}
$$

Taking conditional expectation of both sides of the above equality given $\mathcal{F}(t)$, we have

$$
\begin{aligned}
\mathbb{E}\left[\|\overline{\mathbf{x}}(t+1)-v\|^{2} \mid \mathcal{F}(t)\right]= & \|\overline{\mathbf{x}}(t)-v\|^{2}+\|\alpha(t) \overline{\mathbf{g}}(t)\|^{2}+\mathbb{E}\left[\|\bar{W}(t+1) \mathbf{x}(t)-\overline{\mathbf{x}}(t)\|^{2} \mid \mathcal{F}(t)\right] \\
& -2 \alpha(t)\langle\overline{\mathbf{g}}(t), \mathbb{E}[\bar{W}(t+1) \mathbf{x}(t)-v \mid \mathcal{F}(t)]\rangle+2\langle(\overline{\mathbf{x}}(t)-v), \mathbb{E}[\bar{W}(t+1) \mathbf{x}(t)-\overline{\mathbf{x}}(t) \mid \mathcal{F}(t)]\rangle \\
= & \|\overline{\mathbf{x}}(t)-v\|^{2}+\|\alpha(t) \overline{\mathbf{g}}(t)\|^{2}+\mathbb{E}\left[\|\bar{W}(t+1) \mathbf{x}(t)-\overline{\mathbf{x}}(t)\|^{2} \mid \mathcal{F}(t)\right]-2\langle\alpha(t) \overline{\mathbf{g}}(t), \overline{\mathbf{x}}(t)-v\rangle
\end{aligned}
$$

The last equality follows from the assumption that, $W(t+1)$ is doubly stochastic in-expectation and hence,

$$
\mathbb{E}[\bar{W}(t+1) \mid \mathcal{F}(t)]=\frac{1}{n} e^{T},
$$

which implies

$$
\langle\overline{\mathbf{g}}(t), \mathbb{E}[\bar{W}(t+1) \mathbf{x}(t)-v \mid \mathcal{F}(t)]\rangle=\langle\alpha(t) \overline{\mathbf{g}}(t), \overline{\mathbf{x}}(t)-v\rangle
$$

and

$$
\mathbb{E}[\bar{W}(t+1) \mathbf{x}(t)-\overline{\mathbf{x}}(t) \mid \mathcal{F}(t)]=0
$$

Note that $\bar{W}(t+1)$ is a stochastic vector (almost surely), therefore, due to the convexity of norm-square $\|\cdot\|^{2}$, we get

$$
\begin{aligned}
\|\bar{W}(t+1) \mathbf{x}(t)-\overline{\mathbf{x}}(t)\|^{2} & \leq \sum_{i=1}^{n} \bar{W}_{i}(t+1)\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\|^{2} \\
& \leq \sum_{i=1}^{n}\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\|^{2}
\end{aligned}
$$

as $\bar{W}_{i}(t+1) \leq 1$ for all $i \in[n]$. Therefore,

$$
\begin{aligned}
\mathbb{E}\left[\|\overline{\mathbf{x}}(t+1)-v\|^{2} \mid \mathcal{F}(t)\right] & =\|\overline{\mathbf{x}}(t)-v\|^{2}+\|\alpha(t) \overline{\mathbf{g}}(t)\|^{2}+\mathbb{E}\left[\|\bar{W}(t+1) \mathbf{x}(t)-\overline{\mathbf{x}}(t)\|^{2} \mid \mathcal{F}(t)\right]-2\langle\alpha(t) \overline{\mathbf{g}}(t), \overline{\mathbf{x}}(t)-v\rangle \\
& \leq\|\overline{\mathbf{x}}(t)-v\|^{2}+\|\alpha(t) \overline{\mathbf{g}}(t)\|^{2}+\sum_{i=1}^{n}\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\|^{2}-2\langle\alpha(t) \overline{\mathbf{g}}(t), \overline{\mathbf{x}}(t)-v\rangle
\end{aligned}
$$

Finally, Lemma 12 and the fact that

$$
\|\overline{\mathbf{g}}(t)\|^{2}=\frac{1}{n^{2}}\left\|\sum_{i=1}^{n} \mathbf{g}_{i}(t)\right\|^{2} \leq \frac{1}{n^{2}}\left(\sum_{i=1}^{n}\left\|\mathbf{g}_{i}(t)\right\|\right)^{2} \leq \frac{L^{2}}{n^{2}}
$$

complete the proof.
Proof of Theorem [1: Recall that Siegmund and Robbins Theorem [26] states that for a non-negative random process $\{V(t)\}$ (adapted to the filtration $\{\mathcal{F}(t)\})$ satisfying

$$
\begin{equation*}
\mathbb{E}[V(t+1) \mid \mathcal{F}(t)]=(1+a(t)) V(t)-b(t)+c(t) \tag{21}
\end{equation*}
$$

where $a(t), b(t), c(t) \geq 0$ for all $t$, and $\sum_{t=0}^{\infty} a(t)<\infty$ and $\sum_{t=1}^{\infty} c(t)<\infty$ almost surely, $\lim _{t \rightarrow \infty} V(x(t))$ exists and $\sum_{t=1}^{\infty} b(t)<\infty$ almost surely. In order to utilize this result and Lemma 13, for all $t \geq 0$, let

$$
\begin{aligned}
V(t) & \triangleq\|\overline{\mathbf{x}}(t)-z\|^{2} \\
a(t) & \triangleq 0 \\
b(t) & \triangleq-\frac{2 \alpha(t)}{n}(F(\overline{\mathbf{x}}(t))-F(z)), \text { and } \\
c(t) & \triangleq \alpha^{2}(t) \frac{L^{2}}{n^{2}}+\sum_{i=1}^{n}\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\|^{2}+\frac{4 \alpha(t)}{n} \sum_{i=1}^{n} L_{i}\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\|
\end{aligned}
$$

where $z \in \mathcal{Z}$. First, note that $a(t), b(t), c(t) \geq 0$ for all $t$. To invoke the Siegmund and Robbins result (21), we need to to prove that $\sum_{t=0}^{\infty} c(t)<\infty$, almost surely. Since $\sum_{t=0}^{\infty} \alpha^{2}(t)<\infty$, it is enough to show that

$$
\sum_{t=0}^{\infty}\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\|^{2}<\infty
$$

and

$$
\sum_{t=0}^{\infty} \alpha(t)\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\|<\infty
$$

almost surely, for all $i \in[n]$. From Lemma 2 (e) and Lemma [8, we have

$$
\mathbb{E}\left[\frac{\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\|}{\alpha(t)}\right] \leq \frac{\mathbb{E}[\mathrm{d}(\mathbf{x}(t))]}{\alpha(t)} \leq \sqrt{\hat{M}}<\infty
$$

for some $\hat{M}>0$. Therefore, we have

$$
\begin{aligned}
\lim _{T \rightarrow \infty} \mathbb{E}\left[\sum_{t=0}^{T} \alpha(t)\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\|\right] & =\lim _{T \rightarrow \infty} \mathbb{E}\left[\sum_{t=0}^{T} \alpha^{2}(t) \frac{\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\|}{\alpha(t)}\right] \\
& =\lim _{T \rightarrow \infty} \sum_{t=0}^{T} \alpha^{2}(t) \mathbb{E}\left[\frac{\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\|}{\alpha(t)}\right] \\
& \leq \sqrt{\hat{M}} \sum_{t=0}^{\infty} \alpha^{2}(t)<\infty
\end{aligned}
$$

Similarly, using Lemma 8 , there exists some $\hat{M}>0$ such that

$$
\frac{\mathbb{E}\left[\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\|^{2}\right]}{\alpha^{2}(t)} \leq \frac{\mathbb{E}\left[\mathrm{d}^{2}(\mathbf{x}(t))\right]}{\alpha^{2}(t)} \leq \hat{M}
$$

for all $t \geq 0$, where the first inequality follows from Lemma 2-(e). Therefore,

$$
\begin{aligned}
\lim _{T \rightarrow \infty} \mathbb{E}\left[\sum_{t=0}^{T}\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\|^{2}\right] & =\lim _{T \rightarrow \infty} \mathbb{E}\left[\sum_{t=0}^{T} \alpha^{2}(t) \frac{\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\|^{2}}{\alpha^{2}(t)}\right] \\
& \leq \hat{M} \sum_{t=0}^{\infty} \alpha^{2}(t)<\infty
\end{aligned}
$$

Therefore, using Monotone Convergence Theorem, we have

$$
\begin{aligned}
& \mathbb{E}\left[\sum_{t=0}^{\infty} \alpha(t)\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\|\right]<\infty, \text { and } \\
& \quad \mathbb{E}\left[\sum_{t=0}^{\infty}\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\|^{2}\right]<\infty
\end{aligned}
$$

which implies $\sum_{t=0}^{\infty} \alpha(t)\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\|<\infty$ and $\sum_{t=0}^{\infty}\left\|\mathbf{x}_{i}(t)-\overline{\mathbf{x}}(t)\right\|^{2}<\infty$, almost surely.
Now that we showed that $c(t)$ is almost surely a summable sequence, Siegmund and Robbins Theorem implies that almost surely

$$
\lim _{t \rightarrow \infty} V(t)=\lim _{t \rightarrow \infty}\|\overline{\mathbf{x}}(t)-z\|^{2} \text { exists }
$$

and

$$
\sum_{t=1}^{\infty} \alpha(t)(F(\overline{\mathbf{x}}(t))-F(z))<\infty
$$

For $z \in \mathcal{Z}$, let's define

$$
\Omega_{z} \triangleq\left\{\omega \left\lvert\, \begin{array}{c}
\lim _{t \rightarrow \infty}\|\overline{\mathbf{x}}(t, \omega)-z\| \text { exists } \\
\sum_{t=1}^{\infty} \alpha(t)\left(F(\overline{\mathbf{x}}(t, \omega))-F^{*}\right)<\infty
\end{array}\right.\right\}
$$

where $F^{*} \triangleq \min _{z \in \mathbb{R}^{m}} F(z)$. Per Sigmund-Robbins result, we know that $P\left(\Omega_{z}\right)=1$. Now, let $\mathcal{Z}_{d} \subset \mathcal{Z}$ be a countable dense subset of $\mathcal{Z}$ and let

$$
\Omega_{d} \triangleq \bigcap_{z \in \mathcal{Z}_{d}} \Omega_{z}
$$

Since $\mathcal{Z}_{d}$ is a countable set, we have $P\left(\Omega_{d}\right)=1$ and for $\omega \in \Omega_{d}$, since $\sum_{t=1}^{\infty} \alpha(t)\left(F(\overline{\mathbf{x}}(t, \omega))-F^{*}\right)<\infty$ and $\alpha(t)$ is not summable, we have

$$
\liminf _{t \rightarrow \infty} F(\overline{\mathbf{x}}(t))=F^{*}
$$

This fact and the fact that $F(\cdot)$ is a continuous function implies that for all $\omega \in \Omega_{d}$, we have $\liminf _{t \rightarrow \infty}\left\|\overline{\mathbf{x}}(t, \omega)-z^{*}(\omega)\right\|=0$ for some $z^{*}(\omega) \in \mathcal{Z}$. To show this, let $\left\{\overline{\mathbf{x}}\left(t_{k}\right)\right\}$ be a sub-sequence that $\lim _{k \rightarrow \infty} F\left(\overline{\mathbf{x}}\left(t_{k}, \omega\right)\right)=F^{*}$ (such a sub-sequence depends on the sample path $\omega$ ). Since $\omega \in \Omega_{d}$ and

$$
\lim _{t \rightarrow \infty}\|\overline{\mathbf{x}}(t, \omega)-\hat{z}\| \quad \text { exists }
$$

for some $\hat{z} \in \mathcal{Z}_{d}$, we conclude that $\{\overline{\mathbf{x}}(t, \omega)\}$ is a bounded sequences. Therefore, $\left\{\overline{\mathbf{x}}\left(t_{k}, \omega\right)\right\}$ is also bounded and it has an accumulation point $z^{*} \in \mathbb{R}^{m}$ and hence, there is a sub-sequence $\left\{\overline{\mathbf{x}}\left(t_{k_{\tau}}, \omega\right)\right\}_{\tau \geq 0}$ of $\left\{\overline{\mathbf{x}}\left(t_{k}, \omega\right)\right\}_{k \geq 0}$ such that

$$
\lim _{\tau \rightarrow \infty} \overline{\mathbf{x}}\left(t_{k_{\tau}}, \omega\right)=z^{*} .
$$

As a result of continuity of $F(\cdot)$, we have

$$
\lim _{\tau \rightarrow \infty} F\left(\overline{\mathbf{x}}\left(t_{k_{\tau}}\right)\right)=F\left(z^{*}\right)=F^{*}
$$

and hence, $z^{*} \in \mathcal{Z}$. Note that the point $z^{*}=z^{*}(\omega)$ depends on the sample path $\omega$.
Since $\mathcal{Z}_{d} \subseteq \mathcal{Z}$ is dense, there is a sequence $\left\{q^{*}(s, \omega)\right\}_{s \geq 0}$ in $\mathcal{Z}_{d}$ such that $\lim _{s \rightarrow \infty}\left\|q^{*}(s, \omega)-z^{*}(\omega)\right\|=0$. Note that since $\omega \in \Omega_{d}, \lim _{t \rightarrow \infty}\left\|\overline{\mathbf{x}}(t, \omega)-q^{*}(s, \omega)\right\|$ exists for all $s \geq 0$ and we have

$$
\begin{aligned}
\lim _{t \rightarrow \infty}\left\|\overline{\mathbf{x}}(t, \omega)-q^{*}(s, \omega)\right\| & =\lim _{t \rightarrow \infty}\left\|\overline{\mathbf{x}}(t, \omega)-z^{*}(\omega)+z^{*}(\omega)-q^{*}(s, \omega)\right\| \\
& \leq \liminf _{t \rightarrow \infty}\left\|\overline{\mathbf{x}}(t, \omega)-z^{*}(\omega)\right\|+\left\|q^{*}(s, \omega)-z^{*}(\omega)\right\| \\
& =\left\|q^{*}(s, \omega)-z^{*}(\omega)\right\| .
\end{aligned}
$$

Therefore, we have

$$
\begin{equation*}
\lim _{s \rightarrow \infty} \lim _{t \rightarrow \infty}\left\|\overline{\mathbf{x}}(t, \omega)-q^{*}(s, \omega)\right\|=0 . \tag{22}
\end{equation*}
$$

On the other hand, we have

$$
\begin{aligned}
\limsup _{t \rightarrow \infty}\left\|\overline{\mathbf{x}}(t, \omega)-z^{*}(\omega)\right\| & =\limsup _{t \rightarrow \infty}\left\|\overline{\mathbf{x}}(t, \omega)-q^{*}(s, \omega)+q^{*}(s, \omega)-z^{*}(\omega)\right\| \\
& \leq \limsup _{t \rightarrow \infty}\left\|\overline{\mathbf{x}}(t, \omega)-q^{*}(s, \omega)\right\|+\left\|q^{*}(s, \omega)-z^{*}(\omega)\right\| \\
& =\left(\lim _{t \rightarrow \infty}\left\|\overline{\mathbf{x}}(t, \omega)-q^{*}(s, \omega)\right\|\right)+\left\|q^{*}(s, \omega)-z^{*}(\omega)\right\| .
\end{aligned}
$$

Therefore,

$$
\begin{align*}
\limsup _{t \rightarrow \infty}\left\|\overline{\mathbf{x}}(t, \omega)-z^{*}(\omega)\right\| & =\lim _{s \rightarrow \infty} \limsup _{t \rightarrow \infty}\left\|\overline{\mathbf{x}}(t, \omega)-z^{*}(\omega)\right\| \\
& \leq \lim _{s \rightarrow \infty} \lim _{t \rightarrow \infty}\left\|\overline{\mathbf{x}}(t, \omega)-q^{*}(s, \omega)\right\|+\lim _{s \rightarrow \infty}\left\|q^{*}(s, \omega)-z^{*}(\omega)\right\| \\
& =0, \tag{23}
\end{align*}
$$

where the last equality follows by combining (22) and $\lim _{s \rightarrow \infty}\left\|q^{*}(s, \omega)-z^{*}(\omega)\right\|=0$. Note that (23), implies that almost surely (i.e., for all $\omega \in \Omega_{d}$ ), we have

$$
\lim _{t \rightarrow \infty} \overline{\mathbf{x}}(t)=z^{*}(\omega)
$$

exists and it belongs to $\mathcal{Z}$.
Finally, according to Assumption Tand 4, we have

$$
\mathrm{d}(\alpha(t) \mathbf{g}(t)) \leq 2 K t^{-\beta} \max _{i \in[n]} L_{i} .
$$

Therefore, from lemma 11, we conclude that $\lim _{t \rightarrow \infty} \mathrm{~d}(\mathbf{x}(t))=0$ almost surely, and hence

$$
\lim _{t \rightarrow \infty}\left\|\overline{\mathbf{x}}(t)-\mathbf{x}_{i}(t)\right\|=0 \quad \text { almost surely }
$$

Since we almost surely have $\lim _{t \rightarrow \infty} \overline{\mathbf{x}}(t)=z^{*}$ for a random vector $z^{*}$ supported in $\mathcal{Z}$, we have $\lim _{t \rightarrow \infty} \mathbf{x}_{i}(t)=z^{*}$ for all $i \in[n]$ almost surely and the proof is complete.

## 6 Conclusion and Future Research

In this work, we showed that the averaging-based distributed optimization solving algorithm over dependent random networks converges to an optimal random point under the standard conditions on the objective function and network formation that is conditionally $B$-connected. To do so, we established a rate of convergence for the second moment of the autonomous averaging dynamics over such networks and used that to study the convergence of the sample-paths and second moments of the controlled variation of those dynamics.

Further extensions of the current work to non-convex settings, accelerated algorithms, and distributed online learning algorithms are of interest for future consideration on this topic.

Proof of Lemma 园 For the proof of part (a), let $\mathbf{x}_{i}^{(k)}$ be the $k$ th coordinate of $\mathbf{x}_{i}$, and define the vector $y=\left[y^{(1)}, \ldots, y^{(m)}\right]^{T}$ where

$$
y^{(k)}=\frac{1}{2}\left(u^{(k)}+U^{(k)}\right),
$$

with $u^{(k)}=\min _{i \in[n]} \mathbf{x}_{i}^{(k)}$ and $U^{(k)}=\max _{i \in[n]} \mathbf{x}_{i}^{(k)}$. Therefore, for $\ell \in[n]$, we have

$$
\begin{align*}
\left\|\mathbf{x}_{\ell}-y\right\| & =\max _{k \in[m]}\left|\mathbf{x}_{\ell}^{(k)}-y^{(k)}\right| \\
& =\max _{k \in[m]}\left|\mathbf{x}_{\ell}^{(k)}-\frac{1}{2}\left(u^{(k)}+U^{(k)}\right)\right| \\
& \stackrel{(a)}{\leq} \max _{k \in[m]} \frac{1}{2}\left|u^{(k)}-U^{(k)}\right| \\
& =\frac{1}{2} \mathrm{~d}(\mathbf{x}) \tag{24}
\end{align*}
$$

where (a) follows from $u^{(k)} \leq \mathbf{x}_{\ell}^{(k)} \leq U^{(k)}$. Also, we have

$$
\begin{aligned}
\mathrm{d}(A \mathbf{x}) & =\max _{i, j \in[n]}\left\|\sum_{\ell=1}^{n} a_{i \ell} \mathbf{x}_{\ell}-\sum_{\ell=1}^{n} a_{j \ell} \mathbf{x}_{\ell}\right\| \\
& =\max _{i, j \in[n]}\left\|\sum_{\ell=1}^{n} a_{i \ell}\left(\mathbf{x}_{\ell}-y\right)-\sum_{\ell=1}^{n} a_{j \ell}\left(\mathbf{x}_{\ell}-y\right)+\sum_{\ell=1}^{n}\left(a_{j \ell}-a_{i \ell}\right) y\right\| \\
& =\max _{i, j \in[n]}\left\|\sum_{\ell=1}^{n}\left(a_{i \ell}-a_{j \ell}\right)\left(\mathbf{x}_{\ell}-y\right)\right\|
\end{aligned}
$$

where the last equality holds as $A$ is a row-stochastic matrix and hence, $\sum_{\ell=1}^{n}\left(a_{j \ell}-a_{i \ell}\right)=0$. Therefore,

$$
\begin{aligned}
\mathrm{d}(A \mathbf{x}) & =\max _{i, j \in[n]}\left\|\sum_{\ell=1}^{n}\left(a_{i \ell}-a_{j \ell}\right)\left(\mathbf{x}_{\ell}-y\right)\right\| \\
& \stackrel{(a)}{\leq} \max _{i, j \in[n]} \sum_{\ell=1}^{n}\left|a_{i \ell}-a_{j \ell}\right|\left\|\mathbf{x}_{\ell}-y\right\| \\
& \stackrel{(b)}{\leq} \max _{i, j \in[n]} \frac{1}{2} \mathrm{~d}(\mathbf{x}) \sum_{\ell=1}^{n}\left|a_{i \ell}-a_{j \ell}\right| \\
& \leq \operatorname{diam}(A) \mathrm{d}(\mathbf{x}),
\end{aligned}
$$

where (a) follows from the triangle inequality, and (b) follow from (24). For the part (b), we have

$$
\begin{aligned}
\mathrm{d}(\mathbf{x}+\mathbf{y}) & =\max _{i, j \in[n]}\left\|(\mathbf{x}+\mathbf{y})_{i}-(\mathbf{x}+\mathbf{y})_{j}\right\| \\
& =\max _{i, j \in[n]}\left\|\mathbf{x}_{i}-\mathbf{x}_{j}+\mathbf{y}_{i}-\mathbf{y}_{j}\right\| \\
& \stackrel{(b)}{\leq} \max _{i, j \in[n]}\left(\left\|\mathbf{x}_{i}-\mathbf{x}_{j}\right\|+\left\|\mathbf{y}_{i}-\mathbf{y}_{j}\right\|\right) \\
& \leq \max _{i, j \in[n]}\left\|\mathbf{x}_{i}-\mathbf{x}_{j}\right\|+\max _{i, j \in[n]}\left\|\mathbf{y}_{i}-\mathbf{y}_{j}\right\| \\
& =\mathrm{d}(\mathbf{x})+\mathrm{d}(\mathbf{y})
\end{aligned}
$$

where (b) follows from the triangle inequality.
For the proof of part (c), we have

$$
\begin{aligned}
\operatorname{diam}(A) & =\max _{i, j \in[n]} \sum_{\ell=1}^{n} \frac{1}{2}\left|a_{i \ell}-a_{j \ell}\right| \\
& =\max _{i, j \in[n]} \sum_{\ell=1}^{n}\left(\frac{1}{2}\left(a_{i \ell}+a_{j \ell}\right)-\min \left\{a_{i \ell}, a_{j \ell}\right\}\right) \\
& \stackrel{(a)}{=} \max _{i, j \in[n]} 1-\sum_{\ell=1}^{n} \min \left\{a_{i \ell}, a_{j \ell}\right\} \\
& =1-\min _{i, j \in[n]} \sum_{\ell=1}^{n} \min \left\{a_{i \ell}, a_{j \ell}\right\} \\
& =1-\Lambda(A)
\end{aligned}
$$

where (a) follows from the fact that $A$ is row-stochastic. The proof of part (d) follows from part (c) and Lemma 1 .

For the part (e), due to the convexity of $\|\cdot\|$, we have

$$
\left\|\mathbf{x}_{i}-\sum_{j=1}^{n} \pi_{j} \mathbf{x}_{j}\right\| \leq \sum_{j=1}^{n} \pi_{j}\left\|\mathbf{x}_{i}-\mathbf{x}_{j}\right\| \leq \sum_{j=1}^{n} \pi_{j} \mathrm{~d}(\mathbf{x})=\mathrm{d}(\mathbf{x})
$$

Proof of Lemma 4 We prove by induction on $K_{2}$. By the assumption, the lemma is true for $K_{2}=K_{1}$. For $K_{2}>K_{1}$, we have

$$
\begin{aligned}
\mathbb{E}\left[\prod_{k=K_{1}}^{K_{2}+1} Y(k) \mid \mathcal{F}\left(K_{1}-1\right)\right] & =\mathbb{E}\left[\mathbb{E}\left[\prod_{k=K_{1}}^{K_{2}+1} Y(k) \mid \mathcal{F}\left(K_{2}\right)\right] \mid \mathcal{F}\left(K_{1}-1\right)\right] \\
& =\mathbb{E}\left[\mathbb{E}\left[Y\left(K_{2}+1\right) \mid \mathcal{F}\left(K_{2}\right)\right] \prod_{k=K_{1}}^{K_{2}} Y(k) \mid \mathcal{F}\left(K_{1}-1\right)\right] \\
& \leq \mathbb{E}\left[a\left(K_{2}+1\right) \prod_{k=K_{1}}^{K_{2}} Y(k) \mid \mathcal{F}\left(K_{1}-1\right)\right] \\
& \leq \prod_{k=K_{1}}^{K_{2}+1} a(k) .
\end{aligned}
$$

Proof of Lemma $\bar{\gamma}$. Consider $\hat{\tau} \geq t_{0}$ such that $\hat{\theta} \triangleq \sup _{t \geq \hat{\tau}} \frac{\beta(t)}{\beta(t+1)} \theta<1$ and let $D(t) \triangleq \sum_{s=\tau}^{t-1} \beta(s) \theta^{t-s}$. Dividing both sides by $\beta(t)>0$, for $t>\hat{\tau}$, we have

$$
\begin{aligned}
\frac{D(t)}{\beta(t)} & =\sum_{s=\tau}^{t-1} \frac{\beta(s)}{\beta(t)} \theta^{t-s} \\
& =\sum_{s=\tau}^{t-1} \prod_{\kappa=s}^{t-1} \frac{\beta(\kappa)}{\beta(\kappa+1)} \theta \\
& \leq \sum_{s=\tau}^{\hat{\tau}-1} \prod_{\kappa=s}^{t-1} \frac{\beta(\kappa)}{\beta(\kappa+1)} \theta+\sum_{s=\hat{\tau}}^{t-1} \hat{\theta}^{t-s} \\
& =\sum_{s=\tau}^{\hat{\tau}-1} \prod_{\kappa=s}^{t-1} \frac{\beta(\kappa)}{\beta(\kappa+1)} \theta+\sum_{k=1}^{t-\hat{\tau}} \hat{\theta}^{k} \\
& \leq \sum_{s=\tau}^{\hat{\tau}-1} \prod_{\kappa=s}^{t-1} \frac{\beta(\kappa)}{\beta(\kappa+1)} \theta+\frac{\hat{\theta}}{1-\hat{\theta}} .
\end{aligned}
$$

Let

$$
M_{1} \triangleq \sup _{t>\hat{\tau}} \sup _{\hat{\tau} \geq \tau \geq t_{0}} \sum_{s=\tau}^{\hat{\tau}-1} \prod_{\kappa=s}^{t-1} \frac{\beta(\kappa)}{\beta(\kappa+1)} \theta+\frac{\hat{\theta}}{1-\hat{\theta}}
$$

Note that

$$
\lim _{t \rightarrow \infty} \prod_{\kappa=s}^{t-1} \frac{\beta(\kappa)}{\beta(\kappa+1)} \theta \leq \lim _{t \rightarrow \infty} \hat{\theta}^{t-\hat{\tau}} \prod_{\kappa=s}^{\hat{\tau}-1} \frac{\beta(\kappa)}{\beta(\kappa+1)} \theta=0
$$

Therefore, $\sup _{t \geq \tau} \prod_{\kappa=s}^{t-1} \frac{\beta(\kappa)}{\beta(\kappa+1)} \theta<\infty$, and hence, $M_{1}<\infty$. Thus, $D(t) \leq \max \left\{M_{1}, M_{2}\right\} \beta(t)$, where $M_{2} \triangleq \max _{\hat{\tau} \geq t \geq \tau \geq t_{0}} \sum_{s=\tau}^{t-1} \frac{\beta(s)}{\beta(t)} \theta^{t-s}$, and the proof is complete.

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[^1]:    ${ }^{1}$ The graphs do not need to be time-invariant, and this example can be extended to processes over underlying time-varying graphs.

