Understanding the Capability of PD Control for Uncertain Stochastic Systems

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***Abstract*— In this article, we focus on the global stabilizability problem for a class of second order uncertain stochastic control systems, where both the drift term and the diffusion term are nonlinear functions of the state variables and the control variables. We will show that the widely applied proportional-derivative(PD) control in engineering practice has the ability to globally stabilize such systems in the mean square sense, provided that the upper bounds of the partial derivatives of the nonlinear functions satisfy a certain algebraic inequality. It will also be proved that the stabilizing PD parameters can only be selected from a two dimensional bounded convex set, which is a significant difference from the existing literature on PD controlled uncertain stochastic systems. Moreover, a particular polynomial on these bounds is introduced, which can be used to determine under what conditions the system is not stabilizable by the PD control, and thus demonstrating the fundamental limitations of PD control.**

***Index Terms*— PD control, stochastic systems, nonlinear dynamics, uncertain structure, global stabilizability.**

I. Introduction

Feedback is a basic concept in automatic control, which has had a revolutionary influence in practically all areas. Its primary objective is to reduce the effects of the plant uncertainty on the desired control performance(e.g., stability, optimality of tracking, etc). Plenty of control methods have been developed for dealing with uncertainties over the past sixty years, such as adaptive control [11], robust control [25], active disturbance rejection control [5], [8] and sliding mode control, etc. However, the classical proportional-integral-derivative(PID) control, perhaps the most basic form of feedback, has been at the heart of control engineering practice for several decades [3]. In fact, the PID control is used in more than 90% of industrial processes [2]. One may naturally believe that such basic controller has been deeply understood in both theory and practice. However, as mentioned in [15], many practical PID loops are poorly tuned, and there is strong evidence that its rationale remains to be unclear.

Recently, the PID control has attracted more and more attention from the research community. For example, the stabilization problems of PD(or PID) controlled linear systems with time-delay are investigated, see e.g., [12]–[14], [16]. There are also abundant works on PD controlled mechanical systems(see e.g., [4], [7], [17]), among which [17] is probably the most notable, where a PD controller was constructed to globally stabilize fully actuated robot manipulators. For more general class of nonlinear uncertain systems without special structures, some rigorous mathematical investigations have been made on the theory and design of PID in recent years(see, e.g., [20], [22]–[24]). For instance, it has been shown that for a class of second order single-input-single-output(SISO) affine nonlinear system, one can select the three PID parameters to globally stabilize the closed-loop system and at the same time to make the output of the controlled system converge to any given setpoint, provided that the partial derivatives of the system nonlinear functions are bounded [22]. Extensions to MIMO non-affine systems without stochastic disturbances are discussed in [23]. More recently, the authors investigate the performance and design of PID control for non-affine stochastic systems in [21], where the diffusion term does not depend on the control input.

As a special case of PID, the PD controller has also attracted many scientists and scholars, see [7], [13], [17], [21], [22]. In order to understand the mechanism of the linear PD control, it is of vital importance to take nonlinearity, uncertainty and randomness into consideration. Moreover, efforts must be taken to investigate the limitations of PD control in a general framework. But, to the best of the our knowledge, these issues have not been fully explored. In this article, we are devoted to this fundamental problem by considering a basic class of MIMO stochastic nonlinear uncertain systems, where both the drift term and the diffusion term are functions of the state variables and the control variables. The main contributions are summarized as follows:

1) We have shown that the PD control has the ability to globally stabilize such systems in mean square, if the upper bounds of the partial derivatives of the nonlinear functions satisfy a certain algebraic inequality. Moreover, a particular polynomial is introduced, which can be used to determine under what conditions the system is not stabilizable by PD control, and thus demonstrating the fundamental limitations of PD control.

2) Open and bounded parameter sets for the controller gains are also constructed, which are based on some knowledge of both the drift and diffusion functions. Besides, it will be shown that the PD parameters cannot be chosen arbitrarily large, which is also a significant difference from the existing literature on PD controlled nonlinear uncertain systems, see

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e.g. [9], [22], [6], [21].

The rest of this article is organized as follows. In Section II, we will introduce the mathematical formulation. The main results are presented in Section III. Section IV contains the proofs of the main theorems. Section V will conclude the article with some remarks.

II. Mathematical Formulation

A. Notations and Definitions

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ be the space of $m\times n$ real matrices. Denote $\|x\|$ as the Euclidean norm of a vector x, and x^{T} as the transpose of a vector or matrix x. The norm of a matrix $P\in\mathbb{R}^{m\times n}$ is defined by $\|P\|=\sup_{x\in\mathbb{R}^n,\|x\|=1}\|Px\|$. For a square matrix $P\in\mathbb{R}^{n\times n}$, denote $P^{\text{sym}}:=(P+P^{\mathsf{T}})/2$ as the symmetrization of P, and $\text{tr}(P)$ as the trace of P. For a symmetric matrix $S\in\mathbb{R}^{n\times n}$, we denote $\lambda_{\min}(S)$ and $\lambda_{\max}(S)$ as the smallest and the largest eigenvalues of S, respectively. For two symmetric matrices S_1 and S_2 in $\mathbb{R}^{n\times n}$, the notation $S_1>S_2$ implies that S_1-S_2 is a positive definite matrix; $S_1\geq S_2$ implies that S_1-S_2 is a positive semi-definite matrix. Let $C^1(\mathbb{R}^n,\mathbb{R}^m)$ be the space of continuously differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}^m$, denoted as $C^1(\mathbb{R}^n)$ for simplicity when $m=1$. Denote $C^k(\mathbb{R}^n)$ as the space of functions from $\mathbb{R}^n$ to $\mathbb{R}$ with $k-$times continuous partial derivatives.

B. The Control System

Consider a basic class of second order nonlinear uncertain stochastic control system:

$$\begin{aligned}\mathrm{d}x_1 &= x_2\mathrm{d}t\\ \mathrm{d}x_2 &= f(x_1,x_2,u)\mathrm{d}t+g(x_1,x_2,u)\mathrm{d}B_t,\end{aligned}\tag{1}$$

where $x_1,x_2\in\mathbb{R}^n$ are the system state vector, $u\in\mathbb{R}^n$ is the control input, $B_t\in\mathbb{R}^1$ is an one-dimensional standard Brownian motion, and the nonlinear functions f and g belong to $C^1(\mathbb{R}^{3n},\mathbb{R}^n)$, which may contain unknown dynamics.

In this article, we aim to study the capability together with a design method of the classical PD control(also abbreviated as "the PD control"):

$$u(t)=k_pe(t)+k_d\dot{e}(t),\quad e(t)=y^*-x_1(t),\tag{2}$$

where $y^*\in\mathbb{R}^n$ is the setpoint, $e(t)$ is the regulation error, k_p and k_d are the PD parameters.

The objective is to design suitable PD parameters to globally stabilize and regulate system (1) in mean square, i.e.,

$$\lim_{t\to\infty}\mathbb{E}\left[\|e(t)\|^2+\|\dot{e}(t)\|^2\right]=0,\ \forall(x_1(0),x_2(0))\in\mathbb{R}^{2n},\tag{3}$$

where $\mathbb{E}$ denotes the expectation of a random variable.

We first introduce a basic assumption that will be used throughout the article.

Assumption 1: The setpoint $y^*\in\mathbb{R}^n$ is an equilibrium of the uncontrolled stochastic system (1). To be precise,

$$f(y^*,0,0)=0,\quad g(y^*,0,0)=0.\tag{4}$$

It is worth noting that Assumption 1 is necessary for the existence of (k_p,k_d) to achieve the control objective (3). Specifically, we have the following proposition.

Proposition 1: Consider the PD controlled system (1)-(2), where the functions f and g are Lipschitz continuous. Suppose that there exist some PD parameters k_p, k_d and some $(x_1(0),x_2(0))\in\mathbb{R}^{2n}$, such that the solution of the closed-loop system satisfies $\lim_{t\to\infty}\mathbb{E}\big[\|e(t)\|^2+\|\dot{e}(t)\|^2\big]=0$, then $f(y^*,0,0)=0$ and $g(y^*,0,0)=0$.

The proof of Proposition 1 is given in Appendix B.

Note that both $f(\cdot)$ and $g(\cdot)$ are uncertain functions, we need to find a suitable measure to quantitatively describe the size of uncertainty. The upper bounds of the partial derivatives of the uncertain functions, which reflect the "sensitivity" to their variables, are a natural choice for such measurement, see e.g. [19], [22]. In addition, in order to enable the input signal to affect the state of the controlled system, the control gain matrix $\frac{\partial f}{\partial u}$ should not vanish. These natural intuitions inspired us to introduce the following assumption.

Assumption 2: The drift function $f(\cdot)\in\mathcal{F}_{L_1,L_2}$, where

$$\mathcal{F}_{L_1,L_2}:=\Big\{f:\|\frac{\partial f}{\partial x_i}\|\leq L_i;\ \frac{\partial f}{\partial u}\geq I_n,\ \forall x_1,x_2,u\Big\},\tag{5}$$

where L_1, L_2 are positive constants, I_n is the $n\times n$ identity matrix, $\frac{\partial f}{\partial x_i}$, $\frac{\partial f}{\partial u}$ are the $n\times n$ Jacobian of f with respect to x_i and u, respectively. Moreover, the diffusion function $g(\cdot)$ belongs to

$$\mathcal{G}_{N_1,N_2,M}:=\Big\{g:\|\frac{\partial g}{\partial x_i}\|\leq N_i,\|\frac{\partial g}{\partial u}\|\leq M,\ \forall x_1,x_2,u\Big\},\tag{6}$$

where the constants N_1, N_2 are positive and M is nonnegative.

Next, we introduce the following definition.

Definition 1: We say that the uncertain stochastic system (1) is (globally) *stabilizable* by the PD control (2), if there exist some PD parameters $(k_p,k_d)\in\mathbb{R}^2$, such that the control performance (3) is satisfied for all functions f and g that satisfy Assumptions 1 and 2. Otherwise, we say that system (1) is not stabilizable by the PD control (2).

Remark 1: It is known that, if system (1) has the following special form(the diffusion term does not depend on u):

$$\begin{aligned}\mathrm{d}x_1 &= x_2\mathrm{d}t\\ \mathrm{d}x_2 &= f(x_1,x_2,u)\mathrm{d}t+g(x_1,x_2)\mathrm{d}B_t,\end{aligned}\tag{7}$$

where $f(y^*,0,0)=g(y^*,0)=0$ and

$$\|\frac{\partial f}{\partial x_i}\|\leq L_i,\ \|\frac{\partial g}{\partial x_i}\|\leq N_i;\quad \frac{\partial f}{\partial u}\geq I_n,\ \forall x_1,x_2,u,$$

then for any positive quadruple (L_1,L_2,N_1,N_2), the uncertain stochastic system (7) is globally stabilizable by the PD control (2), see Theorem 3.9 in [21]. Moreover, the selection of the PD parameters k_p and k_d has wide flexibility, since they can be arbitrarily chosen from an open and unbounded set in $\mathbb{R}^2$. Thus, one might naturally conjecture that this result can be extended to the more general system (1) considered in this article, where $g(\cdot)$ is a function of both the state variables and the control variables. To be precisely, for any given positive constants L_1,L_2,N_1,N_2 and M, an open and unbounded

PD parameter set could be constructed, from which the PD control (2) has the ability to globally stabilize the system (1), for all functions f and g that satisfy Assumptions 1 and 2. Surprisingly, the answer of the above problem is no. In fact, these five constants have to meet suitable constraints before such stabilizing PD parameters can be found.

III. Main Results

A. Uncertain Nonlinear Stochastic System

For given positive constants L_1, L_2, N_1 and N_2, we first define a family of parameter set $\{\Omega_0(M),\ M \geq 0\}$ as follows:

$$\Omega_0 := \left\{ (k_p, k_d) \in \mathbb{R}_+^2 \;\middle|\; \begin{array}{c} k_p^2 > \bar{k} + k_d T_1^2 \\ k_d^2 - k_p > \bar{k} + k_d T_2^2 \end{array} \right\}, \tag{8}$$

where $\bar{k} := (L_1 + L_2)(k_p + k_d)$, and T_1, T_2 are defined by

$$T_1 := N_1 + M k_p, \quad T_2 := N_2 + M k_d. \tag{9}$$

Next, we list some geometric properties of the set Ω_0:

- If $M = 0$, Ω_0 is an open and unbounded set in $\mathbb{R}^2$;
- The range of Ω_0 will shrink as M increase, i.e.
$$\Omega_0(M_1) \subset \Omega_0(M_2), \quad \text{if } 0 \leq M_2 < M_1;$$
- $\Omega_0 = \emptyset$ if $M \geq M_0^*$, where M_0^* is the unique positive solution of $16L_1 s^4 + 16N_1 s^3 + 4L_2 s^2 + 4N_2 s = 1$.

Let M_1^* be the supremum of the set consisting of M that makes Ω_0 nonempty. More precisely,

$$M_1^* := \sup\left\{M > 0 : \Omega_0(M) \neq \emptyset\right\}.$$

Theorem 1: Consider the nonlinear stochastic system (1)-(2), where Assumptions 1-2 are satisfied.

(i) If $0 \leq M < M_1^*$, system (1) is stabilizable by the PD control (2). Moreover, the stabilizing PD parameters can be selected from Ω_0.

(ii) If $M \geq M_2^*$, where M_2^* is the unique positive root of the following quartic polynomial:

$$4L_1 s^4 + 4N_1 s^3 + 2L_2 s^2 + 2N_2 s - 1 = 0, \tag{10}$$

system (1) is not stabilizable by the PD control (2).

The proof of Theorem 1 will be provided in the next section.

Remark 2: Note that $M \geq M_2^*$ is equivalent to

$$U := 4L_1 M^4 + 4N_1 M^3 + 2L_2 M^2 + 2N_2 M \geq 1. \tag{11}$$

Hence, it can be seen from Theorem 1(ii) that, if we regard the quantity U as a *measure* of system uncertainty, the PD control (2) will have fundamental limitations in dealing with the uncertain nonlinear stochastic system (1), once the uncertainty of the system is too large, namely, $U \geq 1$.

Remark 3: The constant $M_1^* > 0$ and satisfies

$$16L_1 M_1^{*4} + 16N_1 M_1^{*3} + 4L_2 M_1^{*2} + 4N_2 M_1^* \leq 1. \tag{12}$$

Therefore, $M_1^* < M_2^*$. So, one naturally ask, whether system (1) is stabilizable if $M_1^* \leq M < M_2^*$? Further, does it exist a positive constant M^*, such that system (1) is stabilizable by the PD control (2) if and only if $M < M^*$? In general, these problems can be very challenging due to the inherent nonlinearity, uncertainties and the strong coupling of high-dimensional state variables and non-affine control input, and remains open. However, the stabilizability problems have been solved in this article, when system (1) has a specific linear structure. In fact, the necessary and sufficient condition for a class of uncertain linear stochastic systems to be stabilized by the PD control (2) is $U < 1$, see Theorem 2 for details.

B. Uncertain Linear Stochastic System

Consider the following uncertain linear stochastic system:

$$\begin{aligned} \mathrm{d}x_1 &= x_2 \mathrm{d}t \\ \mathrm{d}x_2 &= (ax_1 + bx_2 + u)\mathrm{d}t + (cx_1 + dx_2 + eu)\mathrm{d}B_t, \end{aligned} \tag{13}$$

where $x_1 \in \mathbb{R}^n$, $u \in \mathbb{R}^n$, and a, b, c, d and e are unknown $n \times n$ constant matrices with known upper bounds, namely,

$$\|a\| \leq L_1, \|b\| \leq L_2, \|c\| \leq N_1, \|d\| \leq N_2, \|e\| \leq M, \tag{14}$$

where L_1, L_2, N_1, N_2 are positive constants, and M is non-negative. Without loss of generality, assume that $y^* = 0$, then the PD control (2) takes the following form:

$$u(t) = -k_p x_1(t) - k_d x_2(t). \tag{15}$$

Next, we will present a necessary and sufficient condition on the five constants, under which the uncertain stochastic system (13) is stabilizable by the PD control (15). Moreover, necessary and sufficient conditions for the choice of PD parameters are also provided.

Theorem 2: Consider the system (13), where (14) is satisfied. Then, the necessary and sufficient condition for system (13) to be stabilizable by the PD control (15) is

$$4L_1 M^4 + 4N_1 M^3 + 2L_2 M^2 + 2N_2 M < 1. \tag{16}$$

Moreover, under (16), the closed-loop system will satisfy (3) for all constant matrices a, b, c, d and e satisfying (14), if and only if k_p and k_d are chosen from the following set:

$$\Omega := \left\{ (k_p, k_d) \;\middle|\; k_p > L_1,\ 2\bar{k}_1 \bar{k}_2 > T_1^2 + \bar{k}_1 T_2^2 \right\}, \tag{17}$$

where T_1, T_2 are defined in (9) and $\bar{k}_1$, $\bar{k}_2$ are defined by

$$\bar{k}_1 := k_p - L_1, \qquad \bar{k}_2 := k_d - L_2. \tag{18}$$

Remark 4: We provide some geometric properties of the parameter set Ω defined by (17):

- The set Ω_0 defined in (8) is a subset of Ω;
- If $M = 0$, Ω is an open and unbounded subset in $\mathbb{R}^2$ for any positive constants L_1, L_2, N_1 and N_2;
- If $M > 0$ and (16) holds, Ω is an open and *bounded convex* subset in $\mathbb{R}^2$.

Hence, the controller gains k_p and k_d cannot be chosen sufficiently large for the case $M > 0$, which is a significant difference from the existing literature on PD or PID controlled nonlinear uncertain systems, see e.g. [9], [22], [6], [21].

IV. Proofs of the Main Results

A. Proof of Theorem 1

First, we prove the first half of Theorem 1 in three steps.

Step 1: (Some properties of the PD parameter set Ω_0)

Firstly, for given positive constants L_1, L_2, N_1, N_2 and $M \geq 0$, we define a set Ω' as follows:

$$\Omega' := \left\{ (k_p, k_d) \mid k_p > L_1,\ \bar{k}_1 \bar{k}_2 > T_1^2 + \bar{k}_1 T_2^2 \right\}, \tag{19}$$

where T_1, T_2, $\bar{k}_1$ and $\bar{k}_2$ are defined in (9) and (18).

Property 1: The sets Ω_0, Ω' and Ω satisfy

$$\Omega_0 \subset \Omega' \subset \Omega, \tag{20}$$

where Ω_0 and Ω are defined in (8) and (17), respectively.

The inclusion $\Omega' \subset \Omega$ is obvious by the definitions of Ω' and Ω. We only need to show $\Omega_0 \subset \Omega'$. Indeed, if $(k_p, k_d) \in \Omega_0$, then by definition (8), we know

$$k_p^2 > \bar{k} := (L_1 + L_2)(k_p + k_d) > k_p(L_1 + L_2), \tag{21}$$

which yields $k_p > L_1$. Moreover, combine $k_p^2 > \bar{k} + k_d T_1^2$ with $\bar{k} > k_p L_1$, it can be obtained that $k_p^2 - k_p L_1 > k_d T_1^2$, hence $k_p - L_1 > k_d T_1^2 / k_p$. Recall $\bar{k}_1 := k_p - L_1$, we have

$$\bar{k}_1 > k_d T_1^2 / k_p. \tag{22}$$

On the other hand, since $k_d^2 - k_p > \bar{k} + k_d T_2^2 > k_d L_2 + k_d T_2^2$, we have $k_d(k_d - L_2) > k_p + k_d T_2^2$. Therefore, we have

$$\bar{k}_2 := k_d - L_2 > k_p / k_d + T_2^2. \tag{23}$$

Combine (22) with (23), it is easy to obtain

$$\bar{k}_1 \left(\bar{k}_2 - T_2^2 \right) > T_1^2. \tag{24}$$

From (24) and recall $k_p > L_1$, we conclude that $(k_p, k_d) \in \Omega'$.

Property 2: Ω_0 will become smaller when M increase, i.e.

$$\Omega_0(M_1) \subset \Omega_0(M_2), \quad \text{if } 0 \leq M_2 < M_1. \tag{25}$$

In fact, let $0 \leq M_2 < M_1$, and suppose $(k_p, k_d) \in \Omega_0(M_1)$, then by (8) and note that $k_p > 0, k_d > 0$, it is easy to obtain

$$\begin{aligned} &k_p^2 > \bar{k} + k_d(N_1 + M_1 k_p)^2 > \bar{k} + k_d(N_1 + M_2 k_p)^2, \\ &k_d^2 - k_p > \bar{k} + k_d(N_2 + M_1 k_d)^2 > \bar{k} + k_d(N_2 + M_2 k_d)^2, \end{aligned}$$

hence, $(k_p, k_d) \in \Omega_0(M_2)$, which yields the relationship (25).

Property 3: If $M = 0$, Ω_0 is an open and unbounded set.

To this end, let $k_p = k_d = k > 0$, then

$$k_p^2 - \bar{k} - k_d T_1^2 = k^2 - 2k(L_1 + L_2) - k N_1^2 > 0, \ \text{ as } k \to \infty.$$

Similarly, $k_d^2 - k_p - \bar{k} - k_d T_2^2 > 0$ for k large enough. Consequently, Ω_0 is open and unbounded when $M = 0$.

Property 4: For given positive constants L_1, L_2, N_1 and N_2, let M_0^* be the unique positive solution of the polynomial

$$16 L_1 s^4 + 16 N_1 s^3 + 4 L_2 s^2 + 4 N_2 s = 1, \tag{26}$$

then $\Omega_0 = \emptyset$ if $M \geq M_0^*$.

First, by the definition of M_0^*, we know that $M \geq M_0^*$ is equivalent to $16 L_1 M^4 + 16 N_1 M^3 + 4 L_2 M^2 + 4 N_2 M \geq 1$. From Lemma 2 in Appendix A, we know Ω' is empty if and only if $M \geq M_0^*$. Besides, by the inclusion relationship (20), we conclude that Ω_0 is also empty if $M \geq M_0^*$.

Let M_1^* be the supremum that makes Ω_0 nonempty, i.e.,

$$M_1^* := \sup \left\{ M > 0 : \Omega_0(M) \neq \emptyset \right\}. \tag{27}$$

Then, it is easy to obtain the following facts:

- The set Ω_0 is not empty, if $0 \leq M < M_1^*$;
- The constant M_1^* depends on L_1, L_2, N_1, N_2 only;
- $M_1^* \leq M_0^*$, i.e., $16 L_1 M_1^{*4} + 16 N_1 M_1^{*3} + 4 L_2 M_1^{*2} + 4 N_2 M_1^* \leq 1$.

Step 2: (Write the closed-loop system into a linearity-like form) Let us denote

$$z_1(t) := -e(t) = x_1(t) - y^*, \quad z_2(t) := -\dot{e}(t) = x_2(t), \tag{28}$$

then the PD control (2) can be rewritten as $u(t) = -k_p z_1(t) - k_d z_2(t)$, and the PD controlled system (1)-(2) turns into

$$\begin{aligned} \mathrm{d}z_1 &= \ z_2 \mathrm{d}t \\ \mathrm{d}z_2 &= \ f(z_1 + y^*, z_2, u)\mathrm{d}t + g(z_1 + y^*, z_2, u)\mathrm{d}B_t. \\ u &= \ -k_p z_1 - k_d z_2 \end{aligned} \tag{29}$$

By Assumption 1, we know that $(z_1, z_2) = (0, 0) \in \mathbb{R}^{2n}$ is an equilibrium of (29). Besides, recall $f(y^*, 0, 0) = 0$ and note that $f \in \mathcal{F}_{L_1, L_2}$, one can obtain(details can be found in [23]):

$$f(z_1 + y^*, z_2, u) = a(z_1) z_1 + b(z_1, z_2) z_2 + \theta(z_1, z_2, u) u, \tag{30}$$

where a, b and θ are $n \times n$ matrices satisfying

$$\|a(z_1)\| \leq L_1, \ \|b(z_1, z_2)\| \leq L_2, \ \theta(z_1, z_2, u) \geq I_n, \tag{31}$$

for all z_1, z_2, u. Similarly, since $g(y^*, 0, 0) = 0$ and $g \in \mathcal{G}_{N_1, N_2, M}$, the function $g(z_1 + y^*, z_2, u)$ can be expressed by

$$g(z_1 + y^*, z_2, u) = c(z_1) z_1 + d(z_1, z_2) z_2 + e(z_1, z_2, u) u, \tag{32}$$

where c, d and e are $n \times n$ matrices satisfying

$$\|c(z_1)\| \leq N_1, \ \|d(z_1, z_2)\| \leq N_2, \ \|e(z_1, z_2, u)\| \leq M. \tag{33}$$

By the expressions of f and g in (30) and (32), the nonlinear system (29) turns into the linearity-like form:

$$\begin{aligned} \mathrm{d}z_1 &= z_2 \mathrm{d}t \\ \mathrm{d}z_2 &= \left[\hat{a} z_1 + \hat{b} z_2 \right] \mathrm{d}t + \left[\hat{c} z_1 + \hat{d} z_2 \right] \mathrm{d}B_t, \end{aligned} \tag{34}$$

where $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ are nonlinear (matrix-valued) functions of $z = (z_1, z_2)$ defined by

$$\hat{a} = a(z_1) - k_p \theta(z, u), \quad \hat{b} = b(z_1, z_2) - k_d \theta(z, u), \tag{35}$$

$$\hat{c} = c(z_1) - k_p e(z, u), \quad \hat{d} = d(z_1, z_2) - k_d e(z, u), \tag{36}$$

with $u = k_p z_1 + k_d z_2$.

Now, suppose that $M < M_1^*$ and $(k_p, k_d) \in \Omega_0$. In the next two steps, we proceed to prove the closed-loop system (1)-(2) will satisfy the performance (3).

Step 3: (Construction of Lyapunov function)

We adopt a similar Lyapunov function $V(z)$ as that used for deterministic system(see [23]),

$$V(z) = k_p k_d z_1^\top z_1 + k_p z_1^\top z_2 + k_d z_2^\top z_2 / 2, \ \ z = (z_1, z_2). \tag{37}$$

Note that $V = \|\sqrt{k_p k_d} z_1 + \frac{1}{2}\sqrt{k_p / k_d} z_2\|^2 + \frac{1}{2}(k_d - \frac{k_p}{2k_d})\|z_2\|^2$ and that $k_d^2 > k_p$, we know that $V(z)$ is positive definite.

Step 4: (Stability analysis based on Lyapunov methods)

By some manipulations, the operator L acting on the function $V(z)$ along the trajectories of (34) is given by

$$LV(z) = \underbrace{\frac{\partial V}{\partial z}\begin{bmatrix} z_1 \\ \hat{a}z_1+\hat{b}z_2 \end{bmatrix}}_{\mathrm{I}} + \underbrace{\frac{k_d}{2}\left\|\hat{c}z_1+\hat{d}z_2\right\|^2}_{\mathrm{II}}. \tag{38}$$

Denote $A = \begin{bmatrix} 0_n & I_n \\ \hat{a} & \hat{b} \end{bmatrix}$ and $P = \begin{bmatrix} 2k_pk_dI_n & k_pI_n \\ k_pI_n & k_dI_n \end{bmatrix}$, where 0_n is the $n \times n$ zero matrix, then the first term can be estimated as follows(see the proof of Proposition 4.3 in [23]):

$$\begin{aligned} \mathrm{I} =& z^{\mathsf{T}}\left(PA + A^{\mathsf{T}}P\right)z \\ \le& -(k_p^2 - \bar{k})\|z_1\|^2 - (k_d^2 - k_p - \bar{k})\|z_2\|^2. \end{aligned} \tag{39}$$

By the definitions of $\hat{c}$, $\hat{d}$ in (36) and the properties (33), we know that $\|\hat{c}\| \le N_1+k_pM = T_1$ and $\|\hat{d}\| \le N_2+k_dM = T_2$. Therefore, the second term has the following upper bound:

$$\mathrm{II} = \frac{k_d}{2}\left\|\hat{c}z_1+\hat{d}z_2\right\|^2 \le k_d\left(T_1^2\|z_1\|^2 + T_2^2\|z_2\|^2\right). \tag{40}$$

Combine (39) and (40), we obtain the upper bounds of $LV(z)$:

$$LV(z) \le -(k_p^2-\bar{k}-k_dT_1^2)\|z_1\|^2 - (k_d^2-k_p-\bar{k}-k_dT_2^2)\|z_2\|^2.$$

Since $(k_p, k_d) \in \Omega_0$, we know that

$$LV(z) \le -\eta\|z\|^2, \quad \forall z \in \mathbb{R}^{2n}. \tag{41}$$

for some positive constant η. Recall $z_1(t) = -e(t),\ z_2(t) = -\dot{e}(t)$, we conclude that the PD control system (1)-(2) will satisfy the control objective (3).

Next, we prove the second half of Theorem 1. For this, it suffices to show the following statement:

If there exist some $(k_p, k_d) \in \mathbb{R}^2$, such that the closed-loop system (1)-(2) satisfies the performance (3) for all $f(\cdot)$ and $g(\cdot)$ satisfying Assumptions 1-2, then $M < M_2^*$.

By the definition of M_2^* in (10), one can see that

$$M < M_2^* \Longleftrightarrow 4L_1M^4 + 4N_1M^3 + 2L_2M^2 + 2N_2M < 1.$$

Now, suppose that the functions $f(\cdot)$ and $g(\cdot)$ are given by

$$f(x_1, x_2, u) = a(x_1 - y^*) + bx_2 + u,\ x_1, x_2, u \in \mathbb{R}^n, \tag{42}$$
$$g(x_1, x_2, u) = c(x_1 - y^*) + dx_2 - eu,\ x_1, x_2, u \in \mathbb{R}^n, \tag{43}$$

where a, b, c, d, e are five real numbers satisfying $|a| \le L_1$, $|b| \le L_2$, $|c| \le N_1$, $|d| \le N_2$ and $|e| \le M$. Then it is easy to see that Assumptions 1 and 2 hold.

Denote $z_1(t) = x_1(t) - y^*$ and $z_2(t) = x_2(t)$,

$$z(t) := \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix},\ A := \begin{bmatrix} 0_n & I_n \\ a_0I_n & b_0I_n \end{bmatrix},\ B := \begin{bmatrix} 0_n & 0_n \\ c_0I_n & d_0I_n \end{bmatrix},$$

where

$$a_0 = a - k_p, b_0 = b - k_d, c_0 = c + k_pe, d_0 = d + k_de, \tag{44}$$

then closed-loop equation (1)-(2) with f and g defined by (42) and (43) turns into:

$$\mathrm{d}z = Az\mathrm{d}t + Bz\mathrm{d}B_t. \tag{45}$$

Define a $2n \times 2n$ time-varying matrix $P(t) := \begin{bmatrix} p(t) & r(t) \\ r^{\mathsf{T}}(t) & q(t) \end{bmatrix}$, where $p(t)$, $r(t)$ and $q(t)$ are $n \times n$ matrix defined by

$$p(t) := \mathbb{E}\left[z_1(t)z_1^{\mathsf{T}}(t)\right],\ r(t) := \mathbb{E}\left[z_1(t)z_2^{\mathsf{T}}(t)\right], \tag{46}$$
$$q(t) := \mathbb{E}\left[z_2(t)z_2^{\mathsf{T}}(t)\right], \tag{47}$$

then it can be seen that $P(t) = \mathbb{E}\left[z(t)z^{\mathsf{T}}(t)\right]$. From Theorem 8.5.5 in [1], we know that $P(t)$ is the unique nonnegative-definite symmetric solution of the equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = AP(t) + P(t)A^{\mathsf{T}} + BP(t)B^{\mathsf{T}}. \tag{48}$$

From (48), it can be obtained that

$$\begin{aligned} \dot{p} &= r + r^{\mathsf{T}} \\ \dot{r} &= a_0p + b_0r + q \\ \dot{q} &= c_0^2p + (a_0 + c_0d_0)\left(r + r^{\mathsf{T}}\right) + \left(2b_0 + d_0^2\right)q. \end{aligned} \tag{49}$$

Let $r_0(t) = (r(t) + r^{\mathsf{T}}(t))/2$ and define

$$Q = \begin{bmatrix} 0 & 2 & 0 \\ a_0 & b_0 & 1 \\ c_0^2 & 2(a_0 + c_0d_0) & 2b_0 + d_0^2 \end{bmatrix}, \tag{50}$$

then it follows from (49) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} p(t) \\ r_0(t) \\ q(t) \end{bmatrix} = Q \otimes I_n \begin{bmatrix} p(t) \\ r_0(t) \\ q(t) \end{bmatrix}, \tag{51}$$

where $\otimes$ denotes the Kronecker product.

Since for any initial state $(z_1(0), z_2(0)) \in \mathbb{R}^{2n}$, the solution of (45) satisfies $\lim_{t\to\infty}\mathbb{E}\left[\|z_1(t)\|^2 + \|z_2(t)\|^2\right] = 0$, which implies that $\lim_{t\to\infty}\|P(t)\| = 0$ for all initial state $(z_1(0), z_2(0))$. We conclude that $Q \otimes I_n$ is a Hurwitz matrix. Note that the matrix $Q \otimes I_n$ shares the same spectrum with Q. Hence, Q is also Hurwitz.

From the expression of Q in (50), the characteristic polynomial of Q can be calculated as follows:

$$\det(\lambda I_3 - Q) = \lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0, \tag{52}$$

where α_0, α_1 and α_2 are given by

$$\alpha_1 = b_0d_0^2 + 2b_0^2 - 4a_0 - 2c_0d_0 \tag{53}$$
$$\alpha_0 = 2(2a_0b_0 + a_0d_0^2 - c_0^2),\ \alpha_2 = -(3b_0 + d_0^2). \tag{54}$$

From the Routh-Hurwitz stability criterion for third order polynomials, the matrix Q is Hurwitz if and only if the following inequalities holds:

$$\alpha_2 > 0, \quad \alpha_0 > 0, \quad \alpha_1\alpha_2 > \alpha_0. \tag{55}$$

We next proceed to show the following statement:

Suppose that the matrix Q defined in (50) is Hurwitz for all $|a| \le L_1$, $|b| \le L_2$, $|c| \le N_1$, $|d| \le N_2$, $|e| \le M$, then the parameters k_p and k_d belong to the set Ω defined in (17).

Proof. First, from the definitions of α_2 and b_0 in (54) and (44), we have $\alpha_2 = -(3b_0+d_0^2) = 3(k_d-b) - d_0^2$. In addition, since $\alpha_2 > 0$, it follows that $3(k_d - b) \ge \alpha_2 > 0$. Choose $b = L_2$, we conclude that

$$k_d > L_2. \tag{56}$$

Next, suppose for all $|a| \leq L_1, |b| \leq L_2, |c| \leq N_1, |d| \leq N_2, |e| \leq M$, there is

$$\alpha_0 = 2(a-k_p)(b-k_d)+(a-k_p)(d-ek_d)^2-(c-ek_p)^2 > 0,$$

Choose $c=d=e=0$, it follows from (56) that $k_p > L_1$. Moreover, if we choose $a=L_1$, $b=L_2$, $c=-N_1$, $d=-N_2$ and $e=M$, then we have

$$2\bar{k}_1\bar{k}_2 - \bar{k}_1(N_2+Mk_d)^2 - (N_1+Mk_p)^2 > 0. \quad (57)$$

Combine (56)-(57), we conclude that (k_p, k_d) belongs to Ω.

Finally, by Lemma 1 in Appendix A, we know that the necessary and sufficient condition for the set Ω to be non-empty is $4L_1M^4 + 4N_1M^3 + 2L_2M^2 + 2N_2M < 1$. Hence, if $M \geq M_2^*$, where M_2^* is the unique positive solution of the equation (10), then Ω is empty, and thus there does not exist k_p and k_d such that Q is Hurwitz for all $|a| \leq L_1$, $|b| \leq L_2$, $|c| \leq N_1$, $|d| \leq N_2$ and $|e| \leq M$. Therefore, the system (1) is not stabilizable by the PD control (2). □

B. Proof of Theorem 2

Sufficiency: Suppose that (16) is satisfied. From Lemma 1 in Appendix A, we know that the set Ω defined by (17) is not empty. Now, suppose $(k_p, k_d) \in \Omega$ and the matrices a, b, c, d and e satisfy (14), we proceed to show that the closed-loop system (13) and (15) will satisfy $\lim_{t\to\infty} \mathbb{E}\left[\|x_1(t)\|^2+\|x_2(t)\|^2\right] = 0$ for all initial states. Substituting (15) into (13), we have

$$\begin{aligned} \mathrm{d}x_1 &= x_2 \mathrm{d}t \\ \mathrm{d}x_2 &= \left[a_0x_1 + b_0x_2\right]\mathrm{d}t + \left[c_0x_1 + d_0x_2\right]\mathrm{d}B_t, \end{aligned} \quad (58)$$

where a_0, b_0, c_0, d_0 are $n \times n$ constant matrices defined by

$$a_0 = a - k_pI_n, b_0 = b - k_dI_n, c_0 = c - k_pe, d_0 = d - k_de. \quad (59)$$

Define

$$A := \begin{bmatrix} 0_n & I_n \\ a_0 & b_0 \end{bmatrix}, \quad X := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad B := \begin{bmatrix} 0_n & 0_n \\ c_0 & d_0 \end{bmatrix}, \quad (60)$$

then (58) can be rewritten in a more compact form:

$$\mathrm{d}X = AX\mathrm{d}t + BX\mathrm{d}B_t. \quad (61)$$

Let us define a $2n \times 2n$ matrix P as follows:

$$P := \begin{bmatrix} m & rI_n \\ rI_n & I_n \end{bmatrix}, \quad (62)$$

where m is an $n \times n$ matrix defined by

$$m := -rb_0 - a_0^\top - c_0^\top d_0, \quad (63)$$

and $r > 0$ is a constant defined by

$$r := \left(2\bar{k}_1\bar{k}_2 + T_1^2 - \bar{k}_1T_2^2\right) / \left(4\bar{k}_1\right). \quad (64)$$

Now, let $V(X) = X^\top PX$, where P is defined in (62), then the differential operator L acting on V is

$$\begin{aligned} &LV(X) = X^\top\left(PA + A^\top P + B^\top PB\right)X \\ =&X^\top \begin{bmatrix} 2ra_0^{\text{sym}} + c_0^\top c_0 & m + rb_0 + a_0^\top + c_0^\top d_0 \\ m^\top + rb_0^\top + a_0 + d_0^\top c_0 & 2rI_n + 2b_0^{\text{sym}} + d_0^\top d_0 \end{bmatrix} X, \end{aligned}$$

where

$$a_0^{\text{sym}} = (a_0 + a_0^\top)/2, \; b_0^{\text{sym}} = (b_0 + b_0^\top)/2$$

are the symmetrization matrices of a_0 and b_0. Denote $Q = PA + A^\top P + B^\top PB$, then from (63), it is easy to obtain

$$Q = \begin{bmatrix} 2ra_0^{\text{sym}} + c_0^\top c_0 & 0_n \\ 0_n & 2r + 2b_0^{\text{sym}} + d_0^\top d_0 \end{bmatrix}. \quad (65)$$

Note that $\|a\| \leq L_1$ and $\|b\| \leq L_2$, it follows that

$$\lambda_{\max}\left[a_0^{\text{sym}}\right] \leq L_1 - k_p = -\bar{k}_1, \quad \lambda_{\max}\left[b_0^{\text{sym}}\right] \leq -\bar{k}_2. \quad (66)$$

Moreover, since $\|c\| \leq N_1$, $\|d\| \leq N_2$, $\|e\| \leq M$, we have

$$\|c_0\| = \|c - k_pe\| \leq T_1, \; \|d_0\| = \|d - k_de\| \leq T_2,$$

where T_1, T_2 are defined in (9). From (66), and recall $(k_p, k_d) \in \Omega$, it can be seen that

$$\begin{aligned} &\lambda_{\max}\left[2ra_0^{\text{sym}} + c_0^\top c_0\right] \leq -2\bar{k}_1r + T_1^2 \\ =&\left(\bar{k}_1T_2^2 + T_1^2 - 2\bar{k}_1\bar{k}_2\right)/2 < 0, \end{aligned} \quad (67)$$

$$\begin{aligned} &\lambda_{\max}\left[2r + 2b_0^{\text{sym}} + d_0^\top d_0\right] < 2r - 2\bar{k}_2 + T_2^2 \\ =&\left(\bar{k}_1T_2^2 + T_1^2 - 2\bar{k}_1\bar{k}_2\right)/(2\bar{k}_1) < 0, \end{aligned} \quad (68)$$

which implies $LV(X)$ is a negative definite function.

Finally, we prove $V(X) = X^\top PX$ is positive definite. By the definition of P in (62), it suffices to show $m^{\text{sym}} - r^2I_n > 0$. Note that $m^{\text{sym}} = -rb_0^{\text{sym}} - a_0^{\text{sym}} - [c_0^\top d_0]^{\text{sym}}$, we have

$$\lambda_{\min}\left[m^{\text{sym}}\right] \geq -r\lambda_{\max}\left[b_0^{\text{sym}}\right] - \lambda_{\max}\left[a_0^{\text{sym}}\right] - \left\|[c_0^\top d_0]^{\text{sym}}\right\|.$$

Therefore, it follows from (66) that

$$\begin{aligned} &\lambda_{\min}\left[m^{\text{sym}} - r^2I_n\right] \;\geq\; r\bar{k}_2 - r^2 + \bar{k}_1 - T_1T_2 \\ =&-(r - \bar{k}_2/2)^2 + \bar{k}_2^2/4 + \bar{k}_1 - T_1T_2 \\ =&-\left(T_1^2/(4\bar{k}_1) - T_2^2/4\right)^2 + \bar{k}_2^2/4 + \bar{k}_1 - T_1T_2. \end{aligned} \quad (69)$$

Consequently,

$$\begin{aligned} &\lambda_{\min}\left[16\bar{k}_1^2(m^{\text{sym}} - r^2I_n)\right] \\ \geq&-\left(T_1^2 - T_2^2\bar{k}_1\right)^2 + 4\bar{k}_1^2\bar{k}_2^2 + 16\bar{k}_1^3 - 16\bar{k}_1^2T_1T_2 \\ =&4\bar{k}_1^2\bar{k}_2^2 - T_1^4 - T_2^4\bar{k}_1^2 + 2T_1^2T_2^2\bar{k}_1 + 16\bar{k}_1^3 - 16\bar{k}_1^2T_1T_2 \\ >&4T_1^2T_2^2\bar{k}_1 + 16\bar{k}_1^3 - 16\bar{k}_1^2T_1T_2 = 4\bar{k}_1\left(T_1T_2 - 2\bar{k}_1\right)^2 \\ \geq&0, \end{aligned} \quad (70)$$

which implies that $V(X)$ is positive definite. As a consequence, the PD control system (13) and (15) will satisfy (3) exponentially, for all initial values $x_1(0)$, $x_2(0) \in \mathbb{R}^n$.

Necessity: The necessity of Theorem 2 is similar to the proof of Theorem 1(ii), we omit it here due to page limitation. □

V. Conclusion

This article investigates the capability and limitations of the classical PD control for a class of nonaffine uncertain stochastic systems. We have shown that the nonaffine uncertain stochastic system can be globally stabilized by the PD control in the mean square sense, if the upper bounds of the partial derivatives of the system nonlinear functions satisfy a certain algebraic inequality. Moreover, we have shown that the PD control has fundamental limitations in stabilizing the considered stochastic systems, once the size of the system uncertainty exceeds a critical value. Furthermore, based on some prior knowledge of both the drift and diffusion terms, necessary and sufficient conditions on the selection of the controller gains are also provided for a class of linear uncertain stochastic systems. For further investigation, it would be meaningful to optimize the PD parameters to get better transient performance, and to consider more practical situations including time-delay and saturation, etc.

Appendix

A. Auxiliary results

We provide two lemmas that are used in the proof of the main results.

Lemma 1: A necessary and sufficient condition for the set Ω defined by (17) to be non-empty is

$$4L_1M^4+4N_1M^3+2L_2M^2+2N_2M-1<0. \tag{71}$$

Sufficiency: First, suppose that (71) holds, we will show $\Omega\neq\emptyset$ by verifying $(k_p^*,k_d^*)\in\Omega$, where

$$k_p^*:=\frac{1-2N_2M-2L_2M^2-2N_1M^3}{2M^4},\ k_d^*:=\frac{1-N_2M}{M^2}. \tag{72}$$

First, note that

$$\begin{aligned}\bar{k}_1=&k_p^*-L_1\\=&(1-2N_2M-2L_2M^2-2N_1M^3-2L_1M^4)/(2M^4),\end{aligned} \tag{73}$$

then it follows from (71) that $\bar{k}_1>0$. Moreover, it can be obtained that

$$\bar{k}_2=k_d^*-L_2=\left(1-N_2M-L_2M^2\right)/M^2, \tag{74}$$
$$T_1=N_1+Mk_p^*=\left(1-2N_2M-2L_2M^2\right)/(2M^3), \tag{75}$$
$$T_2=N_2+Mk_d^*=1/M. \tag{76}$$

It follows from (74)–(76) that

$$2\bar{k}_2-T_2^2=(1-2N_2M-2L_2M^2)/M^2=2MT_1. \tag{77}$$

From (73) and (75), we obtain the following identity:

$$\begin{aligned}&2M^3\left(2M\bar{k}_1-T_1\right)=4M^4\bar{k}_1-2M^3T_1\\=&2-4N_2M-4L_2M^2-4N_1M^3-4L_1M^4\\&-(1-2N_2M-2L_2M^2)\\=&1-2N_2M-2L_2M^2-4N_1M^3-4L_1M^4.\end{aligned} \tag{78}$$

Thus, we have $2M\bar{k}_1-T_1>0$. Consequently, it follows from (77) and (78) that

$$\begin{aligned}&2\bar{k}_1\bar{k}_2-T_1^2-\bar{k}_1T_2^2=\bar{k}_1(2\bar{k}_2-T_2^2)-T_1^2\\=&2MT_1\bar{k}_1-T_1^2=(2M\bar{k}_1-T_1)T_1>0.\end{aligned} \tag{79}$$

From (73) and (79), we know that $(k_p^*,k_d^*)\in\Omega$, which implies the non-empty property of Ω.

Necessity: Suppose that Ω is non-empty, we proceed to show that (71) holds. It suffices to consider the case $M>0$, since (71) is automatically satisfied when $M=0$.

Let $\bar{\Omega}$ be the closure of Ω, i.e.,

$$\bar{\Omega}=\left\{(k_p,k_d)|\ k_p\geq L_1,\quad 2\bar{k}_1\bar{k}_2-T_1^2-\bar{k}_1T_2^2\geq 0\right\}. \tag{80}$$

First, we show that $\bar{\Omega}$ is bounded(hence it is compact).

Suppose that $(k_p,k_d)\in\bar{\Omega}$, then $2\bar{k}_1\bar{k}_2-\bar{k}_1T_2^2\geq 0$, which yields $2k_d>2\bar{k}_2\geq T_2^2\geq M^2k_d^2$. Hence, $k_d<2/M^2$. Also, from $2\bar{k}_1\bar{k}_2>0$, we know that $\bar{k}_2>0$, i.e., $k_d>L_2$.

Next, we estimate the bounds of k_p. It is easy to obtain

$$4k_p/M^2>4\bar{k}_1/M^2>2\bar{k}_1k_d>2\bar{k}_1\bar{k}_2\geq T_1^2>M^2k_p^2, \tag{81}$$

therefore $L_1<k_p<4/M^4$. Combine this with the bounds of k_d, we find that $\bar{\Omega}$ is bounded.

Define a function $H(\cdot)$ as follows:

$$H(k_p,k_d)=2\bar{k}_1\bar{k}_2-T_1^2-\bar{k}_1T_2^2,\quad (k_p,k_d)\in\bar{\Omega}.$$

By the definition (80) of $\bar{\Omega}$, we know that $H(k_p,k_d)\geq 0$, for $(k_p,k_d)\in\bar{\Omega}$ and $H(k_p,k_d)>0$, for $(k_p,k_d)\in\Omega$.

Since $\bar{\Omega}$ is compact, we know that $H(\cdot)$ can attain its maximum value. Note that $H(k_p,k_d)=0$ on the boundary of $\bar{\Omega}$, thus the maximum point $(k_p^*,k_d^*)\in\Omega$, and therefore $\frac{\partial H}{\partial k_p}\big|_{(k_p^*,k_d^*)}=\frac{\partial H}{\partial k_d}\big|_{(k_p^*,k_d^*)}=0$. By simple manipulations, we have

$$\frac{\partial H}{\partial k_p}\Big|_{(k_p^*,k_d^*)}=2\bar{k}_2-2T_1M-T_2^2=0, \tag{82}$$
$$\frac{\partial H}{\partial k_d}\Big|_{(k_p^*,k_d^*)}=2\bar{k}_1(1-T_2M)=0. \tag{83}$$

It follows from (82) and (83) that

$$M(N_2+Mk_d^*)=1,\ \ 2(N_1+Mk_p^*)M+(N_2+Mk_d^*)^2=2\bar{k}_2.$$

Hence, it can be obtained that $k_d^*=(1-MN_2)/M^2$ and

$$k_p^*=(1-2MN_2-2M^2L_2-2M^3N_1)/(2M^4),$$

which is exactly the formula given in (72).

Note that $H(k_p^*,k_d^*)>0$, and from (78)–(79), we know

$$\begin{aligned}&1-2N_2M-2L_2M^2-4N_1M^3-4L_1M^4\\=&2M^3(2M\bar{k}_1-T_1)=2M^3H(k_p^*,k_d^*)/T_1>0.\end{aligned} \tag{84}$$

Hence, Lemma 1 is proved. □

Similar to the proof of Lemma 1, we can obtain:

Lemma 2: A necessary and sufficient condition for the set Ω' defined by (19) to be non-empty is

$$16L_1M^4+16N_1M^3+4L_2M^2+4N_2M-1<0. \tag{85}$$

B. Proof of Proposition 1.

Without loss of generality, we assume that $y^*=0$. Suppose that for some k_p and k_d and for some initial state $(x_1(0),x_2(0))\in\mathbb{R}^n$, the closed-loop equation (1)-(2) satisfies

$$\lim_{t\to\infty}\mathbb{E}\|x_1(t)\|^2=0\quad\text{and}\quad\lim_{t\to\infty}\mathbb{E}\|x_2(t)\|^2=0, \tag{86}$$

we proceed to show $f(0,0,0)=g(0,0,0)=0$.

Note that $u(t)=-k_px_1(t)-k_dx_2(t)$, it follows from (86) that $\lim_{t\to\infty}\mathbb{E}\|u(t)\|^2=0$. Recall $\mathrm{d}x_2=f(x_1,x_2,u)\mathrm{d}t+g(x_1,x_2,u)\mathrm{d}B_t$, it follows that

$$x_2(t+1)-x_2(t)=X_t+Y_t, \tag{87}$$

where

$$X_t = \int_t^{t+1} f(x_1(s), x_2(s), u(s))\mathrm{d}s, \tag{88}$$

$$Y_t = \int_t^{t+1} g(x_1(s), x_2(s), u(s))\mathrm{d}B_s. \tag{89}$$

Next, we proceed to show that

$$\mathbb{E}\left[X_t^{\mathsf{T}} Y_t\right] \to 0, \text{ as } t \to \infty. \tag{90}$$

To this end, we first need to prove the following two facts:

$$\lim_{t\to\infty} \mathbb{E}\|X_t - f(0,0,0)\|^2 = 0, \tag{91}$$

$$\lim_{t\to\infty} \mathbb{E}\big\|Y_t - g(0,0,0)(B_{t+1} - B_t)\big\|^2 = 0. \tag{92}$$

From the Cauchy-Schwarz inequality, we can obtain

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}\|X_t - f(0,0,0)\|^2 \\
=&\lim_{t\to\infty} \mathbb{E}\Big\| \int_t^{t+1} \Big[f(x_1(s), x_2(s), u(s)) - f(0,0,0)\Big]\mathrm{d}s\Big\|^2 \\
\le&\lim_{t\to\infty} \mathbb{E}\int_t^{t+1} \|f(x_1(s), x_2(s), u(s)) - f(0,0,0)\|^2 \mathrm{d}s.
\end{aligned} \tag{93}$$

Moreover, from (93) and the Lipschitz property of f, we have

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}\|X_t - f(0,0,0)\|^2 \\
\le&\lim_{t\to\infty} \mathbb{E}\int_t^{t+1} C\left[\|x_1(s)\|^2 + \|x_2(s)\|^2 + \|u(s)\|^2\right]\mathrm{d}s \\
=&\lim_{t\to\infty} C\int_t^{t+1} \mathbb{E}\left[\|x_1(s)\|^2 + \|x_2(s)\|^2 + \|u(s)\|^2\right]\mathrm{d}s \\
=&0,
\end{aligned} \tag{94}$$

for some constant $C > 0$. Hence, (91) is proved. Similarly, by the Ito's isometry and the Lipschitz property of g, one can prove (92) in a similar way. From (92), we know that $\mathbb{E}\|Y(t)\|^2$ is a bounded function of $t \in [0, \infty)$.

By applying (91)-(92) again, it can be obtained that

$$\lim_{t\to\infty} \mathbb{E}\left[X_t^{\mathsf{T}} Y_t - f^{\mathsf{T}}(0,0,0)g(0,0,0)(B_{t+1} - B_t)\right] = 0. \tag{95}$$

On the other hand, note that

$$\mathbb{E}\left[f^{\mathsf{T}}(0,0,0)g(0,0,0)(B_{t+1} - B_t)\right] = 0, \ \forall t \ge 0. \tag{96}$$

Consequently, (90) follows from (95) and (96).

From (87), we know that

$$\mathbb{E}\left[\|X_t\|^2 + \|Y_t\|^2 + 2X_t^{\mathsf{T}} Y_t\right] = \mathbb{E}\left[\|x_2(t+1) - x_2(t)\|\right]^2.$$

Recall $\lim_{t\to\infty} \mathbb{E}\|x_2(t)\|^2 = 0$, we conclude that

$$\lim_{t\to\infty} \mathbb{E}\left[\|X_t\|^2 + \|Y_t\|^2 + 2X_t^{\mathsf{T}} Y_t\right] = 0. \tag{97}$$

From (90), we have $\lim_{t\to\infty} \mathbb{E}\|X_t\|^2 + \|Y_t\|^2 = 0$. Combine this with (91) and (92), we can obtain $f(0,0,0) = 0$ and $g(0,0,0) = 0$. $\square$

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