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# Hybrid Observer-based Asymptotic Stabilization of Non-uniformly Observable Systems: a Case Study

M. Maghenem   W. Pasillas-Lépine   A. Loria   M. Aguado-Rojas

**Abstract**—We analyze a case study of output-feedback stabilization of an unobservable equilibrium. The problem involves achieving two conflicting goals: to estimate the unmeasured states, for which the system must *a priori* operate in an oscillatory regime, and to stabilize an equilibrium asymptotically. To overcome this quandary we propose a novel hybrid controller that generates a piecewise constant dwindling reference designed to make the system operate in an oscillatory, yet asymptotically stable, regime. The controller relies on a previously proposed switching observer, for which we provide an original analysis of exponential convergence.

**Index Terms**—Non-uniformly observable systems, observers, hybrid control, ABS.

## I. INTRODUCTION

Observer-based output-feedback stabilization of systems that are unobservable for certain inputs is a problem that remains largely open in its full generality—see [1] for a recent account of the literature. Moreover, beyond its academic interest [2]–[4], it is well justified by engineering applications in which the system is not observable at the target equilibrium [5]. Some examples include sensorless motor control [6], bioreactor systems [7], and automotive applications [8].

We address this problem for the system

$$\dot{z}_1 = -az_1z_2 + u \quad (1a)$$

$$\dot{z}_2 = (cz_2 + d)z_1, \quad z_1, z_2 \in \mathbb{R}, \quad (1b)$$

where  $a, c$ , and  $d > 0$ ,  $u$  is the control input,  $z_2$  is the main state of interest to be controlled, which is unmeasurable, and  $y = Cz = [1 \ 0]z = z_1$  is the measured output.

The system defined by Eqs. (1) is representative of a class of systems well-studied in the context of observer design, that of bilinear systems [9], [10]. Moreover, Eqs. (1) model the dynamics of the so-called extended-braking stiffness (XBS) [8], [11] in automotive braking systems. The XBS, which is represented by  $z_2$  in (1b), is a state whose regulation translates into maximizing the braking force [12]. The XBS, however, is not measurable and, as it is clear from (1), the system loses observability for inputs that make  $z_1 \rightarrow 0$ , which is the control goal.

We propose an original observer-based output-feedback hybrid controller that uses state estimates provided by an observer originally proposed in [8] and further developed in [11] and [13]. Essentially, it is of Luenberger type, with a

gain designed to switch between values that are appropriate for either of two dynamics, depending on the sign of  $z_1$ . To overcome the unobservability obstacle, the observer hinges on making the output  $z_1$  oscillate persistently. This, however, is in clear conflict with the goal of stabilizing the origin  $\{z = 0\}$ . To simultaneously achieve the two competing processes of estimation (which requires persistent oscillations) and asymptotic stabilization, we design a piecewise-constant switching reference (for  $z_1$ ), whose amplitude is decreased by half every time the estimation errors decrease by a certain amount. The estimation errors are not measurable, but one of our results is an original proof of exponential stability for the estimation dynamics under the observer of [8], [11], with a guaranteed known convergence rate, which may be used to determine the switching instants.

Our main contribution, however, is the design of an original hybrid controller that makes the origin attract the solutions semi-globally and asymptotically. To some extent, our stabilization method is reminiscent of other switching strategies, *e.g.*, in which the input switches between non-singular inputs to enhance observability [14]. Other efficient methods for semi-global stabilization include perturbing the input with a term that is proportional to the distance to the target [1].

This Technical Note builds upon a long-standing work by the second author, devoted entirely to the observer-design problem [8], [11], [13], [15]; and on [16], which is a preliminary version of this Note—the latter reference contains numerical simulations in place of the proofs of our main results. We start our exposition by revisiting our observer and giving an original statement of exponential stability.

## II. A SWITCHED-OBSERVER REVISITED

Let  $\hat{z} := [\hat{z}_1 \ \hat{z}_2]^\top$  denote the estimate of  $z := [z_1 \ z_2]^\top$  and consider the Luenberger-type observer—see [8],

$$\dot{\hat{z}}_1 = -a\hat{z}_1\hat{z}_2 - u + k_1(z_1)z_1(z_1 - \hat{z}_1) \quad (2a)$$

$$\dot{\hat{z}}_2 = cz_1\hat{z}_2 + dz_1 + k_2(z_1)z_1(z_1 - \hat{z}_1), \quad (2b)$$

where  $k_1, k_2 : \mathbb{R} \rightarrow \mathbb{R}$  are functions to be defined. Then, the dynamics of the estimation error  $\tilde{z}_i := \hat{z}_i - z_i$  is given by

$$\begin{bmatrix} \dot{\tilde{z}}_1 \\ \dot{\tilde{z}}_2 \end{bmatrix} = z_1(t) \begin{bmatrix} -k_1(z_1(t)) & -a \\ -k_2(z_1(t)) & c \end{bmatrix} \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix}, \quad (3)$$

which is a linear system with state  $\tilde{z}$ , and depends on time through the measurable output trajectory  $t \mapsto z_1(t)$ . That is, the latter is part of a solution to (1), with initial conditions  $(t_o, z_o) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2$ , and is defined on  $[t_o, t_f)$  for any  $t_o \geq 0$  and some  $t_f \leq \infty$ . For the sake of argument, we assume in this section that  $t_f = +\infty$ —*cf.* Remark 2.

M. Maghenem is with GIPSA-Lab, University of Grenoble Alpes, CNRS, France (e-mail: mohamed.maghenem@cnrs.fr). W. Pasillas-Lépine is with L2S, CNRS, Univ Paris-Saclay, CentraleSupélec, France (e-mail: william.pasillas-lepine@centralesupelec.fr). A. Loria is with L2S, CNRS, (e-mail: antonio.loria@cnrs.fr). M. Aguado-Rojas is with Hitachi Astemo. The work of M. Maghenem and of A. Loria is supported by the French ANR via project HANDY, contract number ANR-18-CE40-0010.

For the purpose of estimating  $z_2$ , the goal is to define  $k_1$  and  $k_2$  so that  $\tilde{z} \rightarrow 0$ . To that end, following [8], we define

$$k_i(z_1) := \begin{cases} k_i^+ & \text{if } z_1 > 0 \\ k_i^- & \text{if } z_1 < 0 \\ 0 & \text{if } z_1 = 0, \end{cases} \quad i \in \{1, 2\},$$

so, for all  $z_1 \neq 0$ , the matrix on the right-hand side of (3) can only be equal to

$$A_1 = \begin{bmatrix} -k_1^+ & -a \\ -k_2^+ & c \end{bmatrix}, \quad \text{or} \quad A_2 = \begin{bmatrix} k_1^- & a \\ k_2^- & -c \end{bmatrix};$$

these matrices are both Hurwitz if  $k_1^+ > c$ ,  $k_2^+ < -\frac{c}{a}k_1^+$ ,  $k_1^- < c$ ,  $k_2^- < -\frac{c}{a}k_1^-$ —see [11]. Furthermore, the pairs  $(A_1, C)$  and  $(A_2, C)$  are observable and, if  $k_1^- = 2c - k_1^+$ ,  $ck_1^+ + ak_2^+ = ck_1^- + ak_2^-$ , there exists a positive definite symmetric matrix  $P \in \mathbb{R}^{2 \times 2}$ , such that—cf. [8]

$$A_i^\top P + PA_i = -C^\top C \quad \forall i \in \{1, 2\}. \quad (4)$$

The fact that (4) holds is significant because, *albeit* an appropriate change of time-scale [15], the estimation error dynamics (3) may be analyzed as a switched time-invariant system of the class considered in [17]. To better see this, let

$$\tau := \int_{t_o}^t |z_1(s)| ds =: f_{z_1}(t) \quad (5)$$

and

$$A(w_1) := \begin{cases} A_1 & \text{if } w_1 > 0 \\ A_2 & \text{if } w_1 < 0 \\ 0 & \text{if } w_1 = 0. \end{cases}$$

Then, Eq. (3) is equivalent to

$$\tilde{w}' := \frac{d\tilde{w}}{d\tau} = A(w_1(\tau))\tilde{w} \quad \forall \tau \in \text{Im}(f_{z_1}), \quad (6)$$

where  $\text{Im}(\cdot)$  stands for the image of  $(\cdot)$ ,  $w_1 : \text{Im}(f_{z_1}) \rightarrow \mathbb{R}$  and  $\tilde{w} : \text{Im}(f_{z_1}) \rightarrow \mathbb{R}^2$  are given by

$$w_1(\tau) := \begin{cases} z_1(f_{z_1}^{-1}(\tau)) & \text{if } \text{card}(f_{z_1}^{-1}(\tau)) = 1 \\ 0 & \text{otherwise,} \end{cases} \quad (7)$$

$$\tilde{w}(\tau) := \begin{cases} \tilde{z}(f_{z_1}^{-1}(\tau)) & \text{if } \text{card}(f_{z_1}^{-1}(\tau)) = 1 \\ \tilde{z}(\min\{f_{z_1}^{-1}(\tau)\}) & \text{otherwise,} \end{cases} \quad (8)$$

where  $\tilde{z}^\top := [\tilde{z}_1^\top \ \tilde{z}_2^\top]$ , and  $\text{card}(\cdot)$  means cardinality. That is, from Eqs. (5)–(8), we have  $\tau(t_o) := 0$  and, for all initial conditions satisfying  $w_1(0) = \tilde{z}_1(t_o)$  and  $\tilde{w}(0) = \tilde{z}(t_o)$ , we have  $w_1(\tau) = z_1(t)$  and  $\tilde{w}(\tau) = \tilde{z}(t)$  for all  $\tau \geq 0$  and  $t \geq t_o$ . But if  $z_1(t) = 0$  the  $\tau$ -clock freezes while the  $t$ -clock goes on. That is,  $\dot{\tau}(t) \equiv 0$  for all  $t \in \mathcal{T}^0 := \{t \in \mathbb{R}_{\geq t_o} : z_1(t) = 0\}$  and  $\dot{\tau}(t) > 0$  for all  $t \notin \mathcal{T}^0$ . For the  $\tau$ -clock,  $\tau(\mathcal{T}^0)$  is an instant;  $f_{z_1}^{-1}(\tau(\mathcal{T}^0))$  does not exist, so we set  $w_1(\tau(\mathcal{T}^0)) = \tilde{w}(\tau(\mathcal{T}^0)) = 0$ . Thus, since  $w_1(\tau) = 0$  on a null-measure set,

$$A(w_1(\tau)) \in \{A_1, A_2\} \quad \text{for almost all } \tau \in \text{Im}(f_{z_1}),$$

and system (6) corresponds to a linear system switching between two modes.

Now, because  $w_1(\tau)$  and  $\tilde{w}(\tau)$  coincide, respectively, with  $z_1(t)$  and  $\tilde{z}(t)$ , the origin for (3) is asymptotically stable if and only if so is the origin for (6). This fact is at the basis of Lemma 1 below. In that regard, consider the following hypothesis, which is later proved to hold by design.

*Assumption 1:* There exist positive constants  $\tau_d$ ,  $\tau_s$ ,  $\underline{z}$ , and  $\bar{z}$ , and an infinite union of disjoint intervals  $I_d$ , such that: (i)  $|z_1(t)| \geq \underline{z}$  for all  $t \in I_d$ , (ii)  $|z_1(t)| \leq \bar{z}$  for all  $t \in \mathbb{R}_{\geq 0} \setminus I_d$ , (iii) the length of each connected interval in  $I_d$  is no smaller than  $\tau_d$ , and (iv) the length of each connected interval in  $\mathbb{R}_{\geq 0} \setminus I_d$  is smaller than  $\tau_s$ .

*Lemma 1:* If Assumption 1 holds, there exists  $\mu > 0$  and, for any  $P$  solving (4), there exist  $\kappa_1$  and  $\kappa_2$ , such that

$$|\tilde{z}(t)| \leq \kappa_1 |\tilde{z}(t_o)| e^{-\kappa_2 \mu (t-t_o)} \quad \forall t \geq t_o + T, \quad t_o \geq 0. \quad (9)$$

*Remark 1:* The proof of Lemma 1 is constructive and is provided in Appendix A. The lemma improves over the main result in [15], by establishing exponential stability and, more importantly, an explicit stability bound. This is primordial for the control design, which relies on the knowledge of the rate of decrease of the estimation errors.

### III. OBSERVER-BASED HYBRID CONTROL ALGORITHM

Let  $z^* : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  be a given, piecewise-constant, reference trajectory (to be defined) for  $z_1$  and consider the simple certainty-equivalence control law

$$u := az_1 \hat{z}_2 - kz_{1e}, \quad z_{1e} := z_1 - z^*. \quad (10)$$

Then, the tracking-error dynamics corresponds to

$$\dot{z}_{1e} = -(k + a\tilde{z}_2)z_{1e} + az^* \tilde{z}_2. \quad (11)$$

This system is input-to-state stable with respect to  $z^*$  uniformly in balls of initial conditions. To better see this, let  $R > 0$  be arbitrarily fixed. Then, after [8], let  $P \in \mathbb{R}^{2 \times 2}$  be a positive definite matrix such that the time derivative of

$$V_{obs}(\tilde{z}) := \tilde{z}^\top P \tilde{z}, \quad (12)$$

along the solutions to (3), verifies

$$\dot{V}_{obs}(\tilde{z}(t)) \leq 0 \quad \forall t \geq 0 \quad (13)$$

—see the proof of Lemma 1 in Appendix A. Therefore, for any  $R > 0$ , and all  $|z(0)| \leq R$ , one can compute  $\tilde{R}$  such that  $V_{obs}(\tilde{z}(0)) \leq \lambda_{\max}(P)\tilde{R}^2$ . In turn, we have

$$\tilde{z}(t)^2 \leq \gamma^2 \tilde{z}(0)^2 \leq \gamma^2 \tilde{R}^2 \quad \forall t \geq 0, \quad (14)$$

with  $\gamma := \sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)} \geq 1$ . Then, using the function  $V(z_{1e}) := (1/2)z_{1e}^2$  and setting

$$k := \gamma a \tilde{R} + k', \quad k' > 0, \quad (15)$$

we see that, in view of (14), the derivative of  $V$  along the trajectories of (11) satisfies

$$\dot{V}(z_{1e}) \leq -k' z_{1e}^2 + a|z^*|_\infty |\tilde{z}_2|_\infty |z_{1e}|, \quad (16)$$

where  $|\phi|_\infty := \text{ess sup}_{t \geq 0} |\phi(t)|$ . It follows from (16) that the tracking error converges provided that so do  $|\tilde{z}_2|$  and  $|z^*|$ . On the other hand, under Assumption 1, for  $|\tilde{z}_2|$  to converge, it is required that  $|z^*(t)|$  dwells a certain amount of time separated from zero. To achieve these antagonistic objectives, we design a succession of cycles indexed by  $i \in \{1, 2, \dots\}$ , during each of which,  $z^*$  takes values in  $\left\{-\frac{z_{in}^*}{2^i}, \frac{z_{in}^*}{2^i}\right\}$ , so let

$$S^* := \bigcup_{i=0}^{\infty} \left\{-\frac{z_{in}^*}{2^i}, \frac{z_{in}^*}{2^i}\right\}, \quad (17)$$

where  $z_{in}^* > 0$  is fixed by design (see below). That is,  $z^*(t)$  undergoes a sequence of commutations between two constant values during each cycle (this guarantees the decrease of  $|\tilde{z}_2|$ ) and the said constants decrease as the index  $i$  increases.

*Initialization step:* Let  $z_{in}^*$  and  $R > 0$  be given. Then, initially, we set  $z^*(t) = z_{in}^*$  for all  $t \in [0, t_1]$ , where  $t_1$  is to be defined, and  $\hat{z}_2(0)$  is chosen such that (14) holds. Then,

$$|\tilde{z}_2(t)| \leq \gamma \tilde{R} \quad \forall t \geq 0.$$

In view of (14) and (16), there exists  $T > 0$  such that

$$|z_{1e}(t)| \leq \frac{2a\tilde{R}}{\sqrt{k'}} |z^*(t)| \quad \forall t \geq T,$$

so, by setting  $k' \geq 16a^2\tilde{R}^2$ , it follows that

$$|z_{1e}(t)| \leq z_{in}^*/2 \quad \forall t \geq T,$$

$z_1(t) \in [\frac{z_{in}^*}{2}, z_{in}^*]$ , and, consequently, Assumption 1 holds. On the other hand, there exist  $\kappa_{1o}, \kappa_{2o} > 0$  such that—see (9),

$$|\tilde{z}(t)| \leq |\tilde{z}(0)|\kappa_{1o} \exp\left(-\frac{\kappa_{2o}z_{in}^*t}{2}\right) \quad \forall t \geq 0,$$

so, for any  $\varepsilon > 0$ , there exists  $T_o \geq T > 0$  such that

$$|\tilde{z}(t)| \leq g(0)(\varepsilon/\gamma) \quad \forall t \geq T_o, \quad g(0) := 1,$$

and, since  $\gamma \geq 1$ —see below (14)—we have  $|\tilde{z}_2(t)| \leq \varepsilon$  for all  $t \geq T_o$ .

*First cycle:* From  $t_1 := T_o$ , we set  $z^*$  to satisfy  $|z^*| = \frac{z_{in}^*}{2}$ , moreover, the tracking error  $z_{1e}$  satisfies (16) with  $|\tilde{z}_2| \leq \varepsilon$ . Therefore, a limit cycle is generated by switching  $z^*$  between  $-z_{in}^*/2$  and  $z_{in}^*/2$  each time  $\hat{z}_2(t)$  reaches  $d/2c$  or  $-d/2c$ , as follows:

- 1) If  $\hat{z}_2(t_1) \leq 0$ ,  $z^*(t_1)$  is set to  $\frac{z_{in}^*}{2}$ . Then, at  $t'_1 \geq t_1$  such that  $\hat{z}_2(t'_1) = \frac{d}{2c}$ , which means that  $z_2(t'_1) \in [\frac{d}{2c} - \varepsilon, \frac{d}{2c} + \varepsilon]$ , the reference  $z^*$  is set to  $z^*(t'_1) = -\frac{z_{in}^*}{2}$ . Then, at  $t''_1 \geq t'_1$  such that  $\hat{z}_2(t''_1) = -\frac{d}{2c}$ , which means that  $z_2(t''_1) \in [-\varepsilon - \frac{d}{2c}, \varepsilon - \frac{d}{2c}]$ , the reference  $z^*$  is set back to  $\frac{z_{in}^*}{2}$ .
- 2) If  $\hat{z}_2(t_1) \geq 0$ , the reference is set to  $z^*(t_1) = -\frac{z_{in}^*}{2}$  and the same switching rules as above apply *mutatis mutandis*.

Along the first cycle, Assumption 1 holds on  $[t_1, +\infty)$ ; thus, there exist positive constants  $(\kappa_{11}, \kappa_{21})$  such that

$$|\tilde{z}(t)| \leq \kappa_{11} |\tilde{z}(t_1)| e^{-\kappa_{21} \int_{t_1}^t |z_1(s)| ds} \quad \forall t \geq t_1.$$

The first cycle ends at  $t_2 := T_o + T_1 > 0$ , such that

$$|\hat{z}_2(t_2)| \leq d/2c, \quad |\tilde{z}(t_2)| \leq (\varepsilon/\gamma)g(1), \quad g(1) \in (0, g(0)),$$

*idem* for each succeeding cycle indexed  $i \geq 2$ .

*ith cycle:* Let  $\{g(i)\}_{i=0}^\infty \subset (0, 1)$ ,  $g(0) := 1$  be a decreasing sequence. From  $t_i = T_o + T_1 + \dots + T_{i-1}$ , the reference  $z^*$  is set to satisfy  $|z^*| = \frac{z_{in}^*}{2^i}$ , and  $z_{1e}$  satisfies (16) with  $|\tilde{z}_2| \leq \varepsilon g(i-1)$  for some  $g(i-1) \in (0, 1)$ . Hence, a limit cycle is generated by making  $z^*$  switch between  $-\frac{z_{in}^*}{2^i}$  and  $\frac{z_{in}^*}{2^i}$  each time  $\hat{z}_2(t) = d/2^i c$  or  $\hat{z}_2(t) = -d/2^i c$ , as follows:

- 1) If  $\hat{z}_2(t_i) \leq 0$ ,  $z^*(t_i)$  is set to  $\frac{z_{in}^*}{2^i}$ . Then, at  $t'_i \geq t_i$  such that  $\hat{z}_2(t'_i) = \frac{d}{2^i c}$ , which means that  $z_2(t'_i) \in [\frac{d}{2^i c} - \varepsilon, \frac{d}{2^i c} + \varepsilon]$ , the reference  $z^*$  is set to  $z^*(t'_i) = -\frac{z_{in}^*}{2^i}$ . Then, at  $t''_i \geq$

$t'_i$  such that  $\hat{z}_2(t''_i) = -\frac{d}{2^i c}$ , which means that  $z_2(t''_i) \in [-\frac{d}{2^i c} - \varepsilon, -\frac{d}{2^i c} + \varepsilon]$ , the reference  $z^*$  is set to  $\frac{z_{in}^*}{2^i}$ .

- 2) If  $\hat{z}_2(t_i) \geq 0$ , the reference is set to  $z^*(t_i) = -\frac{z_{in}^*}{2^i}$ , etc.

During the  $i$ th cycle, Assumption 1 holds on  $[t_i, +\infty)$ , so there exist positive constants  $(\kappa_{1i}, \kappa_{2i})$  such that

$$|\tilde{z}(t)| \leq \kappa_{1i} |\tilde{z}(t_i)| e^{-\kappa_{2i} \int_{t_i}^t |z_1(s)| ds} \quad \forall t \geq t_i.$$

The cycle ends at  $t_{i+1} := T_o + T_1 + \dots + T_i > 0$ , such that

$$|\hat{z}_2(t_{i+1})| \leq d/2^i c \text{ and } |\tilde{z}(t_{i+1})| \leq (\varepsilon/\gamma)g(i) \quad \forall t \geq t_{i+1},$$

A new cycle starts over and so on.

*Remark 2:* Inequality (16), all of the analysis above, and consequently that in Section II, only hold on the maximal interval of solutions—say on  $[t_o, t_f]$  with  $t_f \leq \infty$ . To show that  $t_f = +\infty$  we assume otherwise. Then, we replace  $|\tilde{z}_2|_\infty$  with  $|\tilde{z}_2|_{t_f} =: c$  in (16), so we have  $\dot{V}(z_{1e}(t)) \leq ac|z^*|_\infty V(z_{1e}(t))$  for all  $t$  such that  $|z_{1e}(t)| \geq 2$ . That is, as  $t \rightarrow t_f$  we have  $|z_{1e}(t)| \rightarrow \infty$  and  $V(z_{1e}(t)) \rightarrow \infty$ , but integrating on both sides of  $\dot{V}(z_{1e}(t)) \leq ac|z^*|_\infty V(z_{1e}(t))$ , we obtain  $+\infty = ac|z^*|_\infty [t_f - t_o]$ , which contradicts  $t_f < +\infty$ .

#### IV. MAIN STATEMENT

To analyze formally the stability of the closed-loop system composed of the plant (1), the controller (10), and the observer (2), we rely on expressing it as a hybrid system that consists in the combination of a constrained differential and a constrained difference equations, as per in [18], *i.e.*,

$$\mathcal{H} : \begin{cases} \dot{x} = F(x) & x \in C \\ x^+ = G(x) & x \in D, \end{cases} \quad (18)$$

where the state variable  $x \in \mathcal{X} \subset \mathbb{R}^n$  has a continuous evolution while in the flow set  $C \subset \mathcal{X}$  and it is allowed to jump if in the jump set  $D \subset \mathcal{X}$ . The continuous- and the discrete-time evolution of  $x$  are governed by the flow and the jump maps  $F : C \rightarrow \mathbb{R}_{\geq 0} \times \{0\} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \{0\}$  and  $G : D \rightarrow \mathcal{X}$ , respectively. Furthermore, the closed-loop state is defined as

$$x := (\tau, i, z, \tilde{z}, z^*) \in \mathcal{X}, \\ \mathcal{X} := \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R} \times \left(-\frac{d}{c}, +\infty\right) \times \mathbb{R}^2 \times S^*.$$

Then, the jump and flow sets are defined as follows. The flow set  $C := \text{cl}(\mathcal{X} \setminus D)$ , where  $\text{cl}(\cdot)$  denotes the closure relative to  $\mathcal{X}$  and the jump set  $D := D_c \cup D_{nc}$ . The set  $D_c$ , which determines the jump conditions *within* the  $i$ th cycle, is

$$D_c := \left\{ x \in \mathcal{X} : |\hat{z}_2| \geq \frac{d|z^*|}{cz_{in}^*}, \hat{z}_2 z^* \geq 0 \right\} \quad (19)$$

and the set  $D_{nc}$ , which determines the jump condition *from* the  $i$ th to the  $(i+1)$ th cycle, is given by

$$D_{nc} := \left\{ x \in \mathcal{X} : |\hat{z}_2| \leq \frac{d|z^*|}{cz_{in}^*}, \hat{z}_2 z^* \leq 0, \right. \\ \left. |\Phi_i(\tau, 0)^\top P \Phi_i(\tau, 0)|^{\frac{1}{2}} \leq \lambda_{\min}(P)^{\frac{1}{2}} h(i) \right\}, \quad (20)$$

where the transition matrix  $\Phi_i$  is obtained by integrating (e.g., numerically) the equation—cf. Eq. (6)

$$\frac{d\Phi_i}{d\tau} = A(w_1(\tau + \tau_i))\Phi_i \quad \tau \geq 0, \quad (21)$$

where  $\tau_i := \int_0^{t_i} |z_1(s)| ds$ , with  $t_i$  being the moment when the  $i$ th cycle starts,

$$h(i) := \frac{g(i)}{g(i-1)} \in (0, 1), \quad h(0) := \varepsilon/(\gamma\tilde{R}), \quad (22)$$

and we recall that  $\{g(i)\}_{i=0}^\infty \subset (0, 1)$ , with  $g(0) := 1$ , is a decreasing sequence.

The definition of the jump sets  $D_c$  and  $D_{nc}$  follows the rationale developed in the previous section, but certain technical aspects are also considered in order to cast the analysis in the framework of [18]. The respective first inequalities in  $D_c$  and  $D_{nc}$  correspond to the switch conditions explained in Section III. The constraint  $\hat{z}_2 z^* \leq 0$ , which requires that the signs of  $\hat{z}_2$  and  $z^*$  be different, is imposed in the definition of  $D_{nc}$ , while the opposite is used to define  $D_c$ , to render the intersection of these sets empty (the apparent intersection  $\{\hat{z}_2 = z^* = 0\}$  is void since  $z^* \neq 0$  by design). Defining the jump sets  $D_{nc}$  and  $D_c$  by simply imposing a strict inequality in either set would be in better concordance with the algorithm described in the previous section, but such definition would lead to the hybrid system being not well-posed [18].

The third inequality in the definition of  $D_{nc}$ ,

$$|\Phi_i(\tau, 0)^\top P \Phi_i(\tau, 0)| \leq \lambda_{\min}(P) h(i)^2, \quad (23)$$

is a conservative, yet verifiable, condition that essentially tests the size of the otherwise non-measurable estimation errors  $\tilde{z}(t) \equiv \tilde{w}(\tau)$ . To better see this, consider the function  $V_{obs}$  in (12). Its total derivative along the solutions to (6) satisfies  $\dot{V}_{obs}(\tilde{w}(\tau)) \leq 0$ , so  $V_{obs}(\tilde{w}(\tau)) \leq V_{obs}(\tilde{w}(0))$  for all  $\tau \geq 0$ . Hence, equivalently,

$$\tilde{w}(\tau)^\top P \tilde{w}(\tau) \leq \tilde{w}(0)^\top P \tilde{w}(0).$$

Therefore, using the fact that  $\tilde{w}(\tau) = \Phi_i(\tau, 0)\tilde{w}(0)$ , we see that (23) implies that, for any  $\tilde{w}(0) \in \mathbb{R}^2$ ,

$$\tilde{w}(\tau)^\top P \tilde{w}(\tau) \leq \lambda_{\min}(P) h(i)^2 \tilde{w}(0)^2,$$

that is,  $|\tilde{w}(\tau)|^2 \leq h(i)^2 |\tilde{w}(0)|^2$ .

Then, we introduce the flow map

$$F(x) := \begin{bmatrix} |z_1| \\ 0 \\ -(k + a\tilde{z}_2)z_{1e} + az^*\tilde{z}_2 \\ (cz_2 + d)z_1 \\ z_1 \begin{bmatrix} -k_1(z_1) & -a \\ -k_2(z_1) & c \end{bmatrix} \tilde{z} \\ 0 \end{bmatrix}. \quad (24)$$

Note that in the definition of  $F$  the dynamics of the discrete variables  $(i, z^*)$  is null, the dynamics of  $\tau$  corresponds to (5), and the dynamics of  $z$  and  $\tilde{z}$  are simply repeated from (11) and (3), respectively.

On the other hand, the jump map is given by

$$G(x) := \begin{bmatrix} \begin{cases} 0 & \text{if } x \in D_{nc} \\ \tau & \text{if } x \in D_c \\ i+1 & \text{if } x \in D_{nc} \\ i & \text{if } x \in D_c \end{cases} \\ z \\ \tilde{z} \\ \begin{cases} z^*/2 & \text{if } x \in D_{nc} \\ -z^* & \text{if } x \in D_c \end{cases} \end{bmatrix}. \quad (25)$$

The map  $G$  is designed to reset the value of  $\tau$  to 0 each time a new cycle starts and updates the cycle index  $i$ . The variables  $z$  and  $\tilde{z}$  are continuous variables, so they do not change their values during jumps. According to the algorithm previously explained, the variable  $z^*$  halves its size in absolute value whenever a jump to a new cycle occurs. Otherwise, while switching within a cycle,  $z^*$  only alternates sign. It is important to note that since  $D_c \cap D_{nc} = \emptyset$ , then the map  $G$  is continuous on  $D$ . This is important for the system to be well-posed [18].

In addition, the initial state  $x_o := (\tau_o, i_o, z_o, \tilde{z}_o, z_o^*) \in \mathcal{X}$  is defined as follows. By assumption, a number  $R$  is known such that  $|z_o| \leq R$ . Then, the estimates  $\hat{z}_o$  are set so that  $|\tilde{z}_o| \leq \tilde{R}$  for some  $\tilde{R} > 0$  known. Hence, when a reliable estimate of  $|z_o|$  is available, the Initialization step described on p. 3 may be skipped by defining the initial cycle index as  $i_o := \max\{0, \kappa_1(\tilde{R})\}$ , where

$$\kappa_1(\tilde{R}) := \max \left\{ i \in \mathbb{Z} : \tilde{R} \leq \frac{\varepsilon g(i-1)}{\gamma} \right\}.$$

Furthermore, according to (5),  $\tau_o = 0$ . Finally, the reference trajectory  $z^*$  is initialized to

$$z_o^* := \begin{cases} \begin{cases} \frac{z_{in}^*}{2^{i_o}} & \text{if } \hat{z}_{2o} < 0 \\ -\frac{z_{in}^*}{2^{i_o}} & \text{otherwise} \end{cases} & \text{if } i_o \geq 1 \\ z_{in}^* & \text{if } i_o = 0. \end{cases}$$

Our main statement establishes semi-global attractivity of the set  $\mathcal{A} := \{x \in \mathcal{X} : z = \tilde{z} = 0\}$  for the closed-loop system. That is, for any ball of initial conditions of radius  $R$ , there exists a control gain  $k(R)$ , as defined in (15), such that all trajectories converge to the set  $\mathcal{A}$ . In particular, the domain of attraction may be enlarged by increasing the control gain.

**Theorem 1:** Consider the closed-loop hybrid system  $\mathcal{H} = (\mathcal{C}, F, D, G)$  defined by (18)–(20), (22), (24), and (25). Let  $R, \tilde{R} > 0$  be such that  $|z_o| \leq R$  and  $|\tilde{z}_o| \leq \tilde{R}$ , and let  $(i_o, z_o^*, \tau_o)$  be defined as above. Then, for each  $R$  and  $\tilde{R}$ , we can find  $k > 0$  such that

- (i) each solution to<sup>1</sup>  $\mathcal{H}$  satisfies  $\lim_{(t+j) \rightarrow +\infty} |x(t, j)|_{\mathcal{A}} = 0$ , provided that  $\lim_{i \rightarrow \infty} g(i) = 0$ ;
- (ii) there exists  $\kappa \in \mathcal{K}$  and  $\delta^* > 0$ , such that, for any  $\delta \in (0, \delta^*)$ , if  $|\tilde{z}_o| \leq \delta$ , the system's trajectories satisfy the bound  $|(z, \tilde{z})|_\infty \leq \kappa(|z_o| + \delta)$ ;
- (iii) system  $\mathcal{H}$  is well posed—see [18], and its solutions are uniformly non-Zeno, that is, there exist  $T > 0$  and  $J \in \mathbb{N}$

<sup>1</sup>Note that  $(t, j) \mapsto x(t, j)$  are defined as absolutely continuous functions mapping their hybrid domain,  $\text{dom } x \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ , into  $\mathbb{R}^2$ . See [18] for details.

such that, on any time period of length  $T$ , at most  $J$  jumps can occur.  $\square$

Statement (i) establishes attractivity of the origin, provided that the algorithm is initialized as shown above Theorem 1. In that regard, note that the initialization cycle  $i_o = 0$ , described in Section III, may be avoided if one has approximate knowledge of the initial condition  $z_o$ , so as to set  $\hat{z}_o$   $\delta$ -close for  $\delta > 0$  sufficiently small. In this case, a bound on the overshoot of the trajectories is guaranteed—see Statement (ii). Note that this bound does not require convergence of  $g$  to 0.

Now, in general, the assumption in (ii) is restrictive, but not for commercial ABS systems, for which the initial condition  $z_{1o}$  is often approximately known.

### Proof of Theorem 1

*Proof of item (i):* To guarantee asymptotic convergence of  $z$  to zero, we first show that Assumption 1 holds on the  $i$ th cycle, for all  $i \in \{1, 2, \dots\}$ . Let  $i \in \{1, 2, \dots\}$  be arbitrarily fixed and consider the behavior of the solutions to  $\mathcal{H}$  for all  $t \in \mathcal{I}_i$ , that is, during the duration of the  $i$ th cycle. Over the interval  $\mathcal{I}_i$ , the solutions to  $\mathcal{H}$  coincide with those of the hybrid system  $\mathcal{H}_i := (F_i, G_i, C_i, D_i)$ , with state vector

$$x := (z, \tilde{z}, z^*) \in \mathcal{X}_i := \mathbb{R}^2 \times \mathbb{R}^2 \times \left\{ -\frac{z_{in}^*}{2^i}, \frac{z_{in}^*}{2^i} \right\},$$

$$\text{flow map } F_i(x) := \begin{bmatrix} -(k + a\tilde{z})z_{1e} + az^*\tilde{z}_2 \\ (cz_2 + d)z_1 \\ z_1 \begin{bmatrix} -k_1(z_1) & -a \\ -k_2(z_1) & c \end{bmatrix} \tilde{z} \\ 0 \end{bmatrix},$$

jump map  $G_i(x) := [z^\top \ \tilde{z}^\top \ -z^*]^\top$ , jump set  $D_i := D_{i1} \cup D_{i2}$ , where

$$D_{i1} := \left\{ x \in \mathcal{X}_i : \hat{z}_2 \geq \frac{d/c}{2^i}, z^* = \frac{z_{in}^*}{2^i} \right\},$$

$$D_{i2} := \left\{ x \in \mathcal{X}_i : \hat{z}_2 \leq -\frac{d/c}{2^i}, z^* = -\frac{z_{in}^*}{2^i} \right\},$$

flow set  $C_i := \text{cl}(\mathcal{X}_i \setminus D_i)$ , and  $x_o := (z_o, \tilde{z}_o, z_o^*)$ , such that

$$|\tilde{z}_o| \leq \frac{\varepsilon g(i-1)}{\gamma} \quad \text{and} \quad z_o^* = \begin{cases} \frac{z_{in}^*}{2^i} & \text{if } \hat{z}_{o2} < 0 \\ -\frac{z_{in}^*}{2^i} & \text{otherwise.} \end{cases} \quad (26)$$

The solutions to  $\mathcal{H}_i$  (and consequently to  $\mathcal{H}$  over  $\mathcal{I}_i$ ), within the  $i$ th cycle, jump according to the conditions defining  $D_{i1} \cup D_{i2}$  and satisfy the following.

*Lemma 2:* Consider the hybrid system  $\mathcal{H}_i$  such that (26) holds and let the parameter  $k$  satisfy (15) with  $2k' \geq a^2\varepsilon^2$ . Then,

(i) the set

$$D_i := \left\{ x \in \mathcal{X}_i : |z_1| \leq \frac{z_{in}^*}{2^{i-1}} + \frac{z_{in}^*}{2^{i-1}} g(i-1) \right\} \quad (27)$$

is forward invariant and finite-time attractive.

(ii) Let  $x_o \in D_i$  and let the parameter  $k$  satisfy (15) with  $2k' \geq a^2\varepsilon^2$ . Then, there exists  $T_{lmin} > 0$ , independent of  $i$ , such that the time between each pair of consecutive

jumps of the solution starting from  $x_o$  is larger than  $T_{lmin}$ .

Furthermore, after Lemma 2 the following also holds (see the appendix for the proofs).

*Lemma 3:* Consider the hybrid system  $\mathcal{H}_i(C_i, F_i, D_i, G_i)$  such that (26) holds and the parameter  $k$  satisfy (15). Then, for  $k'$  sufficiently large and independent of  $i$ , there exist positive constants  $(\tau_{di}, \tau_{si}, \bar{z}_i, \underline{z}_i)$  so that Assumption 1 holds.

By Lemma 3, Assumption 1 holds. Therefore, from Lemma 1 it follows that there exist positive constants  $\kappa_{1i}, \kappa_{2i}, T_i$ , and  $\mu_i$  such that

$$|\tilde{z}(t)| \leq \kappa_{1i} |\tilde{z}(t_i)| e^{-\kappa_{2i}\mu_i(t-t_{oi})} \quad \forall t \geq t_i + T_i,$$

where  $t_i$  is the beginning the interval  $\mathcal{I}_i$ . Hence, in view of the second condition in (20), the interval of duration of the  $i$ th cycle,  $\mathcal{I}_i$ , is finite. Now, we use Lemmata 2 and 3 to complete the proof of Item (i) of the theorem. We show that, for each  $i_o \in \{1, 2, \dots\}$ , there exists  $i^* \geq i_o$  and  $t_{i^*} \in \mathcal{I}_{i^*}$ , i.e., during the Cycle  $i^*$ , such that  $x(t_{i^*}) \in D_{i^*}$ . By the definition of  $D_i$ , the convergence of  $z_1(t)$  follows.

Let  $i_o \in \{1, 2, \dots\}$  and  $t_{i_o} \geq 0$  be the time at which Cycle  $i_o$  starts. Assume, without loss of generality, that  $z_1(t_{i_o}) > 0$ , but  $x(t) \notin D_{i_o}$  for all  $t \in \mathcal{I}_{i_o}$ , that  $z_2(t_{i_o}) \leq 0$ , and that  $z^*(t_{i_o}) > 0$ —the same reasoning that will follow applies to any other choice of initial conditions. For the considered choice of initial conditions,  $\hat{z}_2(t)$  increases until one of the following two scenarios occurs:

- 1) There exist a time instant when  $\hat{z}_2(t) = \frac{d/c}{2^{i_o}}$ , in which case,  $\text{sign}(z^*(t))$  becomes negative, so the jump to Cycle  $i_o + 1$  does not occur before  $\hat{z}_2(t)$  becomes, again, smaller or equal than  $\frac{d/c}{2^{i_o}}$ . For this to happen,  $z_2(t)$  must decrease, that is,  $z_1(t)$  must become negative—see (1b)—and, consequently,  $x(t)$  must enter the set  $D_{i_o}$ .
- 2) A jump to Cycle  $i_o + 1$  occurs before  $\hat{z}_2$  passes  $\frac{d/c}{2^{i_o}}$ . In this case, either the previous scenario occurs with  $i_o$  replaced by  $i_o + 1$  and  $x(t)$  enters  $D_{i_o+1}$  within Cycle  $i_o + 1$ , or a jump to Cycle  $i_o + 2$  occurs before  $\hat{z}_2$  passes  $\frac{d/c}{2^{i_o+1}}$ . However, at some point, there must exist  $i^* \geq i_o$  such that  $x(t)$  enters  $D_{i^*}$  within Cycle  $i^*$ .

Next, we show that  $z_2(t)$  also converges, by establishing an upperbound in the latter for all  $t \in \mathcal{I}_{i^*}$  such that  $z_1(t) \in D_{i^*}$  and when  $\hat{z}_2(t_{i^*}) = \frac{d/c}{2^{i^*-1}}$  and  $z^*(t_{i^*}) = -\frac{z_{in}^*}{2^{i^*}}$ . The latter must happen at some point while in Cycle  $i^*$ . Note that after the proof of Lemma 3 the overshoot of  $z_2(t)$  occurs during the interval  $[0, T_{lmin}]$ , where  $T_{lmin}$  corresponds to the time it takes  $z_1(t)$  to acquire the same sign as  $z^*(t)$ —in this case, to becomes negative. By virtue of the comparison Lemma, it is enough to construct a bound on the solution of

$$\dot{z}_2 = \left[ \max_{z_1 \in D_{i^*}} |z_1| \right] [cz_2 + d], \quad z_2(0) = \frac{d/c + \varepsilon}{2^{i^*-1}},$$

over the interval  $[0, T_{lmin}]$ . Clearly, we deduce an upperbound on  $z_2$  that converges to zero as  $i^*$  goes to infinity.

*Proof of item (ii):* By definition, the control algorithm is initiated at Cycle  $i_o$  with  $i_o := \max\{0, \kappa_1(\delta)\}$ . Furthermore, when  $\delta$  is sufficiently small, we conclude that  $i_o := \kappa_1(\delta)$ . Therefore, by definition of  $\kappa_1$  and (14), we conclude that

$$|\tilde{z}(t, j)| \leq \min\{\varepsilon g(i_o - 1), \gamma\delta\} \quad \forall (t, j) \in \text{dom } \tilde{z}.$$

Next, to find an upper bound for  $z_1$ , we distinguish between two cases:

- 1) If  $x_o \in \mathcal{D}_{i_o}$ , where  $\mathcal{D}_{i_o}$  is defined in (27) and is forward invariant, then we know that there exists a class  $\mathcal{K}$  function  $\kappa_2$  such that

$$z_1(t, j) \subset \kappa_2(|\delta|)[-1, 1] \quad \forall (t, j) \in \text{dom } z_1.$$

Indeed, it is easy to see that when  $\delta$  goes to zero,  $i_o$  goes to infinity, and thus  $\mathcal{D}_{i_o}$  reduces to  $\{0\}$ .

- 2) If  $\mathcal{D}_{i_o} \subset \{x \in \mathcal{X} : z_1 \in [-|z_{1o}|, |z_{1o}|]\}$  we use the fact that  $\mathcal{D}_{i_o}$  is finite-time attractive—see Item (i) in Lemma 2. Furthermore, since the flows are unique and  $z_1$  is a continuous variable, we conclude that  $[-|z_{1o}|, |z_{1o}|]$  must be forward invariant. Hence, we obtain that

$$|z_1(t, j)| \leq |z_{1o}| \quad \forall (t, j) \in \text{dom } z_1. \quad (28)$$

Finally, to complete the proof, we establish an upper bound on  $z_2$ . Assume, without loss of generality, that  $z_{2o} > 0$  and  $z_o^* = -2^{-i_o} z_{in}^* = -2^{-\kappa_1(\delta)} z_{in}^* =: \kappa_3(\delta)$ . Then, consider the following two possibilities:

- 1) If  $z_{1o} \in \mathcal{D}_{i_o}$  we conclude that the overshoot of  $|z_2|$  occurs only on the interval  $[0, T_{1*}]$ , on which  $|z_1| \leq \kappa_2(\delta)$ , and before  $z_1$  becomes negative.
- 2) If  $\mathcal{D}_{i_o} \subset \{x \in \mathcal{X} : z_1 \in [-|z_{1o}|, |z_{1o}|]\}$ , we conclude that (28) holds. Hence, the overshoot of  $z_2$  occurs only on the interval  $[0, T_{2*}]$ , on which  $|z_1| \leq |z_{1o}|$ , and before  $z_1$  becomes negative.

Thus, after the comparison Lemma, it suffices to assess the behavior of the solutions of

$$\dot{z}_2 = \max\{|z_{1o}|, \kappa_2(|\delta|)\}(cz_2 + d) \quad \text{with } z_2(0) = |z_{2o}|,$$

over the interval  $[0, T_*]$ , where  $T_* := \max\{T_{1*}, T_{2*}\}$  is an upper bound on the time that  $|z_1|$  takes to flow from  $\max\{|z_{1o}|, \kappa_2(|\delta|)\}$  to zero.

To complete the proof, we show that  $T_*$  can be chosen as a class  $\mathcal{K}$  function of  $|(z_o, \tilde{z}_o)|$ . To do so, we use the Lyapunov function  $v(z_{1e}) := z_{1e}^2$ , whose time derivative along the solutions to  $\mathcal{H}_i$  satisfies

$$\dot{v} = -2k'v + 2az^* \tilde{z}_2 z_{1e} \leq -2k'v + 2a\epsilon \frac{z_{in}^*}{2^{i_o}} g(i_o - 1) |z_{1e}|.$$

Let  $k' \geq 1$ . After the triangle inequality and  $g(i_o - 1) < 1$ ,

$$\dot{v} \leq -k'v + \frac{z_{in}^{*2}}{2^{2i_o}} \frac{a^2 \epsilon^2}{4\kappa_1(\delta)} \leq -k'v + a^2 \kappa_4(\delta).$$

Then,  $T_*$  corresponds to the time elapsed for the solution of  $\dot{v} = -k'v + a^2 \kappa_4(\delta)$  to attain the value  $v(t) = \frac{z_{in}^{*2}}{2^{2i_o}}$  from  $v(0) := \left[ \max\{|z_{1o}|, \kappa_2(\delta)\} + \frac{z_{in}^*}{2^{i_o}} \right]^2$ . We obtain that, since  $i_o := \max\{0, \kappa_1(\delta)\}$ ,  $T_*$  is upper bounded by a class  $\mathcal{K}$  function of  $(|z_o| + \delta)$ .

*Proof of item (iii):* After [18], system  $\mathcal{H}$  is well-posed if the sets  $C$  and  $D$  are closed relative to  $\mathcal{X}$  and  $F$  and  $G$  are continuous on  $C$  and  $D$ , respectively. It is easy to conclude that our closed-loop hybrid system  $\mathcal{H}$  satisfies the hybrid basic conditions which require the sets  $C$  and  $D$  to be closed and the maps  $F$  and  $G$  to be continuous. Note that both  $C$  and

$D$  are closed subsets relative to  $\mathcal{X}$ ,  $F$  is smooth and  $G$  is continuous on  $D = D_c \cup D_{nc}$  since both  $D_c$  and  $D_{nc}$  are closed relative to  $\mathcal{X}$  and their intersection is empty.

Next, we show that the closed-loop solutions are uniformly non-Zeno. To do so, we note that within a same Cycle  $i$ , and between each two consecutive jumps,  $\hat{z}_2(t)$  flows from  $-\frac{d}{2^{i-1}c}$  to  $\frac{d}{2^{i-1}c}$  back and forth. The latter flow phase takes a time we denote by  $T_{li}$ . After Item (ii) in Lemma 2, there exists a uniform lower bound  $T_{lmin} > 0$  such that  $T_{li} \geq T_{lmin}$  for all  $i \in \{1, 2, \dots\}$ , provided that  $z_{1o} \in \mathcal{D}_i$ . In general, after Item (i) of Lemma 2,  $z_1$  must reach  $\mathcal{D}_i$  in finite time while in Cycle  $i$ ; otherwise, only one jump occurs within Cycle  $i$ .

On the other hand, for a jump from Cycle  $i$  to Cycle  $i+1$  to occur, the variable  $\tau$  must flow so that  $|\Phi_i(\tau, 0)^\top P \Phi_i(\tau, 0)|^{\frac{1}{2}}$  decreases from  $|P|^{\frac{1}{2}}$  to  $\frac{\lambda_{min}(P)^{\frac{1}{2}}}{2}$ , where  $\Phi_i$  is defined by (21). We show the existence of a strictly positive lower bound on the time the latter decrease process takes. To that end, we use  $V = |\tilde{w}|^2$  and the fact that

$$\tilde{w}^\top [A(w_1(\tau)) + A(w_1(\tau))^\top] \tilde{w} \geq -\eta |\tilde{w}|^2 \quad \forall \tau \geq 0, \tilde{w} \in \mathbb{R}^2,$$

where  $\eta := \max_{i \in \{1, 2\}} \{A_i + A_i^\top\}$ . Then,  $V'(\tau) \geq -aV(\tau)$  for all  $\tau \geq 0$ , which, by using the comparison Lemma, implies that  $V(\tau) \geq e^{-a\tau} V(0)$  and, in turn, for each  $\tilde{w}(0) \in \mathbb{R}^2$ ,  $\tilde{w}(0)^\top \Phi_i(\tau, 0)^\top P \Phi_i(\tau, 0) \tilde{w}(0) \geq e^{-a\tau} |\tilde{w}(0)|^2$ , hence, since for our case  $|P| > \lambda_{min}(P)$  there exists  $\tau^* > 0$  such that, for each  $i \in \{1, 2, \dots\}$ , and for all  $\tau \in [0, \tau^*]$ ,

$$|\Phi_i(\tau, 0)^\top P \Phi_i(\tau, 0)| \geq \lambda_{min}(P) \geq \lambda_{min}(P) h(i).$$

## V. CONCLUSION AND FUTURE WORK

Simultaneous estimation and stabilization at an equilibrium where observability is lost is, in general, an open problem that we solved for a particular bilinear system using a switching-observer-based hybrid controller. It is important to explore the applicability of our approach to other non-uniformly observable systems. Other challenging improvements include the design of a *smooth* output-feedback controller as well as analyzing its robustness (with respect to measurement noise) in the sense of input-to-state stability. Furthermore, beyond these theoretical questions, a deeper study regarding control implementation is required to determine different cycle-jump conditions that deliver good performance, while satisfying the technical conditions imposed by the analysis.

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## APPENDIX

### A. Proof of Lemma 1

Consider system (6), which is equivalent to (3). It is a linear system that switches between two modes defined by the matrices  $A_1$  and  $A_2$ , which are both Hurwitz and the pairs  $(A_1, C)$  and  $(A_2, C)$  are observable. Let [8] generate a positive definite matrix  $P$  such that (4) holds. Then, the derivative of

$$V_{obs}(\tilde{w}) := \tilde{w}^\top P \tilde{w}$$

along the solutions to the switched linear system in (6) verifies

$$V'_{obs}(\tilde{w}) = -\tilde{w}^\top C^\top C \tilde{w} \leq 0,$$

which implies uniform global stability (*i.e.*, uniform stability and uniform global boundedness) of the origin for (6). Furthermore, under Assumption 1,  $\text{Im}(f_{z_1}) = \mathbb{R}_{\geq 0}$  and there exist an infinite union of disjoint intervals, denoted  $\bar{I}_d := f_{z_1}(I_d)$ , such that: (i)  $|w_1(\tau)| > \underline{z}$  for all  $\tau \in \bar{I}_d$ , (ii) the length of each connected interval in  $\bar{I}_d$  is no smaller than  $\tau_d \underline{z}$ , and (iii) the length of each connected interval in  $\mathbb{R}_{\geq 0} \setminus \bar{I}_d$  is smaller than  $\tau_s \bar{z}$ .

Now, inspired by [17], let  $\lambda > 0$  and  $\bar{c} := e^{\max\{|A_1|, |A_2|\} \bar{z} \tau_s}$ , and let [17, Lemma 9] generate  $K_1 \in \mathbb{R}^2$  and  $K_2 \in \mathbb{R}^2$  such that, for each  $i \in \{1, 2\}$ ,

$$|e^{(A_i + K_i C)\tau}| \leq \frac{1}{\bar{c}} e^{-2\lambda(\tau - \frac{\tau_d \bar{z}}{2})} \quad \forall \tau \geq \frac{\tau_d \bar{z}}{2}.$$

Furthermore, let  $k := \max\{|K_1|, |K_2|\}$ ,  $\bar{k} := p_M \bar{c}^2 k^2 / \lambda$ ,  $\gamma := p_M / p_m$ , where  $p_m I \leq P \leq p_M I$ , and  $\kappa_1 := \gamma[\bar{k} + 2\bar{c}^2 \gamma] / [\rho(1 + \bar{k})]$ . Finally, let  $L > 0$  be such that

$$\rho := \frac{2\gamma \bar{c}^2 e^{-2\lambda L} + \bar{k}}{1 + \bar{k}} < 1.$$

According to the proof of [17, Lemma 5], there exists a map  $w_1 \mapsto K(w_1) \in \{K_1, K_2, 0\}$  such that, along each map  $\tau \mapsto w_1(\tau)$  enjoying the properties (i)–(iii) listed above,

$$|\Phi_{\bar{z}_1}(\tau_1, \tau_o)| \leq \bar{c} e^{-\lambda(\tau_1 - \tau_o)} \quad \forall \tau_1 \geq \tau_o \geq 0,$$

where  $\Phi_{\bar{z}_1}$  is the transition matrix of the system

$$\tilde{w}' = [A(w_1(\tau)) + K(w_1(\tau))C] \tilde{w}.$$

Now, after Assumption 1, there exists  $\mu > 0$  such that

$$\int_{t_o}^t |z_1(s)| ds \geq \mu(t - t_o) \quad \forall t \geq t_o + T, \quad \forall t_o \geq 0,$$

so, using the proof of Theorem 4 in [17], we conclude that

$$|\tilde{w}(\tau_1)| \leq \kappa_1 \rho^{\frac{\tau_1 - \tau_o}{L}} |\tilde{w}(\tau_o)| \quad \forall \tau_1 \geq \tau_o \geq 0.$$

Therefore, the statement follows defining  $\kappa_2 := -\frac{\ln(\rho)}{L}$  and using the fact that  $\tilde{z}(t) \equiv \tilde{w}(t)$ .

## B. Proof of Lemma 2

1) *Proof of Item (i)*: We first use the fact that  $|\tilde{z}_o| \leq \frac{\varepsilon g(i-1)}{\gamma}$ , together with (14), to conclude that

$$|\tilde{z}(t, j)| \leq \varepsilon g(i-1) \quad \forall (t, j) \geq \text{dom } \tilde{z}.$$

Next, we use the Lyapunov function  $v(z_1) := z_1^2$ , whose time derivative along the solutions to  $\mathcal{H}_i$  satisfies

$$\dot{v} \leq -k'v + \frac{2k'z_{in}^*{}^2}{4^i} + \frac{2a^2\varepsilon^2 z_{in}^*{}^2}{4^i} g(i-1)^2.$$

As a result, we obtain

$$v(t, j) \leq v(0, 0)e^{-k't} + \frac{2z_{in}^*{}^2}{2^{2i}} \left[ 1 - e^{-k't} \right] + \frac{a^2\varepsilon^2 z_{in}^*{}^2}{(2k')4^{(i-1)}} g(i-1)^2 \left[ 1 - e^{-k't} \right],$$

so, by choosing  $k'$  such that  $k' \geq \frac{a^2\varepsilon^2}{2}$ , we get

$$v(t, j) \leq v(0, 0)e^{-k't} + \left[ \frac{z_{in}^*{}^2}{2^{2(i-1)}} + \frac{z_{in}^*{}^2}{4^{(i-1)}} g(i-1)^2 \right] \left[ 1 - e^{-k't} \right] \leq \max \left\{ v(0, 0), \left[ \frac{z_{in}^*{}^2}{2^{2(i-1)}} + \frac{z_{in}^*{}^2}{2^{2(i-1)}} g(i-1)^2 \right] \right\}.$$

Hence, the set  $\mathcal{D}_i$  is finite-time attractive and forward invariant.

2) *Proof of Item (ii)*: If  $x_{oi} \in \mathcal{D}_i$ , using the comparison lemma, a lower bound on the time between each two consecutive jumps of  $\mathcal{H}_i$  can be obtained by computing the time the solution to

$$\dot{z}_2 = - \left( \frac{2z_{in}^*}{2^{i-1}} \right) (cz_2 + d), \quad (29)$$

with initial conditions  $z_{o2} = \frac{d/c - \varepsilon}{2^{i-1}}$ . That is, the time that takes to reach  $-\frac{d/c + \varepsilon}{2^{i-1}}$ . To compute such time, we introduce the new time scale  $\tau := \left( \frac{2z_{in}^*}{2^{i-1}} \right) t$ , to obtain in the new time scale  $\dot{z}'_2 = -cz_2 - d$ . By solving the latter equation, we obtain

$$z_2(\tau) = \sigma_i e^{-c\tau} - \frac{d}{c} [1 - e^{-c\tau}], \quad \sigma_i := \left[ \frac{d/c - \varepsilon}{2^{i-1}} \right]$$

and we use the latter to solve  $z_2(\tau) = \sigma_i$  for  $\tau$ . Reordering terms, we obtain  $e^{-c\tau} [d/c + \sigma_i] = d/c - \sigma_i$ . Hence,

$$\tau = \frac{1}{c} \ln \left[ \frac{d/c + \sigma_i}{d/c - \sigma_i} \right] = \frac{1}{c} \ln \left[ 1 + 2 \frac{\sigma_i}{d/c - \sigma_i} \right].$$

This implies that, in the original time scale, the length of the interval  $[t_i, t_{i+1}]$  between two jumps of the solution to (29), denoted  $T_{li}$ , satisfies  $T_{li} \geq \frac{2^{i-1}}{2cz_{in}^*} \ln \left[ 1 + 2 \frac{\sigma_i}{d/c - \sigma_i} \right]$ , which is separated from zero, *i.e.*,

$$\lim_{i \rightarrow \infty} T_{li} \geq \frac{1}{dz_{in}^*} \left[ \frac{d}{c} - \varepsilon \right] > 0.$$

## C. Proof of Lemma 3

Let  $x_o \in \mathcal{D}_i$  and let  $\hat{z}_{o2} = \frac{d/c}{2^{i-1}}$ . There is no loss of generality since if  $x_o \notin \mathcal{D}_i$  Assumption 1 trivially holds over the  $i$ th cycle and if  $\hat{z}_{o2} \neq \frac{d/c}{2^{i-1}}$ ,  $\hat{z}_2(t, j) = \frac{d/c}{2^{i-1}}$  for some  $t + j < \infty$ . Moreover, the following reasoning applies *mutatis mutandis* if  $\hat{z}_{o2} = -\frac{d/c}{2^{i-1}}$ . We also use the fact that  $|\tilde{z}_o| \leq g(i-1) \frac{\varepsilon}{\gamma}$  together with (14) to conclude that

$$|\tilde{z}(t, j)| \leq \varepsilon g(i-1) \quad \forall (t, j) \in \text{dom } \tilde{z}. \quad (30)$$

1) At this point, we estimate a lower bound on the flow time that  $\hat{z}_2$  takes to flow from  $\frac{d/c}{2^{i-1}}$  to  $-\frac{d/c}{2^{i-1}}$ . Using (30), we conclude that such a time is lower bounded by the time  $z_2$  takes to flow from  $\frac{d/c}{2^{i-1}} - \frac{\varepsilon}{2^{i-1}}$  to  $-\frac{d/c}{2^{i-1}} + \frac{\varepsilon}{2^{i-1}}$  when

$$z_1 = - \left( \frac{z_{in}^*}{2^{i-1}} + \frac{z_{in}^*}{2^{i-1}} g(i-1) \right).$$

Let us denote such time by  $T_{li}$ , which can be easily obtained by solving the ordinary differential equation

$$\dot{z}_2 = - \left( \frac{z_{in}^*}{2^{i-1}} + \frac{z_{in}^*}{2^{i-1}} g(i-1) \right) (cz_2 + d),$$

with  $z_{2o} = \frac{d/c-\varepsilon}{2^{i-1}}$ . After Item (ii) of Lemma 2 there exists  $T_{lmin} > 0$  such that  $T_{li} \geq T_{lmin}$  for all  $i \in \{1, 2, \dots\}$ .

- 2) During the phase when  $\hat{z}_2$  flows from  $\frac{d/c}{2^{i-1}}$  to  $-\frac{d/c}{2^{i-1}}$ ,  $z^* = -\frac{z_{in}^*}{2^i}$ . Next, we show how to choose  $k' > 0$  to conclude that  $z_{1e}$  must take at most  $\frac{T_{li}}{2}$  units of time to enter the ball of radius  $\frac{|z^*|}{2}$ . To that end, we use the Lyapunov function

$$v(z_{1e}) := z_{1e}^2,$$

whose time derivative along the solutions to  $\mathcal{H}_i$  satisfies

$$\dot{v} \leq -2k'v + \frac{2az_{in}^*}{2^i} \varepsilon g(i-1)|z_{1e}|.$$

By assuming, for example that  $k' \geq 1$ , we conclude that

$$\dot{v} \leq -k'v + \frac{a^2\varepsilon^2 z_{in}^{*2}}{2^{2i}} g(i-1)^2,$$

Then, for each  $t \geq 0$  such that  $(t, 0) \in \text{dom } x$ ,

$$v(t, 0) \leq v(0, 0)e^{-k't} + \left[ \frac{a\varepsilon z_{in}^*}{2^i \sqrt{k'}} g(i-1) \right]^2 [1 - e^{-k't}]$$

so, choosing  $k'$  large such that  $k' \geq 4a^2\varepsilon^2$ , we obtain

$$\begin{aligned} v(t, 0) &\leq v(0, 0)e^{-k't} + \left[ \frac{z_{in}^*}{2^{i+1}} g(i-1) \right]^2 [1 - e^{-k't}] \\ &\leq \left[ \frac{2z_{in}^*}{2^i} + \frac{2z_{in}^*}{2^i} g(i-1) \right]^2 e^{-k't} \\ &\quad + \left[ \frac{z_{in}^*}{2^{i+1}} g(i-1) \right]^2 [1 - e^{-k't}] \\ &\leq 9z_{in}^{*2} e^{-k't} + \frac{z_{in}^{*2}}{4} [1 - e^{-k't}]. \end{aligned}$$

Next, by letting  $k' \geq -\frac{2\ln(2^{-5})}{T_{lmin}}$ , we conclude that

$$z_{1e}(t, 0)^2 \leq \left( \frac{z^*}{2} \right)^2 \quad \forall t \in \left[ \frac{T_{li}}{2}, T_{li} \right]. \quad (31)$$

Hence, during the interval where  $\hat{z}_2$  flows from  $\frac{d/c}{2^{i-1}}$  to  $-\frac{d/c}{2^{i-1}}$ , we have  $z_1 \in [z^*, z^*/2]$  for all  $t$  belonging to a sub interval of length larger than  $T_{li}/2$ .

- 3) Next, we estimate an upper bound on the time that  $\hat{z}_2$  takes to flow from  $\frac{d/c}{2^{i-1}}$  to  $-\frac{d/c}{2^{i-1}}$ . Using (30) and (31), we conclude that such a time is upper bounded by the time that  $z_2$  takes to flow from  $\frac{d/c}{2^{i-1}} + \frac{\varepsilon}{2^{i-1}}$  to  $-\frac{d/c}{2^{i-1}} - \frac{\varepsilon}{2^{i-1}}$  when

$$z_1(t) = \begin{cases} \frac{z_{in}^*}{2^{i-1}} + \frac{z_{in}^*}{2^{i-1}} g(i-1) & \forall t \in [0, \frac{T_{li}}{2}] \\ -\frac{|z^*|}{2} & \forall t \geq \frac{T_{li}}{2}. \end{cases}$$

Let us denote such a time by  $T_{ui}$ . Note that  $T_{ui}$  can be easily obtained by solving the linear switched dynamics:

$$\dot{z}_2 = \begin{cases} \left[ \frac{z_{in}^*}{2^{i-1}} + \frac{z_{in}^*}{2^{i-1}} g(i-1) \right] (cz_2 + d) & \forall t \in [0, \frac{T_{li}}{2}] \\ -\frac{|z^*|}{2} (cz_2 + d) & \forall t \geq \frac{T_{li}}{2}, \end{cases}$$

from the initial condition  $z_{2o} = \frac{d/c+\varepsilon}{2^{i-1}}$ .

Finally, we conclude that Assumption 1 holds on Cycle  $i$  with

$$\begin{aligned} \tau_{di} &:= \frac{T_{li}}{2}, & \tau_{si} &:= T_{ui} - \frac{T_{li}}{2}, \\ \bar{z}_i &:= \frac{z_{in}^*}{2^{i-1}} + \frac{z_{in}^*}{2^{i-1}} g(i-1), & \underline{z}_i &:= \frac{|z^*|}{2}, \end{aligned}$$

which completes the proof.  $\blacksquare$

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