

Output Feedback Consensus for Networked Heterogeneous Nonlinear Negative-Imaginary Systems with Free Body Motion

Kanghong Shi, Ian R. Petersen, *Fellow, IEEE*, and Igor G. Vladimirov

Abstract—This paper provides a protocol to address the robust output feedback consensus problem for networked heterogeneous nonlinear negative-imaginary (NI) systems with free body dynamics. We extend the definition of nonlinear NI systems to allow for systems with free body motion. A new stability result is developed for the interconnection of a nonlinear NI system and a nonlinear output strictly negative-imaginary (OSNI) system. Also, a class of networked nonlinear OSNI controllers is proposed to achieve output feedback consensus for heterogeneous networked nonlinear NI systems. We show that in this control framework, the system outputs converge to the same limit trajectory. This consensus protocol is illustrated by a numerical example.

Index Terms—nonlinear Negative-Imaginary systems, free body motion, heterogeneous systems, consensus, robust control.

I. INTRODUCTION

Negative-Imaginary (NI) systems theory was introduced by Lanzon and Petersen in [1] and [2] to address a robust control problem for flexible structures and has attracted much attention among control theory researchers (see [3]–[9]). Typical NI systems are systems with colocated force actuators and position sensors. Positive-Real (PR) systems theory [10] cannot be applied to the control of such systems in general. An NI system was initially defined in [1] to be a stable system with a frequency response $F(j\omega)$ satisfying $j(F(j\omega) - F(j\omega)^*) \geq 0$ for all $\omega > 0$. Examples of such systems arise in lightly damped structures [11]–[13] and nano-positioning systems [14]–[17]. The stability of an NI system with transfer function matrix $F(s)$ can be guaranteed by applying a strictly Negative-Imaginary (SNI) controller with transfer function matrix $G(s)$ such that the DC gain condition $\lambda_{max}(F(0)G(0)) < 1$ is satisfied [1].

The definition of NI systems in [1], [2] was extended in [9] to include systems with poles in the closed left half of the complex plane except at the origin. The definition has been extended again in [18] to include systems with poles at the origin. Systems with free body motion such as single integrators and double integrators were included in this new definition and a new robust stability result for NI systems in [18]. The original definition of NI systems has also been recently extended to include nonlinear systems [19] and some stability results were established for nonlinear NI systems in [19] and [20]. However, systems with free body motion are excluded in the nonlinear NI definition in [19].

Cooperative control for multi-agent systems has been a highly active research area over the past two decades [21].

This control paradigm enables multiple systems to perform team missions and has been applied on autonomous vehicles including mobile robots, unmanned air vehicles (UAVs), autonomous underwater vehicles (AUVs) and other applications [22]. Consensus, that is convergence to an agreement through information sharing among agents, is one of the most important problems in the area of cooperative control [23]. Consensus algorithms were first studied for first-order dynamics (see [24]–[26], etc) and then extended to second-order systems (see [27]–[30], etc).

As NI properties arise naturally in autonomous vehicles and a wide variety of other applications [31], consensus problems were investigated in [8] for heterogeneous NI systems. Using NI systems theory, the consensus algorithms in [8] only require outputs of the agents, and consensus is guaranteed if a simple DC gain condition is satisfied. The theoretical result presented in [8] has already been applied to real-world cooperative control problems (see [32]–[36]). However, this result is restricted to linear systems. Motivated by nonlinear NI systems theory, the result of [8] has been recently extended in [37], which provides a protocol that achieves output feedback consensus for networked heterogeneous nonlinear NI systems. However, [37] uses the definition of nonlinear NI systems from [19], which excludes systems with free body motion such as integrators.

In this work, we use an alternative definition for a class of nonlinear NI systems to include systems with free body motion. We obtain a stability result for the interconnection of a nonlinear NI system and a nonlinear output strictly negative-imaginary (OSNI) system (see also [38] and [39] for linear OSNI systems), where nonlinear OSNI systems are also redefined to allow direct feedthrough from the input to the output. Then we consider the output feedback consensus problem for multiple nonlinear NI systems with different state-space models. We aim to find a control protocol such that the difference between the outputs of any two agents connected in a network converges to zero. We model the communication between multiple nonlinear NI systems using an undirected connected graph, where the nonlinear NI plants and the nonlinear OSNI controllers correspond to the nodes and the edges of the graph, respectively. Each controller takes the difference between the outputs of the two plants connected to it as input and feeds back its output to the plants. This forms an augmented system whose stability is investigated in order to achieve output consensus for the nonlinear NI plants. We prove the networked nonlinear NI plants act as an augmented nonlinear NI system and the networked nonlinear OSNI controllers act as an augmented nonlinear OSNI system. Therefore, the entire networked control system can be regarded as the interconnection of a nonlinear NI system and a nonlinear

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OSNI system. Similarly to the stability result mentioned above, an extended stability result is established for the networked control system that guarantees output consensus. This shows that for networked heterogeneous nonlinear NI systems satisfying certain assumptions, output feedback consensus can be achieved by using suitable nonlinear OSNI controllers in the proposed control framework. The control framework is robust with respect to bounded perturbations in the system models for both the nonlinear NI plants and the nonlinear OSNI controllers.

This paper contributes an output feedback consensus protocol for networked heterogeneous nonlinear NI systems with free body motion. This work differs from the related previous results in the following aspects: [19] defines nonlinear NI systems but does not allow for systems with free body motion while this work does; [20] only considers consensus for agents with the same model while this work allows agents to have different state-space models; [20] and [37] only deal with consensus problems for nonlinear NI systems without free body motion while this restriction is lifted in this work. This work can also be regarded as an extension of the papers [18] and [8] to nonlinear systems.

Notation: The notation in this paper is standard. \mathbb{R} and \mathbb{C} denote the fields of real and complex numbers, respectively. $\mathbb{R}^{m \times n}$ and $\mathbb{C}^{m \times n}$ denote the spaces of real and complex matrices of dimension $m \times n$, respectively. A^T and A^* denote the transpose and complex conjugate transpose of a matrix A , respectively. $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of a matrix with only real eigenvalues. $A \otimes B$ denotes the Kronecker product of matrices A and B . $\bar{\cdot}$ denotes a constant value for a given vector or scalar signal. $\|\cdot\|$ denotes the Euclidean norm of a vector. I_n is the $n \times n$ identity matrix. For a nonlinear dynamical system H with input u and output y , $y = H(u)$ describes its input-output relationship. $\text{diag}\{a_1, a_2, \dots, a_l\}$ represents a diagonal matrix with the values a_1, a_2, \dots, a_l on its diagonal.

Graph Theory Preliminaries: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$ and $\mathcal{E} = \{e_1, e_2, \dots, e_l\} \subseteq \mathcal{V} \times \mathcal{V}$, describes an undirected graph with N nodes and l edges. The corresponding symmetric adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is defined so that $a_{ii} = 0$, and $\forall i \neq j, a_{ij} = 1$ if $(v_i, v_j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. A sequence of unrepeated edges in \mathcal{E} that joins a sequence of nodes in \mathcal{V} defines a path. An undirected graph is connected if there is a path between every pair of nodes. Given an undirected graph \mathcal{G} , a corresponding directed graph can be obtained by defining a direction for each edge of \mathcal{G} . The incidence matrix $\mathcal{Q} = [q_{ev}] \in \mathbb{R}^{l \times N}$ of a directed graph is defined so that the elements in \mathcal{Q} are given by

$$q_{ev} := \begin{cases} 1 & \text{if } v \text{ is the initial vertex of edge } e, \\ -1 & \text{if } v \text{ is the terminal vertex of edge } e, \\ 0 & \text{if } v \text{ does not belong to edge } e. \end{cases}$$

In this paper, the initial and terminal vertices of an edge in a directed graph can both send information to each other via the corresponding controller. For an undirected graph \mathcal{G} , the choice of a corresponding directed graph is not unique. However, the Laplacian matrix \mathcal{L}_N of \mathcal{G} has the following relationship with

the incidence matrix \mathcal{Q} of any directed graph corresponding to \mathcal{G} : $\mathcal{L}_N = \mathcal{Q}^T \mathcal{Q}$.

II. AN INITIAL STABILITY RESULT

In this section, new definitions of nonlinear NI and nonlinear OSNI systems are provided and a new stability result is established for the interconnection of a single nonlinear NI system and a single nonlinear OSNI system. The new stability result is applicable to nonlinear NI systems with free body motion, which are excluded in the previous stability result given in [20].

Consider the following general nonlinear system:

$$\dot{x}(t) = f(x(t), u(t)), \quad (1)$$

$$y(t) = h(x(t)) + Du(t), \quad (2)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, and $y(t) \in \mathbb{R}^m$ is the output, $f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a Lipschitz continuous function, $h: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a class C^1 function and $D \in \mathbb{R}^{m \times m}$ is a symmetric matrix; i.e., $D = D^T$.

Definition 1: The system (1), (2) is said to be a nonlinear negative-imaginary (NI) system if there exists a positive semidefinite storage function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ of class C^1 such that

$$\dot{V}(x(t)) \leq u(t)^T \dot{\tilde{y}}(t), \quad (3)$$

for all $t \geq 0$, where

$$\tilde{y}(t) = h(x(t)). \quad (4)$$

In contrast to Definition 3 in [19], which excludes linear NI systems with poles at the origin, Definition 1 now includes all linear NI systems satisfying the definition given in [18] by allowing the storage function of the system to be positive semidefinite instead of positive definite.

Definition 2: The system (1), (2) is said to be a nonlinear output strictly negative-imaginary (OSNI) system if there exists a positive semidefinite storage function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ of class C^1 and a scalar $\epsilon > 0$ such that

$$\dot{V}(x(t)) \leq u(t)^T \dot{\tilde{y}}(t) - \epsilon \|\dot{\tilde{y}}(t)\|^2, \quad (5)$$

for all $t \geq 0$, where $\tilde{y}(t)$ is as defined in (4). In this case, we also say that system (1), (2) is nonlinear OSNI with degree of output strictness ϵ .

In this paper, nonlinear OSNI systems defined as in Definition 2 are applied as controllers to achieve the robust output feedback consensus for networked heterogeneous nonlinear NI systems defined as in Definition 1. First, we provide a stability result for a single feedback interconnection of nonlinear NI systems.

Consider a multiple-input multiple-output (MIMO) nonlinear NI system H_1 with the following state-space model:

$$H_1: \quad \dot{x}_1(t) = f_1(x_1(t), u_1(t)), \quad (6)$$

$$y_1(t) = h_1(x_1(t)), \quad (7)$$

where $x_1(t) \in \mathbb{R}^n$ is the state, $u_1(t) \in \mathbb{R}^m$ is the input, and $y_1(t) \in \mathbb{R}^m$ is the output, $f_1: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a Lipschitz continuous function and $h_1: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a class C^1 function.

For the system H_1 with the state-space model (6), (7), we suppose the following assumption is satisfied.

Assumption I: When the system H_1 is in steady state; i.e., $u_1(t) \equiv \bar{u}_1$, $x_1(t) \equiv \bar{x}_1$ and $y_1(t) \equiv \bar{y}_1$, we have $\bar{u}_1^T \bar{y}_1 \geq 0$.

For nonlinear NI systems, Assumption I corresponds to the property of linear NI systems stated in Lemma 2 in [1].

Consider a MIMO nonlinear OSNI system H_2 with the following state-space model:

$$H_2: \quad \dot{x}_2(t) = f_2(x_2(t), u_2(t)), \quad (8)$$

$$y_2(t) = h_2(x_2(t)) + D_2 u_2(t), \quad (9)$$

where $x_2(t) \in \mathbb{R}^n$ is the state, $u_2(t) \in \mathbb{R}^m$ is the input, and $y_2(t) \in \mathbb{R}^m$ is the output, $f_2: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a Lipschitz continuous function, $h_2: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a class C^1 function and $D_2 \in \mathbb{R}^{m \times m}$ is a symmetric matrix; i.e., $D_2 = D_2^T$.

For the system H_2 with the state-space model (8), (9), we suppose that the following assumption is satisfied.

Assumption II: When the system H_2 is in steady state; i.e., $u_2(t) \equiv \bar{u}_2$, $x_2(t) \equiv \bar{x}_2$ and $y_2(t) \equiv \bar{y}_2$, we have $\bar{u}_2^T \bar{y}_2 \leq -\gamma \|\bar{u}_2\|^2$ with $\gamma > 0$.

It might be observed that as nonlinear OSNI systems belong to a subclass of nonlinear NI systems, Assumption II seems to have a conflicting relationship with Assumption I. In fact, Assumption II can be satisfied because of the term $D_2 u_2(t)$ in the output equation (9) and it corresponds to the inequality (61) for linear NI systems in [18].

In addition, both of the systems H_1 and H_2 are assumed to satisfy the following assumptions. For the system H_1 with input $u_1(t)$, state $x_1(t)$ and output $y_1(t) = h_1(x_1(t))$ described by the state-space model (6), (7) and the system H_2 with input $u_2(t)$, state $x_2(t)$ and the auxiliary output $\tilde{y}_2(t) = h_2(x_2(t))$ described by the state-space model (8), (9), we suppose for $i = 1$ and 2, the following conditions are satisfied.

Assumption III: Over any time interval $[t_a, t_b]$ where $t_b > t_a$, $h_i(x_i(t))$ remains constant if and only if $x_i(t)$ remains constant; i.e., $\dot{h}_i(x_i(t)) \equiv 0 \iff \dot{x}_i(t) \equiv 0$. Moreover, $h_i(x_i(t)) \equiv 0 \iff x_i(t) \equiv 0$.

Assumption IV: Over any time interval $[t_a, t_b]$ where $t_b > t_a$, $x_i(t)$ remains constant only if $u_i(t)$ remains constant; i.e., $x_i(t) \equiv \bar{x}_i \implies u_i(t) \equiv \bar{u}_i$. Moreover, $x_i(t) \equiv 0 \implies u_i(t) \equiv 0$.

In the case of linear systems, Assumption III corresponds to observability and Assumption IV corresponds to the B matrix in the realisation (A, B, C, D) of the linear system having full column rank.

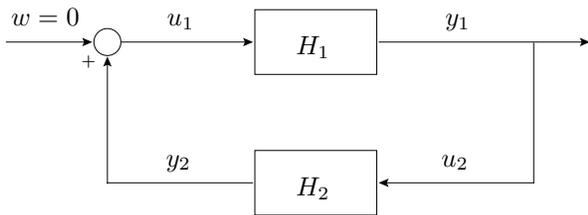


Fig. 1. Closed-loop interconnection of a MIMO nonlinear NI system H_1 and a MIMO nonlinear OSNI system H_2 .

Theorem 1: Consider the closed-loop positive feedback interconnection of the system H_1 with state-space model (6), (7) and H_2 with state-space model (8), (9), as shown in Fig. 1. Suppose that Assumptions I-IV are satisfied, and the storage function, defined as

$$W(x_1, x_2) := V_1(x_1) + V_2(x_2) - h_1(x_1)^T h_2(x_2) - \frac{1}{2} h_1(x_1)^T D_2 h_1(x_1), \quad (10)$$

is positive definite, where $V_1(x_1)$ and $V_2(x_2)$ are positive semidefinite storage functions that satisfy (3) for the system H_1 and (5) for the system H_2 , respectively. Then, the closed-loop interconnection of the systems H_1 and H_2 is asymptotically stable.

Proof: According to the nonlinear NI property (3) for the system H_1 , the nonlinear OSNI property (5) for the system H_2 and the system setting $u_1(t) \equiv y_2(t)$ and $u_2(t) \equiv y_1(t)$ in Fig. 1, we have

$$\begin{aligned} \dot{V}_1(x_1) &\leq u_1^T \dot{y}_1 \\ &= y_2^T \dot{y}_1 \\ &= [h_2(x_2) + D_2 u_2]^T \dot{h}_1(x_1) \\ &= [h_2(x_2) + D_2 y_1]^T \dot{h}_1(x_1) \\ &= [h_2(x_2) + D_2 h_1(x_1)]^T \dot{h}_1(x_1), \end{aligned} \quad (11)$$

and

$$\begin{aligned} \dot{V}_2(x_2) &\leq u_2^T \dot{\tilde{y}}_2 - \epsilon \|\dot{\tilde{y}}_2\|^2 \\ &= y_1^T \dot{\tilde{y}}_2 - \epsilon \|\dot{\tilde{y}}_2\|^2 \\ &= h_1(x_1)^T \dot{h}_2(x_2) - \epsilon \|\dot{h}_2(x_2)\|^2, \end{aligned} \quad (12)$$

where the above equalities also use (7) and (9). We obtain the time derivative of the storage function $W(x_1, x_2)$ in (10) using (11) and (12):

$$\begin{aligned} \dot{W}(x_1, x_2) &= \dot{V}_1(x_1) + \dot{V}_2(x_2) - \dot{h}_1(x_1)^T h_2(x_2) \\ &\quad - h_1(x_1)^T \dot{h}_2(x_2) - h_1(x_1)^T D_2 \dot{h}_1(x_1) \\ &\leq [h_2(x_2) + D_2 h_1(x_1)]^T \dot{h}_1(x_1) \\ &\quad + h_1(x_1)^T \dot{h}_2(x_2) - \epsilon \|\dot{h}_2(x_2)\|^2 \\ &\quad - \dot{h}_1(x_1)^T h_2(x_2) - h_1(x_1)^T \dot{h}_2(x_2) \\ &\quad - h_1(x_1)^T D_2 \dot{h}_1(x_1) \\ &= -\epsilon \|\dot{h}_2(x_2)\|^2 \\ &\leq 0. \end{aligned} \quad (13)$$

From this it follows that $\dot{W}(x_1, x_2) = 0$ is only possible when $\dot{h}_2(x_2) = 0$. Hence, $\dot{W}(x_1, x_2)$ can remain zero only if $\dot{h}_2(x_2)$ remains zero; i.e., $\dot{W}(x_1, x_2) \equiv 0 \implies \dot{h}_2(x_2(t)) \equiv 0$. According to Assumptions III and IV, $\dot{h}_2(x_2(t)) \equiv 0 \implies \dot{x}_2(t) \equiv 0 \implies u_2(t) \equiv \bar{u}_2$. Hence, the system H_2 is in steady-state. According to the system setting in Fig. 1, $y_1(t) \equiv u_2(t)$. Hence, using Assumptions III and IV, $\dot{y}_1(t) \equiv 0 \implies \dot{x}_1(t) \equiv 0 \implies u_1(t) \equiv \bar{u}_1$. Thus, the system H_1 is also in steady-state. Then, according to Assumption II, we have

$$\bar{u}_2^T \bar{y}_2 \leq -\gamma \|\bar{u}_2\|^2.$$

If $\bar{u}_2 = 0$, then $\bar{u}_2^T \bar{y}_2 = 0$. According to Assumptions III and IV, and the system setting in Fig. 1, $\bar{u}_2 = 0 \implies \bar{y}_1 = 0 \implies$

$\bar{x}_1 = 0 \implies \bar{u}_1 = 0 \implies \bar{y}_2 = 0 \implies \bar{x}_2 = 0$. Hence, in this case, the system is in equilibrium. Otherwise, if $\bar{u}_2 \neq 0$, we have

$$\bar{u}_2^T \bar{y}_2 < 0. \quad (14)$$

Also, according to Assumption I, we have

$$\bar{u}_1^T \bar{y}_1 \geq 0. \quad (15)$$

According to the system setting in Fig. 1, we have $\bar{u}_1 = \bar{y}_2$ and $\bar{y}_1 = \bar{u}_2$. Hence, (15) can be rewritten as

$$\bar{u}_2^T \bar{y}_2 \geq 0,$$

which contradicts (14). Thus, we can conclude that $\dot{W}(x_1, x_2)$ cannot remain zero unless $x_1 = x_2 = 0$. Thus, according to LaSalle's invariance principle, $W(x_1, x_2)$ will keep decreasing until $W(x_1, x_2) = 0$. Hence, the equilibrium at $(x_1, x_2) = (0, 0)$ of the closed-loop interconnection is asymptotically stable. ■

III. OUTPUT FEEDBACK CONSENSUS

Consider N heterogeneous nonlinear plants H_{pi} ($i = 1, 2, \dots, N$) described as

$$H_{pi} : \dot{x}_{pi}(t) = f_{pi}(x_{pi}(t), u_{pi}(t)), \quad (16)$$

$$y_{pi}(t) = h_{pi}(x_{pi}(t)), \quad (17)$$

where $x_{pi}(t) \in \mathbb{R}^n$ is the state, $u_{pi}(t) \in \mathbb{R}^m$ is the input, and $y_{pi}(t) \in \mathbb{R}^m$ is the output, $f_{pi} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ are Lipschitz continuous functions and $h_{pi} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are class C^1 functions. These systems operate independently in parallel and each of them has its own input $u_{pi} \in \mathbb{R}^m$ and output $y_{pi} \in \mathbb{R}^m$, ($i = 1, 2, \dots, N$), which is shown in Fig. 2. The subscript “ p ” indicates that this system will play the role of a plant in what follows. We combine the inputs and outputs respectively as the vectors $U_p = [u_{p1}^T, u_{p2}^T, \dots, u_{pN}^T]^T \in \mathbb{R}^{Nm \times 1}$ and $Y_p = [y_{p1}^T, y_{p2}^T, \dots, y_{pN}^T]^T \in \mathbb{R}^{Nm \times 1}$, respectively.

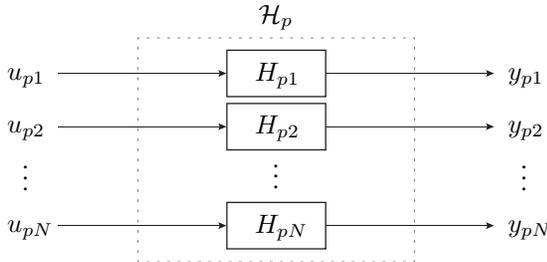


Fig. 2. A plant \mathcal{H}_p consisting of N independent and heterogeneous nonlinear NI systems H_{pi} ($i = 1, 2, \dots, N$) in (16) and (17), with independent inputs and outputs combined as the input and output of the networked system \mathcal{H}_p .

Let us consider the networked plants connected according to the graph network topology $\hat{\mathcal{H}}_p$ as shown in Fig. 3, where \mathcal{Q} is the incidence matrix of a directed graph that represents the communication links between the heterogeneous nonlinear NI plants.

For the system $\hat{\mathcal{H}}_p$ shown in Fig. 3, we have the following lemma:

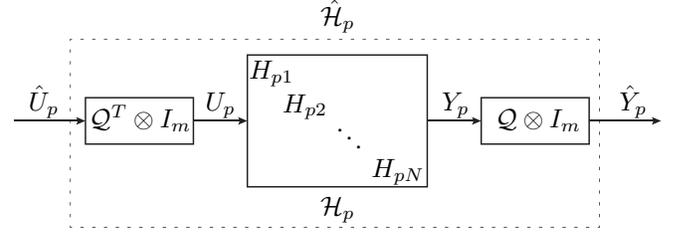


Fig. 3. Heterogeneous nonlinear NI plants connected according to the directed graph network topology.

Lemma 1: If the plants H_{pi} are nonlinear NI systems for all $i = 1, 2, \dots, N$, then the networked plant $\hat{\mathcal{H}}_p$ is also a nonlinear NI system.

Proof: According to Definition 1, each nonlinear NI system H_{pi} ($i = 1, 2, \dots, N$) must have a corresponding positive semidefinite storage function $V_{pi}(x_{pi})$ such that $\dot{V}_{pi}(x_{pi}) \leq u_{pi}^T \dot{y}_{pi}$, where x_{pi} is the state of the system H_{pi} . We define the storage function for the system $\hat{\mathcal{H}}_p$ as $\hat{V}_p = \sum_{i=1}^N V_{pi}(x_{pi})$, which is positive semidefinite. Then

$$\dot{\hat{V}}_p = \sum_{i=1}^N \dot{V}_{pi}(x_{pi}) \leq \sum_{i=1}^N u_{pi}^T \dot{y}_{pi} = U_p^T \dot{Y}_p. \quad (18)$$

Let \hat{U}_p and \hat{Y}_p denote the input and output of the system $\hat{\mathcal{H}}_p$, respectively. According to the system setting in Fig. 3, we have

$$U_p = (Q^T \otimes I_m) \hat{U}_p, \quad \text{and} \quad \hat{Y}_p = (Q \otimes I_m) Y_p.$$

Therefore, we have

$$U_p^T Y_p = [(Q^T \otimes I_m) \hat{U}_p]^T Y_p = \hat{U}_p^T (Q \otimes I_m) Y_p = \hat{U}_p^T \hat{Y}_p. \quad (19)$$

According to (18) and (19), we obtain the nonlinear NI inequality for the system $\hat{\mathcal{H}}_p$:

$$\dot{\hat{V}}_p \leq \hat{U}_p^T \hat{Y}_p. \quad (20)$$

Therefore, $\hat{\mathcal{H}}_p$ is a nonlinear NI system. ■

Now we give a definition of output feedback consensus for a network of systems as shown in Fig. 2.

Definition 3: A distributed output feedback control law achieves output feedback consensus for a network of systems if $|y_{pi}(t) - y_{pj}(t)| \rightarrow 0$ as $t \rightarrow +\infty, \forall i, j \in \{1, 2, \dots, N\}$.

Consider a series of heterogeneous nonlinear OSNI controllers H_{ck} ($k = 1, 2, \dots, l$) applied at the edges in the network. The OSNI controllers have the following state-space models:

$$H_{ck} : \dot{x}_{ck}(t) = f_{ck}(x_{ck}(t), u_{ck}(t)), \quad (21)$$

$$y_{ck}(t) = h_{ck}(x_{ck}(t)) + D_{ck} u_{ck}(t), \quad (22)$$

where $x_{ck}(t) \in \mathbb{R}^q$ is the state, $u_{ck}(t) \in \mathbb{R}^m$ is the input, and $y_{ck}(t) \in \mathbb{R}^m$ is the output, $f_{ck} : \mathbb{R}^q \times \mathbb{R}^m \rightarrow \mathbb{R}^q$ are Lipschitz continuous functions, $h_{ck} : \mathbb{R}^q \rightarrow \mathbb{R}^m$ are class C^1 functions and $D_{ck} \in \mathbb{R}^{m \times m}$ are symmetric matrices. These systems operate independently in parallel and each of them has its own input $u_{ck} \in \mathbb{R}$ and output $y_{ck} \in \mathbb{R}$, $k = 1, 2, \dots, l$, which is shown in Fig. 4. The subscript “ c ”

indicates that this system will play the role of a controller in what follows. We combine the inputs and outputs respectively as the vectors $U_c = [u_{c1}^T, u_{c2}^T, \dots, u_{cl}^T]^T \in \mathbb{R}^{lm \times 1}$ and $Y_c = [y_{c1}^T, y_{c2}^T, \dots, y_{cl}^T]^T = \Pi_c + D_c U_c \in \mathbb{R}^{lm \times 1}$, where

$$\Pi_c = \begin{bmatrix} h_{c1}(x_{c1}) \\ h_{c2}(x_{c2}) \\ \vdots \\ h_{cl}(x_{cl}) \end{bmatrix} \in \mathbb{R}^{lm \times 1}, \quad (23)$$

and

$$D_c = \text{diag}\{D_{c1}, D_{c2}, \dots, D_{cl}\} \in \mathbb{R}^{lm \times lm}. \quad (24)$$

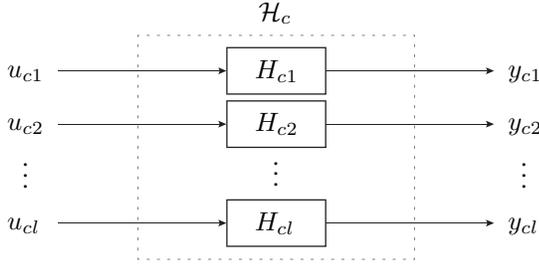


Fig. 4. A controller \mathcal{H}_c consisting of l independent and heterogeneous nonlinear OSNI systems H_{ck} ($k = 1, 2, \dots, l$) in (21) and (22), with independent inputs and outputs combined as the input and output of the networked system \mathcal{H}_c .

Lemma 2: If the controllers H_{ck} are nonlinear OSNI systems for all $k = 1, 2, \dots, l$, then the networked controller \mathcal{H}_c is also a nonlinear OSNI system.

Proof: For every nonlinear OSNI system H_{ck} , we have a positive semidefinite storage function $V_{ck}(x_{ck})$ and a constant $\epsilon_k > 0$ such that

$$\dot{V}_{ck}(x_{ck}) \leq u_{ck}^T \dot{y}_{ck} - \epsilon_k \|\dot{y}_{ck}\|^2, \quad (25)$$

where $\tilde{y}_{ck} = h_{ck}(x_{ck})$ and ϵ_k is the level of output strictness of the system H_{ck} . For the system \mathcal{H}_c , we define its storage function V_c as the sum of the storage functions of all the networked controllers; i.e., $V_c := \sum_{k=1}^l V_{ck}(x_{ck})$, which is positive semidefinite. The time derivative of V_c is:

$$\begin{aligned} \dot{V}_c &= \sum_{k=1}^l \dot{V}_{ck}(x_{ck}) \\ &\leq \sum_{k=1}^l u_{ck}^T \dot{y}_{ck} - \sum_{k=1}^l \epsilon_k \|\dot{y}_{ck}\|^2 \\ &\leq \sum_{k=1}^l u_{ck}^T \dot{y}_{ck} - \epsilon_{\min} \sum_{k=1}^l \|\dot{y}_{ck}\|^2 \\ &= U_c^T \dot{\Pi}_c - \epsilon_{\min} \|\dot{\Pi}_c\|^2, \end{aligned} \quad (26)$$

where $\epsilon_{\min} = \min\{\epsilon_1, \epsilon_2, \dots, \epsilon_l\}$. Hence, the system \mathcal{H}_c satisfies the definition of a nonlinear OSNI system and ϵ_{\min} quantifies a level of output strictness of the system. This completes the proof. ■

Now consider the closed-loop positive feedback interconnection of the networked plants shown in Fig. 3 and the networked controllers shown in Fig. 4, which is depicted in

Fig. 5. In this paper, robust output consensus of heterogeneous nonlinear NI plants is achieved by constructing a control system with the block diagram shown in Fig. 5 and choosing suitable controllers that satisfy certain conditions. Detailed description of the control framework in Fig. 5 can be found in [37], which uses a similar control framework.

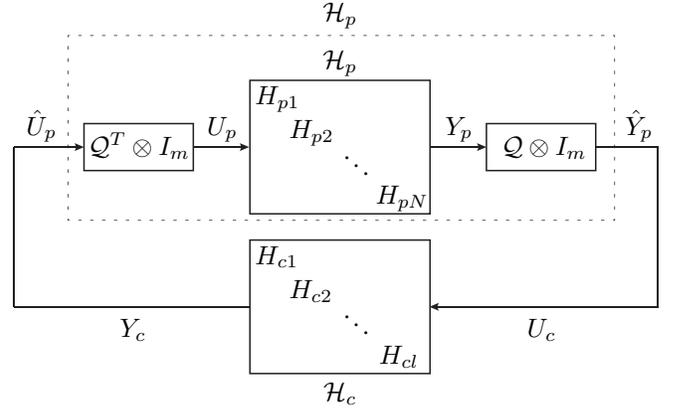


Fig. 5. Positive feedback interconnection of heterogeneous nonlinear NI plants and nonlinear OSNI controllers according to the directed graph network topology.

As is shown in Fig. 5, the control input for the networked plant \mathcal{H}_p is

$$U_p = (Q^T \otimes I_m) \mathcal{H}_c((Q \otimes I_m) Y_p), \quad (27)$$

where Q is the incidence matrix of the directed graph that represents the communication links between the heterogeneous nonlinear NI plants. Equivalently, the distributed control protocol for the plant H_i ($i = 1, 2, \dots, N$) is defined by the equations

$$\dot{x}_{ck}(t) = f_{ck} \left(x_{ck}(t), \sum_{j=1}^N q_{kj} y_{pj} \right), \quad (28)$$

$$y_{ck}(t) = h_{ck}(x_{ck}(t)) + D_{ck} \sum_{j=1}^N q_{kj} y_{pj}, \quad (29)$$

$$u_{pi} = \sum_{k=1}^l q_{ki} y_{ck}, \quad (30)$$

where q_{kj} and q_{ki} are the j -th and i -th elements in the k -th row of the incidence matrix Q , respectively. Here, $\sum_{j=1}^N q_{kj} y_{pj}$ represents the difference between the outputs of the two plants connected by the edge e_k .

Before we present the main result, let us make another assumption. For the system $\hat{\mathcal{H}}_p$ with input $\hat{U}_p(t)$ and output $\hat{Y}_p(t)$, we suppose the following assumption is satisfied.

Assumption V: Given a constant input $\hat{U}_p(t) \equiv \hat{U}_p$ to the system $\hat{\mathcal{H}}_p$, if its output is also constant; i.e., $\hat{Y}_p(t) \equiv \hat{Y}_p$, then \hat{U}_p and \hat{Y}_p satisfy $\hat{U}_p^T \hat{Y}_p \geq 0$.

In fact, Assumption I implies Assumption V in the case that all the plants H_{pi} are in steady state. This is because when all the plants satisfy Assumption I and are in steady state, we have $\hat{U}_p^T \hat{Y}_p \geq 0$ and according to the system

setting in Fig. 5 we have $\bar{U}_p^T \bar{Y}_p = [(\mathcal{Q}^T \otimes I_m)^T \bar{U}_p]^T \bar{Y}_p = \bar{U}_p^T (\mathcal{Q} \otimes I_m) \bar{Y}_p = \bar{U}_p^T \bar{Y}_p$ similarly to (19). Hence $\bar{U}_p^T \bar{Y}_p \geq 0$. However, Assumption V is assumed for the networked plants $\hat{\mathcal{H}}_p$ in the following theorem instead of assuming Assumption I for each individual plant because Assumption V also allows for the situation in which the input and output of the system $\hat{\mathcal{H}}_p$ are constant, but the individual plants H_{pi} are not all in steady state. This situation is possible because the matrix $\mathcal{Q} \otimes I_m$ takes the difference between the outputs of the plants. Under constant inputs, if the plants oscillate with a constant difference between their outputs, then this situation is allowed under Assumption V.

Theorem 2: Consider an undirected connected graph \mathcal{G} that models the communication links for a network of heterogeneous nonlinear NI systems H_{pi} ($i = 1, 2, \dots, N$) as shown in Fig. 2, and any directed graph corresponding to \mathcal{G} with the incidence matrix \mathcal{Q} . Also, consider the heterogeneous nonlinear OSNI control laws H_{ck} ($k = 1, 2, \dots, l$) for all of the edges. Suppose Assumptions III and IV are satisfied for the plants H_{pi} , Assumptions II, III and IV are satisfied for the controllers H_{ck} (with $\gamma = \gamma_k$ for the controller H_{ck} in Assumption II) and Assumption V is satisfied by the system $\hat{\mathcal{H}}_p$. Also, suppose the storage function, defined as

$$\hat{W} := \hat{V}_p + V_c - \hat{Y}_p^T \Pi_c - \frac{1}{2} \hat{Y}_p^T D_c \hat{Y}_p,$$

is positive definite, where \hat{V}_p and V_c are positive semidefinite storage functions that satisfy (20) for the system $\hat{\mathcal{H}}_p$ and (26) for the system $\hat{\mathcal{H}}_c$, respectively. Here, \hat{Y}_p is the output of the system $\hat{\mathcal{H}}_p$. Π_c and D_c are terms in the output Y_c of the system \mathcal{H} and are defined in (23) and (24). Then robust output feedback consensus can be achieved via the protocol (27), or equivalently (28)-(30) in a distributed manner for each plant p_i , as shown in Fig. 5.

Proof: According to (20), (26) and the system setting $\hat{U}_p \equiv Y_c$ and $U_c \equiv \hat{Y}_p$ shown in Fig. 5, we have

$$\dot{\hat{V}}_p \leq \hat{U}_p^T \hat{Y}_p = Y_c^T \hat{Y}_p = \hat{Y}_p^T [\Pi_c + D_c U_c] = \hat{Y}_p^T [\Pi_c + D_c \hat{Y}_p], \quad (31)$$

and

$$\dot{V}_c \leq U_c^T \dot{\Pi}_c - \epsilon_{min} \|\dot{\Pi}_c\|^2 = \hat{Y}_p^T \dot{\Pi}_c - \epsilon_{min} \|\dot{\Pi}_c\|^2. \quad (32)$$

According to (31), (32) and the symmetry of D_c in (24), the time derivative of the storage function \hat{W} satisfies the following inequality:

$$\begin{aligned} \dot{\hat{W}} &= \dot{\hat{V}}_p + \dot{V}_c - \dot{\hat{Y}}_p^T \Pi_c - \hat{Y}_p^T \dot{\Pi}_c - \frac{1}{2} \dot{\hat{Y}}_p^T (D_c + D_c^T) \hat{Y}_p \\ &\leq \hat{Y}_p^T [\Pi_c + D_c \hat{Y}_p] + \hat{Y}_p^T \dot{\Pi}_c - \epsilon_{min} \|\dot{\Pi}_c\|^2 - \hat{Y}_p^T \Pi_c \\ &\quad - \hat{Y}_p^T \dot{\Pi}_c - \hat{Y}_p^T D_c \hat{Y}_p \\ &\leq \epsilon_{min} \|\dot{\Pi}_c\|^2 \\ &\leq 0. \end{aligned} \quad (33)$$

Hence, the closed-loop system is at least Lyapunov stable. Moreover, $\dot{\hat{W}} = 0$ can hold only if $\dot{\Pi}_c = 0$. In other words, $\dot{\hat{W}}$ can remain zero only if $\dot{h}_{ck}(x_{ck}(t))$ remains zero for all $k = 1, 2, \dots, l$. According to Assumptions III and IV,

$\dot{h}_{ck}(x_{ck}(t)) \equiv 0 \implies \dot{x}_{ck}(t) \equiv 0 \implies u_{ck}(t) \equiv \bar{u}_{ck}$. Hence, H_{ck} is in steady-state for all $k = 1, 2, \dots, l$. We have $U_c(t) \equiv \bar{U}_c$ and $Y_c(t) \equiv \bar{Y}_c$. According to the system setting in Fig. 5 that $\hat{U}_p(t) \equiv Y_c(t)$ and $U_c(t) \equiv \hat{Y}_p(t)$, we also have $\hat{U}_p(t) \equiv \bar{U}_p$ and $\hat{Y}_p(t) \equiv \bar{Y}_p$. According to Assumption V, we have

$$\bar{U}_p^T \bar{Y}_p \geq 0. \quad (34)$$

According to Assumption II, we have

$$\bar{U}_c^T \bar{Y}_c = \sum_{k=1}^l \bar{u}_{ck}^T \bar{y}_{ck} \leq - \sum_{k=1}^l \gamma_k \|\bar{u}_{ck}\|^2 \leq -\gamma_{min} \|\bar{U}_c\|^2,$$

where $\gamma_{min} = \min\{\gamma_1, \gamma_2, \dots, \gamma_l\}$. In the case that $\bar{U}_c \neq 0$, we have

$$\bar{U}_c^T \bar{Y}_c < 0 \quad (35)$$

which contradicts (34) because $\bar{U}_c^T \bar{Y}_c = \bar{U}_p^T \bar{Y}_p$. In the case that $\bar{U}_c = 0$, all connected plants have the difference between their system outputs being zero. Hence, output consensus has already been achieved. Otherwise, $\dot{\hat{W}}$ cannot remain zero. According to LaSalle's invariance principle, \hat{W} will keep decreasing until either $\bar{U}_c = 0$ or $\hat{W} = 0$. Thus, output consensus is achieved in both cases. This completes the proof. ■

IV. ILLUSTRATIVE EXAMPLE

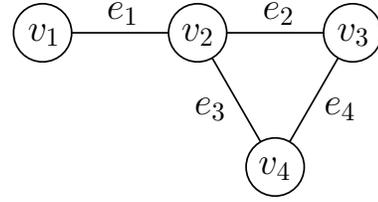


Fig. 6. An undirected and connected graph consisting of four nodes.

Consider four nonlinear NI plants H_{pi} at the vertices v_i of the graph in Fig. 6. We choose directions of the edges as $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, $e_3 = (v_2, v_4)$ and $e_4 = (v_3, v_4)$. Then the incidence matrix of the directed graph corresponding to \mathcal{G} is

$$\mathcal{Q} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

The plants are nonlinear single integrators which have the following state-space models:

$$\begin{aligned} H_{pi} : \quad \dot{x}_{pi}(t) &= \mu_i u_{pi}^3(t), \\ y_{pi}(t) &= x_{pi}(t), \quad i = 1, 2, \dots, 4, \end{aligned}$$

where $\mu_1 = 10$, $\mu_2 = 30$, $\mu_3 = 25$ and $\mu_4 = 5$ are constant coefficients. The storage functions for these four plants are all $V_{pi}(x_{pi}) = 0$. The states x_{p1} , x_{p2} , x_{p3} and x_{p4} of these four plants have initial values 30, 15, -5 and -10 , respectively. We aim to synchronise the outputs of these four plants to the

same limit trajectory by using nonlinear OSNI controllers H_{ck} at the edges e_k of the graph in Fig.6.

$$H_{ck} : \begin{aligned} \dot{x}_{ck}(t) &= -\alpha_k x_{ck}(t) - \beta_k x_{ck}^3(t) + u_{ck}(t), \\ y_{ck}(t) &= x_{ck}(t) - u_{ck}(t), \quad k = 1, 2, \dots, 4, \end{aligned}$$

where α_k and β_k are constant coefficients. $\alpha_1 = 5$, $\alpha_2 = 8$, $\alpha_3 = 7$, $\alpha_4 = 3$; $\beta_1 = 3$, $\beta_2 = 2$, $\beta_3 = 5$ and $\beta_4 = 2$, respectively.

The storage functions of Controllers 1, 2, 3 and 4 are $V_{c1} = \frac{5}{2}x_{c1}^2 + \frac{3}{4}x_{c1}^4$, $V_{c2} = 4x_{c2}^2 + \frac{1}{2}x_{c2}^4$, $V_{c3} = \frac{7}{2}x_{c3}^2 + \frac{5}{4}x_{c3}^4$ and $V_{c4} = \frac{3}{2}x_{c4}^2 + \frac{1}{2}x_{c4}^4$, respectively.

The networked plant system $\hat{\mathcal{H}}_p$ in this example is given by the equations:

$$\begin{aligned} \dot{\hat{x}}_1 &= 10\hat{u}_1^3 + 30(\hat{u}_1 - \hat{u}_2 - \hat{u}_3)^3, \\ \dot{\hat{x}}_2 &= 30(-\hat{u}_1 + \hat{u}_2 + \hat{u}_3)^3 + 25(\hat{u}_2 - \hat{u}_4)^3, \\ \dot{\hat{x}}_3 &= 30(-\hat{u}_1 + \hat{u}_2 + \hat{u}_3)^3 + 5(\hat{u}_3 + \hat{u}_4)^3, \\ \dot{\hat{x}}_4 &= 25(-\hat{u}_2 + \hat{u}_4)^3 + 5(\hat{u}_3 + \hat{u}_4)^3, \\ \hat{y}_1 &= \hat{x}_1, \quad \hat{y}_2 = \hat{x}_2, \quad \hat{y}_3 = \hat{x}_3, \quad \hat{y}_4 = \hat{x}_4. \end{aligned}$$

The storage function of the closed-loop system provided by Theorem 2 takes the form

$$\begin{aligned} \hat{W} &= \frac{5}{2}x_{c1}^2 + \frac{3}{4}x_{c1}^4 + 4x_{c2}^2 + \frac{1}{2}x_{c2}^4 + \frac{7}{2}x_{c3}^2 + \frac{5}{4}x_{c3}^4 + \frac{3}{2}x_{c4}^2 \\ &\quad + \frac{1}{2}x_{c4}^4 - \hat{x}_1 x_{c1} - \hat{x}_2 x_{c2} - \hat{x}_3 x_{c3} - \hat{x}_4 x_{c4} + \frac{1}{2}\hat{x}_1^2 \\ &\quad + \frac{1}{2}\hat{x}_2^2 + \frac{1}{2}\hat{x}_3^2 + \frac{1}{2}\hat{x}_4^2 \\ &= [\hat{x}_1 \quad x_{c1}] \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ x_{c1} \end{bmatrix} + \frac{3}{4}x_{c1}^4 \\ &\quad + [\hat{x}_2 \quad x_{c2}] \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 4 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ x_{c2} \end{bmatrix} + \frac{1}{2}x_{c2}^4 \\ &\quad + [\hat{x}_3 \quad x_{c3}] \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{7}{2} \end{bmatrix} \begin{bmatrix} \hat{x}_3 \\ x_{c3} \end{bmatrix} + \frac{5}{4}x_{c3}^4 \\ &\quad + [\hat{x}_4 \quad x_{c4}] \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \hat{x}_4 \\ x_{c4} \end{bmatrix} + \frac{1}{2}x_{c4}^4, \end{aligned}$$

which is positive definite. Assumptions II-V in Sections II and III are satisfied. Output feedback consensus is achieved, as shown in Fig. 7. Because of the cubic nonlinearity in the plants and the controllers, their outputs have different rates of convergence in the domains where the states are far and close to the limit values. Therefore, a log scale is used for the time axis in the plot shown in Fig. 7.

V. CONCLUSION

This paper provides a control framework to achieve robust output feedback consensus for networked heterogeneous nonlinear NI systems including systems with free body motion, using nonlinear OSNI controllers. New definitions for nonlinear NI systems and nonlinear OSNI systems are given and a stability result is established for the simple feedback interconnection of a nonlinear NI plant and a nonlinear OSNI controller. A networked control framework is then considered by modelling the communication topology between systems as a connected graph, where the plants are nodes and the

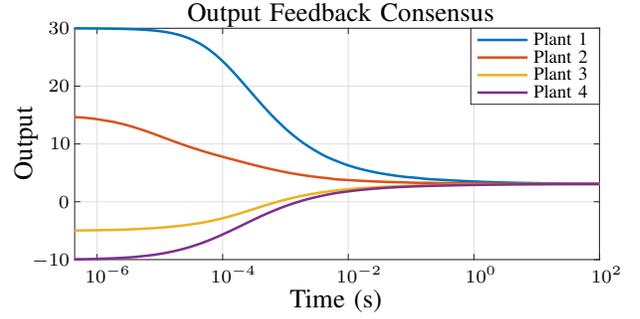


Fig. 7. Output feedback consensus for four nonlinear single integrator plants with different system models. Starting from different initial conditions, the outputs of the plants converge to the same limit trajectory under the effect of the proposed control framework of Section III.

controllers are edges. The network of nonlinear NI plants and the network of OSNI controllers are proved to be a nonlinear NI system and a nonlinear OSNI system, respectively. Under reasonable assumptions, output feedback consensus is established for the networked heterogeneous nonlinear NI systems, and the result is robust against variations in the system models of both the plants and controllers provided that the relevant nonlinear NI and nonlinear OSNI properties are preserved. Finally, an example is given to demonstrate the proposed result on a consensus problem to which earlier results are not applicable.

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