

Cooperative Multi-Agent Reinforcement Learning with Partial Observations

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Abstract—In this paper, we propose a distributed zeroth-order policy optimization method for Multi-Agent Reinforcement Learning (MARL). Existing MARL algorithms often assume that every agent can observe the states and actions of all the other agents in the network. This can be impractical in large-scale problems, where sharing the state and action information with multi-hop neighbors may incur significant communication overhead. The advantage of the proposed zeroth-order policy optimization method is that it allows the agents to compute the local policy gradients needed to update their local policy functions using local estimates of the global accumulated rewards that depend on partial state and action information only and can be obtained using consensus. Specifically, to calculate the local policy gradients, we develop a new distributed zeroth-order policy gradient estimator that relies on one-point residual-feedback which, compared to existing zeroth-order estimators that also rely on one-point feedback, significantly reduces the variance of the policy gradient estimates improving, in this way, the learning performance. We show that the proposed distributed zeroth-order policy optimization method with constant stepsize converges to the neighborhood of a policy that is a stationary point of the global objective function. The size of this neighborhood depends on the agents' learning rates, the exploration parameters, and the number of consensus steps used to calculate the local estimates of the global accumulated rewards. Moreover, we provide numerical experiments that demonstrate that our new zeroth-order policy gradient estimator is more sample-efficient compared to other existing one-point estimators.

Index Terms—Distributed Zeroth-Order Optimization, Multi-Agent Reinforcement Learning, Partial Observation

I. INTRODUCTION

Multi-Agent Reinforcement Learning (MARL) has received a lot of attention in recent years due to its wide applicability in real-world large-scale decision making problems, e.g., cloud autonomous driving, distributed multi-robot planning, and distributed resource allocation, to name a few. The goal is to enable a team of agents to collaboratively determine the global optimal policy that maximizes the sum of their local accumulated rewards. To do so, the agents typically need to communicate with each other in order to obtain information about the global state and action of the team. This is because

their states and rewards are generally affected by the actions of their other peers. However, sharing such information can be undesirable, due to significant communication overhead or privacy concerns. Therefore, there is a great need for MARL algorithms that rely only on partial observations of the global state and action information.

A major challenge in developing cooperative MARL methods under partial observations is that the environment, as perceived by every individual agent when it interacts with the other agents, is non-stationary since it changes as a result of changes in the policies of those agents [1]. In [1]–[3], this challenge is addressed using a centralized Critic function that can mitigate the effect of non-stationarity in the learning process. Then, the trained policies can be executed in a decentralized way. In [4], a distributed offline experience replay technique is developed to enable fully decentralized training, which requires that all agents receive a global reward at each timestep. However, when the global reward is defined as the sum of local agent rewards, as in cooperative MARL, this global reward can not be easily available to the local agents in practice. Cooperative MARL methods that maximize the sum of local rewards are considered in [5]–[8]. These works develop fully decentralized Actor-Critic methods where the agents maintain local estimates of the global value or policy functions, that depend on the states and actions of all other agents and update those estimates until they reach consensus. Then, these local estimates of the global value or policy functions are used to compute the policy gradient estimates needed for optimization. Since these policy gradient estimates require knowledge of the global state and action information, such Actor-Critic methods can not be used for cooperative MARL with partial state and action information.

In this paper, we propose a new distributed zeroth-order estimator of the local policy gradients, which is an extension of the one-point zeroth-order gradient estimator developed in [9] for centralized optimization problems that estimates the gradient using the residual of the function values at two consecutive iterations. As such, it queries the function value only once at each iteration. Specifically, our proposed estimator computes the local policy gradients by locally perturbing the local agent policies, using information about the sum of local accumulated rewards that can be obtained using consensus; the sum of local accumulated rewards is global information that is not otherwise accessible to the local agents. The advantage of our proposed zeroth-order policy optimization method is that it makes it possible to estimate the local policy gradients

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without having access to the global Critic function, which is not available under partial state and action information.

Related work: Zeroth-order policy optimization has been considered in [10], [11] for a special case of single-agent RL problems, namely, Linear Quadratic Regulation (LQR) problems. These results were extended in [12] for distributed LQR problems. All these works use the one-point zeroth-order policy gradient estimator proposed in [13], [14], which is known to have large variance that slows down learning [15]. Instead, our proposed distributed residual-feedback estimator returns policy gradient estimates with significantly lower variance which improves the learning performance. Our work is also related to distributed optimization methods, as in [16]–[20]. In these works, the agents collaborate to find a decision variable that maximizes the sum of local objective functions. For this, the method in [18] assumes that the gradient can be computed, while the methods proposed in [16], [17], [19], [20] do not assume knowledge of the gradient and instead compute the zeroth-order gradient estimates of the local objective functions with respect to all decision variables, even those owned by other agents. However, in MARL with partial observations, the agents can not have access to all decision variables and, therefore, can only compute the zeroth-order gradient estimate of their local objective functions with respect to those partial decision variables to which they have access. Therefore, the methods in [16], [17] cannot be used to solve the MARL problems considered here. In fact, recent work on cooperative MARL [21] relies on the assumption that the local agents can observe the team reward. This is in contrast to the problem considered here where the agents can only observe their local rewards, while the team rewards, which are the summation of local rewards, are not accessible locally. Therefore, the method in [21] cannot be applied to the problem considered in this paper. Related is finally work on multi-agent game formulations of MARL problems [22], [23]. While these problems rely on partial state and action information, they are non-cooperative in nature since the goal of the agents is to optimize their local policies in order to maximize their local accumulated rewards, as opposed to maximizing the sum of their local accumulated rewards. As a result, in these problems, the agents converge to a Nash equilibrium point rather than an optimal policy that maximizes the global accumulated rewards.

Contributions: In this paper, we propose a new distributed zeroth-order policy optimization method for general cooperative MARL problems. Compared to the one-point policy gradient estimators in [10]–[12], our proposed residual-feedback policy gradient estimator reduces the variance of the policy gradient estimates and, therefore, improves the learning performance. Compared to the centralized estimator in [9] that produces unbiased gradient estimates, the proposed distributed policy gradient estimator is biased due to possible consensus errors in distributedly estimating the sum of local accumulated rewards needed for the estimation of the policy gradients. We show that the proposed zeroth-order policy optimization method with constant stepsize converges to a neighborhood of a stationary point (policy) of the global objective function. The size of this neighborhood depends on the number of consensus steps needed to control the

bias in the policy gradient estimates. Moreover, we propose a value tracking method to reduce the number of consensus steps needed to achieve a desired user-specified solution accuracy. Finally, compared to existing distributed zeroth-order optimization methods [16], [17], our method is the first that does not require a perturbation of all decision variables or knowledge of all local objective function values to compute the zeroth-order gradients of the local objective functions, but still converges to a stationary point of the global objective function. To the best of our knowledge, this is the first work that provides convergence guarantees for collaborative multi-agent reinforcement learning problems when global states and actions are partially observable and agents can only receive local rewards.

The rest of the paper is organized as follows. In Section 2, we present the MARL problem under consideration and introduce preliminary results on zeroth-order optimization. In Section 3, we develop our proposed algorithm and present the convergence analysis. In Section 4, we verify the effectiveness of our algorithms using numerical experiments. In Section 5, we conclude the paper.

II. PRELIMINARIES AND PROBLEM FORMULATION

Consider a multi-agent system consisting of N agents. The agent dynamics are governed by a Markov Decision Process (MDP) defined by a tuple $(\mathcal{S}, \mathcal{A}, \mathcal{R}, \mathcal{P}, \gamma)$, where $s_t = [s_{1,t}, s_{2,t}, \dots, s_{N,t}] \in \mathcal{S}$ and $a_t = [a_{1,t}, a_{2,t}, \dots, a_{N,t}] \in \mathcal{A}$ denote the joint state and action spaces of the N agents at time instant t . The reward vector $r = [r_{1,t}, r_{2,t}, \dots, r_{N,t}] \in \mathcal{R}$ denotes the local rewards received by each agent at time t . The local reward $r_{i,t}(s_t, a_t, w_t)$ is affected by the joint state and action of all the agents in the network, and is also subject to noise w_t . The transition function $P(s_t, a_t, s_{t+1}) : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \rightarrow [0, 1] \in \mathcal{P}$ denotes the probability of transitioning to state s_{t+1} when the agents take action a_t at state s_t . Let $o_{i,t} \in \mathcal{O}_i$ represent the local observation received at agent i at time t , which contains partial entries of the joint state and action vectors, s_t and a_t . Agent i selects its action $a_{i,t}$ based on the observation $o_{i,t}$ using its local policy function $\pi_i : \mathcal{O}_i \rightarrow \mathcal{A}_i$. Let π denote the joint policy function which consists of all local policy functions π_i . Then, the accumulated discounted reward received by agent i is defined as $Q_i^\pi(s, a) = \mathbb{E}[\sum_{t=0}^T \gamma^t r_{i,t} | s_0 = s, a_0 = a]$ or $V_i^\pi(s) = \mathbb{E}[\sum_{t=0}^T \gamma^t r_{i,t} | s_0 = s]$, when the agents start from the state-action pair (s, a) or state s , follow the joint policy π , and apply a discount factor $\gamma \leq 1$ to their future rewards¹.

Our goal in this paper is to find an optimal joint policy function π^* that solves the problem $\max_{\pi} \frac{1}{N} \sum_{i=1}^N J_i(\pi)$, where $J_i(\pi) = \mathbb{E}_{(s_0, a_0) \sim \rho_0} [Q_i^\pi(s_0, a_0)]$ and ρ_0 is a distribution that the initial state-action pair is sampled from. To do so, we assume that the local policy function π_i is parameterized as $\pi_i(\theta_{i,k})$, where $\theta_{i,k} \in \mathbb{R}^{d_i}$ is the local policy parameter during episode k . Stacking these local policy parameters into the global policy parameter vector $\theta \in \mathbb{R}^d$, we can rewrite the

¹Although we consider a task with accumulated discounted rewards in this paper, our proposed methods can be easily adapted to task considering averaged rewards.

problem we consider in this paper as

$$\max_{\theta \in \mathbb{R}^d} J(\theta) := \frac{1}{N} \sum_{i=1}^N J_i(\theta). \quad (1)$$

Assumption 2.1: We assume that the local objective function $J_i(\theta)$ is non-convex and non-smooth for all $i = 1, 2, \dots, N$.

Problem (1) can be solved using distributed Actor-Critic methods as in [5], [6]. These methods require that all agents maintain local estimates of the global value function or the global policy function and that these local estimates are parameterized in the same way and depend on the global states and actions of all other agents. Therefore, they cannot be used for MARL with partial state and action information. Instead, in this paper, we propose a new distributed zeroth-order policy optimization method that relies on the stochastic gradient ascent update

$$\theta_{k+1} = \theta_k + \alpha \nabla J(\theta_k) + \epsilon_k \quad (2)$$

to determine optimal policy parameters θ_k that solve Problem (1), where ϵ_k represents the noise in gradient estimate and α denotes the stepsize. The key idea that makes it possible to use partial state and action information in the update (2) is the use zeroth-order estimators $\tilde{\nabla} J(\theta_k)$ of the true policy gradient $\nabla J(\theta_k)$. Zeroth-order gradient estimators have been recently proposed in [15], [24], [25], and take the form

$$\tilde{\nabla} J(\theta_k) = \frac{J(\theta_k + \delta u_k, \xi_k) - J(\theta_k - \delta u_k, \xi_k)}{2\delta} u_k, \quad (3)$$

$$\text{or } \tilde{\nabla} J(\theta_k) = \frac{J(\theta_k + \delta u_k, \xi_k) - J(\theta_k - \delta u_k, \xi_k)}{2\delta} u_k, \quad (4)$$

where $J(\theta_k + \delta u_k, \xi_k)$ is an unbiased noisy sample² of the accumulated rewards under the perturbed policy $\theta_k + \delta u_k$, $\delta \in \mathbb{R}$ is the exploration parameter, $u_k \in \mathbb{R}^d$ is the random exploration direction sampled independently from a standard multivariate normal distribution $\mathcal{N}(0, I_d)$ and I_d is an identity matrix with dimension d . The estimator (3) is called a *one-point estimator* because it requires one policy evaluation at a single perturbed policy, $\theta_k + \delta u_k$. On the other hand, the estimator (4) is called a *two-point estimator* because it requires two policy evaluations at two different perturbed policies, $\theta_k + \delta u_k$ and $\theta_k - \delta u_k$. While the two-point estimator (4) typically produces gradient estimates with lower variance, it is difficult to use in decentralized MARL problems. This is because, to compute zeroth-order policy gradient estimates in MARL, requires coordination between the agents to evaluate their local policies. Specifically, the agents need to initialize an episode, randomly perturb their local policies and implement the perturbed policies until the end of episode. This procedure requires synchronization, which can incur delays. And the more points are used to estimate the zeroth-order gradient, the longer these delays become as multi-point gradient estimation requires synchronization over multiple episodes to implement one update. In practice, this waiting time forces the agents to

²For RL problems, the noise vector ξ_k in the function evaluation is due to noise in the initial state and action samples, the state transition dynamics and the reward signals.

remain at a sub-optimal policy for longer than the proposed one-point gradient estimator.

To address the above limitations akin to estimators (3) and (4), in this paper, we adopt the one-point residual-feedback policy gradient estimator

$$\tilde{\nabla} J(\theta_k) = \frac{J(\theta_k + \delta u_k, \xi_k) - J(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})}{\delta} u_k \quad (5)$$

originally proposed in [9]. Same as the estimator (3), the estimator (5) only requires one policy evaluation at each iteration, but can use the history of policy evaluations to effectively reduce the variance of the current policy gradient estimate and, therefore, improve the learning rate. We note that the estimator (5) cannot be directly used to solve MARL problems where the agents can only observe the value of their local objective functions J_i . To address this challenge, in this paper, we let agents implement a finite number of consensus steps to approximate the value of the global objective function J and analyze the effect of consensus errors on the gradient estimator (5), as we discuss below.³

According to [9], [24], both estimators (3) and (5) provide unbiased gradient estimates of a smoothed function $J_\delta(\theta)$ at θ_k , where $J_\delta(\theta)$ is defined as $J_\delta(\theta) := \mathbb{E}_u [J(\theta + \delta u)]$ and u is subject to a standard multivariate normal distribution. Therefore, updating the policy parameter θ_k as in (2) using the gradient estimates (3) or (5) will in fact converge to a stationary point of the smoothed function $J_\delta(\theta)$ rather than a stationary point of the value function $J(\theta)$ that may be nonsmooth. To ensure that the stationary point found by this process is meaningful for the original MARL problem, we need to define appropriate optimality conditions that additionally ensure that $J_\delta(\theta)$ and $J(\theta)$ are close to each other. Specifically, we consider the following optimality criterion

$$\|\nabla J_\delta(\theta)\|^2 \leq \epsilon, \quad \text{and } |J_\delta(\theta) - J(\theta)| \leq \epsilon_J, \quad (6)$$

which suggests that θ is an ϵ -stationary point of the smoothed value function $J_\delta(\theta)$, and that the smoothed value function $J_\delta(\theta)$ is ϵ_J -close to the true value function $J(\theta)$. In this paper, we use the notation $\|\cdot\|$ to denote the Euclidean vector norm, or the induced matrix norm induced by the Euclidean norm. To bound the distance between the smoothed function $J_\delta(\theta)$ and the original value function $J(\theta)$ in (6), we need the following assumption on the value function $J(\theta)$.

Assumption 2.2: The function $J(\theta)$ is Lipschitz with constant L_0 , that is, $|J(\theta_1) - J(\theta_2)| \leq L_0 \|\theta_1 - \theta_2\|$, for all $\theta_1, \theta_2 \in \mathbb{R}^d$.

We note that Assumption 2.2 can be restrictive in selecting the class of policy functions. However, even with Assumption 2.2, to the best of our knowledge, this work is the first to show convergence of distributed policy optimization under partial observability. Furthermore, it is the first to show convergence of distributed zeroth-order optimization when the local agents can only perturb their local decision variables and observe their

³We note that since the original submission of this paper, a new one-point zeroth-order estimator has been proposed in [26] that improves on the performance of the estimator in (5). However, this method is centralized and does not directly apply to the MARL problem under consideration.

local objective function value. For comparison, the methods in [16], [17] require that the full decision variable is perturbed at every local agent, which imply that they can all observe global states and actions. Given Assumption 2.2, the following result for the smoothed value function $J_\delta(\theta)$ holds.

Lemma 2.1: (Gaussian Approximation [24]) Given Assumption 2.2, the smoothed function $J_\delta(\theta)$ satisfies $|J_\delta(\theta) - J(\theta)| \leq \delta L_0 \sqrt{d}$, for all $\theta \in \mathbb{R}^d$.

According to Lemma 2.1, to control the approximation accuracy of the smoothed function $J_\delta(\theta)$, the parameter δ needs to be selected appropriately. The choice of this parameter will be discussed in Section III. Note that although the random exploration direction $u_k \sim \mathcal{N}(0, I_d)$ needed to evaluate the estimator (5) can be sampled in a fully decentralized way, the global value $J(\theta_k + \delta u_k) = \frac{1}{N} \sum_{i=1}^N J_i(\theta_k + \delta u_k, \xi_k)$ is not accessible by the local agents. In the next section, we design a new algorithm that relies on partial state and action information only to produce a fully decentralized implementation of the estimator (5).

III. ALGORITHM DESIGN AND THEORETICAL ANALYSIS

In this section, we propose a fully distributed zeroth-order policy optimization algorithm for MARL that employs the residual-feedback zeroth-order policy gradient estimator (5). Specifically, we first introduce a consensus step so that the global value $J(\theta_k + \delta u_k, \xi_k)$ in the estimator (5) can be computed locally. Given a finite number of consensus iterations, the local estimates of $J(\theta_k + \delta u_k, \xi_k)$ will be inexact and, therefore, the local policy gradient estimates will be biased. To control this bias, we then introduce a value tracking technique that reduces the bias at the current episode using the local estimates of $J(\theta_k + \delta u_k, \xi_k)$ from previous episodes. Finally, we provide convergence results showing that the proposed distributed zeroth-order policy optimization method with constant stepsize converges to a neighborhood of the stationary point of the smoothed global objective function. The size of this neighborhood is controlled by the number of consensus steps during each episode. Proofs of all theoretical results that follow can be found in the Appendix.

Our proposed algorithm is summarized in Algorithm 1. In what follows, we also assume that the N agents form a communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the index set of agents and \mathcal{E} represents the set of edges. The edge $(i, j) \in \mathcal{E}$ if agents i and $j \in \mathcal{N}$ can directly send information to each other. Moreover, we define by $W \in \mathbb{R}^{N \times N}$ a weight matrix associated with the graph \mathcal{G} such that the entry $W_{ij} > 0$ when $(i, j) \in \mathcal{E}$ and $W_{ij} = 0$ otherwise. Note that the communication graph \mathcal{G} is independent of the coupling between agents in their state transition function $\mathcal{P}(s_t, a_t, s_{t+1})$ and reward functions $\{r_i(s_t, a_t)\}$ defined in Section II.

A. Distributed Residual-Feedback Zeroth-Order Policy Optimization

In this section, we describe and analyze our proposed residual-feedback zeroth-order policy optimization algorithm in the absence of value tracking, i.e., when DoTracking =

Algorithm 1: Distributed Residual-Feedback Zeroth-Order Policy Optimization

Input: Exploration parameter δ , stepsize α , consensus matrix W , number of consensus steps N_c , initial policy parameter θ_0 , discount ratio γ , maximum number of time steps run per episode t_{\max} , number of episodes K , and the logic variable DoTracking.

- 1 Set $\mu_i^{-1}(N_c) = 0$ for all $i = 1, 2, \dots, N$;
- 2 **for** episode $k = 0, 1, 2, \dots, K$ **do**
- 3 For agents $i = 1, 2, \dots, N$, let agent i sample a random exploration direction $u_{i,k}$ from the standard multivariate normal distribution ;
- 4 Let all agents implement their perturbed policy $\pi_i(\theta_{i,k} + \delta u_{i,k})$ for t_{\max} time steps and construct unbiased estimates of their local accumulated rewards $\{J_i(\theta_k + \delta u_k, \xi_k)\}$;
- 5 For all agent $i = 1, 2, \dots, N$,
- 6 **if** DoTracking = False **or** $k == 0$ **then**
- 7 | set $\mu_i^k(0) = J_i(\theta_k + \delta u_k, \xi_k)$;
- 8 **else**
- 9 | set $\mu_i^k(0) = \mu_i^{k-1}(N_c) + J_i(\theta_k + \delta u_k, \xi_k) - J_i(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})$;
- 10 **end**
- 11 **for** $m = 0, 1, 2, \dots, N_c - 1$ **do**
- 12 For agents $i = 1, 2, \dots, N$, let agent i send $\mu_i^k(m)$ to its direct neighbors $j \in \mathcal{N}_i$ and conduct local averaging by computing $\mu_i^k(m+1) = \sum_{j \in \mathcal{N}_i} W_{ij} \mu_j^k(m)$;
- 13 **end**
- 14 For agents $i = 1, 2, \dots, N$, let agent i update its current policy parameter $\theta_{i,k}$ by

$$\theta_{i,k+1} = \theta_{i,k} + \alpha \frac{\mu_i^k(N_c) - \mu_i^{k-1}(N_c)}{\delta} u_{i,k}. \quad (7)$$
- 15 **end**

Output: Uniformly sample an integer k within the interval $[0, K]$ and output θ_k .

False in Algorithm 1 (line 6). Specifically, at the beginning of episode k , the agents randomly perturb their current policy parameters θ_k using a random exploration direction u_k and conduct on-policy local policy evaluation to obtain an unbiased estimate of the local accumulated rewards $\{J_i(\theta_k + \delta u_k)\}_{i=1,2,\dots,N}$ (lines 3-4). To conduct local policy evaluations, existing MARL methods [5], [6] usually assume that the global state-action pairs (s_t, a_t) are available to all local agents. Under this assumption, it is possible to update the local Critic functions $Q_i^\pi(s_t, a_t)$ in [5], [6] to reduce the variance of policy evaluations and, therefore, the variance of the policy gradient estimates [27]. However, when the agents only have access to local observations $o_{i,t}$ which contain partial entries of (s_t, a_t) , these methods cannot be used. Therefore, in this paper, evaluate the local policies as $J_i(\theta_k + \delta u_k, \xi_k) = r_i(1) + \gamma r_i(2) + \gamma^2 r_i(3) + \dots + \gamma^{t_{\max}-1} r_i(t_{\max})$, same as in REINFORCE [28]. This policy evaluation method can be implemented in a fully decentralized way but is subject to large variance, which increases the variance of the zeroth-order policy gradient estimates and degrades the convergence speed of the algorithm. The residual-feedback policy gradient estimator (5) can effectively reduce this variance as we will discuss later. In what follows, we make the following assumption on the local policy value estimator.

Assumption 3.1: For all agents, the local policy evaluation is subject to bounded variance. That is, $\mathbb{E}[(J_i(\theta, \xi) - J_i(\theta))^2] \leq \sigma^2$ for $i = 1, 2, \dots, N$.

After all agents compute local policy values $J_i(\theta_{i,k} + \delta u_k, \xi_k)$ as $J_i(\theta_k + \delta u_k, \xi_k) = r_i(1) + \gamma r_i(2) + \gamma^2 r_i(3) + \dots + \gamma^{t_{max}-1} r_i(t_{max})$, which is an unbiased estimation according to [28], they conduct N_c rounds of local averaging on their local policy values $\{J_i(\theta_k + \delta u_k, \xi_k)\}_{i=1,2,\dots,N}$ (lines 7, 11-13). As a result, they obtain inexact estimates $\mu_i^k(N_c + 1)$ of the global accumulated rewards $J(\theta_k + \delta u_k, \xi_k)$. To bound this estimation error, we need the following two assumptions.

Assumption 3.2: The undirected communication graph \mathcal{G} is connected and fixed for all episodes. In addition, the associated weight matrix W is doubly stochastic. That is, $W\mathbf{1}_N = \mathbf{1}_N$ and $W^T\mathbf{1} = \mathbf{1}_N$, where $\mathbf{1}_N$ denotes a N dimensional vector of elements 1.

For simplicity of notations, in the following we neglect the subscript N in $\mathbf{1}_N$ and write it as $\mathbf{1}$. The dimension of the vector can be taken as appropriate within its context. Assumption 3.2 is introduced to ensure that agents' local estimates μ_i^k of the global objective function value can reach consensus on their average $\frac{1}{N} \sum_i \mu_i^k$ within an error that depends on the number of consensus steps. In this paper we assume that the communication networks is fixed. If it is not, then consensus methods for time-varying graphs, such as that proposed in [29] can be used to estimate the global objective function value. The analysis in this case is left for future research

Assumption 3.3: The local values $J_i(\theta, \xi)$ are upper bounded by J_u and lower bounded by J_l for all $i = 1, 2, \dots, N$ and all policy parameters θ .

Assumption 3.3 can be easily satisfied for episodic RL problems with bounded rewards over all state-action pairs. Let $\bar{\mu}^k(m) = [\mu_1^k(m), \dots, \mu_N^k(m)]^T$. Then, we can show the following lemma.

Lemma 3.1: Given Assumptions 3.2 and 3.3, we have that $\|\bar{\mu}^k(N_c) - J(\theta_k + \delta u_k, \xi_k)\mathbf{1}\| \leq \rho_W^{N_c} \sqrt{N}(J_u - J_l)$, where $\rho_W = \|W - \frac{1}{N}\mathbf{1}\mathbf{1}^T\| < 1$.

Lemma 3.1 shows that the bias in the local estimate $\bar{\mu}^k(N_c)$ can be controlled by choosing a large enough N_c , when the local policy values are upper and lower bounded by J_u and J_l . Using the estimates $\bar{\mu}^k(N_c)$, the agents can then construct the decentralized policy gradients (7) and update their policy parameters $\theta_{i,k}$ (line 14). This completes episode k . The decentralized residual-feedback estimator (7) can reduce the variance of the policy gradient estimates, since the value estimate of the last policy iterate $\bar{\mu}_i^{k-1}(N_c)$ can provide a baseline to compare $\bar{\mu}_i^k(N_c)$ to. Effectively, the value estimate of the last policy iterate has an analogous variance reduction effect to the state value $V^\pi(s)$ that is used as a baseline for the action value $Q^\pi(s, a)$ in Actor-Critic methods [27].³ Next, we show how to select N_c so that the optimality criterion (6) is satisfied.

Theorem 3.2: (Learning Rate of Algorithm 1 without Value Tracking) Let Assumptions 2.2, 3.1, 3.2 and

³Note that the fact that the value of past policy iterates can also be used as a baseline to reduce the variance of policy gradient estimates may be of interest in its own right in the development of policy gradient methods.

3.3 hold and define $\delta = \frac{\epsilon_J}{\sqrt{d}L_0}$, $\alpha = \frac{\epsilon_J^{1.5}}{4d^{1.5}L_0^2\sqrt{K}}$, and $N_c \geq \log(\frac{\sqrt{\epsilon}\epsilon_J}{\sqrt{2}d^{1.5}L_0(J_u - J_l)}) / \log(\rho_W)$. Then, running Algorithm 1 with `DoTracking = False`, we have that $\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[\|\nabla J_\delta(\theta_k)\|^2] \leq \mathcal{O}(d^{1.5}\epsilon_J^{-1.5}K^{-0.5}) + \frac{\epsilon}{2}$, where the expectation is taken over the trajectory of sampled vector u_k and evaluation noise ξ_k .

As shown in Theorem 3.2, Algorithm 1 converges to a neighborhood of the stationary point of the smoothed global objective function. Specifically, according to the optimality conditions in (6), the approximation error on the smoothed objective function $J_\delta(\theta_k)$ compared to the original function $J(\theta_k)$ is controlled by the parameter ϵ_J . And the neighborhood around the stationary solution is characterized by the parameter ϵ . Given the user specified parameters ϵ_J and ϵ , Theorem 3.2 says that the number of iterations K can be selected according to the bound in (32). Note that computing this bound requires knowledge of the problem parameters $J_\delta^* - J_\delta(\theta_0)$, $\mathbb{E}[\|g_\delta(\theta_0)\|^2]$ and L_0 . In practice, we can replace these parameters in (32) by bounds selected sufficiently large. Note also that selecting the number K according to (32) can be conservative. We leave the study of tighter theoretical bounds on K for our future research.

The size of the neighborhood ϵ can be controlled by choosing the number of consensus steps N_c . Specifically, according to Lemma 3.1, the number of steps N_c controls the consensus error $\|\bar{\mu}^k(N_c) - J(\theta_k + \delta u_k, \xi_k)\mathbf{1}\|$, which in turn bounds the error ϵ of the solution; see the proof of Theorem 3.2 in the Appendix. Moreover, the number of consensus steps N_c depends not only on the user-specified accuracy level ϵ and ϵ_J , but also on the range of the policy bounds J_u and J_l . This is because the consensus iteration at each episode is independent of those at previous episodes. Therefore, to select N_c to control the estimation bias $|\mu_i^k(N_c) - J(\theta_k + \delta u_k, \xi_k)|$, we need to select the term $J_u - J_l$ in the definition of N_c in Theorem 3.2 as the difference between the initial estimates $\mu_i^k(0) = J_i(\theta_k + \delta u_k, \xi_k) \in [J_l, J_u]$ that have the maximum and minimum value.

B. Distributed Residual-Feedback Zeroth-Order Policy Optimization with Value Tracking

As discussed in Section III-A, the estimation bias $|\mu_i^k(N_c) - J(\theta_k + \delta u_k, \xi_k)|$ at episode k can be reduced by using local policy estimates from previous episodes. Specifically, rather than resetting $\mu_i^k(0) = J_i(\theta_k + \delta u_k, \xi_k)$ in line 7 of Algorithm 1, we update it using the estimate $\mu_i^{k-1}(N_c)$ from the last episode as $\mu_i^k(0) = \mu_i^{k-1}(N_c) + J_i(\theta_k + \delta u_k, \xi_k) - J_i(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})$. Then, we run N_c consensus iterations on $\mu_i^k(0)$ as before. Let $\bar{\mu}^k(m) = \frac{1}{N} \sum_{i=1}^N \mu_i^k(m)$. The following lemma shows that the value tracking updates preserve the global information $J(\theta_k + \delta u_k, \xi_k)$.

Lemma 3.3: Let Assumption 3.2 hold. Then, running Algorithm 1 with `DoTracking = True`, we have that $\bar{\mu}^k(m) = J(\theta_k + \delta u_k, \xi_k) = \frac{1}{N} \sum_{i=1}^N J_i(\theta_{i,k}, \xi_k)$, for all $m = 1, 2, \dots, N_c$ and all k .

Lemma 3.3 implies that the local estimation bias $|\mu_i^k(N_c) - J(\theta_k + \delta u_k, \xi_k)|$ is equal to the consensus error $|\mu_i^k(N_c) - \bar{\mu}^k(N_c)|$. Using value tracking, the bias at episode k can be

controlled by the consensus steps of past episodes. This is formally shown in the following lemma.

Lemma 3.4: Let Assumptions 2.2, 3.1, 3.2 hold and define $E_\mu^k = \|\bar{\mu}_k(N_c) - \bar{\mu}_k(N_c)\mathbf{1}\|$. Then, running Algorithm 1 with `DoTracking = True`, we have that $\mathbb{E}[(E_\mu^k)^2] \leq \left(2\mathbb{E}[(E_\mu^{k-1})^2] + 32dL_0^2\frac{\alpha^2}{\delta^2}\mathbb{E}[(E_\mu^{k-1})^2\|u_{k-1}\|^2] + 32d^2L_0^2\frac{\alpha^2}{\delta^2}\mathbb{E}[(E_\mu^{k-2})^2]\right)\rho_W^{2N_c} + 16NL_0^2\alpha^2\mathbb{E}[\|\tilde{\nabla}J(\theta_{k-1})\|^2]\rho_W^{2N_c} + 32NdL_0^2\delta^2\rho_W^{2N_c} + 16N\sigma^2\rho_W^{2N_c}$.

Compared to Lemma 3.1, the proposed value tracking technique makes it possible to bound the consensus error at episode k with the consensus errors from episodes $k-1$ and $k-2$. Furthermore, at each episode, this error is perturbed by the second order momentum of the policy gradient estimate (5) which can be controlled by choosing a small stepsize α and a large number N_c . To see the benefit of this result, when the consensus errors from episodes $k-1$ and $k-2$ are small, value tracking needs fewer consensus iterations to achieve small consensus error at episode k . This is in contrast to the case without value tracking, where N_c is selected regardless of previous consensus errors. The following result shows convergence of Algorithm 1 using value tracking.

Theorem 3.5: (Learning Rate of Algorithm 1 with Value Tracking) Let Assumptions 2.2, 3.1, 3.2 hold and define $\delta = \frac{\epsilon_J}{\sqrt{dL_0}}$, $\alpha = \frac{\epsilon_J^{1.5}}{4d^{1.5}L_0^2\sqrt{K}}$, and $N_c \geq \max(\log(\rho_W)^{-1}\log(\frac{1}{2\sqrt{2}}), \log(\rho_W)^{-1}\log(\frac{\epsilon}{4(G^2\epsilon_J + 32(d+4)^2dL_0^2 + 16d^3L_0^2\sigma^2/\epsilon_J^2)}))$ where $G^2 = \max\left(\mathbb{E}[\|\tilde{\nabla}J(\theta_0)\|^2], \frac{2\epsilon_J\epsilon}{dK} + 32L_0^2(d+4)^2 + 16d^2L_0^2\frac{\sigma^2}{\epsilon_J^2}\right)$. Then, running Algorithm 1 with `DoTracking = True`, we have that $\frac{1}{K}\sum_{k=0}^{K-1}\mathbb{E}[\|\nabla J_\delta(\theta_k)\|^2] \leq \mathcal{O}(d^{1.5}\epsilon_J^{-1.5}K^{-0.5}) + \frac{\epsilon}{2}$, where the expectation is taken over the trajectory of the sampled vector u_k and evaluation noise ξ_k .

In Theorem 3.5, the constant G^2 represents the uniform bound on $\mathbb{E}[\|\tilde{\nabla}J(\theta_k)\|^2]$ for all $k = 1, 2, \dots, K$. Moreover, from the bounds on N_c in Theorems 3.2 and 3.5, when ϵ and ϵ_J are close to 0, we obtain that $N_c \sim \mathcal{O}(\log(\frac{\sqrt{\epsilon\epsilon_J}}{d^{1.5}\sigma})/\log(\rho_W))$ in Theorem 3.5 and $N_c \sim \mathcal{O}(\log(\frac{\sqrt{\epsilon\epsilon_J}}{d^{1.5}(J_u - J_l)})/\log(\rho_W))$ in Theorem 1. This suggests that the choice of N_c in Theorem (3.5) depends on the variance of function evaluation σ^2 , while in Theorem 3.2 the choice of N_c depends on the range of value functions $[J_l, J_u]$. In practice, the standard deviation of function evaluation σ can be much smaller than the range of its value $[J_l, J_u]$. To see this, note that by Assumption 3.3, we have that the noisy sample of the objective function value $J_i(\theta, \xi)$ has bounded support. Then, applying the Popoviciu's inequality on variances, we get that the variance σ^2 satisfies that $\sigma^2 \leq \frac{1}{4}(J_u - J_l)^2$, that is, $\sigma \leq \frac{1}{2}|J_u - J_l|$. Therefore, Algorithm 1 with value tracking requires fewer consensus steps per episode than without value tracking.

Remark 3.4: The proposed value-tracking technique can also be combined with the existing distributed one-point policy gradient estimator [12] to reduce the variance of its gradient estimates. To see this, note that the global value function

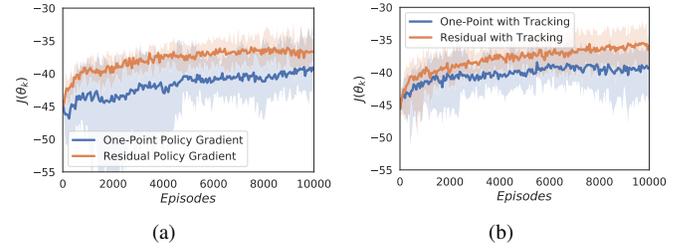


Fig. 1. Distributed zeroth-order policy optimization with the proposed residual-feedback estimator (5) (orange) versus the one-point estimator (3) (blue). In each case, Algorithm 1 is run 10 times. (a): Results without value tracking. (b): Results with value tracking.

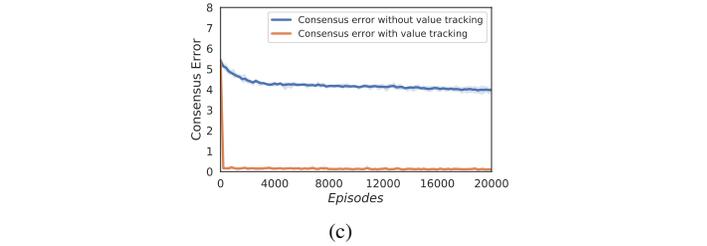
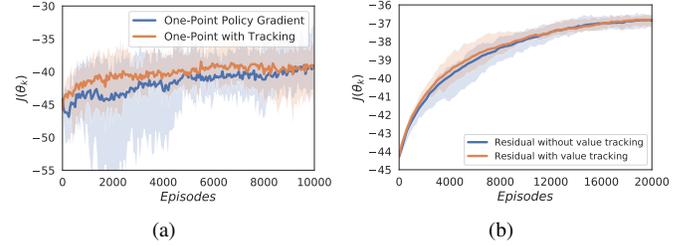


Fig. 2. Distributed zeroth-order policy optimization with value tracking (orange) versus without value tracking (blue). In each case, Algorithm 1 is run 10 times. (a): Comparative results for the one-point estimator (3). (b): Comparative results for the proposed residual-feedback estimator (5). (c) Maximum absolute consensus errors $\max_i |\mu_i^k(N_c) - J(\theta_k + \delta u_k, \xi_k)|$ over episodes.

$J(\theta_k + \delta u_k, \xi_k)$ used in the one-point estimator (3) can be replaced by the local estimate of the value $J(\theta_k + \delta u_k, \xi_k)$, i.e., $\mu_i^k(N_c)$. Then, we obtain the following distributed one-point policy gradient estimator with value tracking: $\tilde{\nabla}_{\theta_{i,k}} J(\theta_k) \approx \frac{\mu_i^k(N_c)}{\delta} u_{i,k} = u_{i,k} (\sum_{j \in \mathcal{N}_i} [W^{N_c}]_{ij} (\mu_j^{k-1}(N_c) + J_j(\theta_k + \delta u_k, \xi_k) - J_j(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1}))) / \delta$. We observe that the estimator $\tilde{\nabla}_{\theta_{i,k}} J(\theta_k)$ has the same structure as the distributed residual-feedback policy gradient estimator without value tracking (7) except for an additional noise term $\frac{\sum_{j \in \mathcal{N}_i} [W^{N_c}]_{ij} \mu_j^{k-1}(N_c)}{\delta} u_{i,k}$. Therefore, the variance of the estimator $\tilde{\nabla}_{\theta_{i,k}} J(\theta_k)$ is reduced through a similar mechanism as that of the distributed residual-feedback policy gradient estimator without value tracking. As a result, the learning performance is improved compared to that of the existing distributed one-point policy gradient estimator [12], as we will demonstrate in the next section.

IV. EXPERIMENTS

In this section, we illustrate our proposed MARL algorithm on stochastic multi-agent multi-stage decision making problems. Specifically, we conduct an ablation study to demonstrate the benefits of applying the decentralized residual-

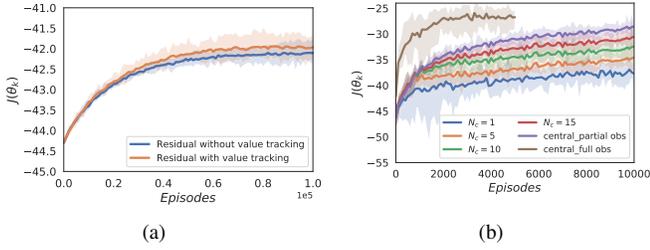


Fig. 3. (a) Algorithm 1 with value tracking (orange) versus without value tracking (blue) under the communication graph which does not respect the coupling relationship among agents. (b) Comparative results for Algorithm 1 with value tracking under different number of consensus steps N_c and the centralized algorithms with partial and full observations. In each case, Algorithm 1 is run 10 times.

feedback zeroth-order policy gradient estimate (7) and the value tracking technique separately. We consider 16 agents that are located on a 4×4 grid. Agent i has state $s_i(t) = [m_i(t), d_i(t)]$, which denotes the resources it stores and the local demand it receives in the amount of $m_i(t)$ and $d_i(t)$ at time t , respectively. In the meantime, agent i takes actions $a_{ij}(t)$, which denote the resources it shares with its neighbors $j \in \mathcal{N}_i$ in the grid. Specifically, $a_{ij}(t) \in [0, 1]$ denotes the fraction of resources agent i sends to its neighbor j at time t . The local resources and demands at agent i are defined as $m_i(t+1) = m_i(t) - \sum_{j \in \mathcal{N}_i} a_{ij}(t)m_i(t) + \sum_{j \in \mathcal{N}_i} a_{ji}(t)m_j(t) - d_i(t)$ and $d_i(t) = A_i \sin(\omega_i t + \phi_i) + w_{i,t}$, where $w_{i,t}$ is the noise in the demand. At time t , agent i receives a local reward $r_i(t)$, such that $r_i(t) = 0$ when $m_i(t) \geq 0$ and $r_i(t) = -m_i(t)^2$ when $m_i(t) < 0$. We consider a partial observation scenario, where agent i can only observe its local resources and demands, that is, $o_i(t) = [m_i(t), d_i(t)]^T$. Agent i determines its actions $\{a_{ij}(t)\}$ using its local policy function $\pi_i(o_i(t)|\theta_i)$, where θ_i is the policy function parameter for agent i . Specifically, we have that $a_{ij} = \exp(z_{ij}) / \sum_j \exp(z_{ij})$, where $z_{ij} = \sum_{p=1}^9 \|o_i - c_p\|^2 \theta_{ij}(p)$ and c_p is the p -th feature parameter. We consider episodes of length $T = 30$, and select the discount factor as $\gamma = 0.75$. The goal of the agents is to find the optimal joint policy parameter θ , so that their total accumulated reward $J(\theta) = \mathbb{E}_{s_0, a_0 \sim \rho_0} [\sum_{t=0}^{30} \gamma^t \sum_{i=1}^N r_i(t) | \pi(\theta)]$ is maximized. The communication graph is assumed to be a chain graph. Moreover, we select the number of consensus steps $N_c = 1$, and show that Algorithm 1 with value tracking can achieve policy improvement even in this challenging scenario. The stepsizes are selected so that the convergence speed is optimized.

First, we compare the performance of Algorithm 1 using the decentralized policy gradients (5) and (3), without value tracking. The learning progress is presented in Figure 1(a). We observe that the decentralized residual-feedback policy gradient estimator has less variance than the existing one-point policy gradient estimator and, therefore, improves faster and finds a better policy in the end of the learning. The same effect is observed in Figure 1(b), when both estimators are implemented with value tracking. This suggests that the residual-feedback zeroth-order policy gradient estimator is superior to the one-point policy gradient estimator for decentralized policy optimization problems.

Next, we demonstrate the merit of using value tracking. Specifically, we first run Algorithm 1 with the decentralized one-point policy gradient estimator (3), with and without value tracking. The difference in the performance is shown in Figure 2(a). We observe that using value tracking results in less variance and also achieves better policies. This is because the decentralized one-point policy gradient estimator with value tracking $\tilde{\nabla}_{\theta_{i,k}} J(\theta_k)$ has the same variance reduction effect on the policy gradient estimates as the residual feedback estimator 5, as we have discussed in Remark 3.4. Figure 2(b) shows the results of using the residual-feedback policy gradient estimator (5) with and without value tracking. We observe that Algorithm 1 with value tracking performs slightly better in the mean than without value tracking. This is because value tracking can track the value of the global objective function better, as shown in Figure 2(c) where the maximum consensus error $\max_i |\mu_i^k(N_c) - J(\theta_k + \delta u_k, \xi_k)|$ at each episode k is presented. The improvement achieved by value tracking is not very significant in Figure 2(b) because the underlying communication graph respects the coupling relationship among agents. Specifically, we say that the communication graph respects the coupling relationship between agents if the action of every agent i that can directly communicate with an agent j also directly affects the reward and transition function of that agent j . In this case, the rewards received from an agent's local neighbors can approximate this agent's contribution to the global reward well even without tracking the information from other distant neighbors in the graph.

To further demonstrate the advantage of combining the decentralized residual-feedback gradient estimator with value tracking, we consider a challenging scenario where the communication graph does not respect the coupling relationship among agents as described above. The performance of Algorithm 1 using the residual-feedback estimator (5) with and without value tracking is presented in Figure 3(a). In this case, using rewards from the local neighbors does not approximate well the local agent's contribution to the global reward. Therefore, value tracking can help obtain a better estimate of the global reward information. As a result, the decentralized residual-feedback policy gradient estimator with value tracking outperforms the one without value tracking, as shown in Figure 3(a). The numerical results presented above show that the performance of Algorithm 1 is affected by the structure of the communication graph among the agents. As shown in Lemma 3.1, the second largest singular value of the matrix W , ρ_W , captures the speed at which the agents can reach consensus on the global objective function value. As shown in (23) in Lemma 1.3, the bound on the second moment of the gradient estimate is related to the consensus error terms, e.g., $\|\bar{\mu}^{k-1} - \bar{\mu}^{k-1} \mathbf{1}\|$. Therefore, the smaller the value of ρ_W , the smaller is the second moment and, therefore, the variance of the gradient estimate. This suggests that the variance in learning can be further reduced for communication graphs that enable fast consensus.

Finally, we demonstrate the effect of the consensus steps N_c on Algorithm 1 by comparing to a centralized algorithm that uses the gradient estimator (5) with both full and partial observations. Specifically, in the centralized algorithm, the value

of the global objective function $J(\theta_k + \delta u_k, \xi_k)$ is directly provided to each local agent at each episode, and the local agents' policy functions receive all agents' states as inputs when full observations are assumed and only receive the neighboring agents' states as inputs when partial observations are assumed. As shown in Figure 3(b), as the number of consensus steps N_c increases, the performance of Algorithm 1 with value tracking approaches that of the centralized algorithm with partial observations. And the performance of the centralized algorithm with partial observations slightly underperforms that of the centralized algorithm with full observations. This is because policy functions learned using partial observations constitute a subset of those that can be learned using full observations. Note that the centralized algorithm requires all the agents to observe the values of all local objective functions, which is not a scalable approach in practice. Its performance is provided here as a benchmark to compare to the performance of the proposed distributed algorithm.

V. CONCLUSION

In this paper, we proposed a new distributed zeroth-order policy optimization method for MARL problems. Compared to existing MARL algorithms that require all the agents' states and actions to be accessible by every local agent, our algorithm can be applied even when each agent only observes partial states and actions. Specifically, we developed a new distributed residual-feedback zeroth-order estimator of the policy gradient and analyzed the effect of bias in the local policy gradient estimates on the convergence of the proposed MARL algorithm. Furthermore, we introduced a value tracking technique to reduce the number of consensus steps needed at each episode to control the bias in the estimation of the policy gradient. Finally, we provided numerical experiments on a stochastic multi-agent multi-stage decision making problem that demonstrated the effectiveness of both the decentralized residual-feedback policy gradient estimator and the value tracking technique.

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APPENDIX

Lemma 3.1. Given Assumptions 3.2 and 3.3, we have that $\|\bar{\mu}^k(N_c) - J(\theta_k + \delta u_k, \xi_k)\mathbf{1}\| \leq \rho_W^c \sqrt{N} (J_u - J_l)$, where $\rho_W = \|W - \frac{1}{N}\mathbf{1}\mathbf{1}^T\| < 1$.

Proof: First, we show that $\frac{1}{N}\mathbf{1}^T \bar{\mu}^k(m) = J(\theta_k + \delta u_k, \xi_k)$ for all $m = 0, 1, \dots, N_c$. Note that the consensus step $\mu_i^k(m+1) = \sum_{j \in N_i} W_{ij} \mu_j^k(m)$ (line 12 in Algorithm 1) can be equivalently written in a compact form as $\bar{\mu}^k(m+1) = W \bar{\mu}^k(m)$. Therefore, we have that

$$\frac{1}{N}\mathbf{1}^T \bar{\mu}^k(m+1) = \frac{1}{N}\mathbf{1}^T W \bar{\mu}^k(m) = \frac{1}{N}\mathbf{1}^T \bar{\mu}^k(m), \quad (8)$$

where the second equality is due to Assumption 3.2, that the matrix W is doubly stochastic. Extending equality (8) from m to 0, we obtain that $\frac{1}{N}\mathbf{1}^T \bar{\mu}^k(m) = \frac{1}{N}\mathbf{1}^T \bar{\mu}^k(0) = \frac{1}{N} \sum_{i=1}^N J_i(\theta_k + \delta u_k, \xi_k) = J(\theta_k + \delta u_k, \xi_k)$, for $m = 0, 1, \dots, N_c$.

Next, we show that $\|\bar{\mu}^k(m) - \frac{1}{N}\mathbf{1}\mathbf{1}^T \bar{\mu}^k(m)\| \leq \|W - \frac{1}{N}\mathbf{1}\mathbf{1}^T\|^m \|\bar{\mu}^k(0) - \frac{1}{N}\mathbf{1}\mathbf{1}^T \bar{\mu}^k(0)\|$, for $m = 1, 2, \dots, N_c$. To see this, we have that

$$\begin{aligned} & \|\bar{\mu}^k(m+1) - \frac{1}{N}\mathbf{1}\mathbf{1}^T \bar{\mu}^k(m+1)\| \\ &= \|W \bar{\mu}^k(m) - \frac{1}{N}\mathbf{1}\mathbf{1}^T W \bar{\mu}^k(m)\| \\ &= \|W \bar{\mu}^k(m) - \frac{1}{N}\mathbf{1}\mathbf{1}^T \bar{\mu}^k(m)\|, \end{aligned} \quad (9)$$

where the second equality is due to Assumption 3.2. According to (9), we have that $\|\bar{\mu}^k(m+1) - \frac{1}{N}\mathbf{1}\mathbf{1}^T \bar{\mu}^k(m+1)\| = \|(W - \frac{1}{N}\mathbf{1}\mathbf{1}^T) \bar{\mu}^k(m) - \frac{1}{N}\mathbf{1}\mathbf{1}^T \bar{\mu}^k(m)\|$. This is because $(W - \frac{1}{N}\mathbf{1}\mathbf{1}^T) \frac{1}{N}\mathbf{1}\mathbf{1}^T \bar{\mu}^k(m) = 0$. Therefore, we get that

$$\begin{aligned} & \|\bar{\mu}^k(m+1) - \frac{1}{N}\mathbf{1}\mathbf{1}^T \bar{\mu}^k(m+1)\| \\ & \leq \|W - \frac{1}{N}\mathbf{1}\mathbf{1}^T\| \|\bar{\mu}^k(m) - \frac{1}{N}\mathbf{1}\mathbf{1}^T \bar{\mu}^k(m)\| \\ & \leq \dots \leq \|W - \frac{1}{N}\mathbf{1}\mathbf{1}^T\|^{m+1} \|\bar{\mu}^k(0) - \frac{1}{N}\mathbf{1}\mathbf{1}^T \bar{\mu}^k(0)\|. \end{aligned} \quad (10)$$

The first inequality is due to the definition of the induced matrix norm $\|W - \frac{1}{N}\mathbf{1}\mathbf{1}^T\|$. According to Assumption 3.2, we have that $\rho_W = \|W - \frac{1}{N}\mathbf{1}\mathbf{1}^T\| < 1$. Furthermore, recalling the fact that $\frac{1}{N}\mathbf{1}^T \bar{\mu}^k(m) = \frac{1}{N}\mathbf{1}^T J_i(\theta_k + \delta u_k, \xi_k) = J(\theta_k + \delta u_k, \xi_k)$ for $m = 0, 1, \dots, N_c$, we obtain that

$$\begin{aligned} & \|\bar{\mu}^k(N_c) - J(\theta_k + \delta u_k, \xi_k)\mathbf{1}\| \\ & \leq \rho_W^c \|\bar{\mu}^k(0) - J(\theta_k + \delta u_k, \xi_k)\mathbf{1}\|. \end{aligned} \quad (11)$$

Since $\mu_i^k(0) = J_i(\theta_k + \delta u_k, \xi_k) \in [J_l, J_u]$ and $J(\theta_k + \delta u_k, \xi_k) = \frac{1}{N} \sum_{i=1}^N J_i(\theta_k + \delta u_k, \xi_k) \in [J_l, J_u]$, we have that $\|\bar{\mu}^k(0) - J(\theta_k + \delta u_k, \xi_k)\mathbf{1}\| \leq \sqrt{N}(J_u - J_l)$. Plugging this inequality into the bound in (11), we complete the proof. ■

In the subsequent proof, we need the following lemma by [24].

Lemma 1.1: (Lipschitz Properties of the Smoothed Function, [24]) Given Assumption 2.2, the smoothed function $J_\delta(\theta)$ is differentiable and its gradient is Lipschitz, that is, $\|\nabla J_\delta(\theta_1) - \nabla J_\delta(\theta_2)\| \leq \frac{\sqrt{d}L_0}{\delta} \|\theta_1 - \theta_2\|$, for all $\theta_1, \theta_2 \in \mathbb{R}^d$. Next, we present a lemma that bounds the squared norm of the gradient of the smoothed function $\nabla J_\delta(\theta_k)$ at iterate θ_k .

Lemma 1.2: Let Assumptions 2.2 and 3.1 hold. Then, for all $k \geq 0$, we have that

$$\begin{aligned} \mathbb{E}[\|\nabla J_\delta(\theta_k)\|^2 | \mathcal{F}_{k-1}] & \leq \frac{2}{\alpha} (\mathbb{E}[J_\delta(\theta_{k+1}) - J_\delta(\theta_k) | \mathcal{F}_{k-1}]) \\ & \quad + 2\sqrt{d}L_0 \frac{\alpha}{\delta} \mathbb{E}[\|g_\delta(\theta_k)\|^2 | \mathcal{F}_{k-1}] \\ & \quad + \frac{d_i}{\delta^2} \mathbb{E}[\|\bar{\mu}^k - \bar{\mu}^k \mathbf{1}\|^2 \|u_k\|^2 | \mathcal{F}_{k-1}] \\ & \quad + 4\sqrt{d}d_i L_0 \frac{\alpha}{\delta^3} \mathbb{E}[\|\bar{\mu}^k - \bar{\mu}^k \mathbf{1}\|^2 \|u_k\|^2 | \mathcal{F}_{k-1}] \\ & \quad + 4d^{1.5}d_i L_0 \frac{\alpha}{\delta^3} \mathbb{E}[\|\mu^{k-1} - \bar{\mu}^{k-1} \mathbf{1}\|^2 | \mathcal{F}_{k-1}], \end{aligned} \quad (12)$$

where the filtration $\mathcal{F}_{k-1} = \sigma(u_t, \xi_t | t \leq k-1)$, $g_\delta(\theta_k) = \frac{J(\theta_k + \delta u_k, \xi_k) - J(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})}{\delta} u_k$ and $\bar{\mu}^k = \frac{1}{N}\mathbf{1}^T \bar{\mu}^k = J(\theta_k + \delta u_k, \xi_k)$.

Proof: According to Assumption 2.2 and Lemma 1.1, we have that the smoothed function $f_\delta(\theta)$ has Lipschitz gradient with the Lipschitz constant $L_{1,\delta} = \frac{\sqrt{d}L_0}{\delta}$. Therefore, using the inequality (6) in [24], we obtain that

$$\begin{aligned} \langle \nabla J_\delta(\theta_k), \theta_{k+1} - \theta_k \rangle & \leq J_\delta(\theta_{k+1}) - J_\delta(\theta_k) + \\ & \quad \frac{L_{1,\delta}}{2} \|\theta_{k+1} - \theta_k\|^2. \end{aligned} \quad (13)$$

Without loss of generality, we assume that each agent's local policy function π_i is parameterized with $\theta_i \in \mathbb{R}^{d_i}$ and $d_i = \frac{d}{N}$ for all i . Then, the update (7) can be written in the compact form

$$\begin{aligned} \theta_{k+1} &= \theta_k + \frac{\alpha}{\delta} \text{diag}([\mu_1^k(N_c) - \mu_1^{k-1}(N_c), \\ & \quad \dots, \mu_N^k(N_c) - \mu_N^{k-1}(N_c)]) \otimes I_{d_i} u_k \\ &= \theta_k + \frac{\alpha}{\delta} \text{diag}(\bar{\mu}^k - \bar{\mu}^{k-1}) \otimes I_{d_i} u_k \end{aligned} \quad (14)$$

where \otimes represents the kronecker product and I_{d_i} is an identity matrix with dimension d_i . To simplify the notation, we use μ_i^k or $\bar{\mu}^k$ to denote $\mu_i^k(N_c)$ and $\bar{\mu}^k(N_c)$, respectively. Then, we equivalently rewrite equality (14) as

$$\begin{aligned} \theta_{k+1} &= \theta_k + \frac{\alpha}{\delta} \text{diag}(\bar{\mu}^k - \bar{\mu}^k \mathbf{1} + J(\theta_k + \delta u_k, \xi_k)\mathbf{1} \\ & \quad - \bar{\mu}^{k-1} + \bar{\mu}^{k-1} \mathbf{1} - J(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})\mathbf{1})) \otimes I_{d_i} u_k, \end{aligned}$$

because $\bar{\mu}^k = \frac{1}{N}\mathbf{1}^T \bar{\mu}^k = J(\theta_k + \delta u_k, \xi_k)$ as in the proof of Lemma 3.1. Rearranging terms in above equality, we have that

$$\begin{aligned} \theta_{k+1} - \theta_k &= \alpha \frac{J(\theta_k + \delta u_k, \xi_k) - J(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})}{\delta} u_k \\ & \quad + \frac{\alpha}{\delta} \text{diag}(\bar{\mu}^k - \bar{\mu}^k \mathbf{1}) \otimes I_{d_i} u_k \\ & \quad - \frac{\alpha}{\delta} \text{diag}(\bar{\mu}^{k-1} - \bar{\mu}^{k-1} \mathbf{1}) \otimes I_{d_i} u_k, \end{aligned} \quad (15)$$

Substituting (15) into the bound in (13) and rearranging terms, we get that

$$\begin{aligned} & \alpha \langle \nabla J_\delta(\theta_k), g_\delta(\theta_k) \rangle \\ & \leq J_\delta(\theta_{k+1}) - J_\delta(\theta_k) - \frac{\alpha}{\delta} \langle \nabla J_\delta(\theta_k), \text{diag}(\bar{\mu}^k - \bar{\mu}^k \mathbf{1}) \otimes I_{d_i} u_k \rangle \\ & \quad + \frac{L_{1,\delta}}{2} \alpha^2 \|g_\delta(\theta_k)\|^2 + \frac{1}{\delta} \text{diag}(\bar{\mu}^k - \bar{\mu}^k \mathbf{1}) \otimes I_{d_i} u_k \\ & \quad - \frac{1}{\delta} \text{diag}(\bar{\mu}^{k-1} - \bar{\mu}^{k-1} \mathbf{1}) \otimes I_{d_i} u_k \|^2 \\ & \quad + \frac{\alpha}{\delta} \langle \nabla J_\delta(\theta_k), \text{diag}(\bar{\mu}^{k-1} - \bar{\mu}^{k-1} \mathbf{1}) \otimes I_{d_i} u_k \rangle, \end{aligned} \quad (16)$$

where $g_\delta(\theta_k) = \frac{J(\theta_k + \delta u_k, \xi_k) - J(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})}{\delta} u_k$. Dividing both sides of (16) by α and taking the expectation of both sides with respect to u_k and ξ_k conditioned on the filtration $\mathcal{F}_{k-1} = \sigma(u_t, \xi_t | t \leq k-1)$, we have that

$$\begin{aligned} \mathbb{E}[\|\nabla J_\delta(\theta_k)\|^2 | \mathcal{F}_{k-1}] &\leq \frac{\mathbb{E}[J_\delta(\theta_{k+1}) - J_\delta(\theta_k) | \mathcal{F}_{k-1}]}{\alpha} \\ &- \frac{1}{\delta} \mathbb{E}[\langle \nabla J_\delta(\theta_k), \text{diag}(\bar{\mu}^k - \bar{\mu}^k \mathbf{1}) \otimes I_{d_i} u_k \rangle | \mathcal{F}_{k-1}] \\ &+ \frac{L_{1,\delta}}{2} \alpha \mathbb{E}[\|g_\delta(\theta_k)\|^2] + \frac{1}{\delta} \text{diag}(\bar{\mu}^k - \bar{\mu}^k \mathbf{1}) \otimes I_{d_i} u_k \\ &- \frac{1}{\delta} \text{diag}(\mu^{k-1} - \bar{\mu}^{k-1} \mathbf{1}) \otimes I_{d_i} u_k \|^2 | \mathcal{F}_{k-1}]. \end{aligned} \quad (17)$$

Note that all the expectation in the rest of this proof shall be conditional on the filtration \mathcal{F}_{k-1} . To simplify notation, in what follows we omit conditioning on the filtration. This is because $\mathbb{E}[g_\delta(\theta_k)] = \nabla J_\delta(\theta_k)$, $\nabla J_\delta(\theta_k)$ and $\text{diag}(\mu^{k-1} - \bar{\mu}^{k-1} \mathbf{1})$ are fixed when conditioned on the filtration \mathcal{F}_{k-1} , and $\mathbb{E}[u_k] = 0$. Next, we provide bounds on the second and third terms in the right hand side (RHS) of (17). Specifically, because $-\langle \nabla J_\delta(\theta_k), \frac{1}{\delta} \text{diag}(\bar{\mu}^k - \bar{\mu}^k \mathbf{1}) \otimes I_{d_i} u_k \rangle \leq \frac{1}{2} \|\nabla J_\delta(\theta_k)\|^2 + \frac{1}{2\delta^2} \|\text{diag}(\bar{\mu}^k - \bar{\mu}^k \mathbf{1}) \otimes I_{d_i} u_k\|^2$, we have that

$$\begin{aligned} &- \frac{1}{\delta} \mathbb{E}[\langle \nabla J_\delta(\theta_k), \text{diag}(\bar{\mu}^k - \bar{\mu}^k \mathbf{1}) \otimes I_{d_i} u_k \rangle] \\ &\leq \frac{1}{2} \mathbb{E}[\|\nabla J_\delta(\theta_k)\|^2] + \frac{1}{2\delta^2} \mathbb{E}[\|\text{diag}(\bar{\mu}^k - \bar{\mu}^k \mathbf{1}) \otimes I_{d_i} u_k\|^2] \\ &\leq \frac{1}{2} \mathbb{E}[\|\nabla J_\delta(\theta_k)\|^2] + \frac{d_i}{2\delta^2} \mathbb{E}[\|\bar{\mu}^k - \bar{\mu}^k \mathbf{1}\|^2 \|u_k\|^2], \end{aligned} \quad (18)$$

where the third inequality is due to the fact that $\|\text{diag}(v_1 v_2)\|^2 \leq \|v_1\|^2 \|v_2\|^2$ for all $v_1, v_2 \in \mathbb{R}^d$. Furthermore, we have that

$$\begin{aligned} \mathbb{E}[\|g_\delta(\theta_k) + \frac{1}{\delta} \text{diag}(\bar{\mu}^k - \bar{\mu}^k \mathbf{1}) \otimes I_{d_i} u_k \\ - \frac{1}{\delta} \text{diag}(\mu^{k-1} - \bar{\mu}^{k-1} \mathbf{1}) \otimes I_{d_i} u_k\|^2] \\ &\leq 2\mathbb{E}[\|g_\delta(\theta_k)\|^2] + \frac{2}{\delta^2} \mathbb{E}[\|\text{diag}(\bar{\mu}^k - \bar{\mu}^k \mathbf{1}) \otimes I_{d_i} u_k \\ - \text{diag}(\mu^{k-1} - \bar{\mu}^{k-1} \mathbf{1}) \otimes I_{d_i} u_k\|^2] \\ &\leq 2\mathbb{E}[\|g_\delta(\theta_k)\|^2] + \frac{4}{\delta^2} \mathbb{E}[\|\text{diag}(\bar{\mu}^k - \bar{\mu}^k \mathbf{1}) \otimes I_{d_i} u_k\|^2] \\ &+ \frac{4}{\delta^2} \mathbb{E}[\|\text{diag}(\mu^{k-1} - \bar{\mu}^{k-1} \mathbf{1}) \otimes I_{d_i} u_k\|^2] \\ &\leq 2\mathbb{E}[\|g_\delta(\theta_k)\|^2] + \frac{4d_i}{\delta^2} \mathbb{E}[\|\bar{\mu}^k - \bar{\mu}^k \mathbf{1}\|^2 \|u_k\|^2] \\ &+ \frac{4dd_i}{\delta^2} \mathbb{E}[\|\mu^{k-1} - \bar{\mu}^{k-1} \mathbf{1}\|^2], \end{aligned} \quad (19)$$

where the first two inequalities are due to the fact that $\|v_1 + v_2\|^2 \leq 2\|v_1\|^2 + 2\|v_2\|^2$ and the third inequality is due to the fact that $\mathbb{E}[\|\text{diag}(\mu^{k-1} - \bar{\mu}^{k-1} \mathbf{1}) \otimes I_{d_i} u_k\|^2] \leq d_i \mathbb{E}[\|\text{diag}(\mu^{k-1} - \bar{\mu}^{k-1} \mathbf{1})\|^2 \|u_k\|^2]$, the fact that $\mu^{k-1} - \bar{\mu}^{k-1} \mathbf{1}$ is independent of u_k and the fact that $\mathbb{E}[\|u_k\|^2] = d$. Substituting the bounds in (18) and (19) into the bound in (17)

and rearranging the terms, we get that

$$\begin{aligned} &\mathbb{E}[\|\nabla J_\delta(\theta_k)\|^2] \\ &\leq \frac{2}{\alpha} (\mathbb{E}[J_\delta(\theta_{k+1}) - J_\delta(\theta_k)]) + 2\sqrt{d}L_0 \frac{\alpha}{\delta} \mathbb{E}[\|g_\delta(\theta_k)\|^2] \\ &+ 4\sqrt{d}d_i L_0 \frac{\alpha}{\delta^3} \mathbb{E}[\|\bar{\mu}^k - \bar{\mu}^k \mathbf{1}\|^2 \|u_k\|^2] \\ &+ 4d^{1.5}d_i L_0 \frac{\alpha}{\delta^3} \mathbb{E}[\|\mu^{k-1} - \bar{\mu}^{k-1} \mathbf{1}\|^2] \\ &+ \frac{d_i}{\delta^2} \mathbb{E}[\|\bar{\mu}^k - \bar{\mu}^k \mathbf{1}\|^2 \|u_k\|^2]. \end{aligned} \quad (20)$$

The proof is complete. \blacksquare

Next, we present a lemma bounding the second moment of the gradient estimate $g_\delta(\theta_k)$.

Lemma 1.3: Let Assumptions 2.2 and 3.1 hold. Then, for all $k \geq 1$, we have that

$$\begin{aligned} \mathbb{E}[\|g_\delta(\theta_k)\|^2] &\leq 8dL_0^2 \frac{\alpha^2}{\delta^2} \mathbb{E}[\|g_\delta(\theta_{k-1})\|^2] \\ &+ 16dd_i L_0^2 \frac{\alpha^2}{\delta^4} \mathbb{E}[\|\mu^{k-1} - \bar{\mu}^{k-1} \mathbf{1}\|^2 \|u_{k-1}\|^2] \\ &+ 16d^2 d_i L_0^2 \frac{\alpha^2}{\delta^4} \mathbb{E}[\|\mu^{k-2} - \bar{\mu}^{k-2} \mathbf{1}\|^2] + 16(d+4)L_0^2 + \frac{8d\sigma^2}{\delta^2}. \end{aligned} \quad (21)$$

Proof: Recalling that $g_\delta(\theta_k) = \frac{J(\theta_k + \delta u_k, \xi_k) - J(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})}{\delta} u_k$, we have that

$$\begin{aligned} \mathbb{E}[\|g_\delta(\theta_k)\|^2] &= \frac{1}{\delta^2} \mathbb{E}[\|J(\theta_k + \delta u_k, \xi_k) - J(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})\|^2 \|u_k\|^2]. \end{aligned} \quad (22)$$

In addition, using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we have that

$$\begin{aligned} &|J(\theta_k + \delta u_k, \xi_k) - J(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})|^2 \\ &= |J(\theta_k + \delta u_k, \xi_k) - J(\theta_{k-1} + \delta u_{k-1}, \xi_k) \\ &+ J(\theta_{k-1} + \delta u_{k-1}, \xi_k) - J(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})|^2 \\ &\leq 2(J(\theta_k + \delta u_k, \xi_k) - J(\theta_{k-1} + \delta u_{k-1}, \xi_k))^2 \\ &+ 2(J(\theta_{k-1} + \delta u_{k-1}, \xi_k) - J(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1}))^2. \end{aligned}$$

Then, by adding and subtracting the term $J(\theta_{k-1} + \delta u_k, \xi_k)$ within the term $(J(\theta_k + \delta u_k, \xi_k) - J(\theta_{k-1} + \delta u_{k-1}, \xi_k))^2$ in above inequality and applying the bound $(a+b)^2 \leq 2a^2 + 2b^2$ similarly as above, we get that

$$\begin{aligned} &|J(\theta_k + \delta u_k, \xi_k) - J(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})|^2 \\ &\leq 4(J(\theta_k + \delta u_k, \xi_k) - J(\theta_{k-1} + \delta u_k, \xi_k))^2 \\ &+ 4(J(\theta_{k-1} + \delta u_k, \xi_k) - J(\theta_{k-1} + \delta u_{k-1}, \xi_k))^2 \\ &+ 2(J(\theta_{k-1} + \delta u_{k-1}, \xi_k) - J(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1}))^2. \end{aligned} \quad (23)$$

According to Assumption 2.2, we have that $(J(\theta_k + \delta u_k, \xi_k) - J(\theta_{k-1} + \delta u_k, \xi_k))^2 \leq L_0^2 \|\theta_k - \theta_{k-1}\|^2$ and $(J(\theta_{k-1} + \delta u_k, \xi_k) - J(\theta_{k-1} + \delta u_{k-1}, \xi_k))^2 \leq L_0^2 \delta^2 \|u_k - u_{k-1}\|^2$. Furthermore, according to Assumption 3.1, we have that $(J(\theta_{k-1} + \delta u_{k-1}, \xi_k) - J(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1}))^2 \leq 4\sigma^2$. Applying the above bounds to the RHS of (23), we get that

$$\begin{aligned} &|J(\theta_k + \delta u_k, \xi_k) - J(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})|^2 \\ &\leq 4L_0^2 \|\theta_k - \theta_{k-1}\|^2 + 4L_0^2 \delta^2 \|u_k - u_{k-1}\|^2 + 8\sigma^2. \end{aligned} \quad (24)$$

Substituting the bound (24) into (22), we have that

$$\begin{aligned} \mathbb{E}[\|g_\delta(\theta_k)\|^2] &\leq \frac{4L_0^2}{\delta^2} \mathbb{E}[\|\theta_k - \theta_{k-1}\|^2 \|u_k\|^2] \\ &\quad + 4L_0^2 \mathbb{E}[\|u_k - u_{k-1}\|^2 \|u_k\|^2] + \frac{8\sigma^2}{\delta^2} \mathbb{E}[\|u_k\|^2] \\ &\leq \frac{4dL_0^2}{\delta^2} \mathbb{E}[\|\theta_k - \theta_{k-1}\|^2] + 16(d+4)^2 L_0^2 + \frac{8d\sigma^2}{\delta^2}. \end{aligned} \quad (25)$$

The second inequality is due to the fact that $\|\theta_k - \theta_{k-1}\|^2$ is independent of u_k , $\mathbb{E}[\|u_k - u_{k-1}\|^2 \|u_k\|^2] \leq 4(d+4)^2$ and $\mathbb{E}[\|u_k\|^2] = d$. Specifically, we have that $\mathbb{E}[\|u_k - u_{k-1}\|^2 \|u_k\|^2] \leq 4(d+4)^2$ because $\|u_k - u_{k-1}\|^2 \leq 2\|u_k\|^2 + 2\|u_{k-1}\|^2$ and that $\mathbb{E}[\|u_k\|^4] \leq (d+4)^2$ according to [24]. Substituting the expression for $\theta_k - \theta_{k-1}$ in (15) into (25) and applying the bound in (19), we obtain that

$$\begin{aligned} \mathbb{E}[\|g_\delta(\theta_k)\|^2] &\leq 8dL_0^2 \frac{\alpha^2}{\delta^2} \mathbb{E}[\|g_\delta(\theta_{k-1})\|^2] \\ &\quad + 16dd_i L_0^2 \frac{\alpha^2}{\delta^4} \mathbb{E}[\|\mu^{k-1} - \bar{\mu}^{k-1} \mathbf{1}\|^2 \|u_{k-1}\|^2] \\ &\quad + 16d^2 d_i L_0^2 \frac{\alpha^2}{\delta^4} \mathbb{E}[\|\mu^{k-2} - \bar{\mu}^{k-2} \mathbf{1}\|^2] + 16(d+4)^2 L_0^2 + \frac{8d\sigma^2}{\delta^2}. \end{aligned}$$

The proof is complete. \blacksquare

Now, we are ready to present the proof for Theorem 3.2.

Theorem 3.2. (Learning Rate of Algorithm 1 without Value Tracking) Let Assumptions 2.2, 3.1, 3.2 and 3.3 hold and define $\delta = \frac{\epsilon_J}{\sqrt{d}L_0}$, $\alpha = \frac{\epsilon_J^{1.5}}{4d^{1.5}L_0^2\sqrt{K}}$, and $N_c \geq \log(\frac{\sqrt{\epsilon\epsilon_J}}{\sqrt{2d^{1.5}L_0(J_u - J_l)}}) / \log(\rho_W)$. Then, running Algorithm 1 with `DoTracking = False`, we have that $\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[\|\nabla J_\delta(\theta_k)\|^2] \leq \mathcal{O}(d^{1.5}\epsilon_J^{-1.5}K^{-0.5}) + \frac{\epsilon}{2}$.

Proof: According to Lemma 3.1, and using Assumptions 3.2 and 3.3, we select $N_c \geq \log(\frac{\sqrt{\epsilon\epsilon_J}}{\sqrt{2d^{1.5}L_0(J_u - J_l)}}) / \log(\rho_W)$ so that $\|\bar{\mu}^k - \bar{\mu}^k \mathbf{1}\| = \|\bar{\mu}^k - J(\theta_k + \delta u_k, \xi_k) \mathbf{1}\| \leq E_\mu$ regardless of u_k for all $k \geq 0$, where E_μ is a small constant such that $E_\mu^2 = \frac{\epsilon\delta^2}{2dd_i}$. Therefore, the bound in (12) can be simplified as

$$\begin{aligned} \mathbb{E}[\|\nabla J_\delta(\theta_k)\|^2 | \mathcal{F}_{k-1}] &\leq \frac{2}{\alpha} (\mathbb{E}[J_\delta(\theta_{k+1}) - J_\delta(\theta_k) | \mathcal{F}_{k-1}]) \\ &\quad + 2\sqrt{d}L_0 \frac{\alpha}{\delta} \mathbb{E}[\|g_\delta(\theta_k)\|^2 | \mathcal{F}_{k-1}] + 8d^{1.5}d_i L_0 \frac{\alpha}{\delta^3} E_\mu^2 + \frac{dd_i}{\delta^2} E_\mu^2. \end{aligned} \quad (26)$$

Applying the tower rule of the conditional expectation and telescoping the above inequality from $k=0$ to $K-1$, we get that

$$\begin{aligned} \sum_{k=0}^{K-1} \mathbb{E}[\|\nabla J_\delta(\theta_k)\|^2] &\leq \frac{2}{\alpha} (\mathbb{E}[J_\delta(\theta_K) - J_\delta(\theta_0)]) \\ &\quad + 2\sqrt{d}L_0 \frac{\alpha}{\delta} \sum_{k=0}^{K-1} \mathbb{E}[\|g_\delta(\theta_k)\|^2] + 8d^{1.5}d_i L_0 \frac{\alpha}{\delta^3} E_\mu^2 K + \frac{dd_i}{\delta^2} E_\mu^2 K, \end{aligned} \quad (27)$$

where the expectation is taken over the trajectory of random samples of u_k and ξ_k . Next, we bound the term $\sum_{k=0}^{K-1} \mathbb{E}[\|g_\delta(\theta_k)\|^2]$ on the RHS of (27). Specifically, since N_c is selected so that $\|\bar{\mu}^k - \bar{\mu}^k \mathbf{1}\| = \|\bar{\mu}^k - J(\theta_k + \delta u_k, \xi_k) \mathbf{1}\| \leq E_\mu$ regardless of u_k for all $k \geq 0$, the bound in (21) can be

simplified as

$$\begin{aligned} \mathbb{E}[\|g_\delta(\theta_k)\|^2 | \mathcal{F}_{k-1}] &\leq 8dL_0^2 \frac{\alpha^2}{\delta^2} \mathbb{E}[\|g_\delta(\theta_{k-1})\|^2 | \mathcal{F}_{k-1}] \\ &\quad + 32d^2 d_i L_0^2 \frac{\alpha^2}{\delta^4} E_\mu^2 + 16(d+4)^2 L_0^2 + \frac{8d\sigma^2}{\delta^2}. \end{aligned} \quad (28)$$

Applying the tower rule of conditional expectation and telescoping the above inequality from $k=1$ to $K-1$, adding $\mathbb{E}[\|g_\delta(\theta_0)\|^2]$ on both sides, adding $8dL_0^2 \frac{\alpha^2}{\delta^2} \mathbb{E}[\|g_\delta(\theta_{K-1})\|^2]$ on the RHS, and rearranging the terms, we have that

$$\begin{aligned} \sum_{k=0}^{K-1} \mathbb{E}[\|g_\delta(\theta_k)\|^2] &\leq \frac{1}{1-\alpha_g} \mathbb{E}[\|g_\delta(\theta_0)\|^2] + \frac{32d^2 d_i L_0^2 \alpha^2}{1-\alpha_g} E_\mu^2 K \\ &\quad + \frac{16(d+4)^2 L_0^2}{1-\alpha_g} K + \frac{8d\sigma^2}{(1-\alpha_g)\delta^2} K, \end{aligned} \quad (29)$$

where $\alpha_g = 8dL_0^2 \frac{\alpha^2}{\delta^2}$. When $\delta = \frac{\epsilon_J}{\sqrt{d}L_0}$ and $\alpha = \frac{\epsilon_J^{1.5}}{4d^{1.5}L_0^2\sqrt{K}}$, we have that $\alpha_g = \frac{\epsilon_J}{2dK} \leq \frac{1}{2}$ when $\epsilon_J \leq d$ and $K \geq 1$. Substituting the bound on α_g into (29), we obtain that

$$\begin{aligned} \sum_{k=0}^{K-1} \mathbb{E}[\|g_\delta(\theta_k)\|^2] &\leq 2\mathbb{E}[\|g_\delta(\theta_0)\|^2] + 64d^2 d_i L_0^2 \frac{\alpha^2}{\delta^4} E_\mu^2 K \\ &\quad + 32(d+4)^2 L_0^2 K + \frac{16d\sigma^2}{\delta^2} K. \end{aligned} \quad (30)$$

Moreover, substituting the bound in (30) into the bound in (27), we get that

$$\begin{aligned} \sum_{k=0}^{K-1} \mathbb{E}[\|\nabla J_\delta(\theta_k)\|^2] &\leq \frac{2}{\alpha} (\mathbb{E}[J_\delta(\theta_K) - J_\delta(\theta_0)]) \\ &\quad + 4\sqrt{d}L_0 \frac{\alpha}{\delta} \mathbb{E}[\|g_\delta(\theta_0)\|^2] + 64(d+4)^2 L_0^3 \frac{\alpha}{\delta} K \\ &\quad + 32d^{1.5}L_0 \sigma^2 \frac{\alpha}{\delta^3} K + 128d^{2.5}d_i L_0^3 \frac{\alpha^3}{\delta^5} E_\mu^2 K \\ &\quad + 8d^{1.5}d_i L_0 \frac{\alpha}{\delta^3} E_\mu^2 K + \frac{dd_i}{\delta^2} E_\mu^2 K. \end{aligned} \quad (31)$$

Recalling that $E_\mu^2 = \frac{\epsilon\delta^2}{2dd_i}$ and substituting the selected values for $\delta = \frac{\epsilon_J}{\sqrt{d}L_0}$ and $\alpha = \frac{\epsilon_J^{1.5}}{4d^{1.5}L_0^2\sqrt{K}}$ into (31), we obtain that

$$\begin{aligned} \sum_{k=0}^{K-1} \mathbb{E}[\|\nabla J_\delta(\theta_k)\|^2] &\leq \frac{8d^{1.5}L_0^2}{\epsilon_J^{1.5}} \mathbb{E}[J_\delta^* - J_\delta(\theta_0)] \sqrt{K} \\ &\quad + \frac{\epsilon_J^{0.5}}{\sqrt{d}K} \mathbb{E}[\|g_\delta(\theta_0)\|^2] + \frac{\epsilon\epsilon_J^{1.5}}{d^{1.5}\sqrt{K}} + 16\frac{(d+4)^2}{d} L_0^2 \epsilon_J^{0.5} \sqrt{K} \\ &\quad + \frac{8d^{1.5}L_0^2 \sigma^2}{\epsilon_J^{1.5}} \sqrt{K} + \frac{\epsilon\epsilon_J^{0.5}}{\sqrt{d}} \sqrt{K} + \frac{\epsilon}{2} K, \end{aligned} \quad (32)$$

where $J_\delta^* \geq J_\delta(\theta)$ for all $\theta \in \mathbb{R}^d$. The upper bound on $J_\delta(\theta)$ exists due to Assumption 3.3. Dividing both sides of (32) by K , we achieve the bound in Theorem 3.2. \blacksquare

Lemma 3.3. Let Assumption 3.2 hold. Then, running Algorithm 1 with `DoTracking = True`, we have that $\bar{\mu}^k(m) = J(\theta_k + \delta u_k, \xi_k) = \frac{1}{N} \sum_{i=1}^N J_i(\theta_k + \delta u_k, \xi_k)$, for all $m = 1, 2, \dots, N_c$ and all k . *Proof:* According to (8), we have that

$$\begin{aligned} \bar{\mu}^k(m) &= \frac{1}{N} \mathbf{1}^T \bar{\mu}^k(m) = \frac{1}{N} \mathbf{1}^T \bar{\mu}^k(m-1) = \dots \\ &= \frac{1}{N} \mathbf{1}^T \bar{\mu}^k(0), \text{ for all } m, k \geq 0. \end{aligned} \quad (33)$$

Next, we show that $\bar{\mu}^k(N_c) = \bar{\mu}^k(0) = \frac{1}{N} \sum_{i=1}^N J_i(\theta_k + \delta u_k, \xi_k)$ for all k . We use mathematical induction to construct the proof. Specifically, suppose that $\bar{\mu}^{k-1}(N_c) = \bar{\mu}^{k-1}(0) = \frac{1}{N} \sum_{i=1}^N J_i(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})$ holds. Then, according to line 9 in Algorithm 1, we have that

$$\begin{aligned} \frac{1}{N} \mathbf{1}^T \bar{\mu}^k(N_c) &= \frac{1}{N} \mathbf{1}^T \bar{\mu}^k(0) \\ &= \frac{1}{N} \mathbf{1}^T (\mu^{k-1}(N_c) + \vec{J}(\theta_k + \delta u_k, \xi_k) \\ &\quad - \vec{J}(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})) \\ &= \frac{1}{N} \sum_{i=1}^N J_i(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1}) + \frac{1}{N} \sum_{i=1}^N J_i(\theta_k + \delta u_k, \xi_k) \\ &\quad - \frac{1}{N} \sum_{i=1}^N J_i(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1}) \\ &= \frac{1}{N} \sum_{i=1}^N J_i(\theta_k + \delta u_k, \xi_k), \end{aligned}$$

where $\vec{J}(\theta_k + \delta u_k, \xi_k) = [\dots, J_i(\theta_k + \delta u_k, \xi_k), \dots]^T$. The second equality above is due to the induction hypothesis. We have that the induction hypothesis is satisfied for $\bar{\mu}^0(N_c)$, according to line 7 in Algorithm 1. Therefore, we have shown that $\frac{1}{N} \mathbf{1}^T \bar{\mu}^k(0) = \frac{1}{N} \sum_{i=1}^N J_i(\theta_k + \delta u_k, \xi_k)$ for all k . And due to (33), we have shown that $\bar{\mu}^k(m) = J(\theta_k + \delta u_k, \xi_k) = \frac{1}{N} \sum_{i=1}^N J_i(\theta_k + \delta u_k, \xi_k)$, for all $m = 1, 2, \dots, N_c$ and all k . The proof is complete. ■

Lemma 3.4. Let Assumptions 2.2, 3.1, 3.2 hold and define $E_\mu^k = \|\bar{\mu}^k(N_c) - \bar{\mu}^k(N_c) \mathbf{1}\|$. Then, running Algorithm 1 with `DoTracking = True`, we have that

$$\begin{aligned} \mathbb{E}[(E_\mu^k)^2] &\leq \left(2\mathbb{E}[(E_\mu^{k-1})^2] + 32dL_0^2 \frac{\alpha^2}{\delta^2} \mathbb{E}[(E_\mu^{k-1})^2 \|u_{k-1}\|^2] \right. \\ &\quad \left. + 32d^2 L_0^2 \frac{\alpha^2}{\delta^2} \mathbb{E}[(E_\mu^{k-2})^2] \right) \rho_W^{2N_c} \\ &\quad + 16NL_0^2 \alpha^2 \mathbb{E}[\|\tilde{\nabla} J(\theta_{k-1})\|^2] \rho_W^{2N_c} \\ &\quad + 32NdL_0^2 \delta^2 \rho_W^{2N_c} + 16N\sigma^2 \rho_W^{2N_c}. \end{aligned} \quad (34)$$

Proof: To simplify notations, in what follows, we denote $\bar{\mu}^k(N_c)$ and $\bar{\mu}^k(N_c)$ by $\bar{\mu}^k$ and $\bar{\mu}^k$ respectively. According to lines 9 and 12 in Algorithm 1, we have that

$$\begin{aligned} \mathbb{E}[(E_\mu^k)^2] &= \mathbb{E}[\|(I - \frac{1}{N} \mathbf{1} \mathbf{1}^T) \bar{\mu}^k\|^2] \\ &= \mathbb{E}[\|(I - \frac{1}{N} \mathbf{1} \mathbf{1}^T) W^{N_c} (\mu^{k-1} + \vec{J}(\theta_k + \delta u_k, \xi_k) \\ &\quad - \vec{J}(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1}))\|^2] \\ &= \mathbb{E}[\|(W^{N_c} - \frac{1}{N} \mathbf{1} \mathbf{1}^T) (\mu^{k-1} + \vec{J}(\theta_k + \delta u_k, \xi_k) \\ &\quad - \vec{J}(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1}))\|^2]. \end{aligned}$$

Since $(W^{N_c} - \frac{1}{N} \mathbf{1} \mathbf{1}^T) \mu^{k-1} \mathbf{1} = 0$, we obtain that

$$\begin{aligned} \mathbb{E}[(E_\mu^k)^2] &= \mathbb{E}[\|(W^{N_c} - \frac{1}{N} \mathbf{1} \mathbf{1}^T) (\mu^{k-1} - \bar{\mu}^{k-1} \mathbf{1} \\ &\quad + \vec{J}(\theta_k + \delta u_k, \xi_k) - \vec{J}(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1}))\|^2] \\ &\leq 2\mathbb{E}[\|W^{N_c} - \frac{1}{N} \mathbf{1} \mathbf{1}^T\|^2 (E_\mu^{k-1})^2] + 2\mathbb{E}[\|W^{N_c} \\ &\quad - \frac{1}{N} \mathbf{1} \mathbf{1}^T\|^2 \|\vec{J}(\theta_k + \delta u_k, \xi_k) - \vec{J}(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})\|^2]. \end{aligned} \quad (35)$$

Moreover, since $(W - \frac{1}{N} \mathbf{1} \mathbf{1}^T)^2 = W^2 - \frac{1}{N} \mathbf{1} \mathbf{1}^T$, we get that $(W - \frac{1}{N} \mathbf{1} \mathbf{1}^T)^3 = (W^2 - \frac{1}{N} \mathbf{1} \mathbf{1}^T)(W - \frac{1}{N} \mathbf{1} \mathbf{1}^T) = W^3 - \frac{1}{N} \mathbf{1} \mathbf{1}^T$. This is because W^2 is also doubly stochastic, which can be shown using the definition. Through induction, we obtain that $(W - \frac{1}{N} \mathbf{1} \mathbf{1}^T)^{N_c} = W^{N_c} - \frac{1}{N} \mathbf{1} \mathbf{1}^T$. Therefore, we have that $\|W^{N_c} - \frac{1}{N} \mathbf{1} \mathbf{1}^T\|^2 = \|(W - \frac{1}{N} \mathbf{1} \mathbf{1}^T)^{N_c}\|^2 \leq \|W - \frac{1}{N} \mathbf{1} \mathbf{1}^T\|^{2N_c} = \rho_W^{2N_c}$. Applying this bound in (35), we get that

$$\begin{aligned} \mathbb{E}[(E_\mu^k)^2] &\leq 2\rho_W^{2N_c} \mathbb{E}[(E_\mu^{k-1})^2] \\ &\quad + 2\rho_W^{2N_c} \mathbb{E}[\|\vec{J}(\theta_k + \delta u_k, \xi_k) - \vec{J}(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})\|^2]. \end{aligned} \quad (36)$$

Following the same procedure used to derive the bound in (24), we get that $|J_i(\theta_k + \delta u_k, \xi_k) - J_i(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})|^2 \leq 4L_0^2 \|\theta_k - \theta_{k-1}\|^2 + 4L_0^2 \delta^2 \|u_k - u_{k-1}\|^2 + 8\sigma^2$, for all $i = 1, 2, \dots, N$. Applying this bound to the RHS in (36), we obtain that

$$\begin{aligned} \mathbb{E}[(E_\mu^k)^2] &\leq 2\rho_W^{2N_c} \mathbb{E}[(E_\mu^{k-1})^2] + 8NL_0^2 \rho_W^{2N_c} \mathbb{E}[\|\theta_k - \theta_{k-1}\|^2] \\ &\quad + 8NL_0^2 \delta^2 \rho_W^{2N_c} \mathbb{E}[\|u_k - u_{k-1}\|^2] + 16N\sigma^2 \rho_W^{2N_c}. \end{aligned} \quad (37)$$

Moreover, we have that $\mathbb{E}[\|u_k - u_{k-1}\|^2] \leq \mathbb{E}[2\|u_k\|^2 + 2\|u_{k-1}\|^2] \leq 4d$. Substituting the expression of $\theta_k - \theta_{k-1}$ for (15) into (37) and applying the bound in (19), we obtain that

$$\begin{aligned} \mathbb{E}[(E_\mu^k)^2] &\leq 2\rho_W^{2N_c} \mathbb{E}[(E_\mu^{k-1})^2] + 16NL_0^2 \rho_W^{2N_c} \alpha^2 \mathbb{E}[\|g_\delta(\theta_{k-1})\|^2] \\ &\quad + 32dNL_0^2 \delta^2 \rho_W^{2N_c} + 16N\sigma^2 \rho_W^{2N_c} \\ &\quad + 32dL_0^2 \rho_W^{2N_c} \frac{\alpha^2}{\delta^2} \mathbb{E}[(E_\mu^{k-1})^2 \|u_{k-1}\|^2] \\ &\quad + 32d^2 L_0^2 \rho_W^{2N_c} \frac{\alpha^2}{\delta^2} \mathbb{E}[(E_\mu^{k-2})^2]. \end{aligned}$$

The proof is complete. ■

First, we present a lemma characterizing the bound on $\mathbb{E}[(E_\mu^k)^2 \|u_k\|^2]$.

Lemma 1.4: Let Assumptions 2.2, 3.1, 3.2 hold. Then, for all $k \geq 1$, we have that

$$\begin{aligned} \mathbb{E}[(E_\mu^k)^2 \|u_k\|^2] &\leq 2d\rho_W^{2N_c} \mathbb{E}[(E_\mu^{k-1})^2] + 16dNL_0^2 \rho_W^{2N_c} \alpha^2 \mathbb{E}[\|g_\delta(\theta_{k-1})\|^2] \\ &\quad + 32d^2 L_0^2 \rho_W^{2N_c} \frac{\alpha^2}{\delta^2} \mathbb{E}[(E_\mu^{k-1})^2 \|u_{k-1}\|^2] \\ &\quad + 32d^3 L_0^2 \rho_W^{2N_c} \frac{\alpha^2}{\delta^2} \mathbb{E}[(E_\mu^{k-2})^2] \\ &\quad + 32(d+4)^2 NL_0^2 \delta^2 \rho_W^{2N_c} + 16dN\sigma^2 \rho_W^{2N_c}. \end{aligned}$$

Proof: According to the bound on $(E_\mu^k)^2$ derived in (35), we have that

$$\begin{aligned} \mathbb{E}[(E_\mu^k)^2 \|u_k\|^2] &\leq 2\rho_W^{2N_c} \mathbb{E}[(E_\mu^{k-1})^2 \|u_k\|^2] \\ &\quad + 2\rho_W^{2N_c} \mathbb{E}[\|\vec{J}(\theta_k + \delta u_k, \xi_k) - \vec{J}(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})\|^2 \|u_k\|^2]. \end{aligned} \quad (38)$$

Since E_μ^{k-1} is independent of u_k , we have that $\mathbb{E}[(E_\mu^{k-1})^2 \|u_k\|^2] = \mathbb{E}[(E_\mu^{k-1})^2] \mathbb{E}[\|u_k\|^2] = d\mathbb{E}[(E_\mu^{k-1})^2]$. Following the same procedure used to derive the bound in (24), we get that $|J_i(\theta_k + \delta u_k, \xi_k) - J_i(\theta_{k-1} + \delta u_{k-1}, \xi_{k-1})|^2 \leq 4L_0^2 \|\theta_k - \theta_{k-1}\|^2 + 4L_0^2 \delta^2 \|u_k - u_{k-1}\|^2 + 8\sigma^2$, for all $i = 1, 2, \dots, N$. Applying this bound to the RHS in (38), we obtain that

$$\begin{aligned} \mathbb{E}[(E_\mu^k)^2 \|u_k\|^2] &\leq 2d\rho_W^{2N_c} \mathbb{E}[(E_\mu^{k-1})^2] \\ &+ 8NL_0^2 \rho_W^{2N_c} \mathbb{E}[\|\theta_k - \theta_{k-1}\|^2 \|u_k\|^2] \\ &+ 8NL_0^2 \delta^2 \rho_W^{2N_c} \mathbb{E}[\|u_k - u_{k-1}\|^2 \|u_k\|^2] + 16dN\sigma^2 \rho_W^{2N_c}. \end{aligned} \quad (39)$$

Moreover, we have that $\mathbb{E}[\|u_k - u_{k-1}\|^2 \|u_k\|^2] \leq \mathbb{E}[2\|u_k\|^4 + 2\|u_{k-1}\|^2 \|u_k\|^2] \leq 4(d+4)^2$. Substituting the expression of $\theta_k - \theta_{k-1}$ in (15) into (39) and applying the bound (19), we obtain that

$$\begin{aligned} \mathbb{E}[(E_\mu^k)^2 \|u_k\|^2] &\leq 2d\rho_W^{2N_c} \mathbb{E}[(E_\mu^{k-1})^2] + 16dNL_0^2 \rho_W^{2N_c} \alpha^2 \mathbb{E}[\|g_\delta(\theta_{k-1})\|^2] \\ &+ 32d^2 L_0^2 \rho_W^{2N_c} \frac{\alpha^2}{\delta^2} \mathbb{E}[(E_\mu^{k-1})^2 \|u_{k-1}\|^2] \\ &+ 32d^3 L_0^2 \rho_W^{2N_c} \frac{\alpha^2}{\delta^2} \mathbb{E}[(E_\mu^{k-2})^2] \\ &+ 32(d+4)^2 NL_0^2 \delta^2 \rho_W^{2N_c} + 16dN\sigma^2 \rho_W^{2N_c}. \end{aligned} \quad (40)$$

The proof is complete. \blacksquare

Now, we are ready to present the proof for Theorem 3.5.

Theorem 3.5. (Learning Rate of Algorithm 1 with Value Tracking) Let Assumptions 2.2,3.1, 3.2 hold and define $\delta = \frac{\epsilon_J}{\sqrt{dL_0}}$, $\alpha = \frac{\epsilon_J^{1.5}}{4d^{1.5}L_0^2\sqrt{K}}$, and $N_c \geq \max(\log(\frac{1}{2\sqrt{2}})/\log(\rho_W), N_c \geq \log(\frac{\epsilon}{\sqrt{4(G^2\epsilon_J + 32(d+4)^2 dL_0^2 + 16d^3 L_0^2 \sigma^2 / \epsilon_J^2)}}/\log(\rho_W)))$

where $G^2 = \max\left(\mathbb{E}[\|\tilde{\nabla} J(\theta_0)\|^2], \frac{2\epsilon_J\epsilon}{dK} + 32L_0^2(d+4)^2 + 16d^2 L_0^2 \frac{\sigma^2}{\epsilon_J^2}\right)$. Then, running Algorithm 1 with `DoTracking = True`, we have that $\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[\|\nabla J_\delta(\theta_k)\|^2] \leq \mathcal{O}(d^{1.5}\epsilon_J^{-1.5}K^{-0.5}) + \frac{\epsilon}{2}$. *Proof:* First, we show that for all $k \geq 0$, we have that $\mathbb{E}[\|g_\delta(\theta_k)\|^2] \leq G^2$, $\mathbb{E}[(E_\mu^k)^2] \leq E_\mu^2$ and $\mathbb{E}[(E_\mu^k)^2 \|u_k\|^2] \leq dE_\mu^2$ when we let $\delta = \frac{\epsilon_J}{\sqrt{dL_0}}$, $\alpha = \frac{\epsilon_J^{1.5}}{4d^{1.5}L_0^2\sqrt{K}}$, and $N_c \geq \max(\log(\frac{1}{2\sqrt{2}})/\log(\rho_W), \log(\frac{\epsilon}{\sqrt{2G^2\epsilon_J + 64(d+4)^2 dL_0^2 + 32d^3 L_0^2 \sigma^2 / \epsilon_J^2}}/\log(\rho_W)))$, where $E_\mu^2 = \frac{\epsilon\delta^2}{2dd_i}$. To prove this, we use mathematical induction. Specifically, suppose we have that $\mathbb{E}[(E_\mu^{k-1})^2] \leq E_\mu^2$, $\mathbb{E}[(E_\mu^{k-2})^2] \leq E_\mu^2$, $\mathbb{E}[(E_\mu^{k-1})^2 \|u_{k-1}\|^2] \leq dE_\mu^2$ and $\mathbb{E}[\|g_\delta(\theta_{k-1})\|^2] \leq G^2$. Then, according to Lemma 1.3, we have that

$$\begin{aligned} \mathbb{E}[\|g_\delta(\theta_k)\|^2] &\leq 8dL_0^2 \frac{\alpha^2}{\delta^2} G^2 + 32d^2 d_i L_0^2 \frac{\alpha^2}{\delta^4} E_\mu^2 \\ &+ 16(d+4)^2 L_0^2 + \frac{8d\sigma^2}{\delta^2}. \end{aligned} \quad (41)$$

Substituting the selected values for δ , α and the constant E_μ^2

in (41), we get that

$$\begin{aligned} \mathbb{E}[\|g_\delta(\theta_k)\|^2] &\leq \frac{\epsilon_J}{2dK} G^2 + \frac{\epsilon\epsilon_J}{dK} + 16L_0^2(d+4)^2 + 8d^2 L_0^2 \frac{\sigma^2}{\epsilon_J^2}, \\ &\leq G^2 \end{aligned} \quad (42)$$

where the second inequality holds because $\frac{\epsilon_J}{2dK} G^2 \leq \frac{1}{2}G^2$ when $\frac{\epsilon_J}{dK} \leq 1$. In addition, we have that $\frac{\epsilon\epsilon_J}{dK} + 16L_0^2(d+4)^2 + 8d^2 L_0^2 \frac{\sigma^2}{\epsilon_J^2} \leq \frac{1}{2}G^2$ due to the choice of G^2 in Theorem 3.5. Furthermore, according to Lemma 3.4, we have that

$$\begin{aligned} \mathbb{E}[(E_\mu^k)^2] &\leq (2 + 64d^2 L_0^2 \frac{\alpha^2}{\delta^2}) \rho_W^{2N_c} E_\mu^2 + 16NL_0^2 G^2 \alpha^2 \rho_W^{2N_c} \\ &+ 32NdL_0^2 \delta^2 \rho_W^{2N_c} + 16N\sigma^2 \rho_W^{2N_c}. \end{aligned} \quad (43)$$

Substituting the selected values for δ , α and E_μ^2 into (43), we get that

$$\begin{aligned} \mathbb{E}[(E_\mu^k)^2] &\leq (2 + 4\frac{\epsilon_J}{K}) \rho_W^{2N_c} E_\mu^2 + (\frac{NG^2\epsilon_J^3}{d^3 L_0^2 K} + 32N\epsilon_J^2 + 16N\sigma^2) \rho_W^{2N_c} \\ &\leq E_\mu^2. \end{aligned} \quad (44)$$

The second inequality is because when $\epsilon_J/K \leq \frac{1}{2}$ and $N_c \geq \log(\frac{1}{2\sqrt{2}})/\log(\rho_W)$, we have that $(2 + 4\frac{\epsilon_J}{K}) \rho_W^{2N_c} E_\mu^2 \leq \frac{1}{2}E_\mu^2$. In addition, when $N_c \geq \log(\frac{\epsilon}{\sqrt{4(G^2\epsilon_J + 32(d+4)^2 dL_0^2 + 16d^3 L_0^2 \sigma^2 / \epsilon_J^2)}}/\log(\rho_W))$, we get that $(\frac{NG^2\epsilon_J^3}{d^3 L_0^2 K} + 32N\epsilon_J^2 + 16N\sigma^2) \rho_W^{2N_c} \leq \frac{1}{2}E_\mu^2$.

Next, according to Lemma 1.4, we have that

$$\begin{aligned} \mathbb{E}[(E_\mu^k)^2 \|u_k\|^2] &\leq (2 + 64d^2 L_0^2 \frac{\alpha^2}{\delta^2}) \rho_W^{2N_c} dE_\mu^2 \\ &+ 16dNL_0^2 G^2 \alpha^2 \rho_W^{2N_c} + 32N(d+4)^2 L_0^2 \delta^2 \rho_W^{2N_c} + 16dN\sigma^2 \rho_W^{2N_c}. \end{aligned} \quad (45)$$

Substituting the selected values for δ , α and E_μ^2 into (45), we get that $\mathbb{E}[(E_\mu^k)^2 \|u_k\|^2] \leq (2 + 4\frac{\epsilon_J}{K}) \rho_W^{2N_c} dE_\mu^2 + (\frac{NG^2\epsilon_J^3}{d^2 L_0^2 K} + 32N\frac{(d+4)^2}{d}\epsilon_J^2 + 16dN\sigma^2) \rho_W^{2N_c} \leq dE_\mu^2$. The second inequality holds for similar reasons as those used to obtain (44). To complete the induction argument, we simply need to verify the induction hypothesis when $k = 1$. It is straightforward to see that $\mathbb{E}[\|g_\delta(\theta_0)\|^2] \leq G^2$ due to the definition of G^2 . In addition, due to the initialization step $\mu^{-1}(N_c) = 0$ in line 1 in Algorithm 1, we have that $\mathbb{E}[(E_\mu^{-1})^2] \leq E_\mu^2$. To satisfy the conditions $\mathbb{E}[(E_\mu^0)^2] \leq E_\mu^2$ and $\mathbb{E}[(E_\mu^0)^2 \|u_0\|^2] \leq dE_\mu^2$, it is sufficient to run many enough consensus steps only at the first iteration of Algorithm 1, according to Lemma 3.1. To summarize, the induction hypothesis is satisfied at the first iteration of Algorithm 1 and we have shown that for all $k \geq 0$, we have that $\mathbb{E}[\|g_\delta(\theta_k)\|^2] \leq G^2$, $\mathbb{E}[(E_\mu^k)^2] \leq E_\mu^2$ and $\mathbb{E}[(E_\mu^k)^2 \|u_k\|^2] \leq dE_\mu^2$ under the choice of parameters specified in Theorem 3.5.

Finally, using the uniform bounds $\mathbb{E}[(E_\mu^k)^2] \leq E_\mu^2$ and $\mathbb{E}[(E_\mu^k)^2 \|u_k\|^2] \leq dE_\mu^2$, we can follow the same procedure as in the proof of Theorem 3.2 and obtain the following

optimality bound

$$\begin{aligned}
\sum_{k=0}^{K-1} \mathbb{E}[\|\nabla J_{\delta}(\theta_k)\|^2] &\leq \frac{8d^{1.5}L_0^2}{\epsilon_J^{1.5}} \mathbb{E}[J_{\delta}^* - J_{\delta}(\theta_0)]\sqrt{K} \\
&+ \frac{\epsilon_J^{0.5}}{\sqrt{dK}} \mathbb{E}[\|g_{\delta}(\theta_0)\|^2] + \frac{\epsilon_J^{1.5}}{d^{1.5}\sqrt{K}} + 16 \frac{(d+4)^2}{d} L_0^2 \epsilon_J^{0.5} \sqrt{K} \\
&+ \frac{8d^{1.5}L_0^2\sigma^2}{\epsilon_J^{1.5}} \sqrt{K} + \frac{\epsilon_J^{0.5}}{\sqrt{d}} \sqrt{K} + \frac{\epsilon}{2} K. \tag{46}
\end{aligned}$$

Dividing both sides by K completes the proof. ■



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