

Distributed Neighbor Selection in Multi-agent Networks

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Abstract—Achieving consensus via nearest neighbor rules is an important prerequisite for multi-agent networks to accomplish collective tasks. A common assumption in consensus setup is that each agent interacts with all its neighbors. This paper examines whether network functionality and performance can be maintained—and even enhanced—when agents interact only with a subset of their respective (available) neighbors. As shown in the paper, the answer to this inquiry is affirmative. In this direction, we show that by exploring the monotonicity property of the Laplacian eigenvectors, a neighbor selection rule with guaranteed performance enhancements, can be realized for consensus-type networks. For distributed implementation, a quantitative connection between entries of Laplacian eigenvectors and the “relative rate of change” in the state between neighboring agents is further established; this connection facilitates a distributed algorithm for each agent to identify “favorable” neighbors to interact with. Multi-agent networks with and without external influence are examined, as well as extensions to signed networks. This paper underscores the utility of Laplacian eigenvectors in the context of distributed neighbor selection, providing novel insights into distributed data-driven control of multi-agent systems.

Index Terms—Distributed neighbor selection; Laplacian eigenvectors; convergence rate; Fiedler vector; block-cut tree; relative tempo; data-driven control.

I. INTRODUCTION

A multi-agent network is composed of a group of agents, interacting with their respective nearest neighbors by following local rules; when such local rules lead to an emerging collective behavior at the network level is of great interest [1], [2], [3]. Achieving consensus via pairwise diffusive interactions between neighboring agents is a prototypical collective behavior of multi-agent systems [4], [5], [6], [7], which also turns out to be a critical prerequisite in disciplines such as distributed control of networked systems [8], [9], distributed estimation over sensor networks [10], synchronization in complex networks [11], large-scale multi-agent machine learning [12], and opinion dynamics [13].

A. Motivation

The functionality and performance of a multi-agent network are dependent on the underlying network topology, realized via

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each agent’s interactions with its nearest neighbors [4], [5], [7]. In practice, information exchange using communication channels amongst agents are often expensive [14]. Here, an important question is whether the network performance can be maintained and even enhanced if agents interact only with a subset of their available neighbors, namely, via a neighbor selection. For instance, in leader-follower multi-robotic networks, it is often assumed that each robot interacts with all robots within a sensing radius; an important observation is that the resultant network is not necessary efficient in diffusion of information from leader robots to the followers [8]. A similar situation arises in distributed optimization and estimation, where the coordination network is realized via the spatial distribution of processors or sensors which may not be optimal for the given tasks, leading to a performance loss [15].

In fact, the neighbor selection is ubiquitous in both natural and artificial networks. For instance, it has been reported that in flocks of starlings, birds interact only with a subset of their nearest neighbors, rather than with all birds within a sensing radius [16]. An analogous scenario is observed in social networks, where an individual often determines the subset of their friends to interact with on online social media; this phenomenon also occurs in real-world social interactions amongst people [17]. Neighbor selection schemes are also employed in peer-to-peer networks, such as BitTorrent, to save traffic overhead [18]. Along the same lines, adaptive neighbor selection has been proposed to enhance the quality of predicted ratings in recommender systems [19]. The k NN imputation methods are designed to select k nearest neighbors to deal with missing data in datasets [20]. Notably, neighbor selection can also be viewed as an attention mechanism (each agent pays more attention to specific agents), which is ubiquitously employed in recently developed learning algorithms [21].

For multi-agent consensus problems, network topology plays a crucial role in both reaching consensus and the corresponding convergence rate [22], [23], [24]. A common assumption in this line of work is that each agent interacts with all its neighbors [1], [2], [3]; however, there may exist excessive interactions that degrade the performance of the multi-agent network. A natural question thereby is whether the importance of agents’ neighbors (with respect to the desired performance) can be inferred from local measurements. This (data-driven) distributed neighbor selection problem is the focus of the present work. Our work is also inspired by the observation that information flow between a pair of neighboring agents does not need to be bidirectional, especially when two neighboring agents are not hierarchical equivalent

[25]. For instance, a rooted tree exhibits a typical hierarchical structure for the efficient spreading of information from the root to other nodes- a bidirectional information exchange on the other hand may lessen the efficiency of the convergence process. In particular, we provide a theoretical framework to reason about the distributed neighbor selection problem as well as a guarantee of its performance. As we will show, specific Laplacian eigenvectors facilitate a systematic treatment for designing and analyzing a novel distributed neighbor selection algorithm for consensus-type networks.

B. Contribution

The contribution of this paper is threefold. First, we will show that neighbor selection (effectively removing a specific subset of edges from the network) can be effective for improving the network performance. In this direction, one of our contributions involves using entries of the Laplacian eigenvector as a criterion for neighbor selection; subsequently, we show how the obtained reduced network maintains, and even enhances the functionality of the original network in terms of network reachability. Secondly, inspired by the observation that bidirectional interactions amongst neighboring agents can hinder the efficiency of information propagation throughout a consensus-type network, we provide theoretical guarantees on the performance enhancement of the reduced network in terms of convergence rate. Finally, as the Laplacian eigenvectors are global network variables, we establish a quantitative connection between the entries of the Laplacian eigenvector and the relative rate of change in state between a pair of neighboring agents, a quantity that we have referred to as the network relative tempo. An important observation is that relative tempo is computable from local measurements. In this direction, we show how the relative tempo can be employed for the distributed online neighbor selection process.

The contributions of this work have several immediate consequences. First, linking local interactions and global collective behaviors is a central topic in complex systems, this work essentially initiates a novel local neighbor selection rule that leads to a reduced network with guaranteed network reachability and enhanced convergence rate at the network level [26], [27]. Second, as consensus-type networks play a central role in distributed algorithms, our work has immediate consequences for distributed control, estimation, optimization, and learning algorithm design [28], [12]. Third, the quantitative connection between entries of Laplacian eigenvectors and relative tempo provides novel insights into distributed online control of multi-agent networks [29]. Lastly, but certainly not least, as opposed to Laplacian eigenvalues (e.g., algebraic connectivity [30]) that have been extensively examined in graph theory literature and for consensus problems [4], [31], [23], this paper underscores the utility of the Laplacian eigenvectors (namely, Fiedler vector and its variant) by unveiling the network reachability that they encode (e.g., further extending the celebrated results of Fiedler in [32]), a novel application of Fiedler vector in addition to spectral clustering [32], [33], [34].

C. Organization

The remainder of this paper is organized as follows. We introduce preliminaries covering notation, graph theory, and network dynamics in §II. A motivational example is then provided and discussed in §III. The main results for semi-autonomous networks in terms of analysis of reachability of reduced networks after neighbor selection process, as well as the corresponding convergence rates, are provided in §IV; this is then followed by parallel results for fully autonomous networks in §V. Extensions of main results to signed networks are discussed in §VI, followed by concluding remarks in §VIII.

II. PRELIMINARIES

First a quick note on the notation. Let \mathbb{R} and \mathbb{Z}_+ denote the set of real numbers and positive integers, respectively. Denote the set $\{1, 2, \dots, n\}$ as \underline{n} , where $n \in \mathbb{Z}_+$; $\mathbf{1}_n$ and $\mathbf{0}_{n \times m}$ denote $n \times 1$ vector and $n \times m$ matrix of all ones and all zeros, respectively. Let I_d denote the $d \times d$ identity matrix and \mathbf{e}_j denote the j th column of I_d where $j \in \underline{d}$. The i th smallest eigenvalue and the corresponding normalized eigenvector of a symmetric matrix $M \in \mathbb{R}^{n \times n}$ is signified by $\lambda_i(M)$ and $\mathbf{v}_i(M)$, respectively. The entry located at the i th row and j th column in a matrix $M \in \mathbb{R}^{n \times n}$ is denoted by $[M]_{ij}$ and the i th entry of a vector \mathbf{x} by $[\mathbf{x}]_i$. Let \mathbf{x}_{ij} denote $\frac{[\mathbf{x}]_i}{[\mathbf{x}]_j}$ for a vector $\mathbf{x} \in \mathbb{R}^n$. The Euclidean norm of a vector $\mathbf{x} \in \mathbb{R}^n$ is designated by $\|\mathbf{x}\| = (\mathbf{x}^\top \mathbf{x})^{\frac{1}{2}}$. A vector $\mathbf{x} \in \mathbb{R}^n$ is positive if $[\mathbf{x}]_i > 0$ for all $i \in \underline{n}$. The spectral radius of a matrix M is denoted by $\rho(M)$.

Next, we provide a few graph-theoretic constructs that will be subsequently used in the paper. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ denote a graph with the node set $\mathcal{V} = \{1, 2, \dots, n\}$ and edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The adjacency matrix $W = (w_{ij}) \in \mathbb{R}^{n \times n}$ is such that the edge weight between agents i and j satisfies $w_{ij} \neq 0$ if and only if $(i, j) \in \mathcal{E}$ and $w_{ij} = 0$ otherwise. A graph \mathcal{G} is undirected if $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$, otherwise \mathcal{G} is directed. A graph \mathcal{G} is a signed graph if there exists an edge $(i, j) \in \mathcal{E}$ such that $w_{ij} < 0$, otherwise \mathcal{G} is unsigned. Let $\mathcal{N}_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$ denote the neighbor set of an agent $i \in \mathcal{V}$. A path from $i_p \in \mathcal{V}$ to $i_1 \in \mathcal{V}$ in \mathcal{G} is a concatenation of edges $(i_1, i_2), (i_2, i_3), \dots, (i_{p-1}, i_p)$, where all nodes i_1, i_2, \dots, i_p are distinct; a node $i \in \mathcal{V}$ is reachable from a node $j \in \mathcal{V}$ if there exists a path from j to i in \mathcal{G} . An undirected graph is connected if each pair of nodes are reachable from each other. Let S_n denote a star graph with $n \in \mathbb{Z}_+$ nodes. A subgraph $\tilde{\mathcal{G}} = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$ of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a graph such that $\tilde{\mathcal{V}} \subset \mathcal{V}$ and $\tilde{\mathcal{E}} \subset \mathcal{E}$. The subgraph obtained by removing a node set $\mathcal{V}' \subset \mathcal{V}$ and all incident edges from a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is denoted by $\mathcal{G} - \mathcal{V}'$. Let $\mathcal{S} \subset \mathcal{V}$ be any subset of nodes in $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Then the induced subgraph $\mathcal{G}(\mathcal{S})$ is the graph whose node set is \mathcal{S} and whose edge set consists of all of the edges incident to nodes in \mathcal{S} .

Lastly, we provide a brief synopsis of multi-agent networks¹. In a multi-agent network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, each agent $i \in \mathcal{V}$ has the state $\mathbf{x}_i(t) \in \mathbb{R}^d$ (or $\mathbf{x}_i \in \mathbb{R}^d$) at time

¹We will use “graphs” and “networks” interchangeably in this paper.

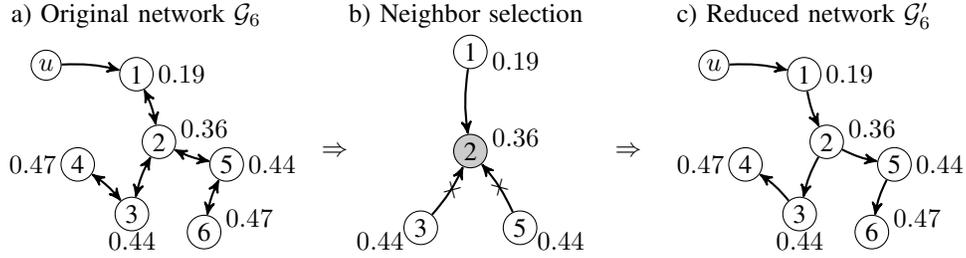


Fig. 1. The orientation of each edge indicates the direction of information flow or influence. For instance, the edge from u to agent 1 implies that the information of u can be transmitted from u to agent 1 and subsequently agent 1 can be influenced by u . The bidirectional edges between neighboring agents are identified by the line with double arrows for simplicity. Entries of $\mathbf{v}_1(L_B(\mathcal{G}_6))$ associated to each agent in \mathcal{G}_6 are indicated by numbers close to each agent.

t . In the sequel we will consider two distinct categories of diffusively coupled networks, namely, fully-autonomous and semi-autonomous.

In a fully autonomous network (FAN), $n \in \mathbb{Z}_+$ agents evolve their respective states through interactions characterized by an unsigned graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$. In particular, each agent updates its state by adopting the diffusive interaction protocol (extensively examined in distributed algorithms and synchronization problems [8], [9], [10], [11], [12]),

$$\dot{\mathbf{x}}_i(t) = - \sum_{j=1}^n w_{ij} (\mathbf{x}_i(t) - \mathbf{x}_j(t)), \quad i \in \mathcal{V}. \quad (1)$$

In relation with the protocol (1), we say the state of agent i is influenced by its neighbors $j \in \mathcal{N}_i$ or, equivalently, agent i follows its neighbors $j \in \mathcal{N}_i$. Denote the graph Laplacian of \mathcal{G} as $L(\mathcal{G}) = (l_{ij}) \in \mathbb{R}^{n \times n}$ where $l_{ii} = \sum_{j=1}^n w_{ij}$ for $i \in \mathcal{V}$ and $l_{ij} = -w_{ij}$ for $i \neq j$. The collective behavior of a FAN can be characterized as,

$$\dot{\mathbf{x}} = -(L(\mathcal{G}) \otimes I_d) \mathbf{x}, \quad (2)$$

where $\mathbf{x} = (\mathbf{x}_1^\top(t), \dots, \mathbf{x}_n^\top(t))^\top \in \mathbb{R}^{nd}$.

In semi-autonomous networks (SANs), a subset of agents (referred to as leaders or informed agents) are selected to receive external control signals so as to steer the entire network towards a desired state. In this direction, consider a SAN consisting of $n \in \mathbb{Z}_+$ agents whose interaction structure is characterized by an unsigned graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$. In a SAN, the leaders, denoted by $\mathcal{V}_{\text{leader}} \subset \mathcal{V}$, can be directly influenced by the external input signals and the remaining agents are referred to as followers, denoted by $\mathcal{V}_{\text{follower}} = \mathcal{V} \setminus \mathcal{V}_{\text{leader}}$. In this paper, the set of external inputs is denoted by $\mathcal{U} = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$, where $\mathbf{u}_l \in \mathbb{R}^d$, $l \in \underline{m}$ and $m \in \mathbb{Z}_+$. Then \mathcal{U} is homogeneous if $\mathbf{u}_i = \mathbf{u}_j$ for all $i \neq j \in \underline{m}$ and heterogeneous if otherwise. In this setup, it is assumed that each leader is at most influenced by one external input. In a SAN, the interaction protocol for each agent $i \in \mathcal{V}$ admits the form,

$$\dot{\mathbf{x}}_i(t) = - \sum_{j=1}^n w_{ij} (\mathbf{x}_i(t) - \mathbf{x}_j(t)) - \sum_{l=1}^m b_{il} (\mathbf{x}_i(t) - \mathbf{u}_l), \quad (3)$$

where $b_{il} = 1$ if and only if $i \in \mathcal{V}_{\text{leader}}$ and $b_{il} = 0$ otherwise². Subsequently, the collective behavior of SAN (3) can be characterized as,

$$\dot{\mathbf{x}} = -(L_B(\mathcal{G}) \otimes I_d) \mathbf{x} + (B \otimes I_d) \mathbf{u}, \quad (4)$$

where $\mathbf{x} = (\mathbf{x}_1^\top(t), \dots, \mathbf{x}_n^\top(t))^\top \in \mathbb{R}^{nd}$, $B = (b_{il}) \in \mathbb{R}^{n \times m}$, $\mathbf{u} = (\mathbf{u}_1^\top, \dots, \mathbf{u}_m^\top)^\top \in \mathbb{R}^{md}$ and

$$L_B(\mathcal{G}) = L(\mathcal{G}) + \mathbf{diag}(B \mathbf{1}_m), \quad (5)$$

which is referred to as perturbed Laplacian since $L_B(\mathcal{G})$ (or L_B for brevity) is obtained from a perturbation on the Laplacian matrix by a diagonal matrix $\mathbf{diag}(B \mathbf{1}_m)$ [31], [35], [23], [36], [37], [38]. The FAN (2) or SAN (4) are said to achieve consensus if $\lim_{t \rightarrow \infty} \|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| = 0$ for all $i, j \in \mathcal{V}$ and some norm on \mathbb{R}^d [4], [36]. We assume throughout this paper that the underlying networks of FAN (2) and SAN (4) are all undirected and connected before implementing neighbor selection.

III. A MOTIVATIONAL SCENARIO

We provide an example to motivate this work. Consider a SAN on a connected unsigned network \mathcal{G}_6 in Figure 1a (referred to as original network), where agent 1 is a leader with an external input $u = 0.9$. We know that reachability (existence of a directed path) from the external input to each agent is a prerequisite for the agents to track this external input u . This observation motivates us to inquire whether the remaining edges, apart from those that can guarantee the leader-follower reachability of the network, are necessary for reaching a consensus. For instance, the network \mathcal{G}'_6 (Figure 1c) is the minimal subgraph of \mathcal{G}_6 (in terms of the number of edges) that can guarantee the reachability from external input u to all the agents, the corresponding convergence performance is significantly enhanced compared with that of the original network \mathcal{G}_6 (see Figure 2). Apparently, the reduced network \mathcal{G}'_6 can be constructed from \mathcal{G}_6 by eliminating one of the bidirectional edges between neighboring agents in \mathcal{G}_6 (see Figure 1b), how the local accessible information of each agent can be employed to guide this neighbor selection process is challenging.

²The SAN (3) is also known as Taylor's model in social network analysis [13].

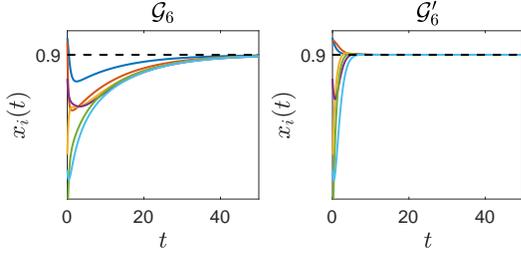


Fig. 2. State trajectories of agents in a SAN (4) evolving on networks \mathcal{G}_6 (left) and \mathcal{G}'_6 (right) in Figure 1, respectively.

Recall that the eigenvector associated with the smallest non-zero eigenvalue of perturbed Laplacian can be chosen to be positive. An important observation is that the entries of this eigenvector, along the directed paths from leader agents to all follower agents, are monotonically increasing (subsequently, we will show that this is not accidental). We will subsequently see that this monotonicity property plays an important role in the distributed neighbor selection process (see Figure 1b). In the sequel we first examine this observation analytically for SANs, followed by its implications for FANs.

IV. SEMI-AUTONOMOUS NETWORKS

In this section, a neighbor selection algorithm, based on the monotonicity of the eigenvector entries associated with the perturbed Laplacian, is proposed. Subsequently, the convergence rate of the multi-agent system on the reduced network, post neighbor selection process, will be examined. Furthermore, the distributed implementation of the neighbor selection process is discussed.

A. Reachability Analysis

We shall first examine the reachability property of SANs, as encoded in a Laplacian eigenvector of the underlying network. The eigen-pair $(\lambda_1(L_B), \mathbf{v}_1(L_B))$ associated with the perturbed Laplacian L_B of a SAN, with input matrix B , turns out to be an important algebraic construct revealing graph-theoretic properties of SANs. As such, we shall unveil the network reachability, encoded in the eigenvector $\mathbf{v}_1(L_B)$, providing useful insights for designing neighbor selection algorithm for SANs.

First, we provide some preliminary properties of $(\lambda_1(L_B), \mathbf{v}_1(L_B))$.

Lemma 1. *Let $\lambda_1(L_B)$ and $\mathbf{v}_1(L_B)$ denote the smallest eigenvalue and the corresponding normalized eigenvector of L_B in (5), respectively. Then, $\lambda_1(L_B) > 0$ is a simple eigenvalue of L_B and $\mathbf{v}_1(L_B)$ can be chosen to be (component-wise) positive.*

Proof. Refer to the Appendix. \square

For a SAN \mathcal{G} with the input matrix B , we proceed to construct a reduced subgraph of \mathcal{G} by eliminating a subset of edges between an agent and its neighboring agents, using information encoded in $\mathbf{v}_1(L_B)$, namely, realizing neighbor selection. We shall refer to this class of reduced subgraphs as

following the slower neighbor (FSN) networks of \mathcal{G} , since it is implied that each agent follows (or chooses to be influenced by) those neighbors whose rate of change in states are relatively slower; this statement will be made rigorous in §IV-C.

Definition 1 (FSN network of SANs). Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be an unsigned SAN with the input matrix B . The FSN network of \mathcal{G} , denoted by $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{W})$, is a subgraph of \mathcal{G} such that $\bar{\mathcal{V}} = \mathcal{V}$, $\bar{\mathcal{E}} \subseteq \mathcal{E}$ and $\bar{W} = (\bar{w}_{ij}) \in \mathbb{R}^{n \times n}$, where $\bar{w}_{ij} = w_{ij}$ if $\mathbf{v}_1(L_B)_{ij} > 1$ and $\bar{w}_{ij} = 0$ when $\mathbf{v}_1(L_B)_{ij} \leq 1$.

According to Definition 1, the FSN network of a SAN is determined by the perturbed Laplacian, specifically by the corresponding eigenvector $\mathbf{v}_1(L_B)$. Note that the construction of the FSN network is essentially achieved by comparing $\mathbf{v}_1(L_B)_{ij}$ and 1; hence this process can be regarded as a “rule” of neighbor selection. We shall now proceed to reveal the leader-to-follower reachability (LF-reachability) of FSN networks.

Theorem 1. *Let $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{W})$ be the FSN network of the SAN (4) on the unsigned connected network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$. Then for each agent $i \in \bar{\mathcal{V}}$, there exists $l \in \underline{m}$ such that i is reachable from u_l in $\bar{\mathcal{G}}$.*

Proof. Let $\mathcal{V}_{\text{leader}}$ and $\mathcal{V}_{\text{follower}}$ be the leader and the follower sets of the SAN (4), respectively. According to Lemma 5 in Appendix, it is sufficient to show that for an arbitrary $i \in \mathcal{V}_{\text{follower}}$, there exists a leader agent $l \in \mathcal{V}_{\text{leader}}$ such that i is reachable from l .

By contradiction, assume that there exists a subset of agents $\{i_1, i_2, \dots, i_s\} \subset \mathcal{V}_{\text{follower}}$ in the FSN network $\bar{\mathcal{G}}$ such that i_k is not reachable from any $l \in \mathcal{V}_{\text{leader}}$, where $k \in \underline{s}$ and $s \in \mathbb{Z}_+$. Let λ_1 be the smallest eigenvalue of the perturbed Laplacian matrix $L_B(\mathcal{G})$ with the corresponding eigenvector \mathbf{v}_1 . According to Lemma 1, one has $\lambda_1 > 0$ and the corresponding eigenvector \mathbf{v}_1 is positive. Now consider the following two cases:

Case 1: There exists an isolated agent $i' \in \{i_1, i_2, \dots, i_s\}$ such that agent i' is not reachable from any leader agent in the FSN network $\bar{\mathcal{G}}$. Then, according to Definition 1, one has,

$$[\mathbf{v}_1]_{i'} \leq [\mathbf{v}_1]_j, \quad (6)$$

for all $j \in \mathcal{N}_{i'}$. Examining the i' th row in eigen-equation $L_B(\mathcal{G})\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ yields,

$$\left(\sum_{j \in \mathcal{N}_{i'}} w_{i'j} \right) [\mathbf{v}_1]_{i'} - \sum_{j \in \mathcal{N}_{i'}} w_{i'j} [\mathbf{v}_1]_j = \lambda_1 [\mathbf{v}_1]_{i'}. \quad (7)$$

Combining (6) and (7), now yields the following inequality,

$$\left(\sum_{j \in \mathcal{N}_{i'}} w_{i'j} \right) [\mathbf{v}_1]_{i'} - \sum_{j \in \mathcal{N}_{i'}} w_{i'j} [\mathbf{v}_1]_{i'} \geq \lambda_1 [\mathbf{v}_1]_{i'}. \quad (8)$$

By eliminating $[\mathbf{v}_1]_{i'} > 0$ from both sides of the above inequality, it follows that $\lambda_1 \leq 0$, establishing a contradiction.

Case 2: There exists a weak connected component $\bar{\mathcal{G}}(\{i_1, i_2, \dots, i_{s_0}\})$ in $\{i_1, i_2, \dots, i_s\}$, such that any agent in

this weak connected component is not reachable from any leader agent, where $s_0 \in \mathbb{Z}_+$ and $s_0 \leq s$. Let

$$[\mathbf{v}_1]_{i'} = \min_{k \in \{i_1, i_2, \dots, i_{s_0}\}} \{[\mathbf{v}_1]_k\}. \quad (9)$$

Then, one has $[\mathbf{v}_1]_j \geq [\mathbf{v}_1]_{i'}$ for all $j \in \mathcal{N}_{i'}$. Again, one can conclude the contradiction $\lambda_1 \leq 0$ by applying a similar procedure as in Case 1. \square

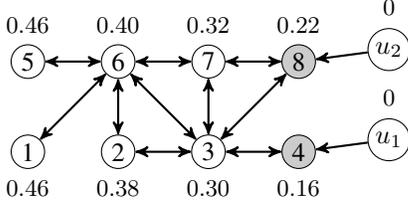


Fig. 3. An eight-node SAN \mathcal{G}_8 . The entries of $\mathbf{v}_1(L_B)$ corresponding to each agent are shown close to each node (with a two decimal point accuracy).

It turns out that the entries of the eigenvector $\mathbf{v}_1(L_B)$ are influenced by the selection of leader agents. We provide an example to demonstrate the utility of Theorem 1.

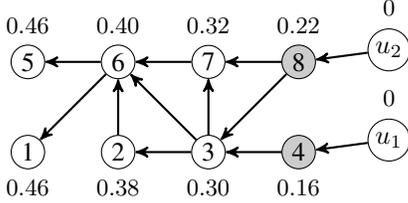


Fig. 4. The FSN network corresponding to the network \mathcal{G}_8 in Figure 3.

Example 1. Consider a SAN on the network \mathcal{G}_8 shown in Figure 3; each agent holds a three-dimensional state and agents 4 and 8 are leaders that are directly influenced by the homogeneous input $\mathbf{u} = (\mathbf{u}_1^\top, \mathbf{u}_2^\top)^\top$, where $\mathbf{u}_1 = \mathbf{u}_2 = (0.7, 0.8, 0.9)^\top \in \mathbb{R}^3$. The initial states of agents are randomly selected from $[0, 1] \times [0, 1] \times [0, 1]$. Computing $\mathbf{v}_1(L_B)$ corresponding to the perturbed Laplacian in this example yields,

$$\mathbf{v}_1(L_B) = (0.46, 0.38, 0.30, 0.16, 0.46, 0.40, 0.32, 0.22)^\top.$$

One can observe from Figure 3 that for each agent $i \in \mathcal{V}$, there exists a directed path from \mathbf{u}_1 or \mathbf{u}_2 to i such that the entries in $\mathbf{v}_1(L_B)$ along this path are monotonically increasing. Therefore, the associated FSN network according to Definition 1, is as shown in Figure 4. One can observe from Figure 6 that each agent tends to track the external input directly in the FSN network (see Figure 6b) instead of aggregating and moving together towards the external input, this is shown in Figure 6a.

Theorem 1 ensures that all agents in the FSN network of a SAN are influenced by the external inputs, namely, LF-reachability can be guaranteed. Therefore, a SAN (4)

can exhibit either consensus or clustering over the corresponding FSN network, depending on heterogeneity of the external input; see Lemma 5 in Appendix. One can verify that constructing the FSN network using eigenvectors other than $\mathbf{v}_1(L_B)$ do not ensure the LF-reachability according to Definition 1.

Inspired by Theorem 1, we postulate that if one reverses the construction of FSN network for SANs (each agent now follows neighbors whose respective rates of change in state are relatively faster), the influence of external input exerted on the network can be weakened or even eliminated. One can refer to the resulting reduced network as following the faster neighbor (FFN) network. In this case, agents in the FFN network are not reachable from the external input. This can be useful when the external input, say, represents epidemics or rumors, and the network structure is rearranged in a distributed manner by each agent to attenuate the spreading process. For example, the FFN network of \mathcal{G}_8 in Figure 3 is shown in Figure 5; in this case, the influence from external inputs to leaders can be eliminated since the rate of change in state of external inputs can be viewed as zero. The trajectory of SAN on FFN network is shown on the right-hand plot in Figure 6. In FFN networks, the influence structure is reversed in contrast to FSN network. Therefore, only leader agents (agents 4 and 8) are influenced by external inputs and as a result, the influence of external inputs on the follower agents have been eliminated. In this paper, we shall concentrate on FSN networks; such networks closely abstract means of enhancing the spreading process on a network.

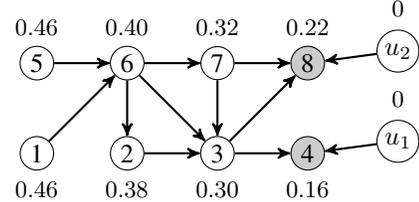


Fig. 5. The FFN network corresponding to the network \mathcal{G}_8 in Figure 3.

B. Convergence Rate Enhancement

In order to evaluate the performance of neighbor selection based on $\mathbf{v}_1(L_B)$, we now proceed to examine the convergence rate of SANs on the resultant FSN networks. Note that the smallest non-zero eigenvalue of the perturbed Laplacian of a SAN characterizes the convergence rate of the multi-agent system towards its steady-state, either consensus or clustering [23], [35], [31]. We provide the following result on the convergence rate of SAN on connected unsigned networks and the corresponding FSN networks.

Theorem 2. Let $\bar{\mathcal{G}} = (\mathcal{V}, \bar{\mathcal{E}}, \bar{W})$ denote the FSN network of a SAN $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ with the input matrix B . Then

$$\lambda_1(L_B(\bar{\mathcal{G}})) \geq \lambda_1(L_B(\mathcal{G})),$$

where equality holds only when all agents are leaders.

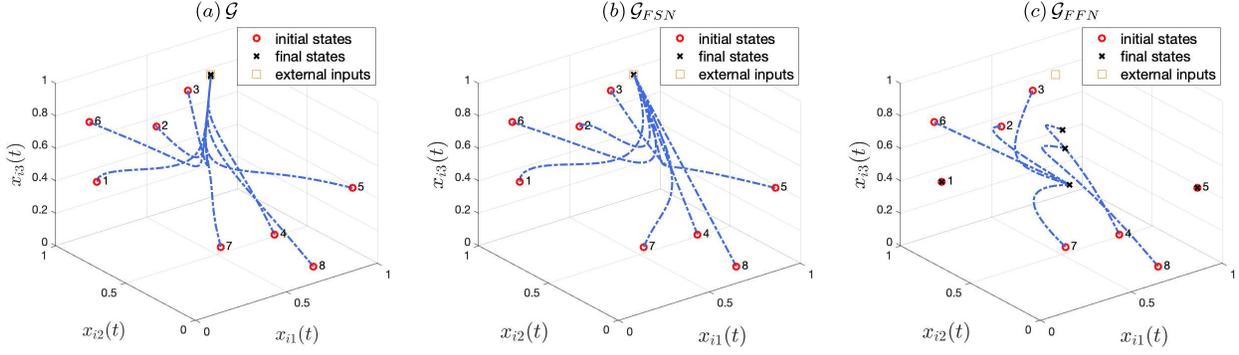


Fig. 6. State trajectories of agents in SAN on the network \mathcal{G}_8 shown in Figure 3 as well as its associated FSN network (Figure 4) and FFN network (Figure 5), respectively.

Proof. Denote the perturbed Laplacian matrix $L_B(\bar{\mathcal{G}})$ and $L_B(\mathcal{G})$ as \bar{L}_B and L_B , respectively. According to Definition 1, the FSN network $\bar{\mathcal{G}}$ is a directed acyclic network. Therefore, one can relabel the agents in $\bar{\mathcal{G}}$ such that \bar{W} is a lower triangular matrix, implying that \bar{L}_B is also lower triangular. Thus, the eigenvalues of \bar{L}_B are exactly the entries on its diagonal.

Note that each agent in the FSN network has at least one in-degree neighbor, namely, the diagonal entries, satisfying that $[\bar{L}_B]_{ii} \geq 1$ for all $i \in \mathcal{V}$, which in turn, implies that $\lambda_1(\bar{L}_B) \geq 1$. In particular, if all agents are leaders ($\mathbf{diag}(B\mathbf{1}_m) = I$), the eigenvector corresponding to $\lambda_1(L_B)$ is $a_0\mathbf{1}_n$ ($a_0 \in \mathbb{R}$). Then according to Definition 1, all the edges in the graph \mathcal{G} will be eliminated. Therefore, $\lambda_1(\bar{L}_B) = 1$. Recall that $L_B = L + \mathbf{diag}(B\mathbf{1}_m)$. Applying Weyl theorem ([39, Theorem 4.3.1, p.239]), one has,

$$\lambda_1(L_B) \leq \lambda_1(L) + \lambda_n(\mathbf{diag}(B\mathbf{1}_m)); \quad (10)$$

due to the fact $\lambda_1(L) = 0$ and $\lambda_n(\mathbf{diag}(B\mathbf{1}_m)) = 1$, it follows that $\lambda_1(L_B) \leq 1$.

On the one hand, if $\mathbf{diag}(B\mathbf{1}_m) = I$, again using Weyl theorem, it follows that,

$$\lambda_1(L_B) \geq \lambda_1(L) + \lambda_1(I) = 1; \quad (11)$$

hence, $\lambda_1(L_B) = 1$. Now suppose that $\mathbf{diag}(B\mathbf{1}_m) \neq I$. Let

$$L_B = L + I + \Delta, \quad (12)$$

where $\Delta \in \mathbb{R}^{n \times n}$ is a non-zero diagonal matrix whose diagonal entries are either -1 or 0 . In fact, (12) produces all possible perturbed Laplacians apart from the case that all agents are leaders. Without loss of generality, we choose $\Delta = \mathbf{diag}(-1, 0, \dots, 0)^\top$. Then, by applying Weyl theorem one more time, we have,

$$\lambda_1(L + I + \Delta) \leq \lambda_1(L + I) + \lambda_n(\Delta) = 1. \quad (13)$$

According to $(L + I)(a\mathbf{1}_n) = a\mathbf{1}_n$, where $a \in \mathbb{R}$, it follows that $\mathbf{span}\{\mathbf{1}_n\}$ is an eigenspace of the matrix $L + I$ corresponding to the eigenvalue $\lambda_1(L + I)$.

For the matrix Δ in (13), assume that there exists $a_0 \in \mathbb{R}$ such that,

$$\Delta(a_0\mathbf{1}_n) = \lambda_n(\Delta)a_0\mathbf{1}_n; \quad (14)$$

as $\Delta(a_0\mathbf{1}_n) = (-a_0, 0, \dots, 0)^\top$ and $\lambda_n(\Delta)a_0\mathbf{1}_n = (0, 0, \dots, 0)^\top$, one has $a_0 = 0$.

Thus, there does not exist a common non-zero eigenvector corresponding to $\lambda_1(L + I + \Delta)$, $\lambda_1(L + I)$ and $\lambda_n(\Delta)$, respectively. According to Weyl theorem, one has,

$$\lambda_1(L + I + \Delta) < 1. \quad (15)$$

Thus, one can conclude that $\lambda_1(\bar{L}_B) > \lambda_1(L_B)$ if $\mathbf{diag}(B\mathbf{1}_m) \neq I$ and $\lambda_1(\bar{L}_B) = \lambda_1(L_B) = 1$ when $\mathbf{diag}(B\mathbf{1}_m) = I$. \square

Theorem 2 provides theoretical guarantees on the convergence rate of FSN network $\bar{\mathcal{G}}$ as compared with the original network \mathcal{G} . Let us continue to employ Example 1 to demonstrate the convergence rate of SAN on the original network \mathcal{G} and its related FSN network $\bar{\mathcal{G}}$. The convergence rate comparison of SAN in Example 1 on both original network and the associated FSN network is demonstrated in Figure 7. In this case, the convergence rate of SAN on the original network \mathcal{G} and FSN network \mathcal{G}_{FSN} are $\lambda_1(L_B(\mathcal{G})) = 0.1414$ and $\lambda_1(L_B(\mathcal{G}_{FSN})) = 1$, respectively.

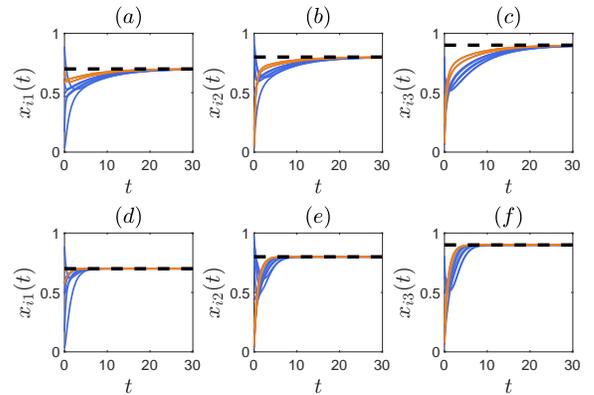


Fig. 7. State trajectories of agents in the SAN (4) on the network in Figure 3 ((a)-(c)). State trajectories of agents in the SAN (4) on the FSN network in Figure 4 ((d)-(f)). The orange and blue lines in each plot are trajectories of leader and follower agents, respectively. The dotted line in each panel represents external input.

C. Neighbor Selection in SANs

So far, we have presented a neighbor selection framework with guaranteed performance using the eigenvector of perturbed Laplacian. However, this eigenvector is a network-level quantity, hindering the direct applicability of this setup for large-scale networks. For such networks, it is desirable that decision-making relies only on local observations [40].

In this section, we establish a quantitative link between the Laplacian eigenvector and the relative rate of change in the state of neighboring agents, referred to as the relative tempo. Using this approach, we connect the global property of the network to a locally measurable quantity—as such, we are able to propose a fully distributed neighbor selection algorithm. In order to simplify our derivation, we first introduce the so-called selection matrix.

The selection matrix of a subset of agents $\mathcal{V}' = \{i_1, \dots, i_s\} \subset \mathcal{V}$ is defined as $\phi(\mathcal{V}') = (e_{i_1}, \dots, e_{i_s})^\top \in \mathbb{R}^{s \times n}$. We now proceed to introduce the notion of relative tempo, characterizing the steady-state of the relative rate of change in state between two subsets of agents.

Definition 2. Let $\mathcal{V}_1 \subset \mathcal{V}$ and $\mathcal{V}_2 \subset \mathcal{V}$ be two subsets of agents in multi-agent network (4) (or (2)). Then the relative tempo between agents in \mathcal{V}_1 and \mathcal{V}_2 is defined as the limiting ratio,

$$\mathbb{L}(\mathcal{V}_1, \mathcal{V}_2) = \lim_{t \rightarrow \infty} \frac{\|\phi(\mathcal{V}_1) \otimes I_d \dot{\mathbf{x}}(t)\|}{\|\phi(\mathcal{V}_2) \otimes I_d \dot{\mathbf{x}}(t)\|}, \quad (16)$$

where $\phi(\mathcal{V}_1)$ and $\phi(\mathcal{V}_2)$ are selection matrices associated with \mathcal{V}_1 and \mathcal{V}_2 , respectively.

The relative tempo in Definition 2 was initially examined in [41], characterizing relative influence of agents in consensus-type networks, and subsequently being employed to construct a centrality measure that can be inferred from network data [42]. This paper provides a more systematic treatment for the application of relative tempo in the distributed neighbor selection problem. As we shall see subsequently, the limit in (16) exists, implying that the relative tempo is well-defined. We now proceed to formally provide a quantitative connection between relative tempo and the Laplacian eigenvector.

Theorem 3. Let $\mathcal{V}_1 \subset \mathcal{V}$ and $\mathcal{V}_2 \subset \mathcal{V}$ be two subsets of agents in the SAN (4). Then

$$\mathbb{L}(\mathcal{V}_1, \mathcal{V}_2) = \frac{\|\phi(\mathcal{V}_1) \mathbf{v}_1(L_B)\|}{\|\phi(\mathcal{V}_2) \mathbf{v}_1(L_B)\|}.$$

Proof. Refer to the Appendix \square

Remark 1. Theorem 3 provides a quantitative connection between the relative tempo (constructed from local observations of each agent) and the Laplacian eigenvector of the underlying network. According to Theorem 1 and Theorem 2, such a connection enables a distributed implementation of neighbor selection for enhancing the convergence rate of the network. We provide an example to illustrate Theorem 3.

Example 2. Consider the following quantity

$$g_{ij}(t) = \frac{\|\dot{\mathbf{x}}_i(t)\|}{\|\dot{\mathbf{x}}_j(t)\|}, \quad i \in \mathcal{V}, j \in \mathcal{N}_i, \quad (17)$$

which satisfies $\lim_{t \rightarrow \infty} g_{ij}(t) = \mathbb{L}(i, j)$ (by Definition 2). Let us continue to examine Example 1. The trajectories of $g_{ij}(t)$ for $i = 7$ and $j \in \{3, 6, 8\}$ are shown in Figure 8. The steady-states of $g_{ij}(t)$ are archived at around $t = 10$, particularly, $g_{73}(10) = 1.057$, $g_{76}(10) = 0.8123$ and $g_{78}(10) = 1.47$, respectively. In the meanwhile, one has $\mathbf{v}_1(L_B)_{73} = 1.0577$, $\mathbf{v}_1(L_B)_{76} = 0.8113$ and $\mathbf{v}_1(L_B)_{78} = 1.4694$. Note that such correspondences are sufficient for the construction of the associated FSN network (shown in Figure 4) using Definition 1.

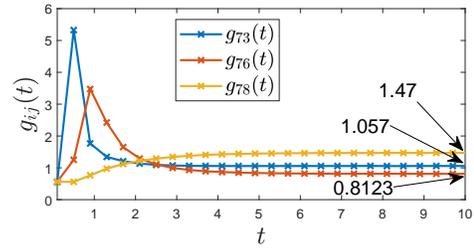


Fig. 8. Trajectories of $g_{ij}(t)$ for agent $i = 7$ and its neighbors $j \in \{3, 6, 8\}$ in the SAN shown in Figure 3.

Remark 2. Examining the convergence rate of agents' states in the original network shown in Figure 7 and that of $g_{ij}(t)$ in Figure 8, one observes that the SAN (4) achieves an "ordered" state characterized by the relative tempo, prior to the final consensus.

By Definition 1 and Theorem 3, the reduced neighbor set for each agent for constructing the FSN network can be defined as follows.

Definition 3. The reduced neighbor set for agent $i \in \mathcal{V}$ to construct the associated FSN network for SAN (4) is defined as

$$\mathcal{N}_i^{\text{FSN}} = \{j \in \mathcal{N}_i \mid \mathbf{v}_1(L_B)_{ij} > 1\} = \{j \in \mathcal{N}_i \mid \mathbb{L}(i, j) > 1\}. \quad (18)$$

According to Definition 2, if $\mathbb{L}(i, j) > 1$ for a pair of neighboring agents $i, j \in \mathcal{V}$, then the state of agent i evolves towards the external input in a relatively faster rate than that of agent j . Therefore, agents in a SAN are involved in a sort of hierarchy encoded in $\mathbf{v}_1(L_B)$ according to Theorem 3. As such, each agent can select a specific group of neighbors to interact with for a given task. The main insight from our discussion is that $\mathbf{v}_1(L_B)_{ij}$ can be estimated for agent $i \in \mathcal{V}$ and $j \in \mathcal{N}_i$ via only local measurements, and the obtained reduced network exhibits LF-reachability property, as stated by Theorem 1.

To end the discussion on SANs, we provide the following algorithm to summarize the flowchart of distributed neighbor selection process for constructing FSN networks. For the algorithm implementation, we choose the sampling step size $\delta > 0$ to discretize agent state evolution as $\tilde{x}_i(k) = x_i(t_0 + k\delta)$ for all $i \in \mathcal{V}$, where $t_0 = 0$ and $k = 0, 1, \dots$

Algorithm 1 Distributed neighbor selection for SANs.

Initialization:

- 1: set $k = 1$
- 2: **for** each agent $i \in \mathcal{V}$ **do**
- 3: choose the termination threshold $\varepsilon_i > 0$
- 4: receives $\tilde{x}_i(0)$ and $\tilde{x}_i(1)$ from $j \in \mathcal{N}_i$
- 5: computes $g_{ij}(k) = \frac{\|\tilde{x}_i(k) - \tilde{x}_i(k-1)\|}{\|\tilde{x}_j(k) - \tilde{x}_j(k-1)\|}$
- 6: **end for**

Loop:

- 7: **repeat**
 - 8: set $k = k + 1$
 - 9: **for** each agent $i \in \mathcal{V}$ **do**
 - 10: receives $\tilde{x}_i(k)$ from $j \in \mathcal{N}_i$
 - 11: computes $g_{ij}(k) = \frac{\|\tilde{x}_i(k) - \tilde{x}_i(k-1)\|}{\|\tilde{x}_j(k) - \tilde{x}_j(k-1)\|}$
 - 12: **end for**
 - 13: **until** $\|g_{ij}(k) - g_{ij}(k-1)\| < \varepsilon_i, \forall j \in \mathcal{N}_i$
 - 14: $\bar{w}_{ij} = \begin{cases} w_{ij}, & g_{ij}(k) > 1, \\ 0, & g_{ij}(k) \leq 1. \end{cases}$
-

Note from Algorithm 1 that each agent only uses local accessible state information to construct FSN networks.

V. FULLY-AUTONOMOUS NETWORKS

In this section, we proceed to investigate parallel results for FANs- the corresponding analysis turns out to be more intricate than those for SANs.

A. Reachability Analysis

Recall that for the eigenvector associated with perturbed Laplacian matrix L_B in SANs (4), all elements of $\mathbf{v}_1(L_B)$ have the same sign. However, in the case of FANs, the entries in eigenvectors of graph Laplacian can be positive, negative or equal to zero; this is also valid for the Fiedler vector $\mathbf{v}_2(L)$, the eigenvector corresponding to the second smallest eigenvalue of graph Laplacian [30], [32], [43]. This situation renders the extension of the aforementioned neighbor selection framework -from SANs to FANs- non-trivial.

In this section, we shall first examine the property of Laplacian eigenvectors related to SANs, specifically the Fiedler vector $\mathbf{v}_2(L)$, and then proceed to provide the neighbor selection algorithm to construct the FSN network of FANs for fast convergence. In the following discussions, we shall refer to a node corresponding to positive, negative or zero entry in $\mathbf{v}_2(L)$ as a positive node, negative node and zero node, respectively.

In FANs, the structural properties of the network turn out to be critical in the analysis and design of the neighbor selection algorithm. We introduce the following results related to the block decomposition of a graph [44], [32].

A cut node $i \in \mathcal{V}$ of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ is a node such that $\mathcal{G} - \{i\}$ is disconnected. A block of a graph \mathcal{G} is a maximal connected subgraph of \mathcal{G} with no cut nodes. Two blocks of \mathcal{G} are the neighboring blocks if they are connected via a cut node. Consider a connected graph \mathcal{G} with blocks $\{B_i\}$ and cut nodes $\{c_j\}$, where $i, j \in \mathbb{Z}_+$. The block-cut graph of \mathcal{G} , denoted by

$\mathcal{B}(\mathcal{G})$, is defined as the graph with node set composed of blocks and cut nodes, namely, $\mathcal{V}(\mathcal{B}(\mathcal{G})) = \{B_i\} \cup \{c_j\}$, where two nodes are adjacent if one corresponds to a block B_i and the other to a cut node such that $c_j \in B_i$.

Lemma 2. [44] *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ be a connected graph. Then the block-cut graph of \mathcal{G} is a tree.*

Here, we provide an example to illustrate the block-cut tree associated with a connected graph.

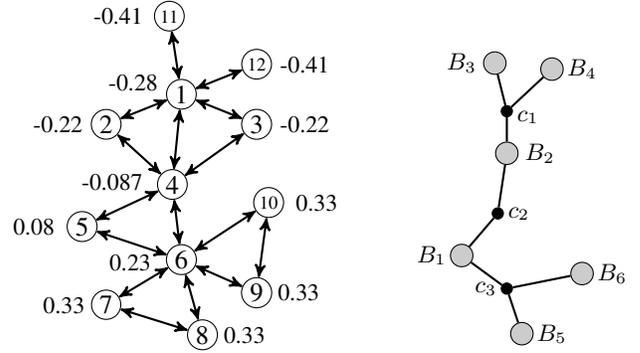


Fig. 9. An unsigned network \mathcal{G}_{12} with 12 nodes (left) and the associated block-cut tree (right). The entry in $\mathbf{v}_2(L)$ corresponding to each agent is shown close to each node. The block decomposition of network \mathcal{G}_{12} is shown in the right where the black nodes represent cut nodes in \mathcal{G}_{12} and the grey nodes represent blocks in \mathcal{G}_{12} .

Example 3. Consider a FAN \mathcal{G}_{12} shown on the left-hand plot of Figure 9. There are six blocks in \mathcal{G}_{12} , that is,

$$\begin{aligned} B_1 &= \mathcal{G}(\{4, 5, 6\}), & B_2 &= \mathcal{G}(\{1, 2, 3, 4\}), \\ B_3 &= \mathcal{G}(\{1, 11\}), & B_4 &= \mathcal{G}(\{1, 12\}), \\ B_5 &= \mathcal{G}(\{6, 7, 8\}), & B_6 &= \mathcal{G}(\{6, 9, 10\}). \end{aligned}$$

There are three cut nodes $c_1 = \{1\}$, $c_2 = \{4\}$, $c_3 = \{6\}$. The block-cut tree of \mathcal{G}_{12} is shown on the right panel in Figure 9. Blocks B_1 and B_2 are neighboring blocks since they are connected via cut node $c_2 = \{4\}$.

The following result reveals the monotonicity property of the entries in $\mathbf{v}_2(L)$ along certain paths in block-cut tree of a graph, which will subsequently be used for constructing the FSN network associated with FANs.

Lemma 3. [32, Theorem 3.12] *Let \mathcal{G} be a connected graph with Laplacian matrix L ; let $\lambda_2(L)$ and $\mathbf{v}_2(L)$ be the second smallest eigenvalue of L and the corresponding eigenvector, respectively. Then exactly one of the following two cases occurs:*

Case 1. There is a single block B_0 in \mathcal{G} which contains both positive and negative nodes. Each other block has either positive nodes only, or negative nodes only, or zero nodes only. Every path P starting in B_0 and containing just one node k in B_0 has the property that the entries in $\mathbf{v}_2(L)$ corresponding to cut nodes contained in P form either an increasing, or decreasing, or a zero sequence along this path according to

whether $[v_2(L)]_k > 0$, $[v_2(L)]_k < 0$ or $[v_2(L)]_k = 0$; in the last case all nodes in P are zero nodes.

Case 2. No block of \mathcal{G} contains both positive and negative nodes. There exists a single zero cut node with a non-zero node neighbor. Each block (with the exception of that zero cut node) has either positive nodes only, or negative nodes only, or zero nodes only. Every path P starting in that zero cut node has the property that the entries in $v_2(L)$ corresponding to cut nodes contained in P form either an increasing, or decreasing, or a zero sequence along this path and in the last case all nodes in P are zero nodes. Every path containing both positive and negative nodes passes through that zero cut node.

As an example, the FAN \mathcal{G}_{12} in Figure 9 satisfies Case 1 in the Lemma 3. In the following discussions, we will refer to the block B_0 in Case 1 and the zero cut node in Case 2 in the Lemma 3 as core block and core node, respectively; we will refer to a block having only positive, negative, or zero nodes (with the exception of that core node) as a positive block, negative block, and zero block, respectively. We are now ready to present the construction of FSN networks for FANs.

Definition 4 (FSN Network of FANs). Let $\{B_1, \dots, B_r\}$ be the block decomposition of an unsigned network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ where $r \in \mathbb{Z}_+$. The FSN network of \mathcal{G} is its subgraph $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{W})$ with node set $\bar{\mathcal{V}} = \mathcal{V}$, edge set $\bar{\mathcal{E}} \subseteq \mathcal{E}$ and adjacency matrix $\bar{W} = (\bar{w}_{ij}) \in \mathbb{R}^{n \times n}$ that satisfies,

- 1) if B_p ($p \in \mathcal{P}$) is a positive or negative block, then for each $i \in B_p$ and $j \in B_p \cap \mathcal{N}_i$,

$$\bar{w}_{ij} = \begin{cases} w_{ij}, & v_2(L)_{ij} > 1 \text{ or } v_2(L)_{ij} < 0, \\ 0, & 1 \geq v_2(L)_{ij} \geq 0; \end{cases}$$

- 2) for all remaining $(i, j) \in \mathcal{E}$, $\bar{w}_{ij} = w_{ij}$.

According to Definition 4, the construction of the FSN network is built upon the block decomposition of a graph; the edges in the core block and zero block will remain unchanged while the other edges can be eliminated depending on the quantity $v_2(L)_{ij}$. We now proceed to examine the reachability of the FSN network associated with FANs.

Theorem 4. Let $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{W})$ be the FSN network of the FAN (2) on $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$. Then, the FAN (2) achieves consensus on the associated FSN network $\bar{\mathcal{G}}$. Moreover, the consensus value is the average of the initial values of either the agents in the core block and zero blocks or the core node and the agents in zero blocks.

Proof. Refer to the Appendix. \square

According to the Theorem 4, the FAN (2) can achieve consensus on the associated FSN network $\bar{\mathcal{G}}$, however, the consensus value is generally not equal to the consensus value achieved by the original network (average of initial states of all agents). In fact, the consensus value achieved on the FSN network $\bar{\mathcal{G}}$ is eventually the average of the initial states of agents belonging to the core block and the zero blocks or the average of the initial states of the core node and agents belonging to zero blocks. We provide the following example

to demonstrate the reachability property of the FSN network $\bar{\mathcal{G}}_{12}$ corresponding to the network \mathcal{G}_{12} in the left plot of Figure 9.

Example 4. The FSN network $\bar{\mathcal{G}}_{12}$ corresponding to the network \mathcal{G}_{12} in Figure 9 is shown in Figure 10. The core block in \mathcal{G}_{12} is $B_0 = \mathcal{G}(\{4, 5, 6\})$. As one can see from Figure 10, all agents except that in the core block B_0 are reachable from agents in the core block.

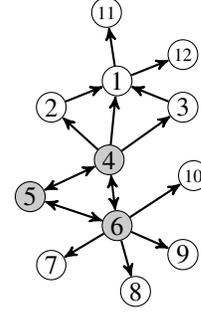


Fig. 10. The FSN network $\bar{\mathcal{G}}_{12}$ corresponding to the network \mathcal{G}_{12} in the left plot of Figure 9. The core block in \mathcal{G}_{12} is $B_0 = \mathcal{G}(\{4, 5, 6\})$ which is highlighted in dark.

Remark 3 (Generalized Monotonicity of Fiedler Vector). Remarkably, Theorem 4 further extends the Lemma 3 (a celebrated result by Fiedler [32]) by revealing the monotonicity property of Fiedler's entries within each block, rather than only on cut nodes.

B. Convergence Rate Enhancement

We proceed to examine the convergence rate enhancement of FANs on the corresponding FSN networks. First, we provide the following result, for general FANs, that characterize the convergence rate of FAN (2) on the associated FSN network.

Proposition 1. Let $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{W})$ be the FSN network of a FAN $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ characterized by (2). Let $\lambda_2(L(\bar{\mathcal{G}}))$ and \bar{v}_2 be the second smallest eigenvalue of $L(\bar{\mathcal{G}})$ and the corresponding normalized eigenvector, respectively. Then $\lambda_2(L(\bar{\mathcal{G}}))$ is lower bounded by

$$\lambda_2(L(\bar{\mathcal{G}})) + \sum_{(i,j) \in \bar{\mathcal{E}} \setminus \bar{\mathcal{E}}} [\bar{v}_2]_i ([\bar{v}_2]_j - [\bar{v}_2]_i). \quad (19)$$

Proof. Refer to the Appendix. \square

One can observe from Proposition 1 that the quantitative relationship between $\bar{\lambda}_2(L(\bar{\mathcal{G}}))$ and $\lambda_2(L(\bar{\mathcal{G}}))$ is a bit vague since the second term in (19) can be negative. This motivates us to examine the special case of tree graphs- in which case, the convergence rate between a FAN and the corresponding FSN network can be well-characterized.

Theorem 5. Let \mathcal{T} be an n -node tree network without zero blocks where $n \geq 4$. Let $\bar{\mathcal{T}}$ be its FSN network characterized by (2). Let $\lambda_2(L(\bar{\mathcal{T}}))$ and $\lambda_2(L(\mathcal{T}))$ be the second smallest eigenvalue of $L(\bar{\mathcal{T}})$ and $L(\mathcal{T})$, respectively. Then $\lambda_2(L(\bar{\mathcal{T}})) > \lambda_2(L(\mathcal{T}))$.

Proof. Refer to the Appendix. \square

Example 5. Consider a 12-node FAN with the tree structure in Figure 11 (left) whose associated FSN network is shown in Figure 11 (right). The agents' initial states are $[\mathbf{x}(0)]_1 = 0.973$, $[\mathbf{x}(0)]_2 = 0.649$, $[\mathbf{x}(0)]_3 = 0.8$, $[\mathbf{x}(0)]_4 = 0.454$, $[\mathbf{x}(0)]_5 = 0.432$, $[\mathbf{x}(0)]_6 = 0.825$, $[\mathbf{x}(0)]_7 = 0.084$, $[\mathbf{x}(0)]_8 = 0.133$, $[\mathbf{x}(0)]_9 = 0.173$, $[\mathbf{x}(0)]_{10} = 0.391$, $[\mathbf{x}(0)]_{11} = 0.831$, $[\mathbf{x}(0)]_{12} = 0.803$.

In this example, the core block in tree \mathcal{T} is the induced subgraph $\mathcal{T}(\{4, 6\})$. According to Figure 12, the FAN (2) on its associated FSN network $\tilde{\mathcal{G}}_{12}$ with the aforementioned initial states achieves consensus on the value 0.6396, which is equal to the average of the initial states of the agents in the core block, namely, $\frac{1}{2}([\mathbf{x}(0)]_4 + [\mathbf{x}(0)]_6) = 0.6396$. Computing $\mathbf{v}_2(L)$ corresponding to the Laplacian matrix L of \mathcal{T} in this example yields, $[\mathbf{v}_2(L)]_1 = 0.333$, $[\mathbf{v}_2(L)]_2 = 0.101$, $[\mathbf{v}_2(L)]_3 = 0.101$, $[\mathbf{v}_2(L)]_4 = 0.079$, $[\mathbf{v}_2(L)]_5 = 0.101$, $[\mathbf{v}_2(L)]_6 = -0.257$, $[\mathbf{v}_2(L)]_7 = -0.327$, $[\mathbf{v}_2(L)]_8 = -0.327$, $[\mathbf{v}_2(L)]_9 = -0.327$, $[\mathbf{v}_2(L)]_{10} = -0.327$, $[\mathbf{v}_2(L)]_{11} = 0.424$, $[\mathbf{v}_2(L)]_{12} = 0.424$. Trajectories of $g_{ij}(t)$ in (17) for $i = 1$ and $j \in \{3, 5, 11, 12\}$ in the FAN \mathcal{T} shown in Figure 13. One can see that $g_{13}(t) \rightarrow \frac{[\mathbf{v}_2(L)]_1}{[\mathbf{v}_2(L)]_3} \approx 3.299$ and $g_{1,11}(t) \rightarrow \frac{[\mathbf{v}_2(L)]_1}{[\mathbf{v}_2(L)]_{11}} \approx 0.785$. Moreover, $g_{15}(t)$ and $g_{1,12}(t)$ exhibit the same tendency due to network symmetry.

The convergence rate associated with networks \mathcal{T} and \mathcal{T}_{FSN} are $\lambda_2(L(\mathcal{T})) = 0.2148$ and $\lambda_2(L(\mathcal{T}_{FSN})) = 1$, respectively.

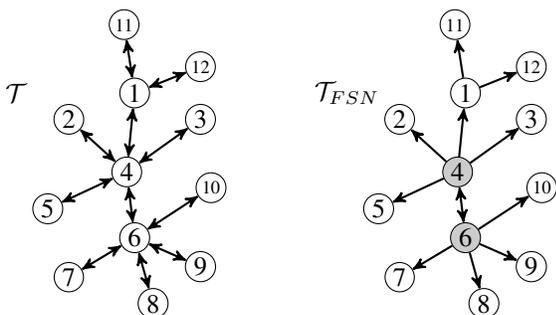


Fig. 11. A tree \mathcal{T} (left) and its corresponding FSN network \mathcal{T}_{FSN} where the core block $B = \{4, 6\}$ is highlighted in dark (right).

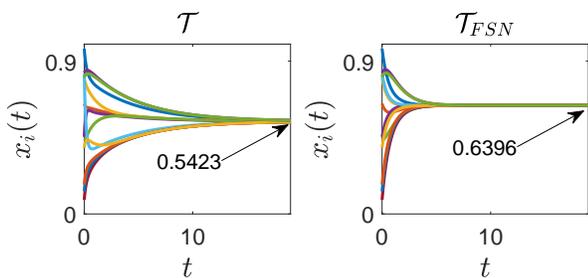


Fig. 12. State trajectories of agents in FAN on the network \mathcal{T} shown in the left plot of Figure 11 as well as its associated FSN network \mathcal{T}_{FSN} in the right plot of Figure 11.

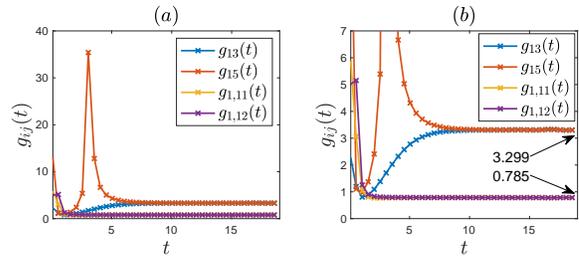


Fig. 13. Trajectories of $g_{ij}(t)$ in (17) for $i = 1$ and $j \in \{3, 5, 11, 12\}$ in the FAN \mathcal{T} shown in Figure 11.

Remark 4. Notably, the convergence rate of FSN network corresponding to trees are always equal to one, however the convergence rate of a tree can decrease dramatically when the diameter of the tree grows. For example, that convergence rate on trees with diameter $\text{diam}(\mathcal{T})$ is upper bounded by $2 \left(1 - \cos\left(\frac{\pi}{\text{diam}(\mathcal{T})+1}\right)\right)$, signifying that $\lim_{\text{diam}(\mathcal{T}) \rightarrow \infty} \lambda_2(L(\mathcal{T})) = 0$.

C. Neighbor Selection in FANs

We now examine networks whose second smallest eigenvalue of the Laplacian matrix is simple. The following result establishes the relationship between the relative tempo and the Fiedler vector of a network.

Theorem 6. Let \mathcal{V}_1 and \mathcal{V}_2 be two subsets of agents in \mathcal{V} . Each agent in \mathcal{V} adopts dynamics (1). If the second smallest eigenvalue of the Laplacian matrix L is simple, then the relative tempo of agents in \mathcal{V}_1 compared to that of \mathcal{V}_2 is

$$\mathbb{L}(\mathcal{V}_1, \mathcal{V}_2) = \frac{\|\phi(\mathcal{V}_1)\mathbf{v}_2(L)\|}{\|\phi(\mathcal{V}_2)\mathbf{v}_2(L)\|}.$$

Proof. According to Lemma 6 in Appendix, the proof follows by choosing $M = -L \otimes I_d$. \square

On the one hand, if the knowledge of both the core block and zero block of a FAN is available, according to the Definition 4 and Theorem 6, one can employ relative tempo to construct the FSN network instead of using the information in $\mathbf{v}_2(L)$. Under the FSN network, a consensus at the value of the average of initial states associated with agents in the core block and zero block, or the average of initial states associated with agents in zero block and initial state of core node, can be reached. In the meanwhile, the FSN network can be constructed in a distributed manner, which is similar to SANs.

On the other hand, if the knowledge of both the core block and zero block of the network \mathcal{G} is unavailable, a natural question is whether one can determine the core block and zero block from the network data. For general networks, this might be challenging. However, such a data-driven approach can be adopted for tree networks, as every node in a tree is a cut node; the core block in a tree network contains at most two nodes.

Here, we further discuss a class of the tree networks without zero blocks and examine how to construct their FSN networks only using local observations similar to the relative tempo. In fact, according to the proof of Lemma 6 in the Appendix, one can see that $\lim_{t \rightarrow \infty} \frac{e_1^\top \dot{\mathbf{x}}_u(t)}{e_1^\top \dot{\mathbf{x}}_v(t)} = \frac{[\mathbf{v}_2]_u}{[\mathbf{v}_2]_v}$, where $u, v \in \mathcal{V}$. Note that for a tree network without zero blocks, either there exists a core block containing two nodes $\{u, v\}$ such that $[\mathbf{v}_2]_u[\mathbf{v}_2]_v < 0$, or there exists a core node $\{w\}$ such that $[\mathbf{v}_2]_w = 0$. For the former case, one can use the quantity $\mathbb{L}'(u, v) = \lim_{t \rightarrow \infty} \frac{e_1^\top \dot{\mathbf{x}}_u(t)}{e_1^\top \dot{\mathbf{x}}_v(t)}$ instead of $\mathbb{L}(u, v)$ to identify nodes u and v in the core block, then the edges between u and v are reserved while the remaining edges shall be treated according to Definition 4. For the latter case, one can also use the quantity $\mathbb{L}'(u, v)$ to construct the FSN network according to Definition 4.

According to Definition 4 and Theorem 6, the reduced neighbor set for constructing FSN network of FAN on tree networks can be defined as follows.

Definition 5. Let $\mathcal{T} = (\mathcal{V}, \mathcal{E}, W)$ be a tree network without zero blocks. The reduced neighbor set for $i \in \mathcal{V}$ to construct the associated FSN network of FAN (2) is

$$\begin{aligned} \mathcal{N}_i^{\text{FSN}} &= \{j \in \mathcal{N}_i \mid \mathbf{v}_2(L(\mathcal{T}))_{ij} > 1 \text{ or } \mathbf{v}_2(L(\mathcal{T}))_{ij} < 0\} \\ &= \{j \in \mathcal{N}_i \mid \mathbb{L}'(i, j) > 1 \text{ or } \mathbb{L}'(i, j) < 0\}. \end{aligned}$$

To sum up, we have established the parallel framework of distributed neighbor selection for FANs with a specific focus on tree networks.

VI. EXTENSION TO SIGNED NETWORKS

In this section, we discuss extensions of the aforementioned results to structurally balanced signed networks. A signed network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ is structurally balanced if there exists a bipartition of the node set \mathcal{V} (hence, $\mathcal{V}_1 \subset \mathcal{V}$ and $\mathcal{V}_2 \subset \mathcal{V}$ such that $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$) such that the edge weights within each subset are positive, but negative for edges between the two subsets [45]. For structurally balanced signed SANs, the eigenvectors of the perturbed Laplacian can be transformed from the unsigned SANs via Gauge transformations [46]. We now discuss the extension of our results for structurally balanced signed networks.

A. Signed Semi-Autonomous Networks

Consider the interaction protocol,

$$\begin{aligned} \dot{\mathbf{x}}_i(t) &= - \sum_{j=1}^n |w_{ij}| (\mathbf{x}_i(t) - \mathbf{sgn}(w_{ij}) \mathbf{x}_j(t)) \\ &\quad - \sum_{l=1}^m |b_{il}| (\mathbf{x}_i(t) - \mathbf{sgn}(b_{il}) \mathbf{u}_l), \quad i \in \mathcal{V}, \end{aligned} \quad (20)$$

where $b_{il} \in \{1, -1\}$ if and only if $i \in \mathcal{V}_{\text{leader}}$ and $b_{il} = 0$ otherwise. The sign function $\mathbf{sgn}(\cdot)$ is such that $\mathbf{sgn}(z) = 1$ for $z > 0$, $\mathbf{sgn}(z) = -1$ for $z < 0$ and $\mathbf{sgn}(z) = 0$ for $z = 0$.

Denote the signed Laplacian matrix of \mathcal{G} as $L^s = (l_{ij}^s) \in \mathbb{R}^{n \times n}$, where $l_{ii}^s = \sum_{j=1}^n |w_{ij}|$ for $i \in \mathcal{V}$ and $l_{ij}^s = -w_{ij}$ for $i \neq j$. The collective dynamics of (20) is then,

$$\dot{\mathbf{x}} = -(L_B^s(\mathcal{G}) \otimes I_d) \mathbf{x} + (B \otimes I_d) \mathbf{u}, \quad (21)$$

where $L_B^s(\mathcal{G}) = L^s + \mathbf{diag}(|B| \mathbf{1}_m)$, $\mathbf{x} = (\mathbf{x}_1^\top, \dots, \mathbf{x}_n^\top)^\top \in \mathbb{R}^{dn}$, $B = [b_{il}] \in \mathbb{R}^{n \times m}$ and $\mathbf{u} = (\mathbf{u}_1^\top, \dots, \mathbf{u}_m^\top)^\top \in \mathbb{R}^{dm}$. Denote by the edge set between external inputs and the leaders and the input set as \mathcal{E}' and $\mathcal{U} = (\mathbf{u}_1, \dots, \mathbf{u}_m)$, respectively. The augmented graph $\widehat{\mathcal{G}} = (\widehat{\mathcal{V}}, \widehat{\mathcal{E}}, \widehat{W})$ is directed with $\widehat{\mathcal{V}} = \mathcal{V} \cup \mathcal{U}$, $\widehat{\mathcal{E}} = \mathcal{E} \cup \mathcal{E}'$ and $\widehat{W} = \begin{pmatrix} W & B \\ \mathbf{0}_{m \times n} & \mathbf{0}_{m \times m} \end{pmatrix}$. The signed Laplacian matrix of the network $\widehat{\mathcal{G}}$ is positive semi-definite if $\widehat{\mathcal{G}}$ is structurally balanced [46].

Lemma 4. Consider the signed SAN (21) on a signed network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$. Suppose that $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ is connected and $\widehat{\mathcal{G}} = (\widehat{\mathcal{V}}, \widehat{\mathcal{E}}, \widehat{W})$ is structurally balanced, and let $\lambda_1(L_B^s)$ and $\mathbf{v}_1(L_B^s)$ be the smallest eigenvalue of L_B^s and the corresponding normalized eigenvector, respectively. Then, $\lambda_1(L_B^s) > 0$ is a simple eigenvalue of L_B^s and $\mathbf{v}_1(L_B^s)$ is positive under a proper Gauge transformation.

Proof. The proof is an immediate extension of Lemma 1 and omitted for brevity. \square

The FSN network for signed SANs can be defined as follows.

Definition 6 (FSN network of signed SANs). Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be a signed SAN characterized by (21). The FSN network of \mathcal{G} , denoted by $\widehat{\mathcal{G}} = (\widehat{\mathcal{V}}, \widehat{\mathcal{E}}, \widehat{W})$, is a subgraph of \mathcal{G} such that $\widehat{\mathcal{V}} = \mathcal{V}$, $\widehat{\mathcal{E}} \subseteq \mathcal{E}$ and $\widehat{W} = (\widehat{w}_{ij}) \in \mathbb{R}^{n \times n}$, where $\widehat{w}_{ij} = w_{ij}$ if $|\mathbf{v}_1(L_B^s)_{ij}| > 1$ and $\widehat{w}_{ij} = 0$ if $|\mathbf{v}_1(L_B^s)_{ij}| \leq 1$.

Theorem 7. Let $\widehat{\mathcal{G}} = (\widehat{\mathcal{V}}, \widehat{\mathcal{E}}, \widehat{W})$ be the FSN network of the signed SAN $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ characterized by (21). Suppose that $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ is connected and $\widehat{\mathcal{G}} = (\widehat{\mathcal{V}}, \widehat{\mathcal{E}}, \widehat{W})$ is structurally balanced; then all agents in $\widehat{\mathcal{G}}$ are reachable from the external input.

Proof. Note that for a structurally balanced signed network, there exists a quantitative connection between traditional Laplacian matrix and signed Laplacian matrix via a Gauge transformation, assuming matrix form $G = \mathbf{diag}\{\sigma_1, \dots, \sigma_n\}$, where $\sigma_i \in \{1, -1\}$ and $i \in \underline{n}$ [46]. The proof (omitted for brevity) follows from Lemma 5, proof of Theorem 1, and applying the Gauge transformation corresponding to the signed network \mathcal{G} . \square

Theorem 8. Let $\mathcal{V}_1 \subset \mathcal{V}$ and $\mathcal{V}_2 \subset \mathcal{V}$ be two subsets of agents in a connected signed SAN $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ characterized by (21). If $\widehat{\mathcal{G}} = (\widehat{\mathcal{V}}, \widehat{\mathcal{E}}, \widehat{W})$ is structurally balanced, then the relative tempo between agents in \mathcal{V}_1 and \mathcal{V}_2 satisfies $\mathbb{L}(\mathcal{V}_1, \mathcal{V}_2) = \frac{\|\phi(\mathcal{V}_1) \mathbf{v}_1(L_B^s)\|}{\|\phi(\mathcal{V}_2) \mathbf{v}_1(L_B^s)\|}$.

Proof. According to Lemma 6, the proof follows by choosing $M = \begin{pmatrix} -L_B^s \otimes I_d & B \otimes I_d \\ \mathbf{0}_{md \times nd} & \mathbf{0}_{md \times md} \end{pmatrix}$ in Lemma 6 in Appendix. \square

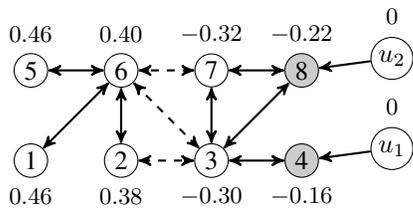


Fig. 14. An eight-node structurally balanced signed network \mathcal{G}_8 with two agents directly influenced by external inputs. The solid lines and dashed lines represent the edges weighted by positive and negative numbers, respectively. The entry in the $\mathbf{v}_1(L_B^s)$ corresponding to each agent is shown close to each node.

Theorem 9. Let $\bar{\mathcal{G}} = (\mathcal{V}, \bar{\mathcal{E}}, \bar{\mathcal{W}})$ be the FSN network of a connected signed SAN $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ characterized by (21). If $\hat{\mathcal{G}} = (\hat{\mathcal{V}}, \hat{\mathcal{E}}, \hat{\mathcal{W}})$ is structurally balanced, then $\lambda_1(L_B^s(\hat{\mathcal{G}})) \geq \lambda_1(L_B^s(\mathcal{G}))$.

Proof. The proof is a straightforward extension of Theorem 2, and omitted for brevity. \square

We now provide an example to illustrate the aforementioned results on signed SANs.

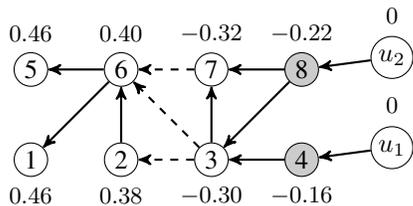


Fig. 15. FSN network of the structurally balanced signed network in Figure 14.

Example 6. Consider a signed SAN on the network \mathcal{G}_8 shown in Figure 14, each agent holds a three-dimensional state and agents 4 and 8 are leaders that are directly influenced by the homogeneous input $\mathbf{u} = (\mathbf{u}_1^\top, \mathbf{u}_2^\top)^\top$, where $\mathbf{u}_1 = \mathbf{u}_2 = (0.7, 0.8, 0.9)^\top \in \mathbb{R}^3$. The associated FSN network is shown in Figure 15. As one can see from Figure 16, the convergence rate of the bipartite consensus is significantly improved on the associated FSN network.

B. Signed Fully-Autonomous Networks

For the case of signed FANs, consider the interaction protocol,

$$\dot{\mathbf{x}}_i(t) = - \sum_{j=1}^n |w_{ij}| (\mathbf{x}_i(t) - \mathbf{sgn}(w_{ij}) \mathbf{x}_j(t)), i \in \mathcal{V}, \quad (22)$$

whose collective dynamics is

$$\dot{\mathbf{x}} = -(L^s(\mathcal{G}) \otimes I_d) \mathbf{x}. \quad (23)$$

Denote the unsigned network corresponding to the signed network \mathcal{G} as $\hat{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, |\mathcal{W}|)$, with Laplacian matrix L . It is

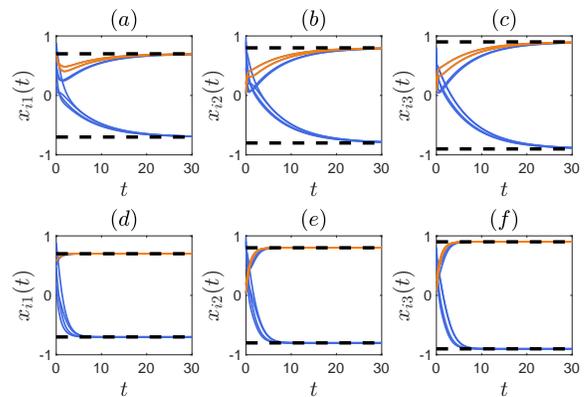


Fig. 16. State trajectories of agents in the signed SAN (21) on the structurally balanced signed network in Figure 14 ((a)-(c)). State trajectories of agents in the signed SAN (21) on FSN network in Figure 15 ((d)-(f)). The orange and blue lines in each plot are trajectories of leader and follower agents, respectively. The dotted lines in each plot represent external inputs.

shown that $L = GL^sG$, where G is the Gauge transformation corresponding to the structurally balanced signed network \mathcal{G} [46]. This correspondence implies that the Laplacian eigenvectors can be respectively transformed via a Gauge transformation. Therefore, the results on the unsigned FANs can be extended to the structurally balanced signed FANs through a proper Gauge transformation on the Fielder vector.

VII. ACKNOWLEDGEMENT

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VIII. CONCLUSIONS REMARKS

This paper addresses distributed neighbor selection problem of multi-agent networks. In this direction, a theoretical framework of distributed neighbor selection for diffusively coupled multi-agent networks has been established. Along the way, we have highlighted the utility of Laplacian eigenvectors to further improve network performance; these eigenvectors encode hierarchical information about the network that in turn, relate to the notion of relative tempo. The latter connection is then used in a data-driven setting for the neighbor selection problem.

Future works in this direction include extensions to directed and/or time-varying networks, multi-agent systems with general individual dynamics, and neighbor selection with noisy and delayed time-series data. Furthermore, a notable feature of multi-agent networks is their robustness to node/link failures. As such, it is often the case that more links are favorable for the functionality of multi-agent systems, e.g., convergence rate of the underlying coordination algorithm. Hence, an interesting problem is to examine the optimal trade-off between network robustness and size of the simplified network.

APPENDIX

The appendix contains the proofs of various results discussed in the paper.

PROOF OF LEMMA 1

Proof. Note that the perturbed Laplacian matrix L_B is symmetric and diagonal dominant; as such, $\lambda_i(L_B) \geq 0$ for all $i \in \underline{n}$. Assume that L_B has an eigenvalue $\lambda_1(L_B) = 0$ with associated eigenvector $\mathbf{v}_1(L_B) \in \mathbb{R}^n$. Then,

$$L_B \mathbf{v}_1(L_B) = L \mathbf{v}_1(L_B) + \mathbf{diag}(B \mathbf{1}_m) \mathbf{v}_1(L_B) = 0. \quad (24)$$

Multiply the above equality by $\mathbf{v}_1^\top(L_B)$ from left yields,

$$\mathbf{v}_1^\top(L_B) L \mathbf{v}_1(L_B) + \mathbf{v}_1^\top(L_B) \mathbf{diag}(B \mathbf{1}_m) \mathbf{v}_1(L_B) = 0. \quad (25)$$

Since, $\mathbf{v}_1^\top(L_B) L \mathbf{v}_1(L_B) \geq 0$, and $\mathbf{v}_1^\top(L_B) \mathbf{diag}(B \mathbf{1}_m) \mathbf{v}_1(L_B) \geq 0$, one has, $\mathbf{v}_1^\top(L_B) L \mathbf{v}_1(L_B) = 0$, and $\mathbf{v}_1^\top(L_B) \mathbf{diag}(B \mathbf{1}_m) \mathbf{v}_1(L_B) = 0$. This however means that $\mathbf{v}_1(L_B) = \mathbf{1}_n$, leading to having $\mathbf{v}_1^\top(L_B) \mathbf{diag}(B \mathbf{1}_m) \mathbf{v}_1(L_B) > 0$. This is a contradiction and therefore $\lambda_i > 0$ for all $i \in \underline{n}$.

We shall proceed to show that $\lambda_1(L_B)$ is simple and $\mathbf{v}_1(L_B)$ is positive. Denote by $L_B = \eta I - M$, where $M \in \mathbb{R}^{n \times n}$ is a non-negative matrix and η is the maximum value of the diagonal entries of L_B . Then $e^{-L_B} = e^{M - \eta I} = e^{-\eta I} e^M$. Note that the matrix M is non-negative, therefore, e^{-L_B} is a non-negative matrix. In addition, since the network \mathcal{G} is connected, M is irreducible, implying that e^{-L_B} is a non-negative irreducible matrix. Thus, according to Perron-Frobenius theorem for irreducible non-negative matrices, the eigenvalue $e^{-\lambda_1(L_B)}$ is simple and the corresponding eigenvector $\mathbf{v}_1(L_B)$ is positive. \square

STEADY-STATE OF SAN ON UNSIGNED NETWORKS

In the case of unsigned networks, the steady-state of SAN (4) is consensus (when the external input is homogeneous) or cluster consensus (when the external input is heterogeneous) [36]. Formally, the steady-state of the SAN (4) is determined by the convex hull spanned by external inputs, namely, $\mathbf{Co}(\mathcal{U}) = \{\sum_{i=1}^m k_i \mathbf{u}_i \mid \mathbf{u}_i \in \mathcal{U}, k_i \geq 0, \sum_{i=1}^m k_i = 1\}$. Then following lemma characterizes the steady-state of the SAN (4) on unsigned networks.

Lemma 5. [6], [7], [36] *Consider the SAN (4) on an unsigned network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$. Then, the state of all agents converge to the convex hull spanned by the external inputs for arbitrary initial conditions if and only if for each agent $i \in \mathcal{V}$, there exists at least one external input $\mathbf{u}_i \in \mathcal{U}$ such that i is reachable from \mathbf{u}_i . Moreover, the steady-state of the SAN (4) admits, $\lim_{t \rightarrow \infty} \mathbf{x}(t) = (L_B^{-1} \otimes I_d)(B \otimes I_d) \mathbf{u} = (L_B^{-1} B) \otimes I_d \mathbf{u}$. Specifically, if \mathbf{u} is homogeneous, then the SAN (4) achieves consensus, namely, $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \frac{1}{m} (\mathbf{1}_n \mathbf{1}_m^\top \otimes I_d) \mathbf{u}$.*

PROOF OF THEOREM 3

In order to show Theorem 3, we need the following lemma.

Lemma 6. *Consider a matrix ordinary differential equation $\dot{\mathbf{x}}(t) = M \mathbf{x}(t)$, where $M \in \mathbb{R}^{n \times n}$ is symmetric and has n linearly independent eigenvectors and $\mathbf{x}(t) = (x_1(t), x_2(t), \dots, x_n(t))^\top$. Denote the ordered eigenvalue of M as $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ with associated mutually perpendicular normalized eigenvectors $\varphi_1, \varphi_2, \dots, \varphi_n$.*

Let $\lambda_{k_1} = \lambda_{k_2} = \dots = \lambda_{k_s}$ be the largest nonzero eigenvalue of M with the algebraic multiplicity $s \in \underline{n}$. Let $\psi(\eta_1) = (e_{i_1}, \dots, e_{i_{s_1}})^\top \in \mathbb{R}^{s_1 \times n}$ and $\psi(\eta_2) = (e_{j_1}, \dots, e_{j_{s_2}})^\top \in \mathbb{R}^{s_2 \times n}$ where $\eta_1 = \{i_1, \dots, i_{s_1}\} \subset \underline{n}$ and $\eta_2 = \{j_1, \dots, j_{s_2}\} \subset \underline{n}$, respectively. Denote $\alpha_{qi} = \psi(\eta_q) \varphi_i = \psi_q \varphi_i \in \mathbb{R}^{s_q}$, $S = [\varphi_1, \varphi_2, \dots, \varphi_n] \in \mathbb{R}^{n \times n}$ and $\beta = [\beta_1, \beta_2, \dots, \beta_n]^\top = S^{-1} \mathbf{x}(0) \in \mathbb{R}^n$ for $q \in \underline{2}$ and $i \in \underline{n}$. Then

$$\lim_{t \rightarrow \infty} \frac{\|\psi_1 \dot{\mathbf{x}}(t)\|}{\|\psi_2 \dot{\mathbf{x}}(t)\|} = \left(\frac{\sum_{i,j=1}^s \lambda_{k_i} \lambda_{k_j} \alpha_{1k_i}^\top \alpha_{1k_j} \beta_{k_i} \beta_{k_j}}{\sum_{i,j=1}^s \lambda_{k_i} \lambda_{k_j} \alpha_{2k_i}^\top \alpha_{2k_j} \beta_{k_i} \beta_{k_j}} \right)^{\frac{1}{2}}. \quad (26)$$

Proof. Note that $M = S J S^{-1}$ where $S = [\varphi_1, \varphi_2, \dots, \varphi_n] \in \mathbb{R}^{n \times n}$ and $J = \mathbf{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\} \in \mathbb{R}^{n \times n}$. According to the solution to the matrix ordinary differential equation $\dot{\mathbf{x}}(t) = M \mathbf{x}(t)$, the derivative of $\mathbf{x}(t)$ is $\dot{\mathbf{x}}(t) = M e^{Mt} \mathbf{x}(0) = S J e^{Jt} S^{-1} \mathbf{x}(0)$. Therefore one has,

$$\begin{aligned} \|\psi_q \dot{\mathbf{x}}(t)\|^2 &= (\dot{\mathbf{x}}(t))^\top \psi_q^\top \psi_q \dot{\mathbf{x}}(t) \\ &= \mathbf{x}(0)^\top (S^{-1})^\top e^{Jt} J S^\top \psi_q^\top \psi_q S J e^{Jt} S^{-1} \mathbf{x}(0) \\ &= \sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j e^{(\lambda_i + \lambda_j)t} \alpha_{qi}^\top \alpha_{qj} \beta_i \beta_j. \end{aligned} \quad (27)$$

The statement of the lemma now follows from straightforward computation, that has been omitted for brevity. \square

We are now in the position to prove Theorem 3.

Proof. Choose $M = \begin{pmatrix} -L_B \otimes I_d & B \otimes I_d \\ \mathbf{0}_{md \times nd} & \mathbf{0}_{md \times md} \end{pmatrix}$ in Lemma 6. Since the algebraic multiplicity of the largest nonzero eigenvalue of the matrix L_B is 1, the proof then follows from a straightforward computation. \square

PROOF OF THEOREM 4

In order to prove Theorem 4, we need the following results.

Lemma 7. *Let $\mathcal{G}(\mathcal{S})$ be a connected induced subgraph of $\mathcal{G}(\mathcal{V})$ whose node set is $\mathcal{S} \in \mathcal{V}$. Denote $\mathcal{E}^{\text{bound}} = \{(i, j) \in \mathcal{E} \mid i \in \mathcal{S} \text{ and } j \in \mathcal{V} \setminus \mathcal{S}\}$ and $\mathcal{V}_S^{\text{out-bound}} = \{j \in \mathcal{V} \setminus \mathcal{S} \mid \exists i \in \mathcal{S} \text{ such that } (i, j) \in \mathcal{E}^{\text{bound}}\}$. If for all $i \in \mathcal{S}$ and $j \in \mathcal{V}_S^{\text{out-bound}}$, $[\mathbf{v}_2]_i$ and $[\mathbf{v}_2]_j$ have the same signs, then there exists an edge $(i, j) \in \mathcal{E}^{\text{bound}}$ such that $|[\mathbf{v}_2]_i| > |[\mathbf{v}_2]_j|$.*

Proof. Assume that $[\mathbf{v}_2]_i$ and $[\mathbf{v}_2]_j$ are positive for all $i \in \mathcal{S}$ and $j \in \mathcal{V}_S^{\text{out-bound}}$ and there does not exist an edge $(i, j) \in \mathcal{E}^{\text{bound}}$ such that $[\mathbf{v}_2]_i > [\mathbf{v}_2]_j$. Let λ_2 be the second smallest eigenvalue of $L(\mathcal{G})$ with the corresponding eigenvector $\mathbf{v}_2 = ([\mathbf{v}_2]_1, [\mathbf{v}_2]_2, \dots, [\mathbf{v}_2]_n)^\top \in \mathbb{R}^n$. Denote the agents in \mathcal{S} as $\{i_1, i_2, \dots, i_p\}$; then examining all the i_k th row in eigen-equation $L(\mathcal{G}) \mathbf{v}_2 = \lambda_2 \mathbf{v}_2$, yields,

$$\left(\sum_{j \in \mathcal{N}_{i_k}} w_{i_k j} \right) [\mathbf{v}_2]_{i_k} - \sum_{j \in \mathcal{N}_{i_k}} w_{i_k j} [\mathbf{v}_2]_j = \lambda_2 [\mathbf{v}_2]_{i_k}, \quad (28)$$

for all $k \in \underline{p}$. Thereby,

$$\sum_{(i,j) \in \mathcal{E}^{bound}} w_{ij}([\mathbf{v}_2]_i - [\mathbf{v}_2]_j) = \lambda_2 \left(\sum_{i \in \mathcal{S}} [\mathbf{v}_2]_i \right). \quad (29)$$

Since $[\mathbf{v}_2]_i - [\mathbf{v}_2]_j \leq 0, \forall (i,j) \in \mathcal{E}^{bound}$, one can conclude that $\lambda_2 \leq 0$, which is a contradiction given the fact that $\lambda_2 > 0$ and there exists an edge $(i,j) \in \mathcal{E}^{bound}$ such that $[\mathbf{v}_2]_i > [\mathbf{v}_2]_j$.

For the case that $[\mathbf{v}_2]_i$ and $[\mathbf{v}_2]_j$ are negative for all $i \in \mathcal{S}$ and $j \in \mathcal{V}_S^{out-bound}$, the proof is analogous. \square

Lemma 8. [32, Theorem 3.3] *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be an unsigned network with associated Laplacian matrix L . Let \mathbf{v}_2 denote the eigenvector corresponding to the second smallest eigenvalue of L . For any $r_1 \geq 0$ and $r_2 \leq 0$, let $M(r_1) = \{i \in \mathcal{V} \mid [\mathbf{v}_2]_i + r_1 \geq 0\}$ and $M(r_2) = \{i \in \mathcal{V} \mid [\mathbf{v}_2]_i + r_2 \leq 0\}$. Then, the subgraph $\mathcal{G}(r_1)$ ($\mathcal{G}(r_2)$) induced by \mathcal{G} on $M(r_1)$ ($M(r_2)$) is connected.*

We are now ready to prove Theorem 4.

Proof. According to Lemma 5, consensus can be examined as a reachability problem; we shall therefore consider the reachability aspect first. According to Lemma 3, we need to discuss two cases.

Case 1: The network \mathcal{G} contains a core block B_0 , we shall discuss the reachability of the positive/negative blocks from B_0 . Without the loss of generality, let B_i be a positive block, then two possible situations may occur in terms of the location of B_i .

First, consider the case where B_i is a positive block connecting with the core block B_0 directly through the cut node i^* . We shall prove that for any node $j \in B_i$, $[\mathbf{v}_2]_j \geq [\mathbf{v}_2]_{i^*}$. By contradiction, assume that there exists a node $j_0 \in B_i$ satisfying $[\mathbf{v}_2]_{j_0} < [\mathbf{v}_2]_{i^*}$. As i^* is a cut node, according to Lemma 8, for any node $p \in B_0$, one has $[\mathbf{v}_2]_p \geq [\mathbf{v}_2]_{i^*}$; this is a contradiction (with the property of B_0). Therefore, one has $[\mathbf{v}_2]_j \geq [\mathbf{v}_2]_{i^*}$ for any node $j \in B_i$.

Second, consider the case where B_i is a positive block such that all its neighboring blocks are positive. Denote by $\mathcal{N}_i^B = \{B_{i_1}, B_{i_2}, \dots, B_{i_{q_i}}\}$ as the neighboring blocks of B_i with the corresponding cut nodes $\{i_1, i_2, \dots, i_{q_i}\}$. We denote $i^* = \underset{k \in \underline{q_i}}{\operatorname{argmin}} [\mathbf{v}_2]_{i_k}$, which connects blocks B_i and B_{i^*} . Then

we show that for any node $j \in B_i$, $[\mathbf{v}_2]_j \geq [\mathbf{v}_2]_{i^*}$. By contradiction, assume that there exists $j_0 \in B_i$ satisfying $[\mathbf{v}_2]_{j_0} < [\mathbf{v}_2]_{i^*}$; then according to Lemma 8, for any node $p \in B_{i^*}$, one has $[\mathbf{v}_2]_p \geq [\mathbf{v}_2]_{i^*}$. Otherwise, assume that there exists a node $p_0 \in B_{i^*}$ such that $[\mathbf{v}_2]_{p_0} < [\mathbf{v}_2]_{i^*}$, and choose $r = \min\{-[\mathbf{v}_2]_{j_0}, -[\mathbf{v}_2]_{p_0}\}$. Note that i^* is a cut node; then the subgraph $\mathcal{G}(r)$ induced by \mathcal{G} on $M(r)$ is disconnected. Therefore, one has $[\mathbf{v}_2]_p \geq [\mathbf{v}_2]_{i^*}$ for any node $p \in B_{i^*}$. In addition, for any $k \in \underline{q_i}$ and $k \neq i^*$, one has $[\mathbf{v}_2]_q \geq [\mathbf{v}_2]_{i_k}$ for any node $q \in B_{i_k}$. This is due to having i_k as a cut node and $[\mathbf{v}_2]_{i_k} > [\mathbf{v}_2]_{i^*}$. Therefore, for any $k \in \underline{q_i}$, one has $[\mathbf{v}_2]_q \geq [\mathbf{v}_2]_{i_k}$ for any node $q \in B_{i_k}$. However, in view of Lemma 7, this is a contradiction. Therefore, for any node $j \in B_i$, one has $[\mathbf{v}_2]_j \geq [\mathbf{v}_2]_{i^*}$.

Based on the above two scenarios, in the following, let B_i be an arbitrary positive block and denote by $\mathcal{N}_i^B = \{B_{i_1}, B_{i_2}, \dots, B_{i_{q_i}}\}$ as the neighboring blocks of B_i with the corresponding cut nodes $\{i_1, i_2, \dots, i_{q_i}\}$. Let $i^* = \underset{k \in \underline{q_i}}{\operatorname{argmin}} [\mathbf{v}_2]_{i_k}$, connecting blocks B_i and B_{i^*} . We shall prove that any node $j \in B_i$ can be reached by the cut node i^* in the FSN network $\bar{\mathcal{G}}$. Let $\bar{\mathcal{G}}(\{i_1, \dots, i_{s_0}\})$ be weakly connected, not reachable from the cut node i^* in $\bar{\mathcal{G}}$, where $s_0 \in \mathbb{Z}_+$. Denote $\mathcal{E}^{bound} = \{(i,j) \in \mathcal{E} \mid i \in \{i_1, \dots, i_{s_0}\}, j \in \mathcal{V} \setminus \{i_1, \dots, i_{s_0}\}\}$, then according to Definition 4, for any edge $(i,j) \in \mathcal{E}^{bound}$, one has $[\mathbf{v}_2]_i \leq [\mathbf{v}_2]_j$, which is a contradiction in view of Lemma 7. Hence, any node $j \in B_i$, can be reached from the cut node i^* in $\bar{\mathcal{G}}$.

Case 2: For the case that the network \mathcal{G} only contains a core node i_0 ; let nodes j and k be arbitrary positive and negative nodes connecting the core node directly, respectively. By Definition 4, nodes j and k are reachable from the core node i_0 . For the remaining nodes that do not connect the core node directly, the proof is then similar to the Case 1.

Consequently, consensus can be guaranteed for aforementioned two cases according to Lemma 5.

Secondly, we shall prove that the steady-state of agents in $\bar{\mathcal{G}}$ is equal to either the average of initial states of the agents in the core block and zero blocks (Case 1) or that of the agents in zero blocks and the core node (Case 2). To simplify our presentation, we employ a general description for both cases, namely, that there are m agents in the union of either core block and zero blocks (Case 1) or core node and zero blocks (Case 2). Then, denote by $\mathcal{L} = \{1, 2, \dots, m\}$ and $\mathcal{F} = \{m+1, m+2, \dots, n\}$, where $m \in \mathbb{Z}_+$ and $m \leq n$. We can represent the Laplacian matrix of \mathcal{G} as, $L = \begin{bmatrix} L_{11} & 0_{m \times (n-m)} \\ L_{21} & L_{22} \end{bmatrix}$, where $L_{11} \in \mathbb{R}^{m \times m}$, $L_{22} \in \mathbb{R}^{(n-m) \times (n-m)}$ and $L_{21} \in \mathbb{R}^{(n-m) \times m}$, and L_{22} is nonsingular. Due to $L\mathbf{1}_n = \mathbf{0}$, then one has $L_{21}\mathbf{1}_m + L_{22}\mathbf{1}_{n-m} = \mathbf{0}$. Therefore, $L_{22}^{-1}L_{21}\mathbf{1}_m = -\mathbf{1}_{n-m}$. Denote by $\mathbf{x}_{\mathcal{L}} = (\mathbf{x}_1^\top(t), \dots, \mathbf{x}_m^\top(t))^\top \in \mathbb{R}^{md}$ and $\mathbf{x}_{\mathcal{F}} = (\mathbf{x}_{m+1}^\top(t), \dots, \mathbf{x}_n^\top(t))^\top \in \mathbb{R}^{(n-m)d}$. Thereby,

$$\dot{\mathbf{x}}_{\mathcal{L}} = -(L_{11} \otimes I_d)\mathbf{x}_{\mathcal{L}}, \quad (30)$$

and

$$\dot{\mathbf{x}}_{\mathcal{F}} = -(L_{22} \otimes I_d)\mathbf{x}_{\mathcal{F}} - (L_{21} \otimes I_d)\mathbf{x}_{\mathcal{L}}. \quad (31)$$

Let $X = \mathbf{x}_{\mathcal{F}} + (L_{22}^{-1}L_{21} \otimes I_d)\mathbf{x}_{\mathcal{L}}$. Then,

$$\dot{X} = -(L_{22} \otimes I_d)X - (L_{22}^{-1}L_{21}L_{11} \otimes I_d)\mathbf{x}_{\mathcal{L}}. \quad (32)$$

According to the input-to-state stability theory, one has $\lim_{t \rightarrow \infty} X = \mathbf{0}$. Therefore,

$$\lim_{t \rightarrow \infty} \mathbf{x}_{\mathcal{F}} = -(L_{22}^{-1}L_{21} \otimes I_d) \lim_{t \rightarrow \infty} \mathbf{x}_{\mathcal{L}} \quad (33)$$

$$= -\frac{1}{m}(L_{22}^{-1}L_{21}\mathbf{1}_m\mathbf{1}_m^\top \otimes I_d)\mathbf{x}_{\mathcal{L}}(0) \quad (34)$$

$$= \frac{1}{m}(\mathbf{1}_{n-m}\mathbf{1}_m^\top \otimes I_d)\mathbf{x}_{\mathcal{L}}(0), \quad (35)$$

which concludes the proof. \square

PROOF OF PROPOSITION 1

Proof. Let $E = L(\bar{\mathcal{G}}) - L(\mathcal{G})$. Note that $L(\bar{\mathcal{G}})\bar{\mathbf{v}}_2 = \lambda_2(L(\bar{\mathcal{G}}))\bar{\mathbf{v}}_2$ and $\bar{\mathbf{v}}_2^\top L^\top(\bar{\mathcal{G}}) = \lambda_2(L(\bar{\mathcal{G}}))\bar{\mathbf{v}}_2^\top$. Hence, $\bar{\mathbf{v}}_2^\top L(\bar{\mathcal{G}})\bar{\mathbf{v}}_2 = \lambda_2(L(\bar{\mathcal{G}}))\bar{\mathbf{v}}_2^\top \bar{\mathbf{v}}_2$, and $\bar{\mathbf{v}}_2^\top L^\top(\bar{\mathcal{G}})\bar{\mathbf{v}}_2 = \lambda_2(L(\bar{\mathcal{G}}))\bar{\mathbf{v}}_2^\top \bar{\mathbf{v}}_2$. Therefore,

$$\bar{\mathbf{v}}_2^\top (L(\bar{\mathcal{G}}) + L^\top(\bar{\mathcal{G}}))\bar{\mathbf{v}}_2 = 2\lambda_2(L(\bar{\mathcal{G}}))\bar{\mathbf{v}}_2^\top \bar{\mathbf{v}}_2, \quad (36)$$

and

$$\lambda_2(L(\bar{\mathcal{G}})) = \frac{\bar{\mathbf{v}}_2^\top (L(\bar{\mathcal{G}}) + L^\top(\bar{\mathcal{G}}))\bar{\mathbf{v}}_2}{2\bar{\mathbf{v}}_2^\top \bar{\mathbf{v}}_2} \quad (37)$$

$$= \frac{\bar{\mathbf{v}}_2^\top (L(\mathcal{G}) + L^\top(\mathcal{G}))\bar{\mathbf{v}}_2}{2\bar{\mathbf{v}}_2^\top \bar{\mathbf{v}}_2} + \frac{\bar{\mathbf{v}}_2^\top (E + E^\top)\bar{\mathbf{v}}_2}{2\bar{\mathbf{v}}_2^\top \bar{\mathbf{v}}_2} \quad (38)$$

$$= \frac{\bar{\mathbf{v}}_2^\top L(\mathcal{G})\bar{\mathbf{v}}_2}{\bar{\mathbf{v}}_2^\top \bar{\mathbf{v}}_2} + \frac{\bar{\mathbf{v}}_2^\top (E + E^\top)\bar{\mathbf{v}}_2}{2\bar{\mathbf{v}}_2^\top \bar{\mathbf{v}}_2}. \quad (39)$$

By applying Rayleigh theorem [39, Theorem 4.2.2, p.235], one has,

$$\lambda_2(L(\bar{\mathcal{G}})) \geq \lambda_2(L(\mathcal{G})) + \frac{\bar{\mathbf{v}}_2^\top (E + E^\top)\bar{\mathbf{v}}_2}{2\bar{\mathbf{v}}_2^\top \bar{\mathbf{v}}_2}. \quad (40)$$

Moreover, according to the expansion,

$$\frac{\bar{\mathbf{v}}_2^\top (E + E^\top)\bar{\mathbf{v}}_2}{2\bar{\mathbf{v}}_2^\top \bar{\mathbf{v}}_2} = \frac{1}{\bar{\mathbf{v}}_2^\top \bar{\mathbf{v}}_2} \sum_{(i,j) \in \mathcal{E} \setminus \bar{\mathcal{E}}} [\bar{\mathbf{v}}_2]_i ([\bar{\mathbf{v}}_2]_j - [\bar{\mathbf{v}}_2]_i), \quad (41)$$

and $\bar{\mathbf{v}}_2^\top \bar{\mathbf{v}}_2 = 1$, one can conclude that,

$$\lambda_2(L(\bar{\mathcal{G}})) \geq \lambda_2(L(\mathcal{G})) + \sum_{(i,j) \in \mathcal{E} \setminus \bar{\mathcal{E}}} [\bar{\mathbf{v}}_2]_i ([\bar{\mathbf{v}}_2]_j - [\bar{\mathbf{v}}_2]_i). \quad (42)$$

□

PROOF OF THEOREM 5

Lemma 9. [47] *Let \mathcal{T} be a non-star tree network with $n \geq 4$ nodes. Then $\lambda_2(L(\mathcal{T})) < 0.59$.*

We now prove Theorem 5.

Proof. Note that every node is a cut node in a tree network. Then in light of Lemma 3, there are two cases to consider. For the case that there exists one core node in \mathcal{T} , denoted by $w \in \mathcal{V}$, the Laplacian matrix $L(\bar{\mathcal{T}})$ of the FSN network $\bar{\mathcal{T}}$ can be decomposed as $L(\bar{\mathcal{T}}) = \begin{bmatrix} 0 & 0 \\ * & L_{\mathcal{V} \setminus \{w\}} \end{bmatrix}$, where $L_{\mathcal{V} \setminus \{w\}}$ is a lower triangular matrix, with all diagonal elements equal to one. Therefore, the second smallest eigenvalue of $L(\bar{\mathcal{T}})$ is equal to one. Hence, using Lemma 9, it follows that $\lambda_2(L(\bar{\mathcal{T}})) > \lambda_2(L(\mathcal{T}))$.

Now we proceed to consider the case that there exists one core block with two nodes; denote these nodes by $u \in \mathcal{V}$ and $v \in \mathcal{V}$. Then, the Laplacian matrix $L(\bar{\mathcal{T}})$ of the FSN network $\bar{\mathcal{T}}$ can be decomposed as, $L(\bar{\mathcal{T}}) = \begin{bmatrix} L_{\{u,v\}} & 0 \\ * & L_{\mathcal{V} \setminus \{u,v\}} \end{bmatrix}$, where $L_{\{u,v\}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ denotes the Laplacian matrix associated with the core block, and $L_{\mathcal{V} \setminus \{u,v\}}$ is a lower triangular matrix with all diagonal elements equal to one. Therefore, the second smallest eigenvalue of $L(\bar{\mathcal{T}})$ is equal to one. Again, in view of the Lemma 9, we conclude that $\lambda_2(L(\bar{\mathcal{T}})) > \lambda_2(L(\mathcal{T}))$. □

REFERENCES

- [1] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.
- [2] J. Qin, Q. Ma, Y. Shi, and L. Wang, "Recent advances in consensus of multi-agent systems: A brief survey," *IEEE Transactions on Industrial Electronics*, vol. 64, no. 6, pp. 4972–4983, 2016.
- [3] Y. Cao, W. Yu, W. Ren, and G. Chen, "An overview of recent progress in the study of distributed multi-agent coordination," *IEEE Transactions on Industrial Informatics*, vol. 9, no. 1, pp. 427–438, 2013.
- [4] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.
- [5] A. Jadbabaie, J. Lin, and A. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.
- [6] Z. Lin, B. Francis, and M. Maggiore, "Necessary and sufficient graphical conditions for formation control of unicycles," *IEEE Transactions on Automatic Control*, vol. 50, no. 1, pp. 121–127, 2005.
- [7] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 655–661, 2005.
- [8] S.-J. Chung, A. A. Paranjape, P. Dames, S. Shen, and V. Kumar, "A survey on aerial swarm robotics," *IEEE Transactions on Robotics*, vol. 34, no. 4, pp. 837–855, 2018.
- [9] Y. Song, D. J. Hill, and T. Liu, "Network-based analysis of rotor angle stability of power systems," *Foundations and Trends® in Electric Energy Systems*, vol. 4, no. 3, pp. 222–345, 2020.
- [10] P. Barooah and J. P. Hespanha, "Estimation on graphs from relative measurements," *IEEE Control Systems Magazine*, vol. 27, no. 4, pp. 57–74, 2007.
- [11] F. Dörfler, M. Chertkov, and F. Bullo, "Synchronization in complex oscillator networks and smart grids," *Proceedings of the National Academy of Sciences*, vol. 110, no. 6, pp. 2005–2010, 2013.
- [12] A. Nedic, "Distributed gradient methods for convex machine learning problems in networks: Distributed optimization," *IEEE Signal Processing Magazine*, vol. 37, no. 3, pp. 92–101, 2020.
- [13] A. V. Proskurnikov and R. Tempo, "A tutorial on modeling and analysis of dynamic social networks. Part II," *Annual Reviews in Control*, vol. 45, pp. 166–190, 2018.
- [14] G. Vásárhelyi, C. Virágh, G. Somorjai, T. Nepusz, A. E. Eiben, and T. Vicsek, "Optimized flocking of autonomous drones in confined environments," *Science Robotics*, vol. 3, no. 20, p. eaat3536, 2018.
- [15] T. Yang, X. Yi, J. Wu, Y. Yuan, D. Wu, Z. Meng, Y. Hong, H. Wang, Z. Lin, and K. H. Johansson, "A survey of distributed optimization," *Annual Reviews in Control*, vol. 47, pp. 278–305, 2019.
- [16] M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini *et al.*, "Interaction ruling animal collective behavior depends on topological rather than metric distance: Evidence from a field study," *Proceedings of the National Academy of Sciences*, vol. 105, no. 4, pp. 1232–1237, 2008.
- [17] B. Allen, G. Lippner, Y.-T. Chen, B. Fotouhi, N. Momeni, S.-T. Yau, and M. A. Nowak, "Evolutionary dynamics on any population structure," *Nature*, vol. 544, no. 7649, p. 227, 2017.
- [18] R. Bindal, P. Cao, W. Chan, J. Medved, G. Suwala, T. Bates, and A. Zhang, "Improving traffic locality in bittorrent via biased neighbor selection," in *26th IEEE International Conference on Distributed Computing Systems (ICDCS'06)*, 2006.
- [19] S. Ahmadian, M. Meghdadi, and M. Afsharchi, "A social recommendation method based on an adaptive neighbor selection mechanism," *Information Processing & Management*, vol. 54, no. 4, pp. 707–725, 2018.
- [20] S. Zhang, "Nearest neighbor selection for iteratively kNN imputation," *Journal of Systems and Software*, vol. 85, no. 11, pp. 2541–2552, 2012.
- [21] A. Vaswani, N. Shazeer, N. Parmar, J. Uszkoreit, L. Jones, A. N. Gomez, Ł. Kaiser, and I. Polosukhin, "Attention is all you need," *Advances in Neural Information Processing Systems*, vol. 30, 2017.
- [22] A. Olshevsky and J. N. Tsitsiklis, "Convergence speed in distributed consensus and averaging," *SIAM Journal on Control and Optimization*, vol. 48, no. 1, pp. 33–55, 2009.
- [23] A. Clark, Q. Hou, L. Bushnell, and R. Poovendran, "Maximizing the smallest eigenvalue of a symmetric matrix: A submodular optimization approach," *Automatica*, vol. 95, pp. 446–454, 2018.
- [24] Y. Kim and M. Mesbahi, "On maximizing the second smallest eigenvalue of a state-dependent graph Laplacian," *IEEE Transactions on Automatic Control*, vol. 51, no. 1, pp. 116–120, 2006.

- [25] S. DeDeo and E. A. Hobson, "From equality to hierarchy," *Proceedings of the National Academy of Sciences*, vol. 118, no. 21, 2021.
- [26] T. Vicsek and A. Zafeiris, "Collective motion," *Physics Reports*, vol. 517, no. 3, pp. 71–140, 2012.
- [27] G. Beni, "Swarm intelligence," *Complex Social and Behavioral Systems: Game Theory and Agent-Based Models*, pp. 791–818, 2020.
- [28] S. S. Kia, B. Van Scoy, J. Cortes, R. A. Freeman, K. M. Lynch, and S. Martinez, "Tutorial on dynamic average consensus: The problem, its applications, and the algorithms," *IEEE Control Systems Magazine*, vol. 39, no. 3, pp. 40–72, 2019.
- [29] S. L. Brunton and J. N. Kutz, *Data-driven science and engineering: Machine learning, dynamical systems, and control*. Cambridge University Press, 2019.
- [30] M. Fiedler, "Algebraic connectivity of graphs," *Czechoslovak Mathematical Journal*, vol. 23, no. 2, pp. 298–305, 1973.
- [31] M. Pirani and S. Sundaram, "On the smallest eigenvalue of grounded laplacian matrices," *IEEE Transactions on Automatic Control*, vol. 61, no. 2, pp. 509–514, 2016.
- [32] M. Fiedler, "A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory," *Czechoslovak Mathematical Journal*, vol. 25, no. 4, pp. 619–633, 1975.
- [33] U. Von Luxburg, "A tutorial on spectral clustering," *Statistics and Computing*, vol. 17, no. 4, pp. 395–416, 2007.
- [34] R. Merris, "Laplacian graph eigenvectors," *Linear Algebra and Its Applications*, vol. 278, no. 1, pp. 221–236, 1998.
- [35] W. Xia and M. Cao, "Analysis and applications of spectral properties of grounded Laplacian matrices for directed networks," *Automatica*, vol. 80, pp. 10–16, 2017.
- [36] Y. Cao, W. Ren, and M. Egerstedt, "Distributed containment control with multiple stationary or dynamic leaders in fixed and switching directed networks," *Automatica*, vol. 48, no. 8, pp. 1586–1597, 2012.
- [37] A. Chapman and M. Mesbahi, "Semi-autonomous consensus: Network measures and adaptive trees," *IEEE Transactions on Automatic Control*, vol. 58, no. 1, pp. 19–31, 2013.
- [38] F. Dörfler, J. W. Simpson-Porco, and F. Bullo, "Electrical networks and algebraic graph theory: Models, properties, and applications," *Proceedings of the IEEE*, vol. 106, no. 5, pp. 977–1005, 2018.
- [39] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 2012.
- [40] Z. Li, G. Wen, Z. Duan, and W. Ren, "Designing fully distributed consensus protocols for linear multi-agent systems with directed graphs," *IEEE Transactions on Automatic Control*, vol. 60, no. 4, pp. 1152–1157, 2014.
- [41] H. Shao and M. Mesbahi, "Degree of relative influence for consensus-type networks," in *2014 American Control Conference*, 2014, pp. 2676–2681.
- [42] H. Shao, M. Mesbahi, D. Li, and Y. Xi, "Inferring centrality from network snapshots," *Scientific Reports*, vol. 7, p. 40642, 2017.
- [43] H. Shao and M. Mesbahi, "On the fiedler vector of graphs and its application in consensus networks," in *2015 American Control Conference*, 2015, pp. 1734–1739.
- [44] F. Harary and G. Prins, "The block-cutpoint-tree of a graph," *Publ. Math. Debrecen*, vol. 13, no. 103-107, p. 19, 1966.
- [45] F. Harary, "On the notion of balance of a signed graph," *Michigan Mathematical Journal*, vol. 2, no. 2, pp. 143–146, 1953.
- [46] C. Altafini, "Consensus problems on networks with antagonistic interactions," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 935–946, 2013.
- [47] N. M. M. De Abreu, "Old and new results on algebraic connectivity of graphs," *Linear Algebra and Its Applications*, vol. 423, no. 1, pp. 53–73, 2007.