Non-Euclidean Contraction Theory for Monotone and Positive Systems

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Abstract—In this note we study contractivity of monotone systems and exponential convergence of positive systems using non-Euclidean norms. We first introduce the notion of conic matrix measure as a framework to study stability of monotone and positive systems. We study properties of the conic matrix measures and investigate their connection with weak pairings and standard matrix measures. Using conic matrix measures and weak pairings, we characterize contractivity and incremental stability of monotone systems with respect to non-Euclidean norms. Moreover, we use conic matrix measures to provide sufficient conditions for exponential convergence of positive systems to their equilibria. We show that our framework leads to novel results on (i) the contractivity of excitatory Hopfield neural networks, and (ii) the stability of interconnected systems using non-monotone positive comparison systems.

Index Terms—contraction theory, monotone systems, positive systems, stability theory, interconnected systems

I. INTRODUCTION

Problem description and motivation: A dynamical system is monotone if its trajectories preserve a partial order of their initial conditions and is positive if the non-negative orthant is a forward invariant set. Monotonicity appears naturally in real world applications including biological systems [28], transportation and flow networks [4], and epidemic networks [17], as well as in smallgain analysis of large-scale interconnected systems [6], [26]. Positive systems are also abundant in engineering and science, for instance, in population dynamics [13] and queuing systems [9]. While the notions of monotonicity and positivity are identical for linear systems, they are distinct and lead to different transient and asymptotic behaviors for nonlinear systems. Linear and nonlinear monotone systems have been studied extensively in dynamical systems [27] and control theory [25], [28]. Monotonicity of dynamical systems with respect to arbitrary cones are studied in [12] and a theory of monotone systems on partially ordered Banach spaces has been developed in [22].

Contraction theory is a classic framework [2], [3], [8], [18] aimed at establishing rigorous nonlinear stability properties of dynamical systems. A dynamical system is contracting if every two trajectories converge exponentially to one another. Contracting systems exhibit many desirable asymptotic properties: (i) their asymptotic behavior is independent of their initial condition, (ii) when the vector field is time-invariant every trajectory converges to a unique equilibrium point, and (iii) when the vector field is periodic, every trajectory converges to a unique periodic orbit. Contracting systems enjoy also desirable transient behavior and robustness properties including input-to-state stability in the presence of bounded unmodeled dynamics.

While classical approaches mostly focus on contraction with respect to the ℓ_2 -norm, recent works have shown that stability of monotone and positive system can be studied more systematically and efficiently using non-Euclidean norms. It is known that for a monotone system satisfying a conservation law (resp. translational

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symmetry), contractivity naturally arises with respect to ℓ_1 -norms (resp. ℓ_∞ -norms). Contraction of monotone systems with respect to state-dependent non-Euclidean norms has been studied in [5]. Contraction of monotone systems with respect to ℓ_1 -norm has been studied for flow networks in [4], for traffic networks in [4], and for gene translation systems in [20]. Another relevant topic for monotone systems is the search for sum-separable and max-separable Lyapunov functions [10]. Recent works have used contraction with respect to non-Euclidean norms for monotone systems to find separable Lyapunov functions [16], [19]. Despite all these works, a differential and integral characterization of monotone and positive contracting systems with respect to non-Euclidean norms is missing.

Contribution: In this note, we build on the framework proposed in [7] and introduce the notion of conic matrix measure, characterize its properties, and propose efficient methods for computing it. We provide a complete characterization of contractive monotone systems using the one-sided Lipschitz constant of their vector fields and the conic matrix measure of their Jacobians. We also propose a sufficient condition, based on the conic matrix measures, for exponential convergence of positive systems to equilibrium points. As a first application of our monotone contraction framework, we provide a sufficient condition for contractivity of excitatory Hopfield neural networks. We remark that strong contractivity of Hopfield neural networks automatically leads to their global stability for time-invariant inputs, their entrainment to a unique periodic orbit for periodic inputs. and their input-to-state stability for general time-varying inputs. As a second application, we establish a novel framework for studying input-to-state stability of interconnected systems. Our framework is based on comparison with positive dynamical systems and can accommodate both inhibitory and excitatory interconnections between subsystems. By allowing the comparison system to be positive instead of monotone, our framework generalizes the well-known Matrosov-Bellman comparison lemma and unifies several existing small-gain theorems and comparison lemmas in the literature.

II. NOTATION

Functions, norms and matrix measures: Let $f: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function. If f is differentiable, then we denote its derivative by f'. If f is continuous, we denote its upper Dini derivative by D^+f . We say f is of class K if it is strictly increasing and f(0) = 0. We say f is of class \mathcal{K}_{∞} if it belongs to class \mathcal{K} and $\lim_{x\to+\infty} f(x) = \infty$. We say a continuous function $g:[0,a)\times[0,\infty)\to[0,\infty)$ is of class \mathcal{KL} if, for each fixed y, the map $x \mapsto g(x,y)$ is of class \mathcal{K} and, for each fixed x, the map $y \mapsto g(x,y)$ is decreasing such that $\lim_{y\to+\infty} g(x,y)=0$. For vectors $v,w\in\mathbb{R}^n$, the Hadamard product of v and w is the vector $v \circ w \in \mathbb{R}^n$ define by $(v \circ w)_i = v_i w_i$, for every $i \in \{1, ..., n\}$. A matrix $A \in \mathbb{R}^{n \times n}$ is non-negative if $A_{ij} \geq 0$, for every $(i,j) \in \{1,\ldots,n\}$, For every matrix $A \in$ $\mathbb{R}^{n \times n}$, the positive part of A is the matrix $[A]^+ \in \mathbb{R}^{n \times n}_{>0}$ defined by $[A]_{ij}^+ = A_{ij}$ if $A_{ij} \ge 0$ and $[A]_{ij}^+ = 0$ if $A_{ij} < 0$. Given $x, y \in \mathbb{R}^n$, $x \le y$ if we have $x_i \le y_i$, for every $i \in \{1, \ldots, n\}$ and we define $[x,y]\subset\mathbb{R}^n$ as the set of all $z\in\mathbb{R}^n$ such that $x\leq z\leq y$. For a vector $\eta\in\mathbb{R}^n$, the diagonal matrix $[\eta]\in\mathbb{R}^{n\times n}$ is defined by $[\eta]_{ii} = \eta_i$, for every $i \in \{1, \dots, n\}$. Given $A, B \in \mathbb{R}^{n \times n}$, $A \leq B$

if B-A is a positive semi-definite matrix. A norm $\|\cdot\|$ on \mathbb{R}^n is monotonic, if for every $x, y \in \mathbb{R}^n$ such that $|x| \leq |y|$, we have ||x|| < ||y||. For $p \in [1, \infty]$ and $R \in \mathbb{R}^{n \times n}$: if R is positive and diagonal, the R-weighted ℓ_p -norm is a monotonic norm. Let $\|\cdot\|$ be a norm on \mathbb{R}^n , the induced matrix norm on $\mathbb{R}^{n \times n}$ is again denoted by $\|\cdot\|$. Given a matrix $A \in \mathbb{R}^{n \times n}$, the matrix measure of A with respect to $\|\cdot\|$ is defined by $\mu(A) := \lim_{h \to 0^+} \frac{\|I_n + hA\| - 1}{h}$

Weak pairings: We briefly the notion of a weak pairing (WP) on \mathbb{R}^n from [7]. A WP on \mathbb{R}^n is a map $[\cdot,\cdot]$: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying:

- (i) (Sub-additivity and continuity of first argument) $[x_1 + x_2, y] \le$ $[\![x_1,y]\!]+[\![x_2,y]\!]$, for all $x_1,x_2,y\in\mathbb{R}^n$ and $[\![\cdot,\cdot]\!]$ is continuous in its first argument,
- (ii) (Weak homogeneity) $[\![\alpha x,y]\!] = [\![x,\alpha y]\!] = \alpha [\![x,y]\!]$ and $[\![-x, -y]\!] = [\![x, y]\!], \text{ for all } x, y \in \mathbb{R}^n, \alpha \ge 0,$
- (iii) (Positive definiteness) [x, x] > 0, for all $x \neq 0_n$,

(iv) (Cauchy-Schwarz inequality)
$$\| \left[\left[x,y \right] \right] \leq \left[\left[x,x \right] \right]^{1/2} \left[\left[y,y \right] \right]^{1/2}, \text{ for all } x,y \in \mathbb{R}^n.$$

For every norm $\|\cdot\|$ on \mathbb{R}^n , there exists a (possibly not unique) associated WP $[\cdot,\cdot]$ such that $||x||^2 = [x,x]$, for every $x \in$ \mathbb{R}^n . A WP $[\cdot, \cdot]$ satisfies Deimling's inequality if $[x, y] \le \|y\| \lim_{h \to 0^+} \frac{\|y + hx\| - \|y\|}{h}$, for every $x, y \in \mathbb{R}^n$ and satisfies the curve norm derivative formula if, for every differentiable $x: (a, b) \to 0$ \mathbb{R}^n and for almost every $t \in (a,b)$ we have $||x(t)||D^+||x(t)|| =$ $[\![\dot{x}(t),x(t)]\!]$. For every $p\in(1,\infty)$ and invertible $R\in\mathbb{R}^{n\times n}$, we define $[\![\cdot,\cdot]\!]_{p.R}$ by

$$[\![x,y]\!]_{p,R} = |\![y]\!]_{p,R}^{2-p} (Ry \circ |Ry|^{p-2})^{\mathsf{T}} Rx. \tag{1}$$

For invertible $R \in \mathbb{R}^{n \times n}$, we define $[\cdot, \cdot]_{1,R}$ and $[\cdot, \cdot]_{\infty,R}$ by

$$[x, y]_{1,R} = ||Ry||_1 \operatorname{sign}(Ry)^{\mathsf{T}} Rx,$$
 (2)

$$[x, y]_{1,R} = ||Ry||_{1} \operatorname{sign}(Ry)^{\mathsf{T}} Rx,$$

$$[x, y]_{\infty,R} = \max_{i \in I_{\infty}(Ry)} (Ry)_{i} (Rx)_{i},$$
(3)

where $I_{\infty}(x) = \{i \in \{1, ..., n\} \mid x_i = \max_i \{|x|_i\}\}$. It can be shown that, for every $p \in [1, \infty]$ and invertible matrix $R \in \mathbb{R}^{n \times n}$, we have $\|x\|_{p,R}^2 = [\![x,x]\!]_{p,R}$ and $[\![\cdot,\cdot]\!]_{p,R}$ satisfies Deimling's inequality and the curve norm derivative formula. We refer to [7] for a detailed discussion on WPs.

Dynamical systems: Consider the dynamical system $\dot{x} = f(t, x)$ on \mathbb{R}^n . Let $\phi(t, t_0, x)$ denote the flow of f at time t starting at time t_0 from x_0 . The vector field f is *positive* if $\mathbb{R}^n_{>0}$ is a forward invariant set. Let C be a convex forward invariant set for vector field f. The vector field f is monotone on C, if for every $x_0, y_0 \in C$ such that $x_0 \leq y_0$, we have $\phi(t, t_0, x_0) \leq \phi(t, t_0, y_0)$, for every $t \geq t_0$. The Jacobian of f is denoted by Df(t,x). Let $\|\cdot\|$ be a norm with associated WP $[\cdot, \cdot]$. The vector field f is contracting with rate c > 0if, for $x, y \in \mathbb{R}^n$ and every $t_0 \le t \in \mathbb{R}_{\ge 0}$, we have

$$\|\phi(t, t_0, x) - \phi(t, t_0, y)\| \le e^{-c(t - t_0)} \|x - y\|.$$

and it is incrementally exponentially stable with rate c > 0 if, there exists M > 0 such that, for $x, y \in \mathbb{R}^n$ and every $t_0 \le t \in \mathbb{R}_{>0}$

$$\|\phi(t,t_0,x) - \phi(t,t_0,y)\| \le Me^{-c(t-t_0)} \|x-y\|.$$

III. CONIC MATRIX MEASURES

In classical contraction theory, the incremental stability of dynamical systems are ensured by imposing suitable conditions on matrix measures of their Jacobian. Monotonicity or positivity of dynamical systems induces a natural partial order structure on their flows, something which can be used to relax the matrix measure conditions for incremental stability. In this section, we introduce the notion of conic matrix measures and show how it can be used to

prove incremental stability of monotone and exponential convergence of positive systems.

Definition 3.1 (Conic matrix measure): Let $\|\cdot\|$ be a norm. The conic matrix measure of $A \in \mathbb{R}^{n \times n}$, denoted by $\mu^+(A)$, is

$$\mu^{+}(A) := \lim_{h \to 0^{+}} \sup_{x \ge \mathbb{Q}_{n}, x \ne \mathbb{Q}_{n}} \frac{\|(I_{n} + hA)x\|/\|x\| - 1}{h}.$$
we study properties of the conic matrix measure. We first

Now we study properties of the conic matrix measure. We first state the following useful lemma.

Lemma 3.2 (Monotonicity of $[\![\cdot,\cdot]\!]_{p,R}$): Let $p\in[1,\infty]$ and $R\in$ $\mathbb{R}^{n \times n}$ be an invertible non-negative matrix. Then

- $\begin{array}{l} \text{(i) for every } x,y \geq \mathbb{O}_n \text{, we have } \llbracket -x,y \rrbracket_{p,R} \leq 0. \\ \text{(ii) for every } x \leq z \text{ and } y \geq \mathbb{O}_n \text{, we have } \llbracket x,y \rrbracket_{p,R} \leq \llbracket z,y \rrbracket_{p,R}. \end{array}$

Proof: Regarding part (i), since R is non-negative, then we have $Ry \geq \mathbb{O}_n$ and $Rx \geq \mathbb{O}_n$. The result then follows by using formulas (1), (2), and (3). Regarding part (ii), note that since $x \le z$, there exists $w \geq 0_n$ such that x = z - w. Thus, using the subadditivity of the WP, we get

$$[\![x,y]\!]_{p,R}=[\![z-w,y]\!]_{p,R}\leq [\![z,y]\!]_{p,R}+[\![-w,y]\!]_{p,R}\leq [\![z,y]\!]_{p,R}\,.$$

where the last inequality holds because of part (i).

Theorem 3.3 (Properties of the conic matrix measure): Let $\|\cdot\|$ be a norm with associated WP $[\cdot, \cdot]$ satisfying Deimling's inequality. For every $A, B \in \mathbb{R}^{n \times n}$ and every $a \in \mathbb{R}$,

- (i) $\mu^{+}(aA) = |a|\mu^{+}(\text{sign}(a)A);$

(i)
$$\mu^{+}(aA) = |a|\mu^{-}(\operatorname{sign}(a)A);$$

(ii) $\mu^{+}(A+B) \leq \mu^{+}(A) + \mu^{+}(B);$
(iii) $\mu^{+}(A+aI_{n}) = \mu^{+}(A) + a;$
(iv) $\mu^{+}(A) = \sup_{x \geq 0_{n}, x \neq 0_{n}} \frac{[\![Ax, x]\!]}{\|x\|^{2}},$

- (v) $\mu^{+}(A) \leq \mu(A)$;
- (vi) if $\|\cdot\|$ is monotonic and A is Metzler then $\mu^+(A) = \mu(A)$;
- (vii) if $\|\cdot\| = \|\cdot\|_{p,R}$ for $p \in [1,\infty]$ and $R \in \mathbb{R}^{n \times n}$ invertible and non-negative, then $\mu^+(A) \leq \mu^+(A+\Delta)$ for every $\Delta \in \mathbb{R}^{n \times n}_{>0}$.

Proof: Regarding parts (i) (ii), and (iii), the proofs are straightforward using definition of the conic matrix measure in Definition 3.1. Regarding (iv), for every non-zero $x \ge 0_n$ and h > 0,

$$||(I_{n} - hA)x|| \ge \frac{1}{||x||} \left[(I_{n} - hA)x, x \right] \ge (1 - h \frac{\left[Ax, x \right]}{||x||^{2}}) ||x||$$

$$\ge \left(1 - h \sup_{x \ge 0_{n}, x \ne 0_{n}} \frac{\left[Ax, x \right]}{||x||^{2}} \right) ||x||, \tag{4}$$

where first inequality holds by Cauchy-Schwarz, the second inequality is by subadditivity of WP and the fact that h > 0, and the third inequality holds because h > 0. Now consider $F(h) = (I_n - hA)^{-1}$. By some simple algebraic manipulation, we get

$$F(h) = I_n + hA + h^2 A^2 F(h). (5)$$

Note that for h = 0, we have $F(0) = I_n$ and thus, for every non-zero $x \ge 0$, we have $F(h)x = (I_n - hA)^{-1}x = x \ge 0_n$. Since $\mathbb{S}_{\|\cdot\|}^n =$ $\{x \in \mathbb{R}^n \mid ||x|| = 1\}$ is a compact set, using a continuity argument, there exists small enough $h^*>0$ such that $(I_n-hA)^{-1}x\geq \mathbb{O}_n$ for every $x \in \mathbb{S}_{\|\cdot\|}^n$ and $x \geq \mathbb{O}_n$ and every $0 \leq h \leq h^*$. Thus, for every $0 \le h \le h^*$ and every $v \ge \mathbb{O}_n$ such that $v \ne \mathbb{O}_n$, we have

$$||F(h)v||/||v|| = \frac{||x||}{||(I_n - hA)x||} \le \frac{1}{(1 - h\sup_{||x|| = 1, x \ge 0_n} \frac{\|Ax, x\|}{||x||^2})},$$
$$h^2 ||A^2 (I_n - hA)^{-1} v||/||v|| \le h^2 ||A^2 (I_n - hA)^{-1}||,$$

where the first equality holds by the change of coordinate $x = (I_n (hA)^{-1}v$ and the second inequality holds by (4). Therefore, for every $0 \le h \le h^*$, we have

$$\mu^{+}(A) = \lim_{h \to 0^{+}} \sup_{\|v\| = 1, v \ge \mathbb{O}_{n}} \frac{\|(I_{n} + hA)v\|/\|v\| - 1}{h}$$

$$\leq \lim_{h \to 0^{+}} \sup_{\|v\| = 1, v \ge \mathbb{O}_{n}} \frac{\|F(h)v\| + h^{2}\|A^{2}F(h)v\| - \|v\|}{h\|v\|}$$

$$= \lim_{h \to 0^{+}} \sup_{\|v\| = 1, v \ge \mathbb{O}_{n}} \frac{\|F(h)v\| - \|v\|}{h\|v\|}$$

$$\leq \lim_{h \to 0^{+}} \frac{1}{h} \left(\frac{1}{1 - h \sup_{\|x\| = 1, x \ge \mathbb{O}_{n}} \frac{\|Ax, x\|}{\|x\|^{2}}} - 1\right)$$

$$= \sup_{\|x\| = 1, x \ge \mathbb{O}_{n}} \frac{\|Ax, x\|}{\|x\|^{2}} = \sup_{x \ge \mathbb{O}_{n}, x \ne \mathbb{O}_{n}} \frac{\|Ax, x\|}{\|x\|^{2}},$$

where the first equality holds by definition, the second inequality holds by applying triangle inequality to the algebraic equation (5), and the fourth inequality holds by (4). This means that $\mu^+(A) \leq$ $\sup_{x \neq 0} \sup_{n,x \geq 0} \frac{\|A^x,x\|}{\|x\|^2}$. Additionally, using Deimling's inequality,

$$\begin{aligned}
& [Ax, x] \le ||x|| \lim_{h \to 0^+} \frac{||x + hAx|| - ||x||}{h} \\
& \le ||x||^2 \lim_{h \to 0^+} \sup_{x \ge 0_n, x \ne 0_n} \frac{||x + hAx|| / ||x|| - 1}{h} = ||x||^2 \mu^+(A).
\end{aligned}$$

This means that $\sup_{x\geq 0_n, x\neq 0_n} \frac{ \llbracket Ax,x \rrbracket}{\|x\|^2} \leq \mu^+(A)$ and completes the proof of (iv). Regarding (v), the proof is straightforward using the definitions. Regarding part (vi), we show that, if $\|\cdot\|$ is monotonic, then, for small enough h > 0, $\sup_{x>0_n} \|(I_n +$ $\|hA(x)\|/\|x\| = \sup_{x \neq \mathbb{D}_n} \|(I_n + hA(x))\|/\|x\|$. First note that, by definition, $\sup_{x>0_n} \|(I_n + hA)x\|/\|x\| \le \sup_{x\neq 0_n} \|(I_n + hA)x\|/\|x\|$. Our goal in this part is to show the other side of the inequality. Since A is Metzler, for small enough h > 0, the matrix $I_n + hA$ is nonnegative. By triangle inequality, this implies that $|(I_n + hA)x| \le$ $(I_n + hA)|x|$, for every $x \in \mathbb{R}^n$ and for small enough h > 0. Since $\|\cdot\|$ is a monotonic norm, for every $x\in\mathbb{R}^n$ such that $x\neq \mathbb{O}_n$,

$$||(I_n + hA)x||/||x|| \le ||(I_n + hA)|x|||/|||x|||.$$

and thus, for every $x \neq 0_n$ and every small enough h > 0, we have

$$\frac{\|(I_n + hA)x\|/\|x\| - 1}{h} \le \frac{\|(I_n + hA)|x|\|/\||x|\| - 1}{h}.$$

Since $|x| \ge 0_n$, we can define y = |x| and take the sup of both sides of the above inequality over $x \neq 0_n$,

$$\mu(A) = \lim_{h \to 0^+} \sup_{x \neq 0_n} \frac{\|(I_n + hA)x\|/\|x\| - 1}{h}$$

$$\leq \lim_{h \to 0^+} \sup_{y \geq 0_n, y \neq 0_n} \frac{\|(I_n + hA)y\|/\|y\| - 1}{h} = \mu^+(A).$$

Regarding part (vii), by Lemma 3.2(ii), we have

$$\begin{split} \mu_{p,R}^+(A) &= \sup_{x \geq \mathbb{O}_n, x \neq \mathbb{O}_n} \frac{ [\![Ax,x]\!]_{p,R}}{\|x\|_{p,R}^2} \\ &\leq \sup_{x \geq \mathbb{O}_n, x \neq \mathbb{O}_n} \frac{ [\![(A+\Delta)x,x]\!]_{p,R}}{\|x\|_{p,R}^2} = \mu_{p,R}^+(A+\Delta). \end{split}$$

Next, we provide formulas for some useful conic matrix measures. Theorem 3.4 (Computing conic matrix measure): Let $A \in \mathbb{R}^{n \times n}$ be a matrix, $\|\cdot\|$ be a norm with the conic matrix measure μ^+ , $R \in \mathbb{R}^{n \times n}$ be an invertible non-negative matrix, and $\eta \in \mathbb{R}^n_{>0}$.

(i)
$$\mu_R^+(A) \le \mu^+(RAR^{-1});$$

(ii) $\mu_{[n]}^+(A) = \mu^+([\eta]A[\eta]^{-1});$

(iii) $\mu_2^+(A) = p^* - |\mu_2(A)|$, where p^* is the optimal value of following Quadratically Constrained Quadratic Program (QCQP):

$$p^* = \max \ x^{\mathsf{T}}(\frac{1}{2}(A + A^{\mathsf{T}}) + |\mu_2(A)|I_n)x$$

 $x^{\mathsf{T}}x \le 1, \ x \ge \mathbb{O}_n.$ (6)

 $\begin{array}{ll} \text{(iv)} \ \ \mu_1^+(A) = \max_j \{a_{jj} + \sum_{i \neq j} [a_{ij}]^+\}; \\ \text{(v)} \ \ \mu_\infty^+(A) = \max_i \{a_{ii} + \sum_{j \neq i} [a_{ij}]^+\}. \end{array}$

Moreover, if A is Metzler, then the following statements hold:

(vi) $\mu_{1,[\eta]}^+(A) = \mu_{1,[\eta]}(A) = \min\{c \in \mathbb{R} \mid \eta^\mathsf{T} A \le c\eta^\mathsf{T}\};$ (vii) $\mu_{\infty,[\eta]^{-1}}^+(A) = \mu_{\infty,[\eta]^{-1}}(A) = \min\{c \in \mathbb{R} \mid A\eta \le c\eta\};$ (viii) $\mu_{2,[\eta]}^+(A) = \mu_{2,[\eta]}(A) = \min\{c \in \mathbb{R} \mid [\eta]A + A^\mathsf{T}[\eta] \le 2c[\eta]\}.$

Proof: Regarding part (i), we compute

$$\mu_{R}^{+}(A) = \lim_{h \to 0^{+}} \sup_{x \ge \mathbb{Q}_{n}, x \ne \mathbb{Q}_{n}} \frac{\|(R + hRA)x\|/\|Rx\| - 1}{h}$$

$$\le \lim_{h \to 0^{+}} \sup_{y \ge \mathbb{Q}_{n}} \frac{\|(I_{n} + hRAR^{-1})y\|/\|y\| - 1}{h} = \mu^{+}(RAR^{-1}), \quad (7)$$

where the second inequality holds by setting y = Rx and noting that if $x \geq 0_n$, then $y = Rx \geq 0_n$. Regarding part (ii), if $R = [\eta]$, then in equation (7), we have $\sup_{x\geq \mathbb{O}_n, x\neq \mathbb{O}_n} \|(R+hRA)x\|/\|Rx\| = \sup_{y\geq \mathbb{O}_n, y\neq \mathbb{O}_n} \|(I_n+hRAR^{-1})y\|/\|y\|$, since $x\geq \mathbb{O}_n$ if and only if $y \ge 0_n$. Thus, all the relations in equation (7) are equality and the result follows. Regarding part (iii), for a symmetric matrix $M \in$ $\mathbb{R}^{n \times n}$, we denote the the largest eigenvalue of M by $\lambda_{\max}(M)$ and we define $B = \frac{1}{2}(A + A^{\mathsf{T}}) + |\mu_2(A)|I_n$. Therefore,

$$\lambda_{\max}(B) = \lambda_{\max}(\frac{1}{2}(A + A^{\mathsf{T}}) + |\mu_2(A)|I_n)$$

= $\mu_2(A) + |\mu_2(A)| \ge 0$

where the first equality holds by the formula $\mu_2(A) = \lambda_{\max}(\frac{1}{2}(A +$ A^{T})) and last inequality holds because $\mu_2(A) \leq |\mu_2(A)|$. As a result, the matrix B is positive semi-definite. Using Theorem 3.3(iv),

$$\mu_2^+(A) = \sup_{x \ge \mathbb{O}_n, x \ne \mathbb{O}_n} \frac{x^\mathsf{T} A x}{\|x\|_2^2} = \sup_{y \ge \mathbb{O}_n, \|y\|_2 = 1} y^\mathsf{T} A y,$$

where the last equality holds by the change of variable $y = \frac{x}{\|x\|_2}$.

$$\mu_2^+(A) = \mu_2^+(\frac{1}{2}(A+A^{\mathsf{T}})) = \mu_2^+(B-|\mu_2(A)|I_n)$$

= $\mu_2^+(B) - |\mu_2(A)|,$

where the last equality holds by Theorem 3.3(iii). Moreover,

$$\mu_2^+(B) = \max_{y \ge 0_n, ||y||_2 = 1} y^\mathsf{T} B y = \max_{y \ge 0_n, ||y||_2 \le 1} y^\mathsf{T} B y$$

where the last equality holds because B is positive semi-definite and thus $x \mapsto x^{\mathsf{T}} B x$ is convex. Regarding part (iv), using Theo-

$$\mu_{1}^{+}(A) = \sup_{x \ge 0_{n}, x \ne 0_{n}} \frac{\operatorname{sign}(x)^{\mathsf{T}} A x}{\|x\|_{1}} = \sup_{\|y\|_{1} = 1, y \ge 0_{n}} \operatorname{sign}(y)^{\mathsf{T}} A y$$
$$= \sup_{\|y\|_{1} = 1, y \ge 0_{n}} y^{\mathsf{T}} A^{\mathsf{T}} \operatorname{sign}(y),$$

where the second equality holds by the change of variable $y = \frac{x}{\|x\|}$ and the last equality holds because $sign(y)^{\mathsf{T}} A y = y^{\mathsf{T}} A^{\mathsf{T}} sign(y)$. Note that, for every $i \in \{1, \ldots, n\}$ such that $y_i \neq 0$, we have $[A^\mathsf{T} \mathrm{sign}(y)]_i \leq a_{ii} + \sum_{k \neq i} [a_{ki}]^+$. This implies that

$$\sup_{\|y\|_{1}=1, y \geq 0_{n}} y^{\mathsf{T}} A^{\mathsf{T}} \operatorname{sign}(y) \leq \sup_{\|y\|_{1}=1, y \geq 0_{n}} \sum_{i=1}^{n} y_{i} (a_{ii} + \sum_{i \neq k} [a_{ki}]^{+})$$
$$\leq \max_{j} \{a_{jj} + \sum_{i \neq j} [a_{ij}]^{+}\}.$$

Now we show the converse inequality, Assume that $k \in \{1, ..., n\}$ is such that

$$a_{kk} + \sum_{i \neq k} [a_{ik}]^+ = \max_{j} \{a_{jj} + \sum_{i \neq j} [a_{ij}]^+\}$$

For every $\epsilon \in (0,1)$, we define $z(\epsilon) \in \mathbb{R}^n_{\geq 0}$ as follows:

$$[z(\epsilon)]_i = \begin{cases} 1 - \epsilon & i = k, \\ \epsilon & a_{ik} > 0, \\ 0 & a_{ik} = 0 \end{cases}$$

Then by setting $y(\epsilon) = \frac{z(\epsilon)}{\|z(\epsilon)\|_1}$, we get

$$\sup_{\|y\|_1=1, y\geq 0_n} yA^\mathsf{T} \mathrm{sign}(y) \geq \lim_{\epsilon \to 0^+} y(\epsilon)^\mathsf{T} A^\mathsf{T} \mathrm{sign}(y(\epsilon))$$

$$= \lim_{\epsilon \to 0^+} y(\epsilon)_k (a_{kk} + \sum_{i\neq k} [a_{ik}]^+) + \lim_{\epsilon \to 0^+} \sum_{i\neq k} y(\epsilon)_i [A^\mathsf{T} \mathrm{sign}(y)]_i$$

$$= a_{kk} + \sum_{i\neq k} [a_{ik}]^+,$$

where the last equality holds because $\lim_{\epsilon \to 0^+} y(\epsilon)_i = 1$ and $\lim_{\epsilon \to 0^+} y(\epsilon)_k = 0$, for every $k \neq i$. This conclude the proof of part (iv). Regarding part (v), using Theorem 3.3(iv),

$$\mu_{\infty}^{+}(A) = \sup_{x \ge 0_{n}, x \ne 0_{n}} \max_{i \in I_{\infty}(x)} \frac{(Ax)_{i}}{x_{i}} = \sup_{y \ge 0_{n}, ||y||_{\infty} = 1} \max_{i \in I_{\infty}(y)} (Ay)_{i}$$
$$= \sup_{y \ge 0_{n}, ||y||_{\infty} = 1} \max_{i \in I_{\infty}(y)} \sum_{j=1}^{n} a_{ij}y_{j}$$
$$\le \max_{i} \{a_{ii} + \sum_{j \ne i} [a_{ij}]^{+} \}.$$

Now we show the converse inequality. Let $k \in \{1,\ldots,n\}$ be such that

$$a_{kk} + \sum_{i \neq k} [a_{ki}]^+ = \max_i \{a_{ii} + \sum_{j \neq i} [a_{ij}]^+ \}.$$

We define $z \in \mathbb{R}^n_{>0}$ as follows:

$$z_i = \begin{cases} 1 & i = k \text{ or } a_{ki} > 0, \\ 0 & a_{ki} \le 0, \end{cases}$$

Then it is easy to see that

$$\sup_{y \ge 0_n, ||y||_{\infty} = 1} \max_{i \in I_{\infty}(y)} (Ay)_i \ge \max_{i \in I_{\infty}(z)} (Az)_i$$

$$\ge (Az)_k = a_{kk} + \sum_{i \ne k} [a_{ki}]^+$$

$$= \max_i \{a_{ii} + \sum_{i \ne i} [a_{ij}]^+\}.$$

This completes the proof of part (v). Regarding (vi), the fact that $\mu_{1,[\eta]}^+(A)=\mu_{1,[\eta]}(A)$ follows from Theorem 3.3(vi). Let $b\in\mathbb{R}$ be such that $\mu_{1,[\eta]}^+(A)\leq c$. By part (iv),

$$\max(\boldsymbol{\eta}^{\mathsf{T}} A[\boldsymbol{\eta}]^{-1}) = \max(\mathbb{1}_n^{\mathsf{T}} [\boldsymbol{\eta}] A[\boldsymbol{\eta}]^{-1}) = \mu_{1,[\boldsymbol{\eta}]}^+(A) \leq c,$$

which is also equivalent to $\eta^{\mathsf{T}} A \leq \eta^{\mathsf{T}} c$. Therefore $\mu_{1,[\eta]}^+(A) \leq c$ is equivalent to $\eta^{\mathsf{T}} A \leq \eta^{\mathsf{T}} c$ and the proof is complete by taking the min over c. The proof of parts (vii) and (viii) are similar.

Remark 3.5: The following remarks are in order.

(i) (Computing conic matrix measures) Theorem 3.4(iii) presents a QCQP optimization problem for computing the ℓ_2 -norm conic matrix measure. Note that, while the QCQP (6) is not in general

convex, one can find the following convex relaxation of (6) based on semidefinite programming [30, Equation (16)]:

$$\min_{X \in \mathbb{R}^{n \times n}, x \in \mathbb{R}^n} \operatorname{tr}\left(-\left(\frac{1}{2}(A + A^{\mathsf{T}}) + |\mu_2(A)|I_n\right)X\right)
\operatorname{tr}(X) \le 1, \quad -x \le 0_n, \quad \begin{bmatrix} X & x \\ x^{\mathsf{T}} & 1 \end{bmatrix} \succeq 0,$$
(8)

where tr is the trace operator. The semidefinite programming (8) can be solved using CVX [11]. Moreover, the semidefinite program (8) provides an upper bound for the optimal solution of (6).

(ii) (Spectral abscissa and conic matrix measures) As is shown in Theorem 3.3, the conic matrix measure shares several nice features with the matrix measure including, positive homogeneity, subadditivity, and translation properties. Remarkably, unlike the matrix measure, the conic matrix measure is sometimes smaller than the spectral abscissa. For instance, consider the matrix $A = \begin{bmatrix} -1 & -\frac{1}{2} \\ -1 & -2 \end{bmatrix}$ with eigenvalues $\lambda_1(A) = -0.6340$ and $\lambda_2(A) = -2.3660$. Using the formula in Theorem 3.4(v),

$$\mu_{\infty}^{+}(A) = -1 < \max_{i} \operatorname{Re}(\lambda_{i}(A)) < \mu_{\infty}(A) = -0.5.$$

Finally, we establish a generalized version of Coppel's inequality. Theorem 3.6 (Conic Coppel's inequality): Let $\|\cdot\|$ be a norm and $t\mapsto A(t)$ be a continuous map. Consider the dynamical system

$$\dot{x} = A(t)x\tag{9}$$

If A(t) is Metzler for all $t \ge 0$ and $x(0) \ge 0_n$, then

$$||x(t)|| \le \exp\left(\int_0^t \mu^+(A(\tau))d\tau\right)||x(0)||,$$
 for all $t \ge 0$.

$$x(t+h) = x(t) + hA(t)x(t) + \mathcal{O}(h^2) = (I_n + hA(t))x(t) + \mathcal{O}(h^2)$$

and, in turn, $D^+\|x(t)\|=\lim_{h\to 0^+}\frac{\|(I_n+hA(t))x(t)\|-\|x(t)\|}{h}.$ Since A(t) is Metzler for every $t\geq 0$ and $x(0)\geq \mathbb{O}_n$, it is well known that $x(t)\geq \mathbb{O}_n$, for every $t\geq 0$. Therefore, for every $t\in \mathbb{R}_{\geq 0}$, we obtain

$$\lim_{h \to 0^+} \frac{\|(I_n + hA(t))x(t)\| - \|x(t)\|}{h} \le \mu^+(A(t))\|x(t)\|.$$

The result then follows from Grönwall-Bellman Lemma.

IV. CONTRACTING MONOTONE AND POSITIVE SYSTEMS

In this section, we use the notions of conic matrix measure and WP to study contractive monotone systems and converging positive systems. Our first result presents a characterization of contracting monotone systems using conic matrix measures and WPs.

Theorem 4.1 (Contracting monotone systems): Let $\dot{x} = f(t,x)$ be a monotone dynamical system with a convex forward invariant set $\mathcal{C} \subseteq \mathbb{R}^n$ and $\|\cdot\|$ be a norm with associated WP $[\![\cdot,\cdot]\!]$ satisfying Deimling's inequality. If $\|\cdot\|$ is monotonic, then the following statements are equivalent for $b \in \mathbb{R}$:

- (i) $\mu^+(Df(t,x)) \leq b$, for every $(t,x) \in \mathbb{R}_{>0} \times \mathcal{C}$;
- (ii) $\llbracket f(t,x) f(t,y), x y \rrbracket \le b \Vert x y \Vert^2$, for every $(t,x), (t,y) \in \mathbb{R}_{\geq 0} \times \mathcal{C}$ such that $x \geq y$;
- (iii) $\|\phi(t, t_0, x_0) \phi(t, t_0, y_0)\| \le e^{b(t-s)} \|\phi(s, t_0, x_0) \phi(s, t_0, y_0)\|$, for every $t_0 \le s \le t$ and every $x_0, y_0 \in \mathcal{C}$.

Instead, if $\|\cdot\|$ is not monotonic, then conditions (i), (ii) are equivalent and, with $\mathcal{C} = [x_{\min}, x_{\max}]$ for some $x_{\min} < x_{\max}$, they imply

(iv) there exists M>0 such that, for every $x_0,y_0\in[x_{\min},x_{\max}]$ and every $t_0\leq s\leq t$,

$$\|\phi(t, t_0, x_0) - \phi(t, t_0, y_0)\|$$

$$\leq Me^{b(t-s)} \|\phi(s, t_0, x_0) - \phi(s, t_0, y_0)\|.$$

Proof: Regarding (i) \Longrightarrow (ii), compute

$$\begin{aligned}
& [f(t,x) - f(t,y), x - y] \\
&= [[\int_0^1 Df(t, \tau x + (1-\tau)y)d\tau](x-y), x - y] \\
&\leq \int_0^1 [Df(t, \tau x + (1-\tau)y)(x-y), x - y] d\tau \\
&\leq \mu^+ (Df(t, \tau x + (1-\tau)y))||x-y||^2 \leq b||x-y||^2,
\end{aligned}$$

where the first equality is by the Mean Value Theorem, the second inequality is by the subadditivity of the WP and the third inequality holds by Theorem 3.3(iv) and the fact that $x-y \geq \mathbb{O}_n$. Regarding (ii) \Longrightarrow (i), pick x=y+hv, for $v\in\mathbb{R}^n_{>0}$ and h>0. Thus,

$$\begin{split} & [\![f(t,x)-f(t,y),x-y]\!] = [\![f(t,y+hv)-f(t,y),hv]\!] \\ & = h^2 \left[\![\frac{f(t,y+hv)-f(t,y)}{h},v]\!] \le b |\!|x-y|\!|^2 = bh^2 |\!|v|\!|^2. \end{split}$$

In the limit as $h \to 0^+$, for every $y \in \mathbb{R}^n$ and every $v \in \mathbb{R}^n_{>0}$,

$$\llbracket Df(t,y)v,v\rrbracket = \lim_{h\to 0^+} \left\llbracket \frac{f(t,y+hv)-f(t,y)}{h},v\right\rrbracket \le b\lVert v\rVert^2.$$

where the first equality holds by the continuity of WP in the first argument. As a result, by Theorem 3.3(iv), $\mu^+(Df(t,x)) \leq b$, for every $t \in \mathbb{R}_{\geq 0}$ and every $x \in \mathcal{C}$. Regarding (i) \Longrightarrow (iii),since f is monotone, Df(t,x) is Metzler for every $(t,x) \in \mathbb{R}_{\geq 0} \times \mathcal{C}$. Since $\|\cdot\|$ is monotonic, Theorem 3.3(vi) implies $\mu^+(Df(t,x)) = \mu(Df(t,x))$, for every $x \in \mathcal{C}$ and every $t \geq 0$. These conclusions then follow from [7, Theorem 29]. Regarding (iii) \Longrightarrow (ii), note that $\|\phi(t+h,t_0,x_0)-\phi(t+h,t_0,y_0)\| \leq e^{bh}\|\phi(t,t_0,x_0)-\phi(t,t_0,y_0)\|$, for every h > 0. As a result

$$\begin{split} \lim_{h \to 0^+} & \frac{\|\phi(t+h,t_0,x_0) - \phi(t+h,t_0,y_0)\| - \|\phi(t,t_0,x_0) - \phi(t,t_0,y_0)\|}{h} \\ & \leq \lim_{h \to 0^+} \frac{e^{bh} - 1}{h} \|\phi(t,t_0,x_0) - \phi(t,t_0,y_0)\| \\ & = b \|\phi(t,t_0,x_0) - \phi(t,t_0,y_0)\|. \end{split}$$

Thus, by Deimling's inequality, for every $x_0, y_0 \in \mathcal{C}$,

$$[[f(t,\phi(t,t_0,x_0)) - f(t,\phi(t,t_0,y_0)),\phi(t,t_0,x_0) - \phi(t,t_0,y_0)]] \le b||\phi(t,t_0,x_0) - \phi(t,t_0,y_0)||^2.$$

This concludes the proof of (iii) \Longrightarrow (ii). Regarding (iv), first we show that, for $x_0 \geq y_0$, $\|\phi(t,t_0,x_0)-\phi(t,t_0,y_0)\| \leq e^{b(t-s)}\|\phi(s,t_0,x_0)-\phi(s,t_0,y_0)\|$. For $\alpha\in[0,1]$, define $\psi(t,\alpha)=\phi(t,s,\alpha\phi(s,t_0,x_0)+(1-\alpha)\phi(s,t_0,y_0))$ and note $\psi(t_0,\alpha)=\alpha\phi(s,t_0,x_0)+(1-\alpha)\phi(s,t_0,y_0)$ and $\frac{\partial\psi}{\partial\alpha}(s,\alpha)=\phi(s,t_0,x_0)-\phi(s,t_0,y_0)$. We then compute:

$$\begin{split} \frac{\partial}{\partial t} \frac{\partial}{\partial \alpha} \psi(t, \alpha) &= \frac{\partial}{\partial \alpha} \frac{\partial}{\partial t} \psi(t, \alpha) = \frac{\partial}{\partial \alpha} f(t, \psi(t, \alpha)) \\ &= \frac{\partial f}{\partial x} (t, \psi(t, \alpha)) \frac{\partial}{\partial \alpha} \psi(t, \alpha). \end{split}$$

Therefore, $\frac{\partial \psi}{\partial \alpha}(t,\alpha)$ satisfies the linear time-varying differential equation $\frac{\partial}{\partial t}\frac{\partial \psi}{\partial \alpha}=Df(t,\psi)\frac{\partial \psi}{\partial \alpha}$. Moreover, $x_0-y_0\geq 0$ and Df(t,x) is

Metzler, for every $(t, x) \in \mathbb{R}_{>0} \times \mathcal{C}$. Therefore, Theorem 3.6 implies

$$\left\| \frac{\partial \psi}{\partial \alpha}(t, \alpha) \right\| \le \left\| \frac{\partial \psi}{\partial \alpha}(s, \alpha) \right\| \exp\left(\int_{s}^{t} \mu^{+} \left(Df(t, \psi(\tau, \alpha)) \right) d\tau \right)$$

$$\le e^{b(t-s)} \left\| \phi(s, t_{0}, x_{0}) - \phi(s, t_{0}, y_{0}) \right\|,$$
(10)

where we used $\mu^+(Df(t,x)) \leq b$, for every $t \in \mathbb{R}_{\geq 0}$ and $x \in \mathbb{R}^n$. In turn, inequality (10) implies

$$\|\phi(t,t_0,x_0) - \phi(t,t_0,y_0)\| = \|\psi(t,1) - \psi(t,0)\|$$

$$= \left\| \int_0^1 \frac{\partial \psi(t,\alpha)}{\partial \alpha} d\alpha \right\| \le \int_0^1 \left\| \frac{\partial \psi(t,\alpha)}{\partial \alpha} \right\| d\alpha$$

$$\le e^{b(t-s)} \|\phi(s,x_0) - \phi(s,y_0)\|.$$

Now assume that $x_0 \not\geq y_0$. We define $\xi, \eta \in \mathbb{R}^n$ by

$$\xi_i = \max\{(x_0)_i, (y_0)_i\}, \quad \eta_i = \min\{(x_0)_i, (y_0)_i\}, \quad i \in \{1, \dots, n\}$$

Then $\xi, \eta \in [x_{\min}, x_{\max}]$ and it is clear that $\eta \leq x \leq \xi$ and $\eta \leq y \leq \xi$. Moreover, since all the norms are equivalent in \mathbb{R}^n , there exists $M_1, M_2 > 0$ such that $M_1 \|v\| \leq \|v\|_{\infty} \leq M_2 \|v\|$. As a result, we get $\|\eta - \xi\| \leq M_1^{-1} \|\eta - \xi\|_{\infty} = M_1^{-1} \|x_0 - y_0\|_{\infty} \leq M_1^{-1} M_2 \|x_0 - y_0\|$. We set $M^{\frac{1}{2}} = M_1^{-1} M_2 > 0$. Since the vector field f is monotone, for every $t \geq t_0$,

$$\phi(t, t_0, \eta) \le \phi(t, t_0, x) \le \phi(t, t_0, \xi),
\phi(t, t_0, \eta) \le \phi(t, t_0, y) \le \phi(t, t_0, \xi).$$

This means that

$$\|\phi(t,t_0,x) - \phi(t,t_0,y)\| \le M_1^{-1} \|\phi(t,t_0,x) - \phi(t,t_0,y)\|_{\infty}$$

$$\le M_1^{-1} \|\phi(t,t_0,\xi) - \phi(t,t_0,\eta)\|_{\infty}$$

$$\le M_1^{-1} M_2 \|\phi(t,t_0,\xi) - \phi(t,t_0,\eta)\|$$

$$= M^{\frac{1}{2}} \|\phi(t,t_0,\xi) - \phi(t,t_0,\eta)\|.$$

However, we know that $\eta \leq \xi$, thus by the above argument we get $\|\phi(t,t_0,\eta)-\phi(t,t_0,\xi)\| \leq e^{b(t-s)} \|\phi(s,t_0,\eta)-\phi(s,t_0,\xi)\|$. Therefore,

$$\begin{aligned} \|\phi(t,t_0,x_0) - \phi(t,t_0,y_0)\| &\leq M^{\frac{1}{2}} \|\phi(t,t_0,\eta) - \phi(t,t_0,\xi)\| \\ &\leq M^{\frac{1}{2}} e^{b(t-s)} \|\phi(s,t_0,\eta) - \phi(s,t_0,\xi)\| \\ &= M e^{b(t-s)} \|\phi(s,t_0,x_0) - \phi(s,t_0,y_0)\|. \end{aligned}$$

Remark 4.2: For monotonic norms, using Theorem 3.3(vi), the notion of conic matrix measure coincides with the standard matrix measure on Metzler matrices. Therefore, Theorem 4.1 can be completely recovered from [7, Theorem 31]. However, for non-monotonic norms, Theorem 4.1 provides a conic matrix measure condition for incremental exponential stability of the system. The next example elaborates this point in more detail.

Example 4.3: Consider the class of dynamical system on \mathbb{R}^2

$$\dot{x}_1 = -x_1 + \alpha x_2 - \gamma g(x_1),
\dot{x}_2 = \beta x_1 - x_2,$$
(11)

where $\alpha, \beta, \gamma \geq 0$ and $g: \mathbb{R} \to \mathbb{R}_{\geq 0}$ is differentiable with g(0) = 0 and $0 \leq g'(x) \leq \overline{G}$, for every $x \in \mathbb{R}$. It is easy to see that, since $\alpha, \beta \geq 0$, the dynamical system (11) is monotone. For $\alpha = \frac{1}{2}$, $\beta = 1.2$, $\gamma = 1$, and $\overline{G} = 0.1$, we have

$$Df(x_1, x_2) = \begin{bmatrix} -1 - g'(x_1) & \frac{1}{2} \\ 1.2 & -1 \end{bmatrix}$$

For monotonic norm $\|\cdot\|_{\infty}$, we have $\mu_{\infty}^+(Df(x_1,x_2)) = \mu_{\infty}(Df(x_1,x_2)) = 0.2$. Therefore the dynamical system (11) is not

contracting with respect to ℓ_∞ -norm. For the non-monotonic norm $\|\cdot\|_{\infty,R}$ with $R=\begin{bmatrix} -1 & 1\\ 1 & 1 \end{bmatrix}$, we can compute

$$\mu_{\infty,R}^{+}(Df(x_1, x_2)) = \mu_{\infty}^{+}(RDf(x_1, x_2)R^{-1})$$

$$= \mu_{\infty}^{+} \left(\begin{bmatrix} -1.85 - \frac{g'(x_1)}{2} & 0.35 + \frac{g'(x_1)}{2} \\ -0.35 + \frac{g'(x_1)}{2} & -0.15 - \frac{g'(x_1)}{2} \end{bmatrix} \right)$$

$$= -0.15 - \frac{g'(x)}{2} \le -0.15$$
 (12)

Therefore, using Theorem 4.1 and inequality (12), the dynamical system (11) is incrementally exponentially stable with respect to the non-monotonic norm $\|\cdot\|_{\infty,R}$. As a consequence, \mathbb{O}_2 is the globally exponentially stable equilibrium point of the dynamical system (11). On the other hand, we have

$$\mu_{\infty,R}(Df(x_1, x_2)) = \mu_{\infty}(RDf(x_1, x_2)R^{-1})$$

$$= \mu_{\infty} \left(\begin{bmatrix} -1.85 - \frac{g'(x_1)}{2} & 0.35 + \frac{g'(x_1)}{2} \\ -0.35 + \frac{g'(x_1)}{2} & -0.15 - \frac{g'(x_1)}{2} \end{bmatrix} \right)$$

$$\geq 0.2 - g'(x_1) \geq 0.1. \tag{13}$$

However, using [7, Theorem 31] and inequality (13), the dynamical system (11) is not contracting with respect to $\|\cdot\|_{\infty,R}$.

We can also simplify Theorem 4.1 for diagonally-weighted norms. Corollary 4.4 (Diagonally-weighted norms): Let $\dot{x} = f(t,x)$ be a monotone dynamical system and $\eta \in \mathbb{R}^n_{>0}$. Then the following statements about $[\eta]$ -weighted ℓ_1 -norm are equivalent:

- $\begin{array}{ll} \text{(i)} \ \ \mu_{1,[\eta]}(Df(t,x)) \leq b, \ \text{for every} \ (t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n; \\ \text{(ii)} \ \ \eta^\mathsf{T}(f(t,x) f(t,y)) \leq b\eta^\mathsf{T}(x-y), \ \text{for every} \ x \geq y \ \text{and every} \end{array}$
- (iii) $\|\phi(t,t_0,x_0) \phi(t,t_0,y_0)\|_{1,[\eta]} \le e^{b(t-s)} \|\phi(s,t_0,x_0) \phi(s,t_0,y_0)\|_{1,[\eta]}$, for every $x_0,y_0 \in \mathbb{R}^n$ and every $t_0 \le t \le s$.

Similarly, the following statements about $[\eta]^{-1}$ -weighted ℓ_{∞} -norm are equivalent:

- (iv) $\mu_{\infty,[\eta]^{-1}}(Df(t,x)) \leq b$, for every $(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$; (v) $f(t,x) f(t,y) \leq b(x-y)$, for every $t \in \mathbb{R}_{\geq 0}$ and every $x = y + c\eta$ with c > 0;

Proof: Note that $\|\cdot\|_{1,[\eta]}$ is a monotonic norm and its associated WP is given by (2) with $R=[\eta]$. Similarly, $\|\cdot\|_{\infty,[\eta]^{-1}}$ is a monotonic norm and its associated WP is given by (3) with R = $[\eta]^{-1}$. Regarding (i) \Longrightarrow (ii),

$$\eta^{\mathsf{T}}(f(t,x) - f(t,y)) = \eta^{\mathsf{T}} \Big(\int_0^1 Df(t,\tau x + (1-\tau)y) d\tau \Big) (x-y)$$
$$= \int_0^1 \eta^{\mathsf{T}} Df(t,\tau x + (1-\tau)y) (x-y) d\tau \le b\eta^{\mathsf{T}}(x-y),$$

where the first equality is by the Mean Value Theorem and the last inequality holds by Theorem 3.4(vi) and using the fact that $x-y \ge$ \mathbb{O}_n . Regarding (ii) \Longrightarrow (i), pick x = y + hv, for $v \in \mathbb{R}^n_{>0}$ and h > 0. Thus, we get

$$\eta^{\mathsf{T}}(f(t,x) - f(t,y)) = \eta^{\mathsf{T}}(f(t,y + hv) - f(t,y)) \le bh\eta^{\mathsf{T}}v.$$

By taking the limit as $h \to 0^+$, for every $y \in \mathbb{R}^n$ and every $v \in \mathbb{R}^n_{>0}$,

$$\eta^{\mathsf{T}} Df(t,y)v = \lim_{h \to 0^+} \eta^{\mathsf{T}} \frac{f(t,y+hv) - f(t,y)}{h} \le b\eta^{\mathsf{T}} v.$$

The result then follows by Theorem 3.4(vi).

Regarding (iv) \implies (v), note that, for every c > 0 such that $x = y + c\eta$, we have

$$f(t,x) - f(t,y) = \int_0^1 Df(t,(1-\tau)y + \tau x)(x-y)d\tau$$
$$= \int_0^1 Df(t,y + \tau c\eta)(c\eta)d\tau \le b(x-y)$$

where the inequality follows from Theorem 3.4(vii). For $(v) \implies (iv)$,

$$Df(t,x)\eta = \lim_{h \to 0^+} \frac{f(t,x+h\eta) - f(t,x)}{h} \le b\eta.$$

The result follows by Theorem 3.4(vii). The rest of the proof follows from Theorem 4.1.

Next, we use the notion of conic matrix measure and weak pairing to study exponential convergence of positive systems to their equilibrium points.

Theorem 4.5 (Converging positive systems): Let $\dot{x} = f(t,x)$ be a positive system with equilibrium point \mathbb{O}_n , $\|\cdot\|$ be a norm with associated WP $[\cdot,\cdot]$ satisfying Deimling's inequality and the curve norm derivative formula, and $b \in \mathbb{R}$. Consider

- (A1) $[f(t,x),x] \le b||x||^2$, for every $(t,x) \in \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0}^n$;
- (A2) $\|\phi(t,t_0,x_0)\| \le e^{b(t-s)} \|\phi(s,t_0,x_0)\|$, for every $t_0 \le s \le t$ and every $x_0 \in \mathbb{R}^n_{>0}$.
- (A3) $\mu^+(B(t,x)) \leq b$, for every $x \geq 0_n$ and every $t \in \mathbb{R}_{>0}$ where B(t, x) is such that f(t, x) = B(t, x)x.

Then the following statements hold:

- (i) (A1) and (A2) are equivalent;
- (ii) (A3) implies (A1) and (A2).

Proof: Regarding (A1) \iff (A2), the proof is similar to [7, Theorem 33] and we omit it.

Regarding (A3) \Longrightarrow (A2), for every $x \ge 0$, we have

$$[f(t,x),x] = [B(t,x)x,x] \le \mu^+(B(t,x))||x||^2 \le b||x||^2.$$

where the second inequality holds by Theorem 3.3(iv).

In the next example, we investigate the role of conic matrix measures in the sufficient condition for exponential stability of positive systems in Theorem 4.5.

Example 4.6: Consider the following dynamical system on \mathbb{R}^2 :

$$\dot{x}_1 = -2x_1 + x_2 := f_1(x_1, x_2),
\dot{x}_2 = -x_1 \alpha(x_2) - x_2 := f_2(x_1, x_2),$$
(14)

where $\alpha:\mathbb{R}\to\mathbb{R}_{\geq 0}$ is a non-negative non-decreasing function. First note that, $Df(x_1,x_2)=\left[egin{array}{cc} -2 & 1 \\ -\alpha(x_2) & -x_1\alpha'(x_2)-1 \end{array} \right]$. Thus, the vector field f is not monotone on $\mathbb{R}^2_{>0}$ because $-\alpha(r) \leq 0$ for every $r \in \mathbb{R}$. However, the dynamical system (14) is positive with an equilibrium point at $\mathbb{O}_2 \in \mathbb{R}^2_{>0}$. Moreover,

$$f(x_1,x_2) = \begin{bmatrix} -2 & 1 \\ -\alpha(x_2) & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := B(x_1,x_2) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Using Theorem 3.4(v), for every $(x_1, x_2) \in \mathbb{R}^2_{>0}$,

$$\mu_{\infty}^{+}(B(x_1, x_2)) = -1 \le \mu_{\infty}(B(x_1, x_2)) = -1 + \alpha(x_2).$$
 (15)

By Theorem 4.5(ii), every trajectory $t\mapsto [x_1(t),x_2(t)]^\mathsf{T}$ of the positive system (14) starting at $[x_1(0),x_2(0)]^\mathsf{T}\in\mathbb{R}^2_{>0}$ satisfies

$$\|[x_1(t), x_2(t)]^{\mathsf{T}}\|_{\infty} \le e^{-t} \|[x_1(0), x_2(0)]^{\mathsf{T}}\|_{\infty}.$$

It is worth mentioning that, by equation (15), the ℓ_{∞} -matrix measure of $B(x_1, x_2)$ might not be bounded and cannot be used to deduce convergence of trajectories of (14) to \mathbb{O}_2 .

V. APPLICATIONS

In this section, we present two applications for our non-Euclidean contraction framework for monotone and positive systems. As a first application, we show that a Hopfield neural network with excitatory interactions between its neurons is monotone but non-positive. We then use our framework to analyze stability and robustness of excitatory Hopfield neural networks. As a second application, we develop a framework for stability analysis of networks of interconnected systems using positive but non-monotone comparison systems.

A. Excitatory Hopfield neural networks

Hopfield model is a class of recurrent neural networks that can serve as an associative memory system [14]. There has been a recent growing interest in the machine learning community to use variations of Hopfield neural networks to store information or to learn prototypes [24]. However, neural networks are notoriously vulnerable to adversarial perturbations of their input; small changes in their input can cause a large change in their output [29]. In this section, we study stability and input-output robustness of Hopfield neural network with excitatory neuron interactions and provide explicit adversarial robustness guarantees for this class of learning algorithms. The dynamics of the Hopfield neural network is given by

$$\dot{x} = -\Lambda x + Tg(x) + I(t) := F_{H}(x),$$
 (16)

where $x \in \mathbb{R}^n$ is the state of neurons, $\Lambda \in \mathbb{R}^{n \times n}$ is the diagonal positive-definite matrix of dissipation rates, $T \in \mathbb{R}^{n \times n}$ is the interaction matrix, and $I : \mathbb{R}_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is a time-varying input. Assume $g(x) = (g_1(x_1), \ldots, g_n(x_n))^\mathsf{T}$, where the *i*th activation function g_i is Lipschitz continuous, monotonic non-decreasing with $g_i(0) = 0$ and with the finite sector property:

$$0 \le \frac{g_i(x) - g_i(y)}{x - y} := G_i(x, y) \le \overline{G}_i,$$

where $\overline{G} = (\overline{G}_1, \dots, \overline{G}_n)^{\mathsf{T}} \in \mathbb{R}_{>0}^n$. We study excitatory Hopfield networks, i.e., neural networks with Metzler interaction matrix T.

Proposition 5.1 (Contracting Hopfield neural networks):

Consider the Hopfield neural network (16) with an irreducible non-negative interaction matrix T. Assume the Metzler matrix $-\Lambda + T\overline{G}$ is Hurwitz with (-c,v) and (-c,w) its left and the right Perron eigenpair, respectively. For any $p \in [1,\infty]$, define $q \in [1,\infty]$ by $\frac{1}{p} + \frac{1}{q} = 1$ (with convention $1/\infty = 0$) and $\eta \in \mathbb{R}^n_{>0}$ by

$$\eta = \left(v_1^{\frac{1}{p}}/w_1^{\frac{1}{q}}, \dots, v_n^{\frac{1}{p}}/w_n^{\frac{1}{q}}\right)^{\mathsf{T}}.$$

Then the following statements hold for any $p \in [1, \infty]$:

- (i) the Hopfield neural network (16) is monotone and contracting with respect to the norm $\|\cdot\|_{p,[\eta]}$ with rate c;
- (ii) if $I(t) = I^*$ is constant, then the Hopfield neural network (16) has a unique globally exponentially stable equilibrium point x_I^* with the Lyapunov functions $||x x_I^*||_{p,[\eta]}$ and $||F_H(x)||_{p,[\eta]}$;
- (iii) if $t \mapsto x_I(t)$ and $t \mapsto x_J(t)$ are solutions of the Hopfield neural network (16) for input signals $t \mapsto I(t)$ and $t \mapsto J(t)$ respectively, then, for every $t \in \mathbb{R}_{>0}$,

$$||x_I(t) - x_J(t)||_{p,[\eta]} \le e^{-ct} ||x_I(0) - x_J(0)||_{p,[\eta]}$$

$$+ \int_0^t e^{-c(t-s)} ||I(s) - J(s)||_{p,[\eta]} ds.$$

Proof: Let $i \in \{1, ..., n\}$ and consider $x \leq y$ such that $x_i = y_i$. For every $i \neq j$, by the finite sector property of g_i , we have $g_i(x_i) \leq g_i(y_j)$, and thus

$$\begin{split} [F_{\mathrm{H}}(x)]_i &= -\gamma_i x_i + \sum_{j=1}^n T_{ij} g_j(x_j) + I(t) \\ &\leq -\gamma_i y_i + \sum_{j=1}^n T_{ij} g(y_j) + I(t) = [F_{\mathrm{H}}(y)]_i, \end{split}$$

where the inequality holds because the matrix T is Metzler. This means that the Hopfield neural network (16) is monotone. Moreover, for every $x \geq y \geq \mathbb{O}_n$,

$$\begin{aligned} \|x - y\|_{p,[\eta]} D^+ \|x - y\|_{p,[\eta]} \\ &= [-\Lambda(x - y) + T(g(x) - g(y)), x - y]_{p,[\eta]} \\ &\leq [(-\Lambda + T\overline{G})(x - y), x - y]_{p,[\eta]} \\ &\leq \mu_{p,[\eta]} (-\Lambda + T\overline{G}) \|x - y\|_{p,[\eta]}^2 = -c \|x - y\|_{p,[\eta]}^2, \end{aligned}$$

where the first equality is the curve norm derivative formula. Since g is non-decreasing and T is non-negative, we get the bound $T(g(x)-g(y)) \leq T\overline{G}(x-y)$ for every $x \geq y \geq \mathbb{O}_n$. Lemma 3.2(ii) and this bound give us the second inequality. The third inequality holds by definition of $\mu_{p,[\eta]}$ and the fourth equality holds by [1] using the fact that $-\Lambda + TG$ is Metzler, irreducible, and Hurwitz with Perron eigenvalue -c and left and right Perron eigenvectors v and v. Then, parts (i) and (ii) follow from Theorem 4.1. Regarding part (iii),

$$\begin{aligned} \|x_{I} - x_{J}\|_{p,[\eta]} D^{+} \|x_{I} - x_{J}\|_{p,[\eta]} \\ &= [\![F_{H}(x_{I}) - F_{H}(x_{J}) + I(t) - J(t), x_{I} - x_{J}]\!]_{p,[\eta]} \\ &\leq [\![F_{H}(x_{I}) - F_{H}(x_{J}), x_{I} - x_{J}]\!]_{p,[\eta]} \\ &+ [\![I(t) - J(t), x_{I}(t) - x_{J}(t)]\!]_{p,[\eta]} \\ &\leq -c \|x_{I} - x_{J}\|_{p,[\eta]}^{2} + \|x_{I} - x_{J}\|_{p,[\eta]} \|I(t) - J(t)\|_{p,[\eta]}, \end{aligned}$$

where the first equality holds by the curve norm derivative formula, the second inequality holds by subadditive property of WPs, and the third inequality holds by contractivity of $F_{\rm H}$ and the Cauchy-Schwarz inequality. This implies that $D^+\|x_I(t)-x_J(t)\|_{p,[\eta]} \leq -c\|x_I(t)-x_J(t)\|_{p,[\eta]} + \|I(t)-J(t)\|_{p,[\eta]}$, for every $t\in\mathbb{R}_{\geq 0}$. The result follows by Grönwall–Bellman inequality [7, Lemma 11].

Remark 5.2 (Comparison with the literature): We refer to [31] for a review of stability properties of Hopfield neural networks; e.g., it is known that Hurwitzness of $-\Lambda + T\overline{G}$ (as we assume in Proposition 5.1) implies global exponential stability. To the best of our knowledge, the strong contractivity (with respect to appropriately weighted p-norms) in part (i) and the Lyapunov functions in part (ii) are novel. Recall that, as reviewed in the Introduction, strong contractivity is a stronger property than global exponential stability. To the best of our knowledge, the input-to-output stability in part (iii) is novel and is directly applicable to obtain adversarial robustness guarantees of Hopfield neural networks.

B. Non-monotone comparison systems

Comparison principles are well-established techniques in dynamical system theory to infer stability of a dynamical system using properties of a simpler comparison systems. In most of the existing comparison frameworks in the literature, monotonicity of the comparison system plays a crucial role [21], [23]. In this subsection, we develop a novel comparison principle for stability analysis of networks of interconnected systems. Unlike the existing comparison results in the literature, our framework uses positive comparison systems which are not necessarily monotone. Consider the interconnection of n subsystems:

$$\dot{x}_i = f_i(x, u_i), \qquad i \in \{1, \dots, n\}.$$
 (17)

where $x_i \in \mathbb{R}^{N_i}$ is the state and $u_i \in \mathbb{R}^{M_i}$ is the exogenous input for the ith subsystem. We define $N = \sum_{i=1}^n N_i$ and $M = \sum_{i=1}^n M_i$ and $x = (x_1, \dots, x_n)^\mathsf{T} \in \mathbb{R}^N$. We assume that, for every $i \in \{1, \dots, n\}$, we have $f_i(\mathbb{O}_N, \mathbb{O}_{M_i}) = \mathbb{O}_{N_i}$ and the ith subsystem is equipped with the norm $\|\cdot\|_i$ on \mathbb{R}^{N_i} . We assume that, for every

 $i \in \{1, \dots, n\}$, the ith subsystem has a storage function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}^{N_i}$ $\mathbb{R}_{>0}$ such that

- (B1) there exists class \mathcal{K}_{∞} functions \underline{lpha}_i and \overline{lpha}_i such that $\underline{\alpha}_i(\|x_i\|_i) \leq V_i(x_i) \leq \overline{\alpha}_i(\|x_i\|_i);$ (B2) for every $x \in \mathbb{R}^N, u \in \mathbb{R}^M$, the inequality

$$\mathcal{L}_{f_i}V_i(x_i) \le -\alpha_i(V_i(x_i)) + g_i(V(x)) + \gamma_i(u). \tag{18}$$

holds for some function $g_i: \mathbb{R}^N_{\geq 0} \to \mathbb{R}$ with $g_i(\mathbb{O}_N) = 0$ and for some class \mathcal{K}_{∞} functions α_i and γ_i .

We define the maps $\Gamma: \mathbb{R}^N_{\geq 0} \to \mathbb{R}^n$ and $A: \mathbb{R}^N_{\geq 0} \to \mathbb{R}^n$ by:

$$\Gamma(x) = (g_1(x), \dots, g_n(x))^{\mathsf{T}}, \quad A(x) = (\alpha_1(x_1), \dots, \alpha_n(x_n))^{\mathsf{T}}$$

One can also define the non-monotone comparison system by

$$\dot{v}_i = -A_i(v_i) + \Gamma_i(v) + \gamma_i(u), \quad i \in \{1, \dots, n\}.$$
 (19)

Using the inequality (18), one can show that, for $u = \mathbb{O}_M$, the comparison system (19) is a positive dynamical system. However, since g_i can be any arbitrary function, the comparison system (19) is not necessarily monotone.

Remark 5.3 (Input-to-state stability): The inequality (18) can be considered as a generalization of component-wise input-to-state stability (ISS). An interconnected system is component-wise ISS if each of its subsystems is ISS when interconnections between subsystems are considered as the input. In other words, the interconnected system (17) is component-wise ISS if, for every $i \in \{1, ..., n\}$, the storage function V_i satisfies

$$\mathcal{L}_{f_i}V_i(x_i) \le -\alpha_i(V_i(x_i)) + \sum\nolimits_{j \ne i} \gamma_{ij}(V_j(x_j)) + \gamma_{iu}(u_i)$$

for class \mathcal{K}_{∞} function α_i and class \mathcal{K} functions γ_{ij} and γ_{iu} . Indeed, if the interconnected system is component-wise ISS, then the associated comparison system (19) is monotone.

Proposition 5.4 (Stability of interconnection of systems): Consider the interconnected system (17) and suppose that every subsystem satisfies conditions (B1) and (B2) above. Let $p \in [1, \infty]$ and $R \in \mathbb{R}^{n \times n}$ be an invertible non-negative matrix. Suppose that there exists c > 0 such that, for every $v \ge 0_n$,

$$- [-A(v), v]_{p,R} \ge [\Gamma(v), v]_{p,R} + c||v||_{p,R}^{2}$$
 (C₁)

Then, the following statements hold:

- (i) the comparison system (19) converges exponentially to \mathbb{O}_n
- (ii) for $u(t) = \mathbb{O}_M$, every trajectory of the interconnected system (17) converges to \mathbb{O}_N .
- (iii) the system (17) is input-to-state stable in the sense that, for every $i \in \{1, \ldots, n\}$ and $t \ge 0$, there exists $L_i > 0$, such that

$$||x_{i}(t)||_{i} \leq \underline{\alpha}_{i}^{-1} \left(L_{i} e^{-ct} ||V(x(0))||_{p,R} + \frac{L_{i}(1 - e^{-ct})}{c} \max_{\tau \in [0,t]} ||\gamma(u(\tau))||_{p,R} \right).$$
 (20)

Alternatively, if $v \mapsto -A(v) + \Gamma(v)$ is continuously differentiable, then (i), (ii), and (iii) still holds by replacing condition (C_1) with the following stronger condition:

$$\mu_{p,R}^+(B(v)) \le -c \tag{C_2}$$

where $B(v) \in \mathbb{R}^{n \times n}$ satisfies $B(v)v = -A(v) + \Gamma(v)$, for $v \in \mathbb{R}^n_{>0}$.

Proof: Regarding part (i), for $u = \mathbb{O}_M$, we have

for every $v \in \mathbb{R}^n_{\geq 0}$. Since the comparison system is positive, the result follows from Theorem 4.5(i). Regarding part (ii), by setting V(x(t)) = V(t), we get

$$\begin{split} \|V(t)\|D^{+}\|V(t)\|_{p,R} &= \left[\!\!\left[\dot{V}(t),V(t)\right]\!\!\right]_{p,R} \\ &\leq \left[\!\!\left[-A(V(t)) + \Gamma(V(t)) + \gamma(u),V(t)\right]\!\!\right]_{p,R} \\ &\leq \left[\!\!\left[-A(V(t)),V(t)\right]\!\!\right]_{p,R} + \left[\!\!\left[\Gamma(V(t)),V(t)\right]\!\!\right]_{p,R} \\ &+ \left[\!\!\left[\gamma(u),V(t)\right]\!\!\right]_{p,R} \\ &\leq -c\|V(t)\|_{p,R}^{2} + \|\gamma(u)\|\|V(t)\|_{p,R}. \end{split}$$

where the first equality holds by the curve norm derivative formula, the second inequality holds by Lemma 3.2(ii), the third inequality holds by subadditivity of the WPs, and the fourth inequality holds by the Cauchy-Schwarz inequality. This implies that

$$||V(t)||_{p,R} \le e^{-ct} ||V(0)||_{p,R} + \frac{1 - e^{-ct}}{c} \max_{\tau \in [0,t]} ||\gamma(u(\tau))||_{p,R}.$$

Therefore, for $u = 0_M$, we have $t \mapsto V(t)$ converges exponentially to \mathbb{O}_n and thus $\lim_{t\to\infty} x(t) = \mathbb{O}_N$. Regarding part (iii), since R is non-negative and invertible, there exists $L_i > 0$ such that $V_i(x_i) \le$ $L_i||V(x)||_{p,R}$, for every $i \in \{1,\ldots,n\}$. Moreover, we know that $\underline{\alpha}_i(\|x_i\|_i) \leq V_i(x_i)$, for every $i \in \{1, \dots, n\}$. The result then easily follows. Finally, for continuously differentiable $v \mapsto -A(v) + \Gamma(v)$, condition (C_2) implies condition (C_1) by Theorem 4.5(ii).

Remark 5.5 (Small-gain interpretation): (i) For condition (C_1) the term $- [\![-A(v),v]\!]_{p,R}$ captures the incremental dissipation gains of the subsystems while the term $[\Gamma(v), v]_{p,R}$ captures the incremental interconnection gains between subsystems. Therefore, one can interpret the condition (C_1) as a smallgain condition requiring the dissipation gains to dominate the interconnection gains.

(ii) For monotone vector field Γ , one can choose p=1 and R= $[\eta] \in \mathbb{R}^{n \times n}$ for some $\eta \in \mathbb{R}^n_{>0}$ and using Corollary 4.4 to write condition (C_1) as:

$$\eta^{\mathsf{T}} A(v) > \eta^{\mathsf{T}} \Gamma(v) + c \eta^{\mathsf{T}} v,$$

for every $v \geq \mathbb{O}_n$. This result is similar to the small-gain theorem developed in [6], [26].

(iii) Compared to the classical comparison results (see [21], [23], [26]), Proposition 5.4 does not require monotonicity of the comparison system. Instead Proposition 5.4 is based on comparing the interconnected system with a positive comparison system. As a result, contrary to the existing small-gain theorems (see [6], [26]), Proposition 5.4 can take into account both the inhibitory and excitatory nature of the interactions between the subsystems. The next example illustrates this point in more detail.

Example 5.6: Consider the following system on \mathbb{R}^2 :

$$\dot{x}_1 = -x_1 + \beta(x_2)x_1x_2^5 - 2x_1^3x_2^4
\dot{x}_2 = -x_2 + x_1^6x_2 - x_1^4x_2^3,$$
(21)

where $\beta: \mathbb{R} \to \mathbb{R}$ is such that $|\beta(r)| \leq |r|$, for every $r \in \mathbb{R}$. We choose the storage functions $V_i(x_i) = x_i^2$ for $i \in \{1, 2\}$. One can construct a monotone comparison system for the dynamics (21) as

$$\dot{V}_1 = -2V_1 + 2\beta(x_2)x_2^5V_1 - 4V_1^2V_2^2 \le -2V_1 + 2V_2^3V_1,$$

$$\dot{V}_2 = -2V_2 + 2V_1^3V_2 - 2V_1^2V_2^2 \le -2V_2 + 2V_1^3V_2.$$

Therefore, the comparison system has the form $\dot{v}=h(v)$ with $h(v)=\begin{bmatrix} -2v_1+2v_2^3v_1\\ -2v_2+2v_1^3v_2 \end{bmatrix}$. Since the Jacobian of h is Metzler on $\llbracket -A(v) + \Gamma(v), v \rrbracket_{p,R} \leq \llbracket -A(v), v \rrbracket_{p,R} + \llbracket \Gamma(v), v \rrbracket_{p,R} \leq -c \Vert v \Vert_{p,R}^2 \, \mathbb{R}^2_{\geq 0}, \text{ the comparison system } h \text{ is monotone on } \mathbb{R}^2_{\geq 0}. \text{ However, this provides a province of } \mathbb{R}^2_{\geq 0}$

comparison system has two equilibrium points $v_1 = v_2 = 0$ and $v_1 = v_2 = 1$. Therefore, it is not possible to use comparison system h to deduce global stability of \mathbb{O}_2 for the original dynamical system (21). On the other hand, one can construct a positive non-monotone comparison system for the dynamics (21) as follows:

$$\dot{V}_1 = -2V_1 + 2\beta(x_2)x_2^5V_1 - 4V_1^2V_2^2 \le -2V_1 + 2V_1V_2^3 - 4V_1^2V_2^2,$$

$$\dot{V}_2 = -2V_2 + 2V_1^3V_2 - 2V_1^2V_2^2 \le -2V_2 + 2V_2V_1^3 - 2V_1^2V_2^2.$$

Therefore, the comparison system has the from $\dot{v}=A(v)+\Gamma(v)$ with $A(v_1,v_2)=\begin{bmatrix} -2v_1\\-2v_2\end{bmatrix}$ and $\Gamma(v_1,v_2)=\begin{bmatrix} 2v_1v_2^3-4v_1^2v_2^2\\2v_2v_1^3-2v_1^2v_2^2\end{bmatrix}$. We can also define $B(v_1,v_2)=2\begin{bmatrix} -1&v_1v_2^2-2v_2v_1^2\\v_2v_1^2-v_1v_2^2&-1\end{bmatrix}$ where $B(v)v=-A(v)+\Gamma(v)$. Moreover, we get

$$\mu_{2}^{+}(B(v_{1}, v_{2})) = \sup_{x \geq 0_{2}} \frac{x^{\mathsf{T}} B x}{\|x\|_{2}^{2}} = 2 \sup_{x \geq 0_{2}} \frac{x^{\mathsf{T}} \begin{bmatrix} -1 & -v_{1}^{2} v_{2} \\ 0 & -1 \end{bmatrix} x}{\|x\|_{2}^{2}}$$
$$= 2 \sup_{x \geq 0_{2}} \frac{-x_{1}^{2} - x_{2}^{2} - v_{1}^{2} v_{2} x_{1} x_{2}}{\|x\|_{2}^{2}} \leq -2,$$

where the first equality holds by Theorem 3.3(iv) and the second equality holds by the fact that $x^{\mathsf{T}}Bx = \frac{1}{2}x^{\mathsf{T}}(B+B^{\mathsf{T}})x$. Therefore, condition (C_2) holds and, by Proposition 5.4, every trajectory of the system (21) converges to \mathbb{O}_n .

VI. CONCLUSION

In this paper, we used conic matrix measures and weak pairings to characterize contracting monotone systems and to provided sufficient conditions for exponential convergence of positive systems to their equilibriums. As applications, we used our monotone contraction results to study contractivity and robustness of Hopfield neural networks. We also used our positive contraction results to established a novel and less-conservative framework for studying stability of interconnected networks. Future work includes extension of this framework to study monotone and positive systems which are weak-or semi-contracting [15] and to characterize contractivity of systems that are monotone with respect to arbitrary cones.

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