A Note on Stability of Event-Triggered Control Systems with Time Delays

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Abstract

This note studies stability of event-triggered control systems with the event-triggered control algorithm proposed in [1]. We construct a novel Halanay-type inequality, which is used to show that sufficient conditions of the main results in [1] ensure stability of the event-triggered control systems that was missing in [1]. It is also shown that a positive parameter in the proposed event-triggering condition in [1] can be freely selected to exclude Zeno behavior from the event-triggered control system. An illustrative example is investigated to demonstrate the theoretical results of this study with numerical simulations.

Index Terms

Event-triggered control, nonlinear system, time delay, stability

I. INTRODUCTION

In [1], an event-triggered control algorithm was proposed for nonlinear time-delay systems. In the designed event-triggering condition, the exponential function $ae^{-b(t-t_0)}$ with tunable parameters *a* and *b* plays a vital role in ensuring boundedness and attractivity of the system states while excluding Zeno behavior, a phenomenon that the control updates are triggered infinitely many

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times over a finite time period. However, stability criterion with the proposed event-triggered control algorithm was not established in [1] when a > 0.

In this note, we will show that event-triggered control systems with positive a in [1] actually are stable if the sufficient conditions in Theorem 2 (or Theorem 3) of [1] are satisfied. We will also prove that the positive parameter b can be chosen arbitrarily, whereas the results in [1] require the positive b to be upper bounded, in order to exclude Zeno behavior from the control system.

The rest of this note is organized as follows. In Section I, we rephrase the event-triggered control problem for general nonlinear time-delay systems considered in [1]. To show stability of the event-triggered control systems, a new Halanay-type inequality is introduced in Section III. Main results are presented in Section IV with some remarks to further elaborate the improvements achieved in this study. We illustrate the main results by a numerical example in Section V, and finally draw conclusions in Section VI.

II. PROBLEM FORMULATION

To make this note self-contained, we adopt the notations from [1] and briefly introduce the event-triggered control problem formulated in [1] in this section.

Denote by \mathbb{N} the set of positive integers, \mathbb{R} the set of real numbers, \mathbb{R}^+ the set of nonnegative reals, and \mathbb{R}^n the *n*-dimensional real space equipped with the Euclidean norm denoted by $\|\cdot\|$. For $a, b \in \mathbb{R}$ with b > a, we define

 $\mathcal{PC}([a,b],\mathbb{R}^n) = \{\phi : [a,b] \to \mathbb{R}^n \mid \phi \text{ is piecewise right-}$

continuous}

$$\mathcal{P}C([a,\infty),\mathbb{R}^n) = \{\phi : [a,\infty) \to \mathbb{R}^n \mid \phi|_{[a,c]} \in \mathcal{P}C([a,c],\mathbb{R}^n)$$

for all $c > a\}$

where $\phi|_{[a,c]}$ is a restriction of ϕ on interval [a, c]. Let $C(J, \mathbb{R}^n)$ denote the set of continuous functions from interval J to \mathbb{R}^n . Given $\tau > 0$, the linear space $C([-\tau, 0], \mathbb{R}^n)$ is equipped with the supremum norm $\|\phi\|_{\tau} := \sup_{s \in [-\tau, 0]} \|\phi(s)\|$ for $\phi \in C([-\tau, 0], \mathbb{R}^n)$. A function $\alpha : \mathbb{R}^+ \to \mathbb{R}$ is said to be of class \mathcal{K} and we write $\alpha \in \mathcal{K}$, if α is continuous, strictly increasing, and satisfies $\alpha(0) = 0$. If $\alpha \in \mathcal{K}$ and also $\alpha(s) \to \infty$ as $s \to \infty$, we say that α is of class \mathcal{K}_{∞} and we write $\alpha \in \mathcal{K}_{\infty}$. A continuous function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be of class $\mathcal{K}\mathcal{L}$ and we write $\beta \in \mathcal{K}\mathcal{L}$, if the function $\beta(\cdot, t)$ is of class \mathcal{K} for each fixed $t \in \mathbb{R}^+$, and the function $\beta(s, \cdot)$ is decreasing and $\beta(s, t) \to 0$ as $t \to \infty$ for each fixed $s \in \mathbb{R}^+$.

Next, we recall the event-triggered control problem in [1]. Consider the following sampled-data control system

$$\begin{cases} \dot{x}(t) = f(t, x_t, u(t)) \\ u(t) = k(x(t_i)), \ t \in [t_i, t_{i+1}) \\ x_{t_0} = \varphi \end{cases}$$
(1)

where $x \in \mathbb{R}^n$ is the system state; control input $u \in \mathbb{R}^m$ is the state feedback control regulated by the feedback control law $k : \mathbb{R}^n \mapsto \mathbb{R}^m$ satisfying k(0) = 0; the sampling time sequence $\{t_i\}_{i \in \mathbb{N}}$ is to be determined by a certain triggering condition from [1] which will be introduced later; $\varphi \in C([-\tau, 0], \mathbb{R}^n)$ represents the initial function; $f : \mathbb{R}^+ \times C([-\tau, 0], \mathbb{R}^n) \times \mathbb{R}^m \mapsto \mathbb{R}^n$ satisfies f(t, 0, 0) = 0 for all $t \in \mathbb{R}^+$, and hence system (1) admits the zero solution; given a time *t*, the function x_t is defined as $x_t(s) := x(t + s)$ for $s \in [-\tau, 0]$, and $\tau > 0$ represents the maximum involved delay in the system.

Let us denote the sampling error by

$$\epsilon(t) = x(t_i) - x(t), \text{ for } t \in [t_i, t_{i+1}).$$

$$(2)$$

Then system (1) can be written as the following closed-loop system

$$\begin{cases} \dot{x}(t) = f(t, x_t, k(x + \epsilon)) \\ x_{t_0} = \varphi \end{cases}$$
(3)

To introduce the triggering condition for determining the time sequence $\{t_i\}_{i\in\mathbb{N}}$, we recall some concepts related to the Lyapunov functional candidate for time-delay systems. A function $V : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^+$ is said to be of class \mathcal{V}_0 and we write $V \in \mathcal{V}_0$, if, for each $x \in C(\mathbb{R}^+, \mathbb{R}^n)$, the composite function $t \mapsto V(t, x(t))$ is continuous. A functional $V : \mathbb{R}^+ \times C([-\tau, 0], \mathbb{R}^n) \mapsto \mathbb{R}^+$ is said to be of class \mathcal{V}_0^* and we write $V \in \mathcal{V}_0^*$, if, for each function $x \in C([-\tau, \infty), \mathbb{R}^n)$, the composite function $t \mapsto V(t, x_t)$ is continuous in t for all $t \ge 0$, and V is locally Lipschitz in its second argument. Given an input $u \in \mathcal{P}C([t_0, \infty), \mathbb{R}^m)$, we define the upper right-hand Dini derivative of the Lyapunov functional candidate $V(t, x_t)$ with respect to system (1):

$$D^{+}V(t,\phi) = \limsup_{\varepsilon \to 0^{+}} \frac{V(t+\varepsilon, x_{t+\varepsilon}(t,\phi)) - V(t,\phi)}{\varepsilon}$$

where $x(t, \phi)$ denotes the solution to (1) satisfying $x_t = \phi$.

Assumption II.1. There exist functions $V_1 \in \mathcal{V}_0$, $V_2 \in \mathcal{V}_0^*$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\chi \in \mathcal{K}$, and constant $\mu > 0$ such that

- (i) $\alpha_1(\|\phi(0)\|) \le V_1(t,\phi(0)) \le \alpha_2(\|\phi(0)\|)$ and $0 \le V_2(t,\phi) \le \alpha_3(\|\phi\|_{\tau})$;
- (ii) $V(t, \phi) := V_1(t, \phi(0)) + V_2(t, \phi)$ satisfies

$$D^+V(t,\phi) \le -\mu V(t,\phi) + \chi(\|\epsilon\|).$$

Now we are in the position to introduce the triggering condition in [1] for system (1). To enforce ϵ to satisfy the condition

$$\chi(\|\boldsymbol{\epsilon}\|) \le \sigma \alpha_1(\|\boldsymbol{x}\|) + \chi \left(ae^{-b(t-t_0)}\right) \tag{4}$$

where $\sigma \ge 0$, $a \ge 0$, and b > 0 are constants, we update the control input *u* when the following triggering condition is satisfied

$$\chi(\|\boldsymbol{\epsilon}\|) = \sigma \alpha_1(\|\boldsymbol{x}\|) + \chi \left(ae^{-b(t-t_0)}\right).$$
⁽⁵⁾

When condition (5) holds, we say an event occurs and then a control update is executed. Therefore, the sampling times $\{t_i\}_{i \in \mathbb{N}}$ are determined as follows:

$$t_{i+1} = \inf \left\{ t \ge t_i \mid \chi(||\epsilon||) = \sigma \alpha_1(||x||) + \chi \left(a e^{-b(t-t_0)} \right) \right\},$$
(6)

which are also called event times. Since the event times are implicitly determined by the triggering condition (5), it is essential to exclude Zeno behavior, a phenomenon that infinitely many events happen over a finite time interval, from the closed-loop system.

It has been shown in [2] that Assumption II.1 implies the closed-loop system (3) is input-tostate stable with respect to the sampling error ϵ , and the state feedback control u(t) = k(x(t)) for $t \ge t_0$ renders the following closed-loop system

$$\begin{cases} \dot{x}(t) = f(t, x_t, k(x(t))) \\ x_{t_0} = \varphi \end{cases}$$
(7)

globally asymptotically stable. The objective of this study is to show that when a > 0, the sufficient conditions provided in [1] ensures global asymptotic stability of system (3) with the event times determined by (6), rather than just uniform boundedness and global attractivity as proved in [1], and also to show that the closed-loop system (3) does not exhibit Zeno behavior for any positive *b*, while the results in [1] require $b < \mu - \sigma$ to rule out Zeno behavior.

To show stability of the closed-loop system, we will introduce a novel Halanay-type inequality in the next section.

III. HALANAY-TYPE INEQUALITY

For a continuous function $g : \mathbb{R} \to \mathbb{R}$, the Dini-derivatives $D^+g(t)$ and $D_-g(t)$ are defined as follows:

$$D^+g(t) = \limsup_{\varepsilon \to 0^+} \frac{g(t+\varepsilon) - g(t)}{\varepsilon}$$

and

$$D_{-}g(t) = \liminf_{\varepsilon \to 0^{-}} \frac{g(t+\varepsilon) - g(t)}{\varepsilon}$$

The following lemma from [3] provides a relationship between $D^+g(t)$ and $D_-g(t)$, which will be used to construct our Halanay-type inequality.

Lemma 1. Let p and q be continuous functions with $D^+p(t) \le q(t)$ for t in some open interval I. Then $D_-p(t) \le q(t)$ for $t \in I$.

Next, we introduce a new Halanay-type inequality which allows us to bound the states of the event-triggered control system with time delays.

Lemma 2. Let $g: [t_0 - r, t_0 + \Gamma) \rightarrow \mathbb{R}^+$ be a continuous function satisfying

$$D^{+}g(t) \le \gamma_{1}g(t_{0}) + \gamma_{2}||g_{t}||_{r} \quad for \quad t_{0} \le t < t_{0} + \Gamma$$
(8)

where r, Γ , γ_1 , and γ_2 are positive constants. Then

 $g(t) \leq \|g_{t_0}\|_r e^{\lambda(t-t_0)} \quad for \quad t_0 \leq t < t_0 + \Gamma$

where $\lambda = \gamma_1 + \gamma_2$.

Proof. Define

$$w(t) = \begin{cases} ||g_{t_0}||_r e^{\lambda(t-t_0)}, & \text{if } t_0 < t < t_0 + \Gamma \\ ||g_{t_0}||_r, & \text{if } t_0 - r \le t \le t_0 \end{cases}$$

and let K > 1 be an arbitrary constant. Then, for $t \in [t_0 - r, t_0]$, we have

$$g(t) \le ||g_{t_0}||_r = w(t) < Kw(t), \tag{9}$$

that is, g(t) < Kw(t) for $t \in [t_0 - r, t_0]$.

Next, we use a contradiction argument to show that g(t) < Kw(t) for $t \in (t_0, t_0 + \Gamma)$. Suppose there exits some $t \in (t_0, t_0 + \Gamma)$ such that $g(t) \ge Kw(t)$, then we define

$$\bar{t} = \inf \{t \in (t_0, t_0 + \Gamma) \mid g(t) \ge Kw(t)\}$$

From the continuities of g and w, we have

$$g(t) < Kw(t) \quad \text{for} \quad t_0 < t < \overline{t} \tag{10}$$

and

$$g(\bar{t}) = Kw(\bar{t}). \tag{11}$$

By (10) and (11), we conclude that

$$\frac{g(\bar{t} + \varepsilon) - g(\bar{t})}{\varepsilon} > \frac{Kw(\bar{t} + \varepsilon) - Kw(\bar{t})}{\varepsilon}$$

for $\varepsilon < 0$ close to 0. Hence,

$$\mathbf{D}_{-g}(\bar{t}) \ge K \dot{w}(\bar{t}). \tag{12}$$

On the other hand, by Lemma 1 and (8), we get

$$D_{-}g(\bar{t}) \leq \gamma_{1}g(t_{0}) + \gamma_{2}||g_{\bar{t}}||_{r}$$
$$< \gamma_{1}Kw(t_{0}) + \gamma_{2}K||w_{\bar{t}}||_{r}$$
$$< (\gamma_{1} + \gamma_{2})Kw(\bar{t})$$
$$= \lambda Kw(\bar{t})$$
$$= K\dot{w}(\bar{t})$$

where we used (9), (10), (11), and the definition of *w* in the last two inequalities. This is a contradiction to (12). Therefore, we conclude that g(t) < Kw(t) for $t \in (t_0, t_0 + \Gamma)$.

Since K > 1 is arbitrary, we let $K \to 1$ and then $g(t) \le w(t)$ for $t \in [t_0, t_0 + \Gamma)$, that is, the proof is completed.

Discussions on this lemma will be provided in Remark 2 with the main results in the following section.

IV. MAIN RESULTS

Now we are ready to introduce the main results of this study.

Theorem 1. Suppose that Assumption II.1 holds with $V_1 \in \mathcal{V}_0$, $V_2 \in \mathcal{V}_0^*$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\chi \in \mathcal{K}$, and constant $\mu > 0$. The event times $\{t_i\}_{i \in \mathbb{N}}$ are defined by (6) with $0 \le \sigma < \mu$, a > 0, and b > 0. We further assume that

(iii) α_1^{-1} , χ , and k are locally Lipschitz, where α_1^{-1} denotes the inverse of α_1 ;

(iv) f is locally Lipschitz in its second and third arguments, respectively.

Then the closed-loop system (3) is globally asymptotically stable and does not exhibit Zeno behavior.

Proof. Let

$$\eta := \begin{cases} \min\{b, \mu - \sigma\}, & \text{if } b \neq \mu - \sigma\\ \xi, & \text{if } b = \mu - \sigma \end{cases}$$
(13)

where $\xi < b$ is a positive constant, and

$$\bar{M} := \begin{cases} \frac{aL}{|\mu - \sigma - b|}, & \text{if } b \neq \mu - \sigma \\ \frac{aL}{|\mu - \sigma - \xi|}, & \text{if } b = \mu - \sigma \end{cases}$$

then from the proof of Theorem 2 in [1], we have

$$\|x(t)\| \le \alpha_1^{-1} (M e^{-\eta(t-t_0)}) \le \alpha_1^{-1} (M)$$
(14)

for all $t \ge t_0$, where $M = \alpha_2(\|\varphi(0)\|) + \alpha_3(\|\varphi\|_{\tau}) + \overline{M}$. Global attractivity of the zero solution follows from (14), that is, $\lim_{t\to\infty} \|x(t)\| = 0$ for any initial condition $\varphi \in C([-\tau, 0], \mathbb{R}^n)$.

Next, we show stability of the closed-loop system (3). From the system dynamics of (1) on $[t_i, t_{i+1})$ and the Lipschitz conditions on f and k, we have

$$D^{+} ||x(t)|| \le ||\dot{x}(t)|| = ||f(t, x_{t}, k(x(t_{i})))||$$

$$\le L_{2} ||x_{t}||_{\tau} + L_{3} ||x(t_{i})||$$
(15)

which implies (8) holds on $[t_i, t_{i+1})$ with $r = \tau$, g(t) = ||x(t)||, $\gamma_1 = L_3$, and $\gamma_2 = L_2$, where L_2 is the Lipschitz constant of the function $f(t, \cdot, u) : C([-\tau, 0], \mathbb{R}^n) \mapsto \mathbb{R}^n$ on the compact set $\{\phi \in C([-\tau, 0] \mid ||\phi||_{\tau} \leq R\}$, and L_3 is the Lipschitz constant of the composite function $f(t, \phi, k(\cdot)) : \mathbb{R}^n \mapsto \mathbb{R}^n$ on the compact set $\{x \in \mathbb{R}^n \mid ||x|| \leq R\}$ with $R = \alpha_1^{-1}(M)$.

We then conclude from Lemma 2 that

$$||x(t)|| \le ||x_{t_i}||_{\tau} e^{\lambda(t-t_i)} \text{ for } t_i \le t < t_{i+1} \text{ and } i \in \mathbb{Z}^+$$
 (16)

where $\lambda = L_2 + L_3$ and \mathbb{Z}^+ denotes the set of non-negative integers.

We next use mathematical induction to show that $||x(t)|| \le ||\varphi||_{\tau} e^{\lambda(t-t_0)}$ for all $t \ge t_0$. We can see from (16) that this statement holds for $t \in [t_0, t_1)$. Suppose $||x(t)|| \le ||\varphi||_{\tau} e^{\lambda(t-t_0)}$ holds for $t \in [t_0, t_i)$, and we will show this inequality holds for $t \in [t_i, t_{i+1})$. By (16), we get

$$\begin{aligned} \|x(t)\| &\leq \|x_{t_i}\|_{\tau} e^{\lambda(t-t_i)} \\ &= e^{\lambda(t-t_i)} \sup_{s \in [-\tau,0]} \|x(t_i+s)\| \\ &\leq e^{\lambda(t-t_i)} \|\varphi\|_{\tau} e^{\lambda(t_i-t_0)} \\ &= \|\varphi\|_{\tau} e^{\lambda(t-t_0)} \quad \text{for} \quad t_i \leq t < t_{i+1} \end{aligned}$$

that is, $||x(t)|| \le ||\varphi||_{\tau} e^{\lambda(t-t_0)}$ for all $t \in [t_0, t_{i+1})$. By induction, we conclude that

$$||x(t)|| \le ||\varphi||_{\tau} e^{\lambda(t-t_0)}$$
 for all $t \ge t_0$. (17)

Then (17) and (14) imply

$$\alpha_1(||x(t)||) \le \min\left\{\alpha_1\left(||\varphi||_\tau e^{\lambda(t-t_0)}\right), M e^{-\eta(t-t_0)}\right\} \quad \text{for all} \quad t \ge t_0.$$
(18)

Let $\delta_1 = \inf\{s \ge 0 : \alpha_2(s) + \alpha_3(s) \ge \overline{M}\}$, then $\|\varphi\|_{\tau} < \delta_1$ implies $M = \alpha_2(\|\varphi(0)\|) + \alpha_3(\|\varphi\|_{\tau}) + \overline{M} < 2\overline{M}$ and

$$\alpha_1(\|x(t)\|) \le \min\left\{\alpha_1\left(\|\varphi\|_{\tau} e^{\lambda(t-t_0)}\right), 2\bar{M} e^{-\eta(t-t_0)}\right\} \quad \text{for all} \quad t \ge t_0.$$
(19)

Since α_1 is strictly increasing, we have that $\|\varphi\|_{\tau} < \delta_2$ implies $\alpha_1(\|\varphi\|_{\tau}) < 2\bar{M}$ where $\delta_2 = \alpha_1^{-1}(2\bar{M})$.

Next, we consider the initial function φ satisfying $\|\varphi\|_{\tau} < \min\{\delta_1, \delta_2\}$. Then, there exists a unique $\hat{t} > t_0$ such that

$$\alpha_1\left(\|\varphi\|_{\tau}e^{\lambda(\hat{t}-t_0)}\right) = 2\bar{M}e^{-\eta(\hat{t}-t_0)},$$

where we used the fact that both $\alpha_1(\|\varphi\|_{\tau}e^{\lambda(t-t_0)})$ and $2\bar{M}e^{-\eta(t-t_0)}$ are strictly monotonic in *t*. Furthermore, for any $\varepsilon > 0$, there exists a δ_3 , depending on ε , such that $\|\varphi\|_{\tau} < \delta_3$ implies

$$\alpha_1\left(\|\varphi\|_{\tau}e^{\lambda(\hat{t}-t_0)}\right) = 2\bar{M}e^{-\eta(\hat{t}-t_0)} < \alpha_1(\varepsilon),$$

that is, small enough $\|\varphi\|_{\tau}$ leads to large enough \hat{t} so that $2\bar{M}e^{-\eta(\hat{t}-t_0)} < \alpha_1(\varepsilon)$. Note that \bar{M} is independent of $\|\varphi\|_{\tau}$.

$$\alpha_1(\|x(t)\|) \le 2\bar{M}e^{-\eta(\hat{t}-t_0)} < \alpha_1(\varepsilon)$$

for $t \ge t_0$, that is, $||x(t)|| < \varepsilon$. The stability proof is completed.

Therefore, if a > 0 in the triggering condition (5), we can conclude from stability and global attractivity of the zero solution that the closed-loop system (3) is globally asymptotically stable.

Last but not least, we show that system (3) does not exhibit Zeno behavior for any b > 0. It has been shown in [1] that the inter-event times $\{t_i - t_{i-1}\}_{i \in \mathbb{N}}$ are lower bounded by a positive quantity when $b < \mu - \sigma$, that is, system (3) does not exhibit Zeno behavior. Hence, we will focus on the scenario of $b \ge \mu - \sigma$.

It follows from the proof of Theorem 2 in [1] that

$$ae^{-b(t_{i+1}-t_0)} \le \lambda_1 \left(e^{-\eta(t_i-t_0)} - e^{-\eta(t_{i+1}-t_0)} \right) + \lambda_2 \left(t_{i+1} - t_i \right) e^{-\eta(t_i-t_0)}$$
(20)

where $\lambda_1 = L_1 L_2 M e^{\eta \tau} / \eta > 0$, $\lambda_2 = L_1 L_3 M > 0$, and L_1 is the Lipschitz constant of α_1^{-1} on the interval [0, M]. Let $T_{i+1} = t_{i+1} - t_i$, then multiplying both sides of (20) by $e^{\eta(t_i - t_0)}$ yields

$$ae^{-bT_{i+1}}e^{(\eta-b)(t_i-t_0)} \le \lambda_1 \left(1 - e^{-\eta T_{i+1}}\right) + \lambda_2 T_{i+1}.$$
(21)

Next we use contradiction argument to show that system (3) is free of Zeno behavior. Suppose that there exists a $\bar{t} < \infty$ such that $\lim_{i\to\infty} t_i = \bar{t}$, that is, $t_i < \bar{t}$ for all $i \in \mathbb{N}$. The fact $b \ge \mu - \sigma$ and the definition of η imply $\eta < b$. It then follows from (21) that

$$ae^{-bT_{i+1}}e^{(\eta-b)(\bar{t}-t_0)} \le \lambda_1 \left(1 - e^{-\eta T_{i+1}}\right) + \lambda_2 T_{i+1}.$$
(22)

Define a function

$$g(T) = ae^{-bT}e^{(\eta-b)(\bar{t}-t_0)} - \lambda_1 \left(1 - e^{-\eta T}\right) - \lambda_2 T$$

for $T \ge 0$. It can be observed that $g(0) = ae^{(\eta-b)(\bar{t}-t_0)} > 0$, $\lim_{T\to\infty} g(T) = -\infty$, and

$$g'(T) = -bae^{-bT}e^{(\eta-b)(\bar{t}-t_0)} - \eta\lambda_1 e^{-\eta T} - \lambda_2 < 0.$$

Thus, g(T) is a strictly decreasing function, and then the equation g(T) = 0 has a unique solution $T^* > 0$. Moreover, g(T) < 0 if $T > T^*$. On the other hand, it follows from (22) that $g(T_{i+1}) < 0$, and then $T_{i+1} = t_{i+1} - t_i > T^*$, that is, the inter-event times $\{t_{i+1} - t_i\}_{i \in \mathbb{N}}$ are bounded by $T^* > 0$ from below. This contradicts to the assumption $\lim_{i\to\infty} t_i = \overline{t} < \infty$. Therefore, system (3) with (6) does not exhibit Zeno behavior when $b \ge \mu - \sigma$, which completes the proof.

Remark 1. It should be mentioned that the above proof does not reply on the Lipschitz condition on α_1^{-1} . Nevertheless, if α_1^{-1} is locally Lipschitz, then \hat{t} and δ can be obtained explicitly. To be more specific, suppose α_1^{-1} is locally Lipschitz, and we can show stability as follows. Combining (17) and (14) yields

$$\|x(t)\| \le \min\left\{ \|\varphi\|_{\tau} e^{\lambda(t-t_0)}, L_1 M e^{-\eta(t-t_0)} \right\} \quad for \ all \ t \ge t_0,$$
(23)

where L_1 is the Lipschitz constant of α_1^{-1} on interval [0, M]. Consider $\|\varphi\|_{\tau} < \min\{\delta_1, \bar{\delta}_2\}$ with $\bar{\delta}_2 = \alpha_1^{-1}(2L_1\bar{M})$, then $M < 2\bar{M}$, $\|\varphi\|_{\tau} < 2L_1\bar{M}$, and there exists a unique $\hat{t} > t_0$ such that

$$\|\varphi\|_{\tau}e^{\lambda(\hat{t}-t_0)} = 2L_1\bar{M}e^{-\eta(\hat{t}-t_0)},$$

and then \hat{t} can be derived as

$$\hat{t} = \frac{\ln\left(\frac{2L_1M}{\|\varphi\|_{\tau}}\right)}{\lambda + \eta} + t_0.$$

By (23) and the definition of \hat{t} , we have

$$\|x(t)\| < 2L_1 \bar{M} e^{-\eta(t-t_0)}$$

$$= 2L_1 \bar{M} \exp\left(\frac{-\eta}{\lambda+\eta} \ln\left(\frac{2L_1 \bar{M}}{\|\varphi\|_{\tau}}\right)\right)$$

$$= \|\varphi\|_{\tau}^{\frac{\eta}{\lambda+\eta}} \left(2L_1 \bar{M}\right)^{\frac{\lambda}{\lambda+\eta}}$$
(24)

for all $t \ge t_0$. For any $\varepsilon > 0$, let $\delta = \min\{\delta_1, \overline{\delta}_2, \delta_3\}$ with $\delta_3 = \varepsilon^{(\lambda+\eta)/\eta} (2L_1 \overline{M})^{-\lambda/\eta}$. For $\|\varphi\|_{\tau} < \delta$, we can derive from (24) that

$$\|x(t)\| < \|\varphi\|_{\tau}^{\frac{\eta}{\lambda+\eta}} \left(2L_1\bar{M}\right)^{\frac{\lambda}{\lambda+\eta}} < \varepsilon$$

for $t \ge t_0$, that is, the closed-loop system (3) is stable. It can be seen that δ is given explicitly since δ_1 , $\overline{\delta}_2$, and δ_3 are specifically defined.

Remark 2. The upper bound $\alpha_1^{-1}(Me^{-\eta(t-t_0)})$ of the state norm in (14) guarantees attractivity of the closed-loop system. However, $M = \alpha_2(||\varphi(0)||) + \alpha_3(||\varphi||_{\tau}) + \overline{M}$ depends not only on the initial function φ but also on parameter a in \overline{M} . Since \overline{M} is independent of the initial function φ , the stability criterion of the event-triggered control system couldn't be derived from this upper bound solely. The role of Lemma 2 is to provide another bound in (17) for ||x||. Combining these two bounds in (18) or (23) allows stability analysis for the closed-loop system. Lemma 2 is different from the existing Halanay-type inequalities (see, e.g., [3], [4]) in the following sense. In the existing Halanay-type inequalities, the Dini derivative $D^+g(t)$ is bounded by the sum of a function of g(t) and a function of $||g_t||_r$, while in Lemma 2 we bound $D^+g(t)$ by a linear combination of $g(t_0)$ and $||g_t||_r$. This major difference in the dependence of g at the initial time t_0 allows the estimation of the state bound over each interval $[t_i, t_{i+1})$ since the control input is unchanged during two consecutive event times.

For quadratic forms of V_1 , the inverse of the \mathcal{K} class function α_1 in Assumption II.1(i) is not locally Lipschitz in its domain. Hence, Theorem 1 cannot be applied for such types of Lyapunov candidates. Nevertheless, the following result allows us to use quadratic forms of Lyapunov function as V_1 in the Lyapunov candidate V.

Theorem 2. Suppose all the conditions of Theorem 1 are satisfied, and the Lipschitz assumption on α_1^{-1} is replaced by the following condition:

• α_1^{-1} is Lipschitz on any closed and bounded sub-interval of $(0, \infty)$.

Then the closed-loop system (3) with the event times determined by (6) is globally asymptotically stable and does not exhibit Zeno behavior.

Proof. From the discussion in Remark 1, we can conclude that the proof of Theorem 1 also implies global asymptotic stability of the closed-loop system under the conditions of Theorem 2. When $b < \mu - \sigma$, the non-existence of Zeno behavior has been shown in [1]. If $b \ge \mu - \sigma$, the exclusion of Zeno behavior is identical to the contradiction argument in the proof of Theorem 1. Therefore, the detailed proof is omitted.

Remark 3. Compared with the results in [1], the improvements that Theorems 1 and 2 have achieved for a > 0 are as follows. Under the sufficient conditions established in [1], the closedloop system (3) with the sequence of event times determined by (6) is guaranteed to be globally asymptotically stable, while the results in [1] only showed the closed-loop system is uniformly bounded and globally attractive. Furthermore, the positive parameter b can be chosen arbitrarily to rule out Zeno behavior from the closed-loop system, whereas the results in [1] need $b < \mu - \sigma$ to exclude Zeno behavior. In summary, if Assumption II.1 and conditions (iii), (iv) of Theorem 1 hold, then the tunable parameter σ can be chosen so that $\sigma < \mu$, and both parameters a and b can be selected freely in the triggering condition (5). We refer the reader to [1] for a detailed discussion about the effects of different parameter selections on the dynamic performance of the event-triggered control system.

V. AN ILLUSTRATIVE EXAMPLE

Consider the following nonlinear time-delay control system

$$\begin{cases} \dot{x}_1(t) = x_2(t) + \omega_0 x_1^2(t) x_2(t) + u(t) \\ \dot{x}_2(t) = -\omega_2 x_1(t) - \omega_1 x_2(t) - \omega_3 x_1(t-1) - \omega_2 x_1^3(t) \end{cases}$$
(25)

where $x(t) = (x_1(t), x_2(t))^{\top} \in \mathbb{R}^2$, ω_i with i = 0, 1, 2, 3 are non-negative constants, and $u(t) = -x_1(t)$ is the feedback control. System (25) has been widely used to model the machine tool chatter in the cutting process (see, e.g., [5] and references therein). In this example, we consider the following parameters: $\omega_0 = 1$, $\omega_1 = 0.5$, $\omega_2 = 1$, and $\omega_3 = 0.3$.

To show system (25) with the given feedback control is asymptotically stable and to design the event-triggered control implementation, we consider the Lyapunov functional $V(t) = V_1(t) + V_2(t)$ with

$$V_1(t) = x^{\top}(t)x(t)$$

and

$$V_2(t) = \delta \int_{t-1}^t e^{-\zeta(t-s)} x^{\mathsf{T}}(s) x(s) \mathrm{d}s$$

where $\delta = 0.4$ and $\zeta = 0.28$. It can be seen that Assumption II.1(i) holds with $\alpha_1(s) = s^2$, and its inverse function is not locally Lipschitz but Lipschitz on any closed and bounded sub-interval of $(0, \infty)$.

Under the sampled-data implementation, system (25) can be written as the following closedloop system

$$\begin{cases} \dot{x}_1(t) = x_2(t) + \omega_0 x_1^2(t) x_2(t) - x_1(t) - \epsilon_1(t) \\ \dot{x}_2(t) = -\omega_2 x_1(t) - \omega_1 x_2(t) - \omega_3 x_1(t-1) - \omega_2 x_1^3(t) \end{cases}$$
(26)

where $\epsilon_1(t) = x_1(t_i) - x_1(t)$ for $t \in [t_i, t_{i+1})$, and the event times $\{t_i\}_{i \in \mathbb{N}}$ are to be determined by the event-triggering condition (5).

To verify condition (iii) of Theorem 1, it yields from the dynamics of system (25) that

$$\dot{V}_{1}(t) = 2x_{1}(t)\dot{x}_{1}(t) + 2x_{2}(t)\dot{x}_{2}(t)$$

$$= (2 - 2\omega_{2})x_{1}(t)x_{2}(t) + (2 - 2\omega_{2})x_{1}^{3}(t)x_{2}(t) - 2x_{1}^{2}(t)$$

$$- 2\omega_{1}x_{2}^{2}(t) - 2x_{1}(t)\epsilon_{1}(t) - 2\omega_{3}x_{2}(t)x_{1}(t - 1)$$
(27)

and

$$\dot{V}_2(t) = -\zeta V_2(t) + \delta V_1(t) - \delta e^{-\zeta} (x_1^2(t-1) + x_2^2(t-1)),$$
(28)

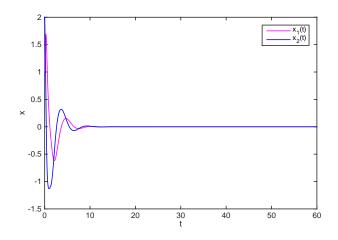


Fig. 1. System trajectories with the state feedback control $u(t) = -x_1(t)$.

then,

$$\dot{V}(t) \leq (2 - 2\omega_2)x_1(t)x_2(t) + (2 - 2\omega_2)x_1^3(t)x_2(t) + \epsilon_1^2(t) - \delta e^{-\zeta}x_2^2(t-1) + (\omega_3 - \delta e^{-\zeta})x_1^2(t-1) - \zeta V(t) + (-2 + 1 + \zeta + \delta)x_1^2(t) + (-2\omega_1 + \omega_3 + \lambda + \delta)x_2^2(t) \leq - 0.28V(t) + ||\epsilon(t)||^2$$
(29)

where $\epsilon(t) = (\epsilon_1(t), \epsilon_2(t))^\top = x(t_i) - x(t)$ for $t \in [t_i, t_{i+1})$. Therefore, condition (iii) is satisfied with $\mu = 0.28$ and $\chi(||\epsilon||) = ||\epsilon(t)||^2$.

Based on the above analysis, the event-triggering condition (5) can be written as follows:

$$\|\epsilon(t)\| = \sigma \|x(t)\|^2 + \left(ae^{-b(t-t_0)}\right)^2$$
(30)

where $\sigma < \mu$, and parameters *a* and *b* can be chosen arbitrarily. Theorem 2 concludes that system (26) with the triggering condition (30) is globally asymptotically stable. Simulation results are shown in Fig. 1, Fig. 2, and Fig. 3 with initial condition $x(s) = (1, 2)^{T}$ for $s \in [-1, 0]$, initial time $t_0 = 0$, and parameters $\sigma = 0.16$, a = 1, b = 0.14. It should be noted that $b > \mu - \sigma = 0.12$ in our simulations, and hence the results in [1] are not applicable for these parameter selections.

VI. CONCLUSIONS

We have revisited the event-triggered control problem for time-delay systems considered in [1]. It has been shown that under the sufficient conditions proposed in [1] the event-triggered

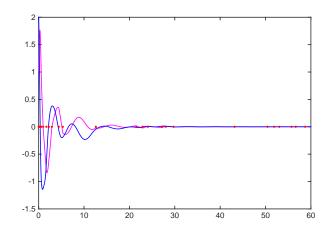


Fig. 2. System trajectories with the proposed event-triggered control mechanism. Red dots on the time axis indicate the event times.

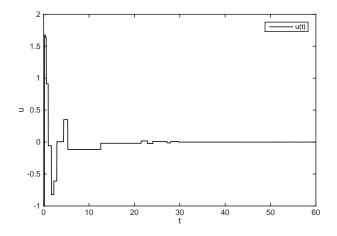


Fig. 3. Event-triggered control input under the triggering condition (30).

control system is globally asymptotically stable rather than just uniformly bounded and globally attractive. Moreover, our analysis has allowed us to arbitrarily choose a parameter in the event-triggering condition to both ensure global asymptotic stability and non-existence of Zeno behavior in the event-triggered time-delay control systems. A numerical example has been presented to verify the theoretical results.

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