Distributed Control of Descriptor Networks: A Convex Procedure for Augmented Sparsity

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***Abstract*— For networks of systems, with possibly improper transfer function matrices, we present a design framework which enables $\mathcal{H}_\infty$ control, while imposing sparsity constraints on the controller's coprime factors. We propose a convex and iterative optimization procedure with guaranteed convergence to obtain distributed controllers. By exploiting the robustness-oriented nature of our proposed approach, we provide the means to obtain sparse representations of our control laws that may not be directly supported by the network's nominal model.**

***Index Terms*— Distributed processes, descriptor systems, sparse $\mathcal{H}_\infty$ control, convex optimization.**

I. Introduction

A. Motivation

When faced with a distributed control problem, one notices an acute lack of dedicated numerical tools, if compared with the classical, centralized design context. Several computational methods, such as those proposed in [1]–[4], aim to exploit specialized techniques, in order to mitigate the numerical complexities inherent to distributed control.

Notably, previous efforts [5] have sought to enforce sparsity constraints directly upon a Finite Impulse Response (FIR) approximation of the Youla parameter, under certain restrictive assumptions, such as Quadratic Invariance (QI) and strong stabilizability (see [6]). However, the technique proposed in Section 5 of [5] cannot cope with enforcing sparsity patterns upon non-sparse *affine expressions* of the Youla parameter.

These issues were tackled in [7], with the introduction of the framework dubbed *System-Level Synthesis* (SLS). Yet the focus on discrete-time systems meant that other architectures, such as the *Network Realization Function* (NRF) representations discussed in [8], [9], have been overshadowed by the FIR approximation methods from the SLS framework.

B. Paper structure and contributions

In this paper, we propose tractable techniques and numerical procedures for the NRF-based framework formalized in [9], which offers distributed control laws in *both* continuous- and discrete-time, without needing to communicate any internal states, *i.e.*, plant or controller states (see Section IV of [9] for a comparison with the SLS framework), thus promoting scalable control laws for large-scale networks. In Section II, we cover a set of preliminary notions, with our paper's problem statement forming Section III-A. Our contributions are structured via the subsequent sections and may be listed as follows:

1) In Section III-B, we show how to impose sparsity in the NRF formalism and how it reduces to a model-matching problem, that is solved via reliable procedures [10]–[12];
2) In Section III-C, we extend the robust stabilization approach from [13] to the distributed, NRF-based setting;
3) In Section IV, we show how to particularize the convex and iterative procedure (with guaranteed convergence) from [14] to obtain robust NRF-based implementations;
4) In Section V, we consider a generalization of the network in [8] and we also show[1] how to employ our robustness-oriented approach to retrieve the same sparse control architecture as in [8] for a more general case.

Finally, Section VI contains a series of concluding remarks.

II. Preliminaries

A. Nomenclature and definitions

Let $\mathbb{C}$, $\mathbb{C}^-$, $j\mathbb{R}$ and $\mathbb{B}$ denote the complex plane, the open left-half plane, the imaginary axis and the set $\{0,1\}$, respectively. Let $\mathbb{M}^{p\times m}$ stand for the set of all $p\times m$ matrices having entries in a set denoted $\mathbb{M}$. We also denote by $P\succ 0$ the fact that $P\in\mathbb{R}^{q\times q}$ is positive definite and by $\overline{\sigma}(Z)$ the maximum singular value of $Z\in\mathbb{C}^{p\times m}$. For any $M\in\mathbb{M}^{p\times m}$, $M^\top$ is its transpose. Let $\mathrm{Ker}(M)$ denote the null space of $M\in\mathbb{M}^{p\times m}$ and let $\|Z\|_*$ denote the sum of the singular values belonging to $Z\in\mathbb{C}^{p\times m}$, which is termed the *nuclear norm*. The operator $\otimes$ denotes the Kronecker product between any two matrices. We define the vectorization of $M\in\mathbb{M}^{p\times m}$ as $\mathrm{vec}(M):=v\in\mathbb{M}^{pm\times 1}$, where $v_{i+(j-1)p}=M_{ij}$, along with the diagonalization of M by $\mathrm{diag}(M):=V\in\mathbb{M}^{pm\times pm}$, where $V_{ii}=(\mathrm{vec}(M))_i$, for $i\in 1:pm$, and $V_{ij}=0$, $\forall i\neq j$.

For $M_i\in\mathbb{M}^{p_i\times m_i}$, with $i\in 1:\ell$, and a natural number g, we define the block-diagonal concatenation operator by $\mathcal{D}(M_1,\dots,M_\ell):=\begin{bmatrix} M_1 & & \\ & \ddots & \\ & & M_\ell\end{bmatrix}\in\mathbb{M}^{\Sigma_{i=1}^{\ell}p_i\times\Sigma_{j=1}^{\ell}m_j}$ and the block-diagonal repetition of M_i by $\mathcal{D}_g(M_i):=\begin{bmatrix} M_i & & \\ & \ddots & \\ & & M_i\end{bmatrix}\in\mathbb{M}^{gp_i\times gm_i}$. For any $R\in\mathbb{M}^{q\times q}$, we denote its symmetric part by $\mathrm{sym}(R):=\frac{1}{2}\left(R+R^\top\right)=\mathrm{sym}\left(R^\top\right)$ and its diagonal part by $R^{\mathrm{diag}}_{ij}:=\begin{cases} R_{ij}, & i=j\\ 0, & i\neq j\end{cases}$, $\forall i,j\in 1:q$.

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[1] All the implementations being compared in this paper are available at the following link: `https://github.com/AndreiSperila/CONPRAS`

The matrix polynomial $A - sE$ is called a pencil, with square ones that have $\det(A - sE) \not\equiv 0$ being termed *regular*. A regular pencil without finite generalized eigenvalues in $\mathbb{C}\backslash\mathbb{C}^-$ and without infinite generalized eigenvalues with partial multiplicities greater than 1 (see [15]) is called *admissible*. Let $\Lambda(A - sE)$ be the collection of generalized eigenvalues (both finite and infinite) belonging to the regular pencil $A - sE$.

In this paper, we will focus on systems described in frequency domain by Transfer Function Matrices (TFMs) of type $(\mathbf{G}(s))_{ij} = \frac{a_{ij}(s)}{b_{ij}(s)}$, with $a_{ij}(s)$ and $b_{ij}(s)$ polynomials with coefficients in $\mathbb{R}$, $i \in 1:p,\ j \in 1:m$. We denote the set of all such TFMs with m inputs and p outputs by $\mathcal{R}^{p\times m}$, with $\mathcal{R}_p^{p\times m}$ being the subset of proper TFMs ($\deg a_{ij} \le \deg b_{ij},\ \forall i \in 1:p,\ j \in 1:m$). Let $\mathcal{B} \in \mathbb{B}^{p\times m}$, which we use to express $\mathcal{S}_\mathcal{B} := \{\mathbf{G} \in \mathcal{R}^{p\times m} | \mathcal{B}_{ij} = 0 \Rightarrow \mathbf{G}_{ij} \equiv 0,\ \forall i \in 1:p,\ \forall j \in 1:m\}$. Note that $\mathbf{G} \in \mathcal{S}_\mathcal{B} \iff (I - \mathrm{diag}(\mathcal{B}))\mathrm{vec}(\mathbf{G}) \equiv 0$. We also define the restriction of $\mathcal{S}_\mathcal{B}$ to proper TFMs $\widehat{\mathcal{S}}_\mathcal{B} := \mathcal{S}_\mathcal{B} \cap \mathcal{R}_p^{p\times m}$.

A TFM without poles (see section 6.5.3 of [16]) located in $\{\mathbb{C}\backslash\mathbb{C}^-\} \cup \{\infty\}$ is called *stable*. Let $\mathcal{RH}_\infty$ denote the set of real-rational and stable TFMs, with the $\mathcal{H}_\infty$ norm of any $\mathbf{G} \in \mathcal{RH}_\infty$ being given by $\|\mathbf{G}\|_\infty := \sup_{s\in j\mathbb{R}} \overline{\sigma}\big(\mathbf{G}(s)\big)$.

The systems considered in this paper are usually represented in the time domain by differential and algebraic equations

$$E\tfrac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t) + Bu(t), \tag{1a}$$
$$y(t) = Cx(t) + Du(t). \tag{1b}$$

The dimension of the regular pencil $A - sE$ and that of x, the vector which contains the realization's *descriptor variables*, is called the order of the realization (1a)-(1b). If its order is the smallest out of all others of its kind, a realization is called minimal (see section 2.4 of [17]). Moreover, we have that

$$\mathbf{G}(s) = C(sE - A)^{-1}B + D =: \left[\begin{array}{c|c} A - sE & B \\ \hline C & D \end{array}\right]. \tag{2}$$

Let the matrix S_∞ span $\mathrm{Ker}E$. A pair $(A - sE, B)$ or a realization (2) for which $[A - sE\ B]$ has full row rank $\forall s \in \mathbb{C}\backslash\mathbb{C}^-$ and $[E\ AS_\infty\ B]$ has full row rank is called strongly stabilizable. By Theorem 1.1 in [18], strong stabilizability is equivalent to the existence of a matrix F, called an admissible feedback, such that the pencil $A + BF - sE$ is admissible. By duality, a pair $(C, A - sE)$ or realization (2) is deemed strongly detectable if $(A^\top - sE^\top, C^\top)$ is strongly stabilizable.

Let both E_r and $D_r^\top D_r$ be invertible and consider

$$\begin{aligned} E_r^\top X_r A_r + A_r^\top X_r E_r + C_r^\top C_r - (E_r^\top X_r B_r + C_r^\top D_r)\times \\ \times (D_r^\top D_r)^{-1}(B_r^\top X_r E_r + D_r^\top C_r) = 0, \end{aligned} \tag{3}$$

the generalized continuous-time algebraic Riccati equation (GCARE, see [19]). A symmetric solution X_r of the GCARE is called stabilizing if $F_r := -(D_r^\top D_r)^{-1}(B_r^\top X_r E_r + D_r^\top C_r)$ is a stabilizing feedback, *i.e.*, $\Lambda(A_r + B_r F_r - sE_r) \subset \mathbb{C}^-$.

B. Parametrization of all stabilizing controllers

To obtain a tractable parametrization for NRF-based control laws, we employ the class of *all* controllers which stabilize a network whose TFM $\mathbf{G}^n \in \mathcal{R}^{(p_u+p)\times(m_u+m)}$ is given by

$$\mathbf{G}^n = \left[\begin{array}{c:c} \mathbf{G}_{11}^n & \mathbf{G}_{12}^n \\ \hdashline \mathbf{G}_{21}^n & \mathbf{G}_{22}^n \end{array}\right] = \left[\begin{array}{c|c:c} A - sE & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hdashline C_2 & D_{21} & D_{22} \end{array}\right], \tag{4}$$

where $A \in \mathbb{R}^{n\times n}$, $D_{11} \in \mathbb{R}^{p_u\times m_u}$, $D_{22} \in \mathbb{R}^{p\times m}$ and all other constant matrices have appropriate dimensions.

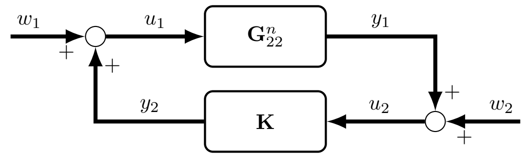

Fig. 1. Closed-loop configuration

Under certain assumptions of strong stabilizability and detectability, the aforementioned class coincides with that of the controllers which render the closed-loop configuration from Fig. 1 well-posed, *i.e.*, $\det(I - \mathbf{G}_{22}^n\mathbf{K}) \not\equiv 0$, and internally stable, *i.e.*, all TFMs from w_1 and w_2 to u_1, u_2, y_1 and y_2 are stable. We now state an extension of the Youla Parametrization, for a class of systems having possibly improper TFMs, by combining the notions from Sections 4.1 and 4.2 of [20].

Theorem II.1. *Let $\mathbf{G}^n \in \mathcal{R}^{(p_u+p)\times(m_u+m)}$ be given as in* (4)*, with $(A - sE, B_2)$ strongly stabilizable and $(C_2, A - sE)$ strongly detectable. Let $(\mathbf{N}, \widetilde{\mathbf{N}}, \mathbf{M}, \widetilde{\mathbf{M}}, \mathbf{X}, \widetilde{\mathbf{X}}, \mathbf{Y}, \widetilde{\mathbf{Y}})$ be a doubly coprime factorization (DCF) of $\mathbf{G}_{22}^n = \mathbf{N}\mathbf{M}^{-1} = \widetilde{\mathbf{M}}^{-1}\widetilde{\mathbf{N}}$ over $\mathcal{RH}_\infty$, with all 8 TFMs being stable and satisfying*

$$\begin{bmatrix} \widetilde{\mathbf{Y}} & -\widetilde{\mathbf{X}} \\ -\widetilde{\mathbf{N}} & \widetilde{\mathbf{M}} \end{bmatrix} \begin{bmatrix} \mathbf{M} & \mathbf{X} \\ \mathbf{N} & \mathbf{Y} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{5}$$

Then, we have that:

(a) *A DCF over $\mathcal{RH}_\infty$ can be obtained, via* (4)*, by*

$$\left[\begin{array}{c:c} \widetilde{\mathbf{Y}} & -\widetilde{\mathbf{X}} \\ \hdashline -\widetilde{\mathbf{N}} & \widetilde{\mathbf{M}} \end{array}\right] := \left[\begin{array}{c|c:c} A_H - sE & -B_2 - HD_{22} & H \\ \hline F & I & 0 \\ \hdashline C_2 & -D_{22} & I \end{array}\right], \tag{6a}$$

$$\left[\begin{array}{c:c} \mathbf{M} & \mathbf{X} \\ \hdashline \mathbf{N} & \mathbf{Y} \end{array}\right] := \left[\begin{array}{c|c:c} A_F - sE & B_2 & -H \\ \hline F & I & 0 \\ \hdashline C_2 + D_{22}F & D_{22} & I \end{array}\right], \tag{6b}$$

with both of the pencils $A_H - sE := A + HC_2 - sE$ and $A_F - sE := A + B_2F - sE$ being admissible;

(b) *The class of all stabilizing controllers is given by*

$$\mathbf{K} = (\mathbf{X} + \mathbf{M}\mathbf{Q})(\mathbf{Y} + \mathbf{N}\mathbf{Q})^{-1} = (\widetilde{\mathbf{Y}} + \mathbf{Q}\widetilde{\mathbf{N}})^{-1}(\widetilde{\mathbf{X}} + \mathbf{Q}\widetilde{\mathbf{M}}), \tag{7}$$

for all $\mathbf{Q} \in \mathcal{RH}_\infty^{m\times p}$ which ensure that $\det(\mathbf{Y} + \mathbf{N}\mathbf{Q}) \not\equiv 0$ and $\det\big(\widetilde{\mathbf{Y}} + \mathbf{Q}\widetilde{\mathbf{N}}\big) \not\equiv 0$;

(c) *For a stabilizing $\mathbf{K}$ given by a DCF over $\mathcal{RH}_\infty$ of $\mathbf{G}_{22}^n$,*

$$\mathbf{G}_{CL} = \mathcal{F}_\ell(\mathbf{G}^n, \mathbf{K}) := \mathbf{G}_{11}^n + \mathbf{G}_{12}^n\mathbf{K}(I - \mathbf{G}_{22}^n\mathbf{K})^{-1}\mathbf{G}_{21}^n$$

is expressed affinely in terms of $\mathbf{Q}$ from (7) *by the identity $\mathbf{G}_{CL} = \mathbf{T}_1 + \mathbf{T}_2\mathbf{Q}\mathbf{T}_3$. Given a realization of the employed DCF, as in* (6a)-(6b)*, we have that*

$$\begin{aligned} \mathbf{T}_1 &:= \mathbf{G}_{11}^n + \mathbf{G}_{12}^n\mathbf{X}\widetilde{\mathbf{M}}\mathbf{G}_{21}^n \\ &= \left[\begin{array}{cc|c} A_F - sE & -B_2F & B_1 \\ 0 & A_H - sE & B_1 + HD_{21} \\ \hline C_1 + D_{12}F & -D_{12}F & D_{11} \end{array}\right], \end{aligned} \tag{8a}$$

$$\mathbf{T}_2 := \mathbf{G}_{12}^n\mathbf{M} = \left[\begin{array}{c|c} A_F - sE & B_2 \\ \hline C_1 + D_{12}F & D_{12} \end{array}\right], \tag{8b}$$

$$\mathbf{T}_3 := \widetilde{\mathbf{M}}\mathbf{G}_{21}^n = \left[\begin{array}{c|c} A_H - sE & B_1 + HD_{21} \\ \hline C_2 & D_{21} \end{array}\right]. \tag{8c}$$

Remark II.1. *The two admissible feedbacks F and H can always be chosen via the two step stabilization algorithm from [18]. Since $A_H - sE$ and $A_F - sE$ are admissible, the TFMs from* (6a)-(6b) *and* (8a)-(8c) *are all stable, and thus proper. State-space realizations for these TFMs can be obtained via the residualization procedure mentioned in Section 3 of [21].*

III. Theoretical results

A. Problem statement

The results presented in this paper tackle the problem of obtaining *sparse* and *robustly stabilizing* control laws of type

$$\mathbf{u}_i = \sum_{j=1}^{m} \mathbf{\Phi}_{ij}\mathbf{u}_j + \sum_{k=1}^{p} \mathbf{\Gamma}_{ik}\mathbf{y}_k,\ \mathbf{\Phi}_{ii} \equiv 0,\ \forall i \in 1:m, \quad (9)$$

discussed in [9]. More specifically, we aim to impose $\mathbf{\Gamma} \in \widehat{\mathcal{S}}_{\mathcal{X}}$ and $\mathbf{\Phi} \in \widehat{\mathcal{S}}_{\mathcal{Y}}$, for some binary matrices $\mathcal{X} \in \mathbb{B}^{m\times p}$ and $\mathcal{Y} \in \mathbb{B}^{m\times m}$, with $\mathcal{Y}^{\text{diag}} = 0$, and to have the control laws from (9) stabilize all network models $\mathbf{G_\Delta} \in \mathcal{C}^{\epsilon}_{\mathbf{G}}$, where the class $\mathcal{C}^{\epsilon}_{\mathbf{G}}$ is of the type discussed in [22], owing to its generality.

In the sequel, we show that this problem reduces to

$$\left\| \widehat{\mathbf{T}}_1 + \sum_{i=1}^{q} \mathbf{x}_i \widehat{\mathbf{T}}_{2i} \right\|_\infty < 1,\ \mathbf{x}_i \in \mathcal{RH}_\infty^{1\times 1},\ i \in 1:q, \quad (10)$$

with $\widehat{\mathbf{T}}_1, \widehat{\mathbf{T}}_{2i} \in \mathcal{RH}_\infty, \forall\, i \in 1:q$, being expressed in terms of (2) and (6a)-(6b). Finally, we particularize convex relaxation-based procedures [14] available in literature to solve (10) and we compose $(\mathbf{\Phi},\mathbf{\Gamma})$ from the obtained $\mathbf{x}_i \in \mathcal{RH}_\infty^{1\times 1}$, $i \in 1:q$.

B. Parametrization of NRF-based control laws

We show here how the problem of obtaining the *sparse* and *stabilizing* distributed control laws of type (9) can be reduced to a readily solvable model-matching problem. As discussed in Section III of [9], this is primarily done by factorizing a stabilizing controller from the class expressed in Theorem II.1 as $\mathbf{K} = (I-\mathbf{\Phi})^{-1}\mathbf{\Gamma}$, where $(\widetilde{\mathbf{Y}}+\mathbf{Q}\widetilde{\mathbf{N}})^{\text{diag}}$ and $(\widetilde{\mathbf{Y}}+\mathbf{Q}\widetilde{\mathbf{N}})$ have proper inverses and the NRF pair $(\mathbf{\Phi},\mathbf{\Gamma})$ is obtained as

$$\mathbf{\Phi} := I - ((\widetilde{\mathbf{Y}}+\mathbf{Q}\widetilde{\mathbf{N}})^{\text{diag}})^{-1}(\widetilde{\mathbf{Y}}+\mathbf{Q}\widetilde{\mathbf{N}}) \in \mathcal{R}_p^{m\times m}, \quad (11a)$$

$$\mathbf{\Gamma} := ((\widetilde{\mathbf{Y}}+\mathbf{Q}\widetilde{\mathbf{N}})^{\text{diag}})^{-1}(\widetilde{\mathbf{X}}+\mathbf{Q}\widetilde{\mathbf{M}}) \in \mathcal{R}_p^{m\times p}. \quad (11b)$$

Remark III.1. *When the realization of $\mathbf{G}_{22}^n \in \mathcal{R}^{p\times m}$ (not necessarily proper) from* (4) *is strongly stabilizable and detectable, the guarantees of closed-loop internal stability and of scalability showcased in Section III of [9] for control laws of type* (9) *will also hold. Thus, since all closed-loop transfers are stable and since the analogues of Lemmas 5.2 and 5.3 in [23] (formulated for descriptor systems) are in effect, then the descriptor variables of both the plant and of the controller's NRF-based implementation (along with their output signals in closed-loop interconnection) will be bounded and will tend to* 0, *when evolving freely from any finite initial conditions.*

With the stability guarantees of (9) clarified in Remark III.1, we now focus on imposing sparsity patterns on the $(\mathbf{\Phi},\mathbf{\Gamma})$ pair. The following result offers a characterization of the stable Youla parameters which, for a given DCF over $\mathcal{RH}_\infty$, produce the desired sparsity structure for the NRF pair in (11a)-(11b).

Proposition III.1. *Let $\mathbf{G} \in \mathcal{R}^{p\times m}$ be given by a DCF over $\mathcal{RH}_\infty$* (6a)-(6b)*, let $\mathcal{X} \in \mathbb{B}^{m\times p}$ and let $\widehat{\mathcal{Y}} \in \mathbb{B}^{m\times m}$, with $\widehat{\mathcal{Y}}^{diag} = I$. Define $F_{\mathcal{X}} := I - diag(\mathcal{X})$ and $F_{\widehat{\mathcal{Y}}} := I - diag(\widehat{\mathcal{Y}})$. If there exist $\mathbf{Q}_0 \in \mathcal{RH}_\infty^{m\times p}$ and $\widehat{\mathbf{Q}} \in \mathcal{RH}_\infty^{m\times p}$ satisfying*

$$\begin{bmatrix} F_{\mathcal{X}}(\widetilde{\mathbf{M}}^\top \otimes I) \\ F_{\widehat{\mathcal{Y}}}(\widetilde{\mathbf{N}}^\top \otimes I) \end{bmatrix} vec(\mathbf{Q}_0) + \begin{bmatrix} F_{\mathcal{X}} vec(\widetilde{\mathbf{X}}) \\ F_{\widehat{\mathcal{Y}}} vec(\widetilde{\mathbf{Y}}) \end{bmatrix} \equiv 0, \quad (12a)$$

$$vec(\widehat{\mathbf{Q}}) \in Ker \begin{bmatrix} F_{\mathcal{X}}(\widetilde{\mathbf{M}}^\top \otimes I) \\ F_{\widehat{\mathcal{Y}}}(\widetilde{\mathbf{N}}^\top \otimes I) \end{bmatrix}, \quad (12b)$$

$$\det\left((\widetilde{\mathbf{Y}} + (\mathbf{Q}_0+\widehat{\mathbf{Q}})\widetilde{\mathbf{N}})(\infty)\right) \neq 0, \quad (12c)$$

$$\det\left((\widetilde{\mathbf{Y}} + (\mathbf{Q}_0+\widehat{\mathbf{Q}})\widetilde{\mathbf{N}})^{diag}(\infty)\right) \neq 0, \quad (12d)$$

then the controller in (7)*, formed via the employed DCF over $\mathcal{RH}_\infty$ of type* (6a)-(6b) *and via $\mathbf{Q} := \mathbf{Q}_0 + \widehat{\mathbf{Q}}$, admits an NRF implementation of* (11a)-(11b) *with $\mathbf{\Gamma} \in \widehat{\mathcal{S}}_{\mathcal{X}}$ and $\mathbf{\Phi} \in \widehat{\mathcal{S}}_{(\widehat{\mathcal{Y}}-I)}$.*

Proof. See the Appendix. □

Remark III.2. *The equation* (12a) *can be solved for a stable $vec(\mathbf{Q}_0)$ as shown in [10]. Moreover, a least order solution can be obtained by employing the generalized minimum cover algorithm from [12]. A benefit of this approach is that it computes a (stable) basis for Ker $\begin{bmatrix} F_{\mathcal{X}}(\widetilde{\mathbf{M}}^\top \otimes I) \\ F_{\widehat{\mathcal{Y}}}(\widetilde{\mathbf{N}}^\top \otimes I) \end{bmatrix}$. Alternatively, a stable basis of least degree can be obtained as in [11].*

Remark III.3. *Selecting a $\mathbf{Q}$ which ensures that* $\det\left((\widetilde{\mathbf{Y}}+\mathbf{Q}\widetilde{\mathbf{N}})(\infty)\right) \neq 0$*, thus guaranteeing that the controller's TFM is well-posed, can be done numerically by using the fact that*

$$\det\left(\widetilde{\mathbf{Y}}(\infty)+\mathbf{Q}(\infty)\widetilde{\mathbf{N}}(\infty)\right) \neq 0 \iff$$
$$\iff \left(\widetilde{\mathbf{Y}}(\infty)+\mathbf{Q}(\infty)\widetilde{\mathbf{N}}(\infty)\right)^\top \left(\widetilde{\mathbf{Y}}(\infty)+\mathbf{Q}(\infty)\widetilde{\mathbf{N}}(\infty)\right) \succ 0. \quad (13)$$

To ensure $\det\left((\widetilde{\mathbf{Y}}+\mathbf{Q}\widetilde{\mathbf{N}})^{diag}(\infty)\right) \neq 0$*, we first denote by e_i the i^{th} vector of the canonical basis of $\mathbb{R}^{m\times 1}$ and impose that*

$$\begin{aligned} &e_i^\top(\widetilde{\mathbf{Y}}(\infty)+\mathbf{Q}(\infty)\widetilde{\mathbf{N}}(\infty))^\top e_i \times \\ &\times e_i^\top(\widetilde{\mathbf{Y}}(\infty)+\mathbf{Q}(\infty)\widetilde{\mathbf{N}}(\infty))e_i \succ 0,\ \forall i \in 1:m. \end{aligned} \quad (14)$$

The bilinear matrix inequalities in (13)-(14) *will be convexified and solved iteratively via the procedure given in Section IV.*

C. Robust stabilization and augmented sparsity

In this subsection, we show how to obtain a controller of type (7) whose NRF implementation (9) stabilizes all network models $\mathbf{G_\Delta}$ in a class $\mathcal{C}^{\epsilon}_{\mathbf{G}}$ and how this technique can be used to obtain a sparse control architecture. However, before this, we begin by defining the aforementioned class of TFMs.

The class $\mathcal{C}^{\epsilon}_{\mathbf{G}}$, introduced in Section III-A, is expressed in terms of a stable right coprime factorization (RCF) of $\mathbf{G} = \widehat{\mathbf{N}}\widehat{\mathbf{M}}^{-1} \in \mathcal{R}^{p\times m}$, *i.e.*, $\widehat{\mathbf{N}},\widehat{\mathbf{M}} \in \mathcal{RH}_\infty$ and $\exists\ \widetilde{\mathbf{X}},\widetilde{\mathbf{Y}} \in \mathcal{RH}_\infty$ so that $\widetilde{\mathbf{Y}}\widehat{\mathbf{M}} - \widetilde{\mathbf{X}}\widehat{\mathbf{N}} = I$, which is additionally normalized, *i.e.*, $\widehat{\mathbf{N}}^\top(-s)\widehat{\mathbf{N}}(s) + \widehat{\mathbf{M}}^\top(-s)\widehat{\mathbf{M}}(s) = I$. With *any* (see [22]) such stable normalized RCF (NRCF) and $\epsilon \in (0,1]$, we define

$$\begin{aligned} \mathcal{C}^{\epsilon}_{\mathbf{G}} := \ &\Big\{ (\widehat{\mathbf{N}}+\mathbf{\Delta}_{\widehat{\mathbf{N}}})(\widehat{\mathbf{M}}+\mathbf{\Delta}_{\widehat{\mathbf{M}}})^{-1}, \mathbf{\Delta}_{\widehat{\mathbf{N}}}, \mathbf{\Delta}_{\widehat{\mathbf{M}}} \in \mathcal{RH}_\infty, \\ &\det(\widehat{\mathbf{M}}+\mathbf{\Delta}_{\widehat{\mathbf{M}}}) \neq 0,\ \left\| \begin{bmatrix} \mathbf{\Delta}_{\widehat{\mathbf{N}}}^\top & \mathbf{\Delta}_{\widehat{\mathbf{M}}}^\top \end{bmatrix}^\top \right\|_\infty < \epsilon \Big\}. \end{aligned} \quad (15)$$

Clearly, in order to manipulate $\mathcal{C}^{\epsilon}_{\mathbf{G}}$, we must first obtain a stable NRCF of $\mathbf{G}$. While (6b) readily provides a stable RCF of $\mathbf{G}$, a stable NRCF can be obtained via the following result.

Lemma III.1. *Let E_r be an invertible matrix and let also $\Lambda(A_r - sE_r) \subset \mathbb{C}^-$. Let the TFM*

$$\begin{bmatrix} \mathbf{N}^\top & \mathbf{M}^\top \end{bmatrix}^\top = \left[\begin{array}{c|c} A_r - sE_r & B_r \\ \hline C_r & D_r \end{array}\right] \in \mathcal{RH}_\infty^{(p+m)\times m} \quad (16)$$

designate a stable RCF of $\mathbf{G} = \mathbf{N}\mathbf{M}^{-1} \in \mathcal{R}^{p\times m}$ and let $H_r \in \mathbb{R}^{m\times m}$ be invertible and satisfy $H_r^\top H_r = D_r^\top D_r$. Then:

- **(a)** *The GCARE from* (3) *has a symmetric stabilizing solution, X_r, along with a stabilizing feedback, F_r;*
- **(b)** *For $\mathbf{G}_0 := \left[\begin{array}{c|c} A_r - sE_r & B_r \\ \hline -H_rF_r & H_r \end{array}\right]$, we get that $\begin{bmatrix} \widehat{\mathbf{N}}^\top & \widehat{\mathbf{M}}^\top \end{bmatrix}^\top := \begin{bmatrix} \mathbf{N}^\top & \mathbf{M}^\top \end{bmatrix}^\top \mathbf{G}_0^{-1}$ designates a stable NRCF of $\mathbf{G}$.*

Proof. For point **(a)**, see the Appendix. Point **(b)** is precisely Proposition 1 in [21]. □

Having now the ability to express the TFMs that make up (15), we turn our attention to characterizing stabilizing controllers whose NRF implementations of type (9) stabilize all TFMs in $\mathcal{C}^{\epsilon}_{\mathbf{G}}$, for a given $\epsilon \in (0,1]$. The following result is central to this section and offers the means to do just so.

Theorem III.1. *Let $\mathbf{G} \in \mathcal{R}^{p\times m}$ be given by a strongly stabilizable and detectable realization* (2) *and let F ensure that $A+BF-sE$ is admissible. Let also $\mathbf{G} = \mathbf{N}\mathbf{M}^{-1}$ be the stable RCF induced by F as in* (6b)*, and for which a realization as in* (16) *is obtained (recall Remark II.1), having E_r invertible and $\Lambda(A_r - sE_r) \subset \mathbb{C}^-$. Let F_r be the stabilizing feedback of the GCARE from* (3) *and let $\epsilon \in (0,1]$ along with $H_r \in \mathbb{R}^{m\times m}$ invertible, such that $H_r^\top H_r = D_r^\top D_r$. Then:*

($\mathbf{a}$) *There exists a class of stabilizing controllers $\mathbf{K} \in \mathcal{R}^{m\times p}$, based upon a DCF over $\mathcal{RH}_\infty$ of $\mathbf{T}^{\epsilon}_{22}$, for the system*

$$\mathbf{T}^{\epsilon} := \left[\begin{array}{cc:c} 0 & -\epsilon\widehat{\mathbf{M}}^{-1} & \epsilon\widehat{\mathbf{M}}^{-1} \\ \hdashline I & -\mathbf{G} & \mathbf{G} \end{array}\right] = \left[\begin{array}{c:c} \mathbf{T}^{\epsilon}_{11} & \mathbf{T}^{\epsilon}_{12} \\ \hdashline \mathbf{T}^{\epsilon}_{21} & \mathbf{T}^{\epsilon}_{22} \end{array}\right]$$
$$= \left[\begin{array}{cc|cc:c} A_r - sE_r & -B_rF & 0 & -B_r & B_r \\ 0 & A - sE & 0 & -B & B \\ \hline -\epsilon H_rF_r & -\epsilon H_rF & 0 & -\epsilon H_r & \epsilon H_r \\ \hdashline 0 & C & I & -D & D \end{array}\right]; \quad (17)$$

($\mathbf{b}$) *Let $\mathbf{K}$ belong to the class from* ($\mathbf{a}$). *If $\|\mathcal{F}_\ell(\mathbf{T}^{\epsilon},\mathbf{K})\|_\infty \leq 1$ and $\mathbf{K}$ admits an NRF implementation as in* (11a)-(11b)*, then the control laws from* (9) *stabilize all $\mathbf{G}_{\mathbf{\Delta}} \in \mathcal{C}^{\epsilon}_{\mathbf{G}}$.*

Proof. See the Appendix. □

Remark III.4. *The key to bypassing the feasibility of the model-matching problem tackled in Proposition III.1 lies with judiciously employing Theorem III.1. Let our network's TFM be $\overline{\mathbf{G}} \in \mathcal{R}^{p\times m}$ and assume that the chosen NRF architecture is either infeasible or difficult to satisfy for the available DCFs over $\mathcal{RH}_\infty$ of $\overline{\mathbf{G}}$. Then, we may resort to an approximation of $\overline{\mathbf{G}}$, denoted $\mathbf{G} \in \mathcal{R}^{p\times m}$, which satisfies $\overline{\mathbf{G}} \in \mathcal{C}^{\epsilon}_{\mathbf{G}}$ and which is described by a DCF over $\mathcal{RH}_\infty$ that supports the desired NRF architecture. By obtaining control laws of type* (9) *with the desired sparsity structure and which stabilize all $\mathbf{G}_{\mathbf{\Delta}} \in \mathcal{C}^{\epsilon}_{\mathbf{G}}$, these sparse control laws will also stabilize $\overline{\mathbf{G}}$. A concrete example of this design procedure will be shown in Section V.*

Although we now possess the means to characterize robustly stabilizing NRF-based implementations of the controller, note that these are obtained by employing a DCF over $\mathcal{RH}_\infty$ whose realization is of the same order as that in (17). The next result shows how to obtain descriptor representations for the DCF over $\mathcal{RH}_\infty$ with the *same order* as that of the network's model.

Proposition III.2. *Let the same framework, hypotheses and notation hold as in the statement of Theorem III.1 and let $\mathbf{T}^{\epsilon}$ be defined as in* (17)*. Then, we have that:*

($\mathbf{a}$) *For any H so that the pencil $A+HC-sE$ is admissible, a DCF over $\mathcal{RH}_\infty$ of $\mathbf{T}^{\epsilon}_{22}$ is given by*

$$\left[\begin{array}{c:c} \widetilde{\mathbf{Y}}^{\epsilon} & -\widetilde{\mathbf{X}}^{\epsilon} \\ \hdashline -\widetilde{\mathbf{N}}^{\epsilon} & \widetilde{\mathbf{M}}^{\epsilon} \end{array}\right] := \left[\begin{array}{c|c:c} A+HC-sE & -B-HD & H \\ \hline F & I & 0 \\ \hdashline C & -D & I \end{array}\right], \quad (18a)$$
$$\left[\begin{array}{c:c} \mathbf{M}^{\epsilon} & \mathbf{X}^{\epsilon} \\ \hdashline \mathbf{N}^{\epsilon} & \mathbf{Y}^{\epsilon} \end{array}\right] := \left[\begin{array}{c|c:c} A+BF-sE & B & -H \\ \hline F & I & 0 \\ \hdashline C+DF & D & I \end{array}\right]; \quad (18b)$$

($\mathbf{b}$) *For any stabilizing controller obtained using* (18a)-(18b) *and an arbitrary $\mathbf{Q} \in \mathcal{RH}^{m\times p}_\infty$, we may express $\mathcal{F}_\ell(\mathbf{T}^{\epsilon},\mathbf{K}) = \mathbf{T}^{\epsilon}_1 + \mathbf{T}^{\epsilon}_2\mathbf{Q}\mathbf{T}^{\epsilon}_3$, where we have*

$$\mathbf{T}^{\epsilon}_1 := \mathbf{T}^{\epsilon}_{11} + \mathbf{T}^{\epsilon}_{12}\mathbf{X}^{\epsilon}\widetilde{\mathbf{M}}^{\epsilon}\mathbf{T}^{\epsilon}_{21}$$
$$= \left[\begin{array}{cc|cc} A_r - sE_r & -B_rF & 0 & -B_r \\ 0 & A+HC-sE & H & -B-HD \\ \hline -\epsilon H_rF_r & -\epsilon H_rF & 0 & -\epsilon H_r \end{array}\right], \quad (19a)$$
$$\mathbf{T}^{\epsilon}_2 := \mathbf{T}^{\epsilon}_{12}\mathbf{M}^{\epsilon} = \left[\begin{array}{c|c} A_r - sE_r & B_r \\ \hline -\epsilon H_rF_r & \epsilon H_r \end{array}\right], \quad (19b)$$
$$\mathbf{T}^{\epsilon}_3 := \widetilde{\mathbf{M}}^{\epsilon}\mathbf{T}^{\epsilon}_{21} = \left[\begin{array}{c|cc} A+HC-sE & H & -B-HD \\ \hline C & I & -D \end{array}\right]. \quad (19c)$$

Proof. See the Appendix. □

IV. Convex procedure for augmented sparsity

A. Procedure setup and norm condition reformulation

Recall that, in order to obtain sparse control laws of type (9), we aim to express controllers of type (7) for $\mathbf{Q} \in \mathcal{RH}^{m\times p}_\infty$ satisfying (12a)-(12d). For robust stability, point ($\mathbf{b}$) of Theorem III.1 argues that we need only satisfy $\|\mathbf{T}^{\epsilon}_1 + \mathbf{T}^{\epsilon}_2\mathbf{Q}\mathbf{T}^{\epsilon}_3\|_\infty \leq 1$, where $\mathbf{T}^{\epsilon}_1$, $\mathbf{T}^{\epsilon}_2$ and $\mathbf{T}^{\epsilon}_3$ are expressed as in (19a)-(19c).

The beginning of this section is dedicated to showing how this norm condition can be converted into (10). Due to this being the setup of the iterative algorithm given in the sequel, this conversion will be given in an ordered sequence of steps:

Step 1. Solve (12a) for $\mathbf{Q}_0 \in \mathcal{RH}^{m\times p}_\infty$ and obtain a basis $\mathbf{B} \in \mathcal{RH}^{mp\times q}_\infty$ for Ker $\begin{bmatrix} F_{\mathcal{X}}(\widetilde{\mathbf{M}}^\top \otimes I) \\ F_{\widehat{\mathcal{Y}}}(\widetilde{\mathbf{N}}^\top \otimes I) \end{bmatrix}$ (recall Remark III.2);

Step 2. Partition $\mathbf{B}$ via its columns, as follows

$$\mathbf{B} := \left[\begin{array}{c:c:c:c:c} \mathbf{B}_1 & \dots & \mathbf{B}_i & \dots & \mathbf{B}_q \end{array}\right], \ \mathbf{B}_i \in \mathcal{RH}^{mp\times 1}_\infty,$$

to obtain minimal realizations $\mathbf{B}_i = \left[\begin{array}{c|c} A^{\mathbf{B}}_i & B^{\mathbf{B}}_i \\ \hline C^{\mathbf{B}}_i & D^{\mathbf{B}}_i \end{array}\right]$, $\forall i \in 1:q$;

Step 3. Using these realizations, write via (6b) a stable RCF of each $\mathbf{B}_i = \mathbf{N}_{\mathbf{B}_i}\mathbf{M}^{-1}_{\mathbf{B}_i}$, which are given explicitly by

$$\begin{bmatrix} \mathbf{M}_{\mathbf{B}_i} \\ \mathbf{N}_{\mathbf{B}_i} \end{bmatrix} := \left[\begin{array}{c|c} A^{\mathbf{B}}_i + B^{\mathbf{B}}_iF^{\mathbf{B}}_i & B^{\mathbf{B}}_i \\ \hline F^{\mathbf{B}}_i & 1 \\ C^{\mathbf{B}}_i + D^{\mathbf{B}}_iF^{\mathbf{B}}_i & D^{\mathbf{B}}_i \end{array}\right] \in \mathcal{RH}^{(mp+1)\times 1}_\infty, \quad (20)$$

with $F^{\mathbf{B}}_i$ ensuring $\Lambda(A^{\mathbf{B}}_i + B^{\mathbf{B}}_iF^{\mathbf{B}}_i - sI) \subset \mathbb{C}^-$ to form

$$\widehat{\mathbf{B}} := \left[\begin{array}{c:c:c:c:c} \mathbf{N}_{\mathbf{B}_1} & \cdots & \mathbf{N}_{\mathbf{B}_i} & \cdots & \mathbf{N}_{\mathbf{B}_q} \end{array}\right] \in \mathcal{RH}^{mp\times q}_\infty; \quad (21)$$

Step 4. Partition $\widehat{\mathbf{B}} = \left[\begin{array}{c:c:c:c:c} \widehat{\mathbf{B}}^\top_1 & \cdots & \widehat{\mathbf{B}}^\top_i & \cdots & \widehat{\mathbf{B}}^\top_p \end{array}\right]^\top$, noting that $\widehat{\mathbf{B}}_i \in \mathcal{RH}^{m\times q}_\infty$, in order to finally define

$$\overline{\mathbf{B}} := \left[\begin{array}{c:c:c} \widehat{\mathbf{B}}_1 & \cdots & \widehat{\mathbf{B}}_p \end{array}\right] = \left[\begin{array}{c|c} A_{\overline{\mathbf{B}}} & B_{\overline{\mathbf{B}}} \\ \hline C_{\overline{\mathbf{B}}} & D_{\overline{\mathbf{B}}} \end{array}\right] \in \mathcal{RH}^{m\times pq}_\infty. \quad (22)$$

Remark IV.1. *Since $\mathbf{B}_i$ are the columns of stable basis of the null space in* (12b)*, then so are $\mathbf{N}_{\mathbf{B}_i} = \mathbf{B}_i\mathbf{M}_{\mathbf{B}_i}$, having realizations of the same order as those of $\mathbf{B}_i$. Thus, $\widehat{\mathbf{B}}$ is a stable basis for the same null space and may also be used to form vec$(\widehat{\mathbf{Q}}) = \widehat{\mathbf{B}}\mathbf{x}$, $\forall \mathbf{x} \in \mathcal{RH}^{q\times 1}_\infty$, as in Proposition III.1.*

This concludes the setup of our procedure and we now move on to converting $\|\mathbf{T}^{\epsilon}_1 + \mathbf{T}^{\epsilon}_2\mathbf{Q}\mathbf{T}^{\epsilon}_3\|_\infty \leq 1$ into (10), through the explicit use of $\overline{\mathbf{B}}$. Recall that $\mathbf{Q}$ can be partitioned additively as $\mathbf{Q} = \mathbf{Q}_0 + \widehat{\mathbf{Q}}$, with $\mathbf{Q}_0$ having been obtained in *Step 1* of the setup and with $\text{vec}(\widehat{\mathbf{Q}})$ formed as in Remark IV.1. Thus, by (22) in *Step 4* of the setup, it is straightforward to obtain

$$\widehat{\mathbf{Q}} = \left[\begin{array}{c:c:c} \widehat{\mathbf{B}}_1\mathbf{x} & \cdots & \widehat{\mathbf{B}}_p\mathbf{x} \end{array}\right] = \overline{\mathbf{B}}\mathcal{D}_p(\mathbf{x}) \in \mathcal{RH}_\infty^{m\times p}.$$

Defining $\mathbf{x} \in \mathcal{RH}_\infty^{q\times 1}$ as $\mathbf{x} := \left[\begin{array}{c|c} A_x & b_x \\ \hline C_x & d_x \end{array}\right]$, we may express

$$\widehat{\mathbf{Q}} = \left[\begin{array}{cc|c} A_{\overline{\mathbf{B}}} & B_{\overline{\mathbf{B}}}\mathcal{D}_p(C_x) & B_{\overline{\mathbf{B}}}\mathcal{D}_p(d_x) \\ 0 & \mathcal{D}_p(A_x) & \mathcal{D}_p(b_x) \\ \hline C_{\overline{\mathbf{B}}} & D_{\overline{\mathbf{B}}}\mathcal{D}_p(C_x) & D_{\overline{\mathbf{B}}}\mathcal{D}_p(d_x) \end{array}\right], \tag{23}$$

whose realization is affine in terms of all variable matrices: A_x, b_x, C_x, d_x, and $A_{\overline{\mathbf{B}}}$ and $C_{\overline{\mathbf{B}}}$, by way of $F_i^{\mathbf{B}}$, for $i \in 1:q$.

It now becomes clear, in terms of (10), that we have $\widehat{\mathbf{T}}_1 = \mathbf{T}_1^\epsilon + \mathbf{T}_2^\epsilon\mathbf{Q}_0\mathbf{T}_3^\epsilon$ and $\widehat{\mathbf{T}}_{2i} = \mathbf{T}_2^\epsilon\widehat{\mathbf{Q}}_i\mathbf{T}_3^\epsilon$, where we have defined

$$\widehat{\mathbf{Q}}_i := \overline{\mathbf{B}}\left[\begin{array}{c:c:c:c:c} \widehat{e}_i & \cdots & \widehat{e}_{jq+i} & \cdots & \widehat{e}_{(p-1)q+i} \end{array}\right], \tag{24}$$

with $i \in 1:q$, $j \in 1:p-2$, and $\widehat{e}_i$ being the i^{th} vector in the canonical basis of $\mathbb{R}^{pq\times 1}$. Moving on, the next section tackles the numerical details of satisfying the inequality from (10).

Remark IV.2. *The free term of the Youla Parametrization is now expressed as* $\mathbf{Q} = \mathbf{Q}_0 + \sum_{i=1}^q \mathbf{x}_i\widehat{\mathbf{Q}}_i$*, for some* $\mathbf{x}_i \in \mathcal{RH}_\infty^{1\times 1}$*,* $\forall i \in 1:q$*. Thus, forming* $\overline{\mathbf{B}}$ *from only a subset of the* q *columns used in* (21) *may prove sufficient to solve* (10)*, which has the benefit of cutting down on computational costs.*

B. Numerical formulation and NRF implementability

In order to formulate a numerical procedure meant to solve (10), we first require a state-space realization of $\mathbf{T}_1^\epsilon + \mathbf{T}_2^\epsilon\mathbf{Q}\mathbf{T}_3^\epsilon$. This can be obtained by first defining the following TFM

$$\mathbf{T}^f := \left[\begin{array}{c:c} \mathbf{T}_1^\epsilon + \mathbf{T}_2^\epsilon\mathbf{Q}_0\mathbf{T}_3^\epsilon & \mathbf{T}_2^\epsilon \\ \hdashline \mathbf{T}_3^\epsilon & 0 \end{array}\right] = \left[\begin{array}{c|c:c} A^f & B_1^f & B_2^f \\ \hline C_1^f & D_{11}^f & D_{12}^f \\ \hdashline C_2^f & D_{21}^f & 0 \end{array}\right], \tag{25}$$

and obtaining a minimal state-space realization as in (25), with $\Lambda\left(A^f - sI\right) \subset \mathbb{C}^-$ due to $\mathbf{T}^f \in \mathcal{RH}_\infty$, via one of $\mathbf{Q}_0$ and via (19a)-(19c), as per Remark II.1. Notice that $\mathbf{T}_1^\epsilon + \mathbf{T}_2^\epsilon\mathbf{Q}\mathbf{T}_3^\epsilon = \mathcal{F}_\ell(\mathbf{T}^f, \widehat{\mathbf{Q}})$ to get, via (23) along with the formulas in Section 10.4 of [23], the realization from (26), given on the next page. Crucially, notice that all the variable matrices which appear in the realization from (26) do so only via *affine terms*.

Before stating the numerical problem which will be tackled by our iterative procedure, we must ensure that the obtained controller is well-defined and can be implemented as in (9), via its NRF pair. As indicated in Remark III.3, this is ensured by satisfying (13)-(14), which can be written generically as

$$\left(Z_1^k + Z_2^k\mathbf{Q}(\infty)Z_3^k\right)^\top\left(Z_1^k + Z_2^k\mathbf{Q}(\infty)Z_3^k\right) \succ 0,\ \forall k \in 1:N_Z, \tag{27}$$

where the various matrices $Z_1^k \in \mathbb{R}^{w_k\times w_k}$, $Z_2^k \in \mathbb{R}^{w_k\times m}$ and $Z_3^k \in \mathbb{R}^{p\times w_k}$ are shown explicitly in (13)-(14). Finally, note that $\mathbf{Q}(\infty) = \mathbf{Q}_0(\infty) + \sum_{i=1}^q d_{xi}\widehat{\mathbf{Q}}_i(\infty)$, where we partition $\mathbf{x}(\infty) = d_x = \left[\begin{array}{c:c:c:c:c} d_{x1} & \cdots & d_{xi} & \cdots & d_{xq} \end{array}\right]^\top$, $i \in 2:q-1$. Thus, we combine (10) and (27) into our numerical problem

$$\begin{cases} \left\|\mathcal{F}_\ell(\mathbf{T}^f, \widehat{\mathbf{Q}})\right\|_\infty < 1,\ \widehat{\mathbf{Q}} \text{ as in (23)}, \\ (\widehat{Z}_1^k)^\top\widehat{Z}_1^k + \sum\limits_{i=1}^q d_{xi}\left((\widehat{Z}_1^k)^\top\widehat{Z}_{2i}^k + (\widehat{Z}_{2i}^k)^\top\widehat{Z}_1^k\right) + \sum\limits_{i=1}^q d_{xi}^2(\widehat{Z}_{2i}^k)^\top\widehat{Z}_{2i}^k + \\ + \sum\limits_{i=1}^{q-1}\sum\limits_{j=i+1}^q d_{xi}d_{xj}\left((\widehat{Z}_{2i}^k)^\top\widehat{Z}_{2j}^k + (\widehat{Z}_{2j}^k)^\top\widehat{Z}_{2i}^k\right) \succ 0,\ \forall k \in 1:N_Z, \\ \widehat{Z}_1^k := Z_1^k + Z_2^k\mathbf{Q}_0(\infty)Z_3^k, \widehat{Z}_{2i}^k := Z_2^k\widehat{\mathbf{Q}}_i(\infty)Z_3^k,\ \forall i \in 1:q. \end{cases} \tag{28}$$

Initialization: Solve the LMI system of (30), given on the next page, along with the equality constraint $\overline{d}_x - d_x = 0$, for $\left(A_x^0, b_x^0, C_x^0, d_x^0, \overline{d}_x^0, \left(F_i^{\mathbf{B}}\right)^0, P^0, \overline{P}^0, \left(\overline{P}^D\right)^0, P_x^0, \overline{P}_x^0, \left(P_i^{\mathbf{B}}\right)^0, \left(\overline{P}_i^{\mathbf{B}}\right)^0\right)$. Using these computed variables, form T_A^0, T_B^0, T_C^0 as in (29d)-(29f) and then set $k = 0$ along with $f^0 = \left\|T_C^0 - T_A^0T_B^0\right\|_* + \|I_{n_T}\|_*$;

repeat

  if k *mod* $2 < 1$ **then**

    Set $k = k+1$ followed by $\Theta^k = T_B - T_B^{k-1}$;

  else

    Set $k = k+1$ followed by $\Theta^k = T_A - T_A^{k-1}$;

  end

  Solve $\mathcal{M}\left(T_A^{k-1}, T_B^{k-1}, \Theta^k\right)$ for $\left(A_x^k, b_x^k, C_x^k, d_x^k, \overline{d}_x^k, (F_i^{\mathbf{B}})^k, P^k, \overline{P}^k, \left(\overline{P}^D\right)^k, P_x^k, \overline{P}_x^k, (P_i^{\mathbf{B}})^k, \left(\overline{P}_i^{\mathbf{B}}\right)^k\right)$ and use them to form T_A^k, T_B^k, T_C^k as in (29d)-(29f);

  Compute $f^k := \left\|T_C^k - T_A^kT_B^k\right\|_* + \|I_{n_T}\|_*$;

until $f^{k-1} - f^k < \eta_1$ *or* $f^k - \|I_{n_T}\|_* < \eta_2$;

Algorithm 1: Convex approach to solving (28)

C. The iterative procedure with guaranteed convergence

We now introduce the most general form (recall Remark IV.2) of our convex and iterative procedure for solving (28), based upon the algorithm with guaranteed convergence in [14].

Theorem IV.1. *Given the realization from* (26) *along with two tolerance values* $0 < \eta_1, 0 < \eta_2 \ll 1$*, define the following:*

$$\overline{A}^f := \begin{bmatrix} A^f & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \overline{C}_2^f := \begin{bmatrix} C_2^f & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{29a}$$

$$\overline{B}_1^f := \left[\left(B_1^f\right)^\top\ 0\ 0\right]^\top, \quad \overline{D}_{21}^f := \left[\left(D_{21}^f\right)^\top\ 0\ 0\right]^\top,$$

$$G := \begin{bmatrix} 2sym\left(P\overline{A}^f + \overline{PC}_2^f\right) & P\overline{B}_1^f + \overline{PD}_{21}^f & \overline{C}^\top \\ \left(\overline{B}_1^f\right)^\top P + \left(\overline{PD}_{21}^f\right)^\top & -I & \overline{D}^\top \\ \overline{C} & \overline{D} & -I \end{bmatrix}, \tag{29b}$$

$$A_S := \begin{bmatrix} B_2^fD_{\overline{\mathbf{B}}}\mathcal{D}_p(d_x) & B_2^fC_{\overline{\mathbf{B}}} & B_2^fD_{\overline{\mathbf{B}}}\mathcal{D}_p(C_x) \\ B_{\overline{\mathbf{B}}}\mathcal{D}_p(d_x) & A_{\overline{\mathbf{B}}} & B_{\overline{\mathbf{B}}}\mathcal{D}_p(C_x) \\ \mathcal{D}_p(b_x) & 0 & \mathcal{D}_p(A_x) \end{bmatrix}, \tag{29c}$$

$$T_A := \mathcal{D}\left(P_1^{\mathbf{B}}, \ldots, P_q^{\mathbf{B}}, P_x, \overline{d}_x, P, 0\right) \in \mathbb{R}^{p_T\times n_T}, \tag{29d}$$

$$T_B := \mathcal{D}\left(\left(F_1^{\mathbf{B}}\right)^\top, \ldots, \left(F_q^{\mathbf{B}}\right)^\top, A_x^\top, d_x^\top, A_S, 0\right) \in \mathbb{R}^{n_T\times m_T}, \tag{29e}$$

$$T_C := \mathcal{D}\left(\overline{P}_1^{\mathbf{B}}, \ldots, \overline{P}_q^{\mathbf{B}}, \overline{P}_x, \overline{P}^D, \overline{P}, \overline{d}_x - d_x\right) \in \mathbb{R}^{p_T\times m_T}. \tag{29f}$$

Then, we have that:

(**a**) *If the problem from* (28) *is feasible, then a solution can be found by the iterative procedure with guaranteed convergence from Algorithm 1 which involves the convex optimization problem from* (30)*, on the next page;*

(**b**) *If, at the proposed iteration's termination, we have that* $\left\|T_C^k - T_A^kT_B^k\right\|_* < \eta_2$*, then* A_x^k*,* b_x^k*,* C_x^k*,* d_x^k *and* $(F_i^{\mathbf{B}})^k$ *can be used to form* $\widehat{\mathbf{Q}}$ *as in* (20)-(23).

Proof. For point (**a**), see the Appendix. Point (**b**) follows directly from the fact that $\left\|T_C^k - T_A^kT_B^k\right\|_* < \eta_2 \ll 1$ indicates that the bilinear equality constraint belonging to the problem (given in the Appendix) that is equivalent to (28) has been satisfied for a feasible tuple, which designates a solution. $\square$

V. Numerical example

A. Design procedure

Consider a set of $\ell = 20$ subsystems which are interconnected in a network with a ring topology, as depicted in Fig. 2. The input-output model of each subsystem can be written as $\mathbf{y}_{(i \bmod \ell)+1} = \overline{\mathbf{G}}_y \mathbf{y}_{((i-1) \bmod \ell)+1} + \overline{\mathbf{G}}_u \mathbf{u}_{(i \bmod \ell)+1}, \forall i \in 1:\ell$, with $\overline{\mathbf{G}}_y(s) := \left[\begin{array}{c|c} A_y - sE_y & B_y \\ \hline C_y & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1-s & 0 & 1 \\ 0 & -1 & 1 \\ \hline 1 & 1 & 0 \end{array}\right]$ and $\overline{\mathbf{G}}_u(s) := \left[\begin{array}{c|c} A_u - sE_u & B_u \\ \hline C_u & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1-s & 0 & 11 \\ 0 & -1 & 4 \\ \hline 1 & 1 & 0 \end{array}\right]$. Define now $\Xi : \mathbb{R} \to \mathbb{R}^{\ell \times \ell}$, $\Xi(\kappa) := \mathcal{D}_\ell(\kappa) \begin{bmatrix} O_{1,\ell-1} & 1 \\ I_{\ell-1} & O_{\ell-1,1} \end{bmatrix}$, to get that

$$\overline{\mathbf{G}}(s) = \left[\begin{array}{cc|c} \mathcal{D}_\ell(A_y - sE_y) + \mathcal{D}_\ell(B_y)\Xi(1)\mathcal{D}_\ell(C_y) & \mathcal{D}_\ell(B_y)\Xi(1)\mathcal{D}_\ell(C_u) & O_{2\ell,\ell} \\ O_{2\ell} & \mathcal{D}_\ell(A_u - sE_u) & \mathcal{D}_\ell(B_u) \\ \hline \mathcal{D}_\ell(C_y) & \mathcal{D}_\ell(C_u) & O_\ell \end{array}\right] \tag{31}$$

is the network's TFM, which is improper, having a strongly stabilizable and detectable realization and whose resulting descriptor vector is the concatenation of the descriptor vectors belonging to the realizations of all $\overline{\mathbf{G}}_y$ and $\overline{\mathbf{G}}_u$ subsystems.

We aim to obtain a control law, for $\mathbf{\Phi}_\mathbf{K}, \mathbf{\Gamma}_\mathbf{K} \in \mathcal{R}_p^{1\times 1}$, with

$$\mathbf{u}_{(i \bmod \ell)+1} = \mathbf{\Phi}_\mathbf{K} \mathbf{u}_{((i-1) \bmod \ell)+1} + \mathbf{\Gamma}_\mathbf{K} \mathbf{y}_{(i \bmod \ell)+1}, \ \forall i \in 1:\ell. \tag{32}$$

Then, approximate $\overline{\mathbf{G}}(s)$ with $\mathbf{G}(s) := \mathcal{D}_\ell(\mathbf{\Psi})\Omega$, where $\Omega := \Xi(2) + I_\ell$ and $\mathbf{\Psi}(s) := \left[\begin{array}{c|c} A_\mathbf{\Psi} - sE_\mathbf{\Psi} & B_\mathbf{\Psi} \\ \hline C_\mathbf{\Psi} & D_\mathbf{\Psi} \end{array}\right] = \left[\begin{array}{cc|c} 1 & -s & 0 \\ 0 & 1 & -1 \\ \hline 1 & 0 & 4 \end{array}\right]$. Note that the latter realization is strongly stabilizable and detectable, such that $F_\mathbf{\Psi} = \begin{bmatrix} 1 & 5 \end{bmatrix}$ and $H_\mathbf{\Psi}^\top = \begin{bmatrix} -5 & -1 \end{bmatrix}$ are admissible feedbacks for it. Thus, we are able to express

$$\mathbf{G}(s) = \left[\begin{array}{c|c} \mathcal{D}_\ell(A_\mathbf{\Psi} - sE_\mathbf{\Psi}) & \mathcal{D}_\ell(B_\mathbf{\Psi})\Omega \\ \hline \mathcal{D}_\ell(C_\mathbf{\Psi}) & \mathcal{D}_\ell(D_\mathbf{\Psi})\Omega \end{array}\right], \tag{33}$$

with $F = \Omega^{-1}\mathcal{D}_\ell(F_\mathbf{\Psi})$ and $H = \mathcal{D}_\ell(H_\mathbf{\Psi})$ being admissible feedbacks. Obtaining stable NRCFs for (31) and (33) via Lemma III.1, we use them to get the maximum stability radius $b_{opt} > 0.9925$ of $\mathbf{G}$ (see [22]) and to compute an upper bound for some $\overline{\mathbf{Q}} \in \mathcal{RH}_\infty^{\ell\times\ell}$ (see Chapter 8 of [24]) denoted $\mu(\overline{\mathbf{Q}}) < 0.5609$ of the directed gap metric between $\mathbf{G}$ and $\overline{\mathbf{G}}$, as given in (4) from [22]. Then, we set $\epsilon = 0.7 > \mu(\overline{\mathbf{Q}})$ and we get, by the same arguments as in the proof of Lemma 2 from [22] applied for $\overline{\mathbf{Q}} \in \mathcal{RH}_\infty^{\ell\times\ell}$, that $\overline{\mathbf{G}} \in \mathcal{C}_\mathbf{G}^\epsilon$. Note that $\mu(\overline{\mathbf{Q}}) < 1$ implies $\det \overline{\mathbf{Q}} \not\equiv 0$. Otherwise, $\exists\, \mathbf{v} \in \operatorname{Ker} \overline{\mathbf{Q}} \cap \mathcal{RH}_\infty^{\ell\times 1}$ with $\mathcal{H}_2$ norm equal to 1 which can be used to obtain that $\mu(\overline{\mathbf{Q}}) \geq 1$.

Use now the realization from (33) and F to compute a stable RCF as in (16). With this stable RCF and H, employ Proposition III.2 to compute $(\mathbf{N}^\epsilon, \widetilde{\mathbf{N}}^\epsilon, \mathbf{M}^\epsilon, \widetilde{\mathbf{M}}^\epsilon, \mathbf{X}^\epsilon, \widetilde{\mathbf{X}}^\epsilon, \mathbf{Y}^\epsilon, \widetilde{\mathbf{Y}}^\epsilon)$ via (18a)-(18b) and $\mathbf{T}_1^\epsilon$, $\mathbf{T}_2^\epsilon$ and $\mathbf{T}_3^\epsilon$ as in (19a)-(19c). Then,

$$\mathbf{K} = (\Omega\widetilde{\mathbf{Y}}^\epsilon + \widetilde{\mathbf{Q}}\widetilde{\mathbf{N}}^\epsilon)^{-1}(\Omega\widetilde{\mathbf{X}}^\epsilon + \widetilde{\mathbf{Q}}\widetilde{\mathbf{M}}^\epsilon), \tag{34a}$$

$$\mathcal{F}_\ell(\mathbf{T}^\epsilon, \mathbf{K}) = \mathbf{T}_1^\epsilon + (\mathbf{T}_2^\epsilon\Omega^{-1})\widetilde{\mathbf{Q}}\mathbf{T}_3^\epsilon, \tag{34b}$$

having defined $\widetilde{\mathbf{Q}} := \Omega\mathbf{Q}$. Note that $(\mathbf{N}^\epsilon\Omega^{-1}, \widetilde{\mathbf{N}}^\epsilon, \mathbf{M}^\epsilon\Omega^{-1}, \widetilde{\mathbf{M}}^\epsilon, \mathbf{X}^\epsilon, \Omega\widetilde{\mathbf{X}}^\epsilon, \mathbf{Y}^\epsilon, \Omega\widetilde{\mathbf{Y}}^\epsilon)$ is also a DCF over $\mathcal{RH}_\infty$ of $\mathbf{T}_{22}^\epsilon = \mathbf{G}$, as all 8 TFMs are stable and they satisfy (5), with the added benefit of $\Omega\widetilde{\mathbf{Y}}^\epsilon, \widetilde{\mathbf{N}}^\epsilon \in \widehat{\mathcal{S}}_{(\Xi(1)+I_\ell)}$ and of $\Omega\widetilde{\mathbf{X}}^\epsilon, \widetilde{\mathbf{M}}^\epsilon \in \widehat{\mathcal{S}}_{I_\ell}$.

We will employ this new DCF over $\mathcal{RH}_\infty$ to form the controller as in (34a) and optimize the $\mathcal{H}_\infty$ norm of (34b). The control laws in (32) can be obtained from a controller's NRF pair with $\mathbf{\Phi} = \mathcal{D}_\ell(\mathbf{\Phi}_\mathbf{K})\Xi(1) \in \widehat{\mathcal{S}}_{\Xi(1)}$ and $\mathbf{\Gamma} = \mathcal{D}_\ell(\mathbf{\Gamma}_\mathbf{K}) \in \widehat{\mathcal{S}}_{I_\ell}$. By Proposition III.1, a solution to (12a) is $\widetilde{\mathbf{Q}}_0 = 0$ and note that a stable basis for the null-space from (12b) is expressed as in (21) with $q = \ell$ and $\mathbf{B}_i = \mathbf{N}_{\mathbf{B}_i} = \widetilde{e}_{1+(\ell+1)(i-1)}$, $\forall i \in 1:\ell$, where $\widetilde{e}_i$ is the i^{th} vector of the canonical basis of $\mathbb{R}^{\ell^2\times 1}$.

We now run Algorithm 1 with MOSEK [27], called through MATLAB via YALMIP [28]. A comparison with other techniques from literature is given in Tab. I, located at the bottom of the next page, and their computational performance will be discussed in the next subsection. Taking $\widetilde{\mathbf{Q}}(s) = \mathcal{D}_\ell(5.9844)$ produces, $\forall i \in 1:\ell$, the distributed control laws of type (32)

$$\mathbf{u}_{(i \bmod \ell)+1} = -2\mathbf{u}_{((i-1) \bmod \ell)+1} + \frac{64.11s + 257.4}{s+4}\mathbf{y}_{(i \bmod \ell)+1}.$$

Remark V.1. *Let $\widetilde{\mathbf{K}} := (I - \widetilde{\mathbf{\Phi}})^{-1}\widetilde{\mathbf{\Gamma}}$, where $\widetilde{\mathbf{\Phi}} := -\Xi(2)$ and $\widetilde{\mathbf{\Gamma}} := 64I_\ell$, and notice that $\begin{bmatrix} \widetilde{\mathbf{\Phi}} & \widetilde{\mathbf{\Gamma}} \end{bmatrix}$ internally stabilizes $\begin{bmatrix} I & \overline{\mathbf{G}}^\top \end{bmatrix}^\top$. Then, the distributed implementation* (9) *of the approximated distributed controller $\widetilde{\mathbf{K}}$ internally stabilizes $\overline{\mathbf{G}}$ even in the presence of communication disturbance (see [9] and recall $\mathbf{b}_{(i \bmod \ell)+1}$ from Fig. 2). Moreover, the control laws from* (9) *implemented with either $(\mathbf{\Phi}, \mathbf{\Gamma})$ or $(\widetilde{\mathbf{\Phi}}, \widetilde{\mathbf{\Gamma}})$ stabilize all $\overline{\mathbf{G}}_\mathbf{\Delta} \in \mathcal{C}_{\overline{\mathbf{G}}}^\delta$ and $\delta = 0.8968$, indicating satisfactory robustness.*

$$\mathbf{T}_1^\epsilon + \mathbf{T}_2^\epsilon\mathbf{Q}\mathbf{T}_3^\epsilon = \left[\begin{array}{c|c} \overline{A} & \overline{B} \\ \hline \overline{C} & \overline{D} \end{array}\right] = \left[\begin{array}{ccc|c} A^f + B_2^f D_{\overline{\mathbf{B}}}\mathcal{D}_p(d_x)C_2^f & B_2^f C_{\overline{\mathbf{B}}} & B_2^f D_{\overline{\mathbf{B}}}\mathcal{D}_p(C_x) & B_1^f + B_2^f D_{\overline{\mathbf{B}}}\mathcal{D}_p(d_x)D_{21}^f \\ B_{\overline{\mathbf{B}}}\mathcal{D}_p(d_x)C_2^f & A_{\overline{\mathbf{B}}} & B_{\overline{\mathbf{B}}}\mathcal{D}_p(C_x) & B_{\overline{\mathbf{B}}}\mathcal{D}_p(d_x)D_{21}^f \\ \mathcal{D}_p(b_x)C_2^f & 0 & \mathcal{D}_p(A_x) & \mathcal{D}_p(b_x)D_{21}^f \\ \hline C_1^f + D_{12}^f D_{\overline{\mathbf{B}}}\mathcal{D}_p(d_x)C_2^f & D_{12}^f C_{\overline{\mathbf{B}}} & D_{12}^f D_{\overline{\mathbf{B}}}\mathcal{D}_p(C_x) & D_{11}^f + D_{12}^f D_{\overline{\mathbf{B}}}\mathcal{D}_p(d_x)D_{21}^f \end{array}\right] \tag{26}$$

$$\mathcal{M}(X, Y, \Theta) := \begin{cases} \displaystyle\min_{A_x, b_x, C_x, d_x, \overline{d}_x, F_i^\mathbf{B}, P, \overline{P}, \overline{P}^D, P_x, \overline{P}_x, P_i^\mathbf{B}, \overline{P}_i^\mathbf{B}} \left\| \begin{bmatrix} T_C + XY - T_AY - XT_B & T_A - X \\ T_B - Y & I_{n_T} \end{bmatrix} \right\|_*, \\ \text{s.t.} \begin{cases} (\widehat{Z}_1^k)^\top \widehat{Z}_1^k + \sum_{i=1}^q d_{xi}\big((\widehat{Z}_1^k)^\top \widehat{Z}_{2i}^k + (\widehat{Z}_{2i}^k)^\top \widehat{Z}_1^k\big) + \sum_{i=1}^q \overline{P}_{ii}^D (\widehat{Z}_{2i}^k)^\top \widehat{Z}_{2i}^k + \\ \qquad + \sum_{i=1}^{q-1}\sum_{j=i+1}^q \overline{P}_{ij}^D\big((\widehat{Z}_{2i}^k)^\top \widehat{Z}_{2j}^k + (\widehat{Z}_{2j}^k)^\top \widehat{Z}_{2i}^k\big) \succ 0, \ \forall k \in 1:N_Z, \\ \overline{P}^D = (\overline{P}^D)^\top, \ P = P^\top \succ 0, \ -G \succ 0, \ P_x = P_x^\top \succ 0, \ -2\text{sym}(\overline{P}_x) \succ 0, \\ P_i^\mathbf{B} = (P_i^\mathbf{B})^\top \succ 0, \ \forall i \in 1:q, \ -2\text{sym}\big(P_i^\mathbf{B}(A_i^\mathbf{B})^\top + \overline{P}_i^\mathbf{B}(B_i^\mathbf{B})^\top\big) \succ 0, \ \forall i \in 1:q, \ \Theta = 0. \end{cases} \end{cases} \tag{30}$$

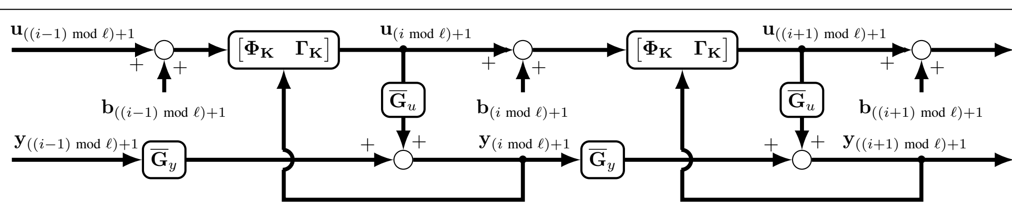

Fig. 2. Interconnection between the network's subsystems and the distributed subcontrollers

B. Computational performance

We conclude this section by presenting a comparative discussion of the results showcased in Tab. I. With respect to our proposed procedure, inspired by [14], we may state that:

1. Algorithm 2 from [14] is slightly more computationally demanding, due to optimizing over all decision variables during each iteration. However, this extra degree of freedom comes at the major cost of guaranteed convergence.
2. Although the individual iterations of Algorithm 1 from [25] are significantly less costly and convergence is initially quite rapid, the latter tapers off on later iterations, similarly to Fig. 2 in [25]. Convergence can be sped up by the judicious choice of ρ_1 and ρ_2 form (3.4) of [25], yet our approach bypasses this empiric decision via the benefits of optimizing the trace heuristic (see [29]).
3. Algorithm 1 from [26] is based upon the same trace optimization heuristic proposed in [29] as our procedure, yet it requires an *explicit* eigenvalue decomposition and orthonormal eigenvector computation at every iteration. For large-scale problems (such as our numerical example) this may prove unreliable, with the accumulation of computational errors noticeably hampering convergence.

VI. Conclusion

In this paper, we have shown that the distributed control of a network (having a possibly improper TFM) can be tackled by imposing constraints upon affine expressions of the Youla parameter. A procedure is given on how to relax this problem, which reduces to solving a structurally-constrained $\mathcal{H}_\infty$ norm contraction. The latter is approached through a convex and iterative optimization algorithm with guaranteed convergence.

Appendix

Proof of Proposition III.1: Let there exist $\mathbf{Q}_0 \in \mathcal{RH}_\infty^{m\times p}$ so that $\widetilde{\mathbf{X}}+\mathbf{Q}_0\widetilde{\mathbf{M}} \in \widehat{\mathcal{S}}_{\mathcal{X}}$ and $\widetilde{\mathbf{Y}}+\mathbf{Q}_0\widetilde{\mathbf{N}} \in \widehat{\mathcal{S}}_{\widehat{\mathcal{Y}}}$. They are equivalent to $F_{\mathcal{X}}\mathrm{vec}(\widetilde{\mathbf{X}}+I\mathbf{Q}_0\widetilde{\mathbf{M}}) \equiv 0$ and $F_{\widehat{\mathcal{Y}}}\mathrm{vec}(\widetilde{\mathbf{Y}}+I\mathbf{Q}_0\widetilde{\mathbf{N}}) \equiv 0$. Using the properties of the vectorization operator (see Lemma 1 in [6]), we retrieve (12a). Pick any $\widehat{\mathbf{Q}} \in \mathcal{RH}_\infty^{m\times p}$ which satisfies (12b) and note that, when replacing $\mathbf{Q}_0$ with $\mathbf{Q} := \mathbf{Q}_0+\widehat{\mathbf{Q}}$ in (12a), the identity with 0 from (12a) will hold. Also, ensuring (12c) is sufficient for the controller from (7) to be well-posed, while ensuring (12d) is sufficient for $\mathbf{\Gamma}$ and $\mathbf{\Phi}$ to be both well-posed and proper. Finally, the sparsity structures of $\mathbf{\Gamma}$ and $\mathbf{\Phi}$ follow from those of $\widetilde{\mathbf{X}}+\mathbf{Q}\widetilde{\mathbf{M}}$ and $\widetilde{\mathbf{Y}}+\mathbf{Q}\widetilde{\mathbf{N}}$, respectively, by the way they are defined in (11a)-(11b). $\square$

Proof of Lemma III.1: To prove point $(\mathbf{a})$, define $\widehat{A}_r := E_r^{-1}A_r$, $\widehat{B}_r := E_r^{-1}B_r$, $\widehat{X}_r := E_r^\top X_r E_r$ to rewrite (3) as

$$\widehat{X}_r\widehat{A}_r + \widehat{A}_r^\top\widehat{X}_r + C_r^\top C_r - (\widehat{X}_r\widehat{B}_r + C_r^\top D_r)\times \times(D_r^\top D_r)^{-1}(\widehat{B}_r^\top\widehat{X}_r + D_r^\top C_r) = 0, \tag{35}$$

which is a standard continuous-time algebraic Riccati equation (see Chapter 13 in [23]). Recall now that $\Lambda(A_r - sE_r) \subset \mathbb{C}^-$ and, thus, $\Lambda(\widehat{A}_r - sI) \subset \mathbb{C}^-$. Then, both $\begin{bmatrix}\widehat{A}_r - sI & \widehat{B}_r\end{bmatrix}$ and $\begin{bmatrix}\widehat{A}_r^\top - sI & C_r^\top\end{bmatrix}$ have full row rank $\forall s \in \mathbb{C}\backslash\mathbb{C}^-$ which, by section 3.2 of [23], means that $(\widehat{A}_r - sI, \widehat{B}_r)$ is stabilizable and $(C_r, \widehat{A}_r - sI)$ is detectable. Note that $\begin{bmatrix}\mathbf{N}^\top(s) & \mathbf{M}^\top(s)\end{bmatrix}^\top = C_r(sI-\widehat{A}_r)^{-1}\widehat{B}_r + D_r$ has full column rank $\forall s \in j\mathbb{R}\cup\{\infty\}$, or else there cannot exist $\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}}$ stable so that $\widetilde{\mathbf{Y}}\mathbf{M} - \widetilde{\mathbf{X}}\mathbf{N} = I$. Then, by point (a) in Corollary 13.23 of [23], (35) has a stabilizing solution, $\widehat{X}_r$. Thus, (3) has a stabilizing solution, $X_r = (E_r^\top)^{-1}\widehat{X}_r E_r^{-1}$, and its stabilizing feedback equals that of (35), $\widehat{F}_r := -(D_r^\top D_r)^{-1}(\widehat{B}_r^\top\widehat{X}_r + D_r^\top C_r) = F_r$. $\square$

Proof of Theorem III.1: To prove point $(\mathbf{a})$, define first $\mathbf{T} := \left[\begin{array}{c|c}\mathbf{T}_{11} & \mathbf{T}_{12}\\ \hline \mathbf{T}_{21} & \mathbf{T}_{22}\end{array}\right] = \left[\begin{array}{cc|c}0 & -\widehat{\mathbf{M}}^{-1} & \widehat{\mathbf{M}}^{-1}\\ \hline I & -\mathbf{G} & \mathbf{G}\end{array}\right]$, where $\widehat{\mathbf{M}}^{-1} = \mathbf{G}_0\mathbf{M}^{-1}$ and $\mathbf{G}_0$ is expressed as in Lemma III.1. Moreover, we have from (6b) in Theorem II.1 that $\mathbf{M}^{-1} = \left[\begin{array}{c|c}A - sE & B\\ \hline -F & I\end{array}\right]$. Expressing $\mathbf{T} = \left[\begin{array}{c|c}\mathbf{G}_0 & 0\\ \hline 0 & I\end{array}\right]\left[\begin{array}{cc|c}0 & -\mathbf{M}^{-1} & \mathbf{M}^{-1}\\ \hline I & -\mathbf{G} & \mathbf{G}\end{array}\right]$ and noticing that $\mathbf{T}^\epsilon = \begin{bmatrix}\epsilon I & 0\\ 0 & I\end{bmatrix}\mathbf{T}$, for an $\epsilon \in (0,1]$, we obtain the realization given in (17). Since $\mathbf{G}$ is given by a strongly stabilizable and detectable realization (2), then it is always possible to find $\widetilde{F}$ and $\widetilde{H}$ so that $A + B\widetilde{F} - sE$ and $A + \widetilde{H}C - sE$ are admissible. Thus, defining $\widehat{F} := \begin{bmatrix}0 & \widetilde{F}\end{bmatrix}$ and $\widehat{H} := \begin{bmatrix}0 & \widetilde{H}^\top\end{bmatrix}^\top$, we extract the realization of $\mathbf{T}_{22}^\epsilon$ from (17), $\mathbf{T}_{22}^\epsilon = \left[\begin{array}{c|c}A_{22} - sE_{22} & B_{22}\\ \hline C_{22} & D_{22}\end{array}\right] := \left[\begin{array}{cc|c}A_r - sE_r & -B_rF & B_r\\ 0 & A - sE & B\\ \hline 0 & C & D\end{array}\right]$, to get that $A_{22} + B_{22}\widehat{F} - sE_{22}$ and $A_{22} + \widehat{H}C_{22} - sE_{22}$ are both admissible, since $\Lambda(A_r - sE_r) \subset \mathbb{C}^-$. Therefore, $\widehat{F}$ and $\widehat{H}$ can be used, as in Theorem II.1, in order to express the class of stabilizing controllers via a DCF over $\mathcal{RH}_\infty$ of $\mathbf{T}_{22}^\epsilon$.

To prove point $(\mathbf{b})$, begin by defining the system

$$\overline{\mathbf{T}} := \left[\begin{array}{cc|cc|c}A_r - sE_r & -B_rF & 0 & -B_r & B_r\\ 0 & A - sE & 0 & -B & B\\ \hline -H_rF_r & -H_rF & 0 & -H_r & H_r\\ \hdashline 0 & 0 & 0 & 0 & I\\ 0 & C & I & -D & D\end{array}\right] = \left[\begin{array}{c|c}\overline{\mathbf{T}}_{11} & \overline{\mathbf{T}}_{12}\\ \hline \overline{\mathbf{T}}_{21} & \overline{\mathbf{T}}_{22}\end{array}\right] \tag{36}$$

and by considering the class of TFMs expressed through $\mathcal{F}_u\left(\overline{\mathbf{T}}, \begin{bmatrix}\mathbf{\Delta}_{\widehat{\mathbf{N}}}\\ \mathbf{\Delta}_{\widehat{\mathbf{M}}}\end{bmatrix}\right) := \overline{\mathbf{T}}_{22} + \overline{\mathbf{T}}_{21}\begin{bmatrix}\mathbf{\Delta}_{\widehat{\mathbf{N}}}\\ \mathbf{\Delta}_{\widehat{\mathbf{M}}}\end{bmatrix}\left(I - \overline{\mathbf{T}}_{11}\begin{bmatrix}\mathbf{\Delta}_{\widehat{\mathbf{N}}}\\ \mathbf{\Delta}_{\widehat{\mathbf{M}}}\end{bmatrix}\right)^{-1}\overline{\mathbf{T}}_{12}$, with $\mathbf{\Delta}_{\widehat{\mathbf{N}}}$ and $\mathbf{\Delta}_{\widehat{\mathbf{M}}}$ as in (15). Denoting now the class of TFMs $\mathbf{G}_{\mathbf{\Delta}} := \left(\widehat{\mathbf{N}} + \mathbf{\Delta}_{\widehat{\mathbf{N}}}\right)\left(\widehat{\mathbf{M}} + \mathbf{\Delta}_{\widehat{\mathbf{M}}}\right)^{-1}$, it is straightforward to check that $\mathcal{F}_u\left(\overline{\mathbf{T}}, \begin{bmatrix}\mathbf{\Delta}_{\widehat{\mathbf{N}}}\\ \mathbf{\Delta}_{\widehat{\mathbf{M}}}\end{bmatrix}\right) = \begin{bmatrix}I & \mathbf{G}_{\mathbf{\Delta}}^\top\end{bmatrix}^\top$. Thus, the proof of point $(\mathbf{b})$ boils down to applying the Small Gain Theorem, as formulated in Chapter 8 of [30], to confirm robust stability.

Note that, since the realization of $\mathbf{T}_{22}^\epsilon$ from (17) is strongly stabilizable and detectable, then so is the one belonging to $\overline{\mathbf{T}}_{22}^\epsilon$ in (36). Now, if $(\mathbf{\Phi}, \mathbf{\Gamma})$ is an NRF implementation of $\mathbf{K}$ as in (11a)-(11b) and $\mathbf{K}$ stabilizes $\mathbf{G}$, then $\begin{bmatrix}\mathbf{\Phi} & \mathbf{\Gamma}\end{bmatrix}$ stabilizes $\overline{\mathbf{T}}_{22} = \begin{bmatrix}I & \mathbf{G}^\top\end{bmatrix}^\top$ (see [9]). Since the latter's realization in (36) is strongly stabilizable and detectable, then $\begin{bmatrix}\mathbf{\Phi} & \mathbf{\Gamma}\end{bmatrix}$ stabilizes $\overline{\mathbf{T}}$ (as in Theorem II.1). Finally, it is straightforward to check that $\mathcal{F}_\ell(\mathbf{T}^\epsilon, \mathbf{K}) = \epsilon\mathcal{F}_\ell\left(\overline{\mathbf{T}}, \begin{bmatrix}\mathbf{\Phi} & \mathbf{\Gamma}\end{bmatrix}\right)$. If $\|\mathcal{F}_\ell(\mathbf{T}^\epsilon, \mathbf{K})\|_\infty \leq 1$ then $\left\|\mathcal{F}_\ell\left(\overline{\mathbf{T}}, \begin{bmatrix}\mathbf{\Phi} & \mathbf{\Gamma}\end{bmatrix}\right)\right\|_\infty \leq \frac{1}{\epsilon}$ and, by applying point (b) of Theorem 8.1 in [30], it follows that the closed-loop interconnection between $\begin{bmatrix}\mathbf{\Phi} & \mathbf{\Gamma}\end{bmatrix}$ and $\mathcal{F}_u\left(\overline{\mathbf{T}}, \begin{bmatrix}\mathbf{\Delta}_{\widehat{\mathbf{N}}}\\ \mathbf{\Delta}_{\widehat{\mathbf{M}}}\end{bmatrix}\right) = \begin{bmatrix}I & \mathbf{G}_{\mathbf{\Delta}}^\top\end{bmatrix}^\top$ will be internally stable and well-posed for any $\mathbf{G}_{\mathbf{\Delta}} \in \mathcal{C}_{\mathbf{G}}^\epsilon$.

Employed procedure	Guaranteed convergence	Runtime	Solution	$\|T_C - T_AT_B\|_*$ at convergence
Alg. 1 (Alg. 1 in [14])	Yes	18.41 sec	$\widetilde{\mathbf{Q}}(s) = \mathcal{D}_\ell(5.9844)$	2.8×10^{-12}
Alg. 2 in [14]	No	20.49 sec	$\widetilde{\mathbf{Q}}(s) = \mathcal{D}_\ell(5.9844)$	2.1×10^{-9}
Alg. 1 in [25]	Yes	timed out after 900 sec	$\widetilde{\mathbf{Q}}(s) = \mathcal{D}_\ell(6.0143)$ at timeout	2.3×10^{2} at timeout
Alg. 1 in [26]	Yes	timed out after 900 sec	$\widetilde{\mathbf{Q}}(s) = \mathcal{D}_\ell(5.7355)$ at timeout	3.2×10^{1} at timeout

TABLE I. Comparison between algorithms which solve convex relaxations of (28)

As shown in the proof of the main result from [9], this ensures that the control laws from (9) will stabilize any $\mathbf{G}_{\boldsymbol{\Delta}} \in \mathcal{C}^{\epsilon}_{\mathbf{G}}$. $\square$

Proof of Proposition III.2: Define first $\widehat{F} := \begin{bmatrix} 0 & F \end{bmatrix}$ and $\widehat{H} := \begin{bmatrix} 0 & H^\top \end{bmatrix}^\top$ and employ these two feedbacks to write, via (6a)-(6b), a DCF of $\mathbf{T}^{\epsilon}_{22}$ from (17). This factorization is indeed a DCF over $\mathcal{RH}_\infty$ due to the fact that $A + BF - sE$ and $A + HC - sE$ are admissible and $\Lambda(A_r - sE_r) \subset \mathbb{C}^-$. The identities from (18a)-(18b) and (19a)-(19c) follow by writing the realizations given by (6a)-(6b) and by (8a)-(8c) in Theorem II.1, and then eliminating all unobservable modes. $\square$

Proof of Theorem IV.1: To prove point $(\mathbf{a})$, we first ensure that the realization from (23) is stable by imposing that $A_x^\top$ and $\left(A_i^{\mathbf{B}} + B_i^{\mathbf{B}} F_i^{\mathbf{B}}\right)^\top$, $\forall\ i \in 1:q$, have eigenvalues only in $\mathbb{C}^-$, along with $A^\top_{\mathbf{B}}$ via (20)-(22). These conditions are equivalent to $\exists\ P_x = P_x^\top \succ 0$ and $P_i^{\mathbf{B}} = \left(P_i^{\mathbf{B}}\right)^\top \succ 0$, $i \in 1:q$, such that $-2\mathrm{sym}(A_x P_x) \succ 0$ and $-2\mathrm{sym}\left(A_i^{\mathbf{B}} P_i^{\mathbf{B}} + B_i^{\mathbf{B}} F_i^{\mathbf{B}} P_i^{\mathbf{B}}\right) \succ 0$, $\forall i \in 1:q$. To remedy the bilinearity induced by $P_x A_x^\top$ and $P_i^{\mathbf{B}} \left(F_i^{\mathbf{B}}\right)^\top$, define $\overline{P}_x := P_x A_x^\top$ and $\overline{P}_i^{\mathbf{B}} := P_i^{\mathbf{B}}(F_i^{\mathbf{B}})^\top$, with $i \in 1:q$, and rewrite the inequalities as $-2\mathrm{sym}(\overline{P}_x) \succ 0$ and $-2\mathrm{sym}\left(P_i^{\mathbf{B}} \left(A_i^{\mathbf{B}}\right)^\top + \overline{P}_i^{\mathbf{B}} \left(B_i^{\mathbf{B}}\right)^\top\right) \succ 0,\ \forall i \in 1:q$.

If these new affine inequalities are satisfied, then due to $\Lambda\left(A^f - sI\right) \subset \mathbb{C}^-$ and to $C_2^f \left(sI - A^f\right)^{-1} B_2^f \equiv 0$, it follows that $\overline{A}$ from (26) has $\Lambda\left(\overline{A} - sI\right) \subset \mathbb{C}^-$. By the equivalence of points (i) and (vii) from Corollary 12.3 in [30], we have that $\|\mathbf{T}_1^\epsilon + \mathbf{T}_2^\epsilon \mathbf{Q} \mathbf{T}_3^\epsilon\|_\infty < 1$ if and only if $\exists\ P = P^\top \succ 0$ such that $-\begin{bmatrix} 2\mathrm{sym}(P\overline{A}) & P\overline{B} & \overline{C}^\top \\ \overline{B}^\top P & -I & \overline{D}^\top \\ \overline{C} & \overline{D} & -I \end{bmatrix} \succ 0$, which contains bilinear products of P with $\overline{A}$ and $\overline{B}$, thus leading to nonconvex optimization. To obtain an affine expression, define A_S as in (29c), in order to introduce $\overline{P} := PA_S$. With this new matrix and the four matrices defined in (29a), notice that $P\overline{A} = \overline{PA}^f + \overline{PC}_2^f$ and that $P\overline{B} = \overline{PB}_1^f + \overline{PD}_{21}^f$. The norm condition is equivalent to $-G \succ 0$, with G from (29b) being affine in all variables.

Recall the inequalities from (28), that contain bilinear terms, to denote $\overline{P}^D := \overline{d}_x d_x^\top$, while imposing that $\overline{d}_x - d_x = 0$. The latter will also induce the additional constraint $\overline{P}^D = \left(\overline{P}^D\right)^\top$, from which we obtain the N_Z LMIs given in (30). Form now the matrices from (29c)-(29f) to note that (28) is equivalent to

$$\begin{cases} (\widehat{Z}_1^k)^\top \widehat{Z}_1^k + \sum\limits_{i\in 1:q} d_{xi}\left((\widehat{Z}_1^k)^\top \widehat{Z}_{2i}^k + (\widehat{Z}_{2i}^k)^\top \widehat{Z}_1^k\right) + \sum\limits_{i \in 1:q} \overline{P}_{ii}^D (\widehat{Z}_{2i}^k)^\top \widehat{Z}_{2i}^k + \\ + \sum_{i=1}^{q-1} \sum_{j=i+1}^{q} \overline{P}_{ij}^D \left((\widehat{Z}_{2i}^k)^\top \widehat{Z}_{2j}^k + (\widehat{Z}_{2j}^k)^\top \widehat{Z}_{2i}^k\right) \succ 0, \forall k \in 1: N_Z, \\ \overline{P}^D = \left(\overline{P}^D\right)^\top,\ P = P^\top \succ 0,\ -G \succ 0,\ P_x = P_x^\top \succ 0, \\ -2\mathrm{sym}(\overline{P}_x) \succ 0,\ P_i^{\mathbf{B}} = (P_i^{\mathbf{B}})^\top \succ 0,\ \forall i \in 1:q, \\ -2\mathrm{sym}\left(P_i^{\mathbf{B}} \left(A_i^{\mathbf{B}}\right)^\top + \overline{P}_i^{\mathbf{B}} \left(B_i^{\mathbf{B}}\right)^\top\right) \succ 0,\ \forall i \in 1:q,\ T_A T_B = T_C. \end{cases} \tag{37}$$

By selecting an artificial scalar $\gamma > 0$ as the cost function and by applying Theorem 1 in [14], we get that (37) is equivalent to the problem in which $T_A I_{n_T} T_B = T_C$ is replaced by $\mathrm{rank} \begin{bmatrix} T_C + XY - T_A Y - X T_B & T_A - X \\ T_B - Y & I_{n_T} \end{bmatrix} = \mathrm{rank}\ I_{n_T}$, for any matrices X and Y. Therefore, by applying Theorem 2 in [14] with a regularization parameter $\lambda > 0$, adapting Algorithm 1 in [14] for the resulting problem, scaling its cost function by $1/\lambda$ and then taking $\lambda \to \infty$, we obtain Algorithm 1, which solves (30) at each iteration. If the initialization is successful (the LMI system along with $\overline{d}_x - d_x = 0$ must be feasible for the original BMI system to be feasible), we then set $\Theta^1 = T_B - T_B^0$ and we employ Theorem 3 in [14] for our algorithm (with the adapted cost function), which guarantees its convergence. $\square$

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