Scale-free Non-collaborative Linear Protocol Design for A Class of Homogeneous Multi-agent Systems

Zhenwei Liu, Ali Saberi, and Anton A. Stoorvogel ***

April 16, 2024

Abstract

In this paper, we have focused on identifying a class of continuous- and discrete-time MAS for which a *scale-free non-collaborative* (i.e., scale-free fully distributed) *linear* protocol design is developed. We have identified conditions on agent models that enable us to design scalable linear protocols. Moreover, we show that these conditions are necessary if the agents are single input and single output. We also provide a complete design of scalable protocols for this class.

1 Introduction

In recent decades, the synchronization problem for multi-agent systems (MAS) has attracted substantial attention due to the wide potential for applications in several areas, see for instance the books [1, 2, 7, 13, 17, 19, 26] and references [8, 15, 16], etc.

In the synchronization literature, the communication between agents is based on measurements of the difference between the output of a specific agent and the

^{*}Zhenwei Liu is with College of Information Science and Engineering, Northeastern University, Shenyang 110819, China (e-mail: liuzhenwei@ise.neu.edu.cn)

[†]Ali Saberi is with School of Electrical Engineering and Computer Science, Washington State University, Pullman, WA 99164, USA (e-mail: saberi@wsu.edu)

[‡]Anton A. Stoorvogel is with Department of Electrical Engineering, Mathematics and Computer Science, University of Twente, P.O. Box 217, Enschede, The Netherlands (e-mail: A.A.Stoorvogel@utwente.nl)

output of neighboring agents:

$$\zeta_i = \sum_{j=1}^N a_{ij} (y_i - y_j)$$

where y_i denotes the output of agent and a_{ij} is constant for $i, j = 1, \dots, N$.

Non-collaborative protocol only uses the relative measurement ζ_i and achieves fully distributed protocols. Collaborative protocols have been traditionally presented in MAS literature (see the above books on MAS). On the other hand, collaborative protocols allows extra information exchange between neighbors. Typically, this additional information exchange consists of relative information about the difference between the state of the protocol of a specific agent and the state of the protocol of a neighboring agent using the same network.

Collaborative protocols which were introduced by [8] has been utilized to somewhat relax the solvability conditions for partial-state coupling as has been documented in the book [19]. Loosely speaking by allowing the extra communication exchange, the solvability conditions for partial-state coupling reduced to solvability conditions for full-state coupling. Using non-collaborative protocols, the solvability conditions for partial-state coupling place strong restrictions on either poles or zeros of the agent model (see [21]) and, in contrast, these conditions are not required in full-state coupling. Moreover, some relaxation on network knowledge also occurs in full-state coupling. This should be apparent since protocol design for partial-state coupling requires a distributed observer which is not needed in full-state coupling.

On the other hand, most of the proposed protocols in the literature for synchronization of MAS requires some knowledge of the communication network such as bounds on the spectrum of the associated Laplacian matrix or the number of agents. As it is pointed out in [22–25], these protocols suffer from *scale fragility* where stability properties are lost when the size of network increases, or when communication network is altered, such as increases or decreases in the size of sensing neighborhoods.

In the past few years, a scale-free protocol design has been the subject of current research for MAS. Scale-free protocol design addresses this issue by designing protocols which do not rely on any knowledge about the communication graph, i.e.,

- 1. The protocol is designed only based on knowledge of the agent model (A, B, C).
- 2. The protocol is designed to work with any fixed communication graph which contains a spanning tree without incorporating knowledge about the graph into the protocol.

Almost all results of scalable protocols available in the literature are collaborative, see [4,9,10,14]. To the best of our knowledge, the scalable non-collaborative protocols are only for MAS with passive or passifiable agents, see [3] and [11].

In this paper, the main objective is to show when it is possible to achieve a scalefree design which is non-collaborative and hence only relies on the original relative measurement ζ_i . We present necessary conditions and design protocols to achieve this objective under assumptions which are very close to these necessary conditions. More specifically we have identified one class of continuous- and discrete-time MAS for which scalable non-collaborative (i.e., scalable fully distributed) linear protocols can be designed.

Notations and Background

Given a matrix $A \in \mathbb{R}^{m \times n}$, A^{T} and A^{*} denote its transpose and conjugate transpose respectively. A square matrix A is said to be Hurwitz stable if all its eigenvalues are in the open left half complex plane, and A is said to be Schur stable if all its eigenvalues are in the open unit disk. $A \otimes B$ depicts the Kronecker product between A and B. I_n denotes the *n*-dimensional identity matrix and 0_n denotes $n \times n$ zero matrix; sometimes we drop the subscript if the dimension is clear from the context.

To describe the information flow among the agents we associate a *weighted* graph \mathcal{G} to the communication network. The weighted graph \mathcal{G} is defined by a triple $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, \ldots, N\}$ is a node set, \mathcal{E} is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is the weighted adjacency matrix with non negative elements a_{ij} . Each pair in \mathcal{E} is called an *edge*, where $a_{ij} > 0$ denotes an edge $(j, i) \in \mathcal{E}$ from node j to node i with weight a_{ij} . Moreover, $a_{ij} = 0$ if there is no edge from node j to node i. We assume there are no self-loops, i.e. we have $a_{ii} = 0$. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, \ldots, k - 1$. A *directed tree* is a subgraph (subset of nodes and edges) in which every node has exactly one parent node except for one node, called the *root*, which has no parent node. A *directed spanning tree* is a subgraph which is a directed tree containing all the nodes of the original graph. If a directed spanning tree exists, the root has a directed path to every other node in the tree [5].

For a weighted graph \mathcal{G} , the matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^{N} a_{ik}, i = j, \\ -a_{ij}, \quad i \neq j, \end{cases}$$

is called the *Laplacian matrix* associated with the graph \mathcal{G} . The Laplacian matrix L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector **1** [5]. Moreover, if the graph contains a

directed spanning tree, the Laplacian matrix L has a single eigenvalue at the origin and all other eigenvalues are located in the open right-half complex plane [17].

A row stochastic matrix D can be associated with a graph \mathcal{G} . N, the dimension of D, is the number of node and an arc $(j,i) \in \mathcal{E}$ if $d_{ij} > 0$. It is shown in [16] that 1 is a simple eigenvalue of D if and only if \mathcal{G} contains a directed spanning tree. Moreover, the other eigenvalues are in the open unit disk if $d_{ij} > 0$ for all i.

2 Problem formulation

Consider a homogeneous MAS composed of N identical linear time-invariant agents of the form,

$$\begin{aligned} x_i^+ &= A x_i + B u_i, \\ y_i &= C x_i, \end{aligned}$$
 (1)

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $y_i \in \mathbb{R}^p$ are the state, input, output of agent *i* for i = 1, ..., N. In the aforementioned presentation, for continuous-time systems, $x_i^+(t) = \dot{x}_i(t)$ for $t \in \mathbb{R}$; while for discrete-time systems, $x_i^+(t) = x_i(t+1)$ for $t \in \mathbb{Z}$.

The communication network is composed of N linear combinations and each combination includes agent's own output relative to that of other agents. Network with **continuous-time** agent is shown as follows for agent *i*:

$$\zeta_{i} = \sum_{j=1}^{N} a_{ij} (y_{i} - y_{j})$$
(2)

where $a_{ij} > 0$ and $a_{ii} = 0$. Here we use a weighted and directed graph G to describe the communication topology of the network, the nodes of network correspond to the agents and the weight of edges given by the coefficient a_{ij} . In the matter of the coefficients of the associated Laplacian matrix $L = [\ell_{ij}]_{N \times N}$, ζ_i can be rewritten as

$$\zeta_i = \sum_{j=1}^N \ell_{ij} y_j. \tag{3}$$

We refer to (3) as *partial-state coupling* since only part of the states are communicated over the network. When C = I, we call it *full-state coupling*.

In the case of networks with **discrete-time** agents, each agent has access to the following information

$$\zeta_i(t) = \frac{1}{1 + \bar{d}_{in}(i)} \sum_{j=1, j \neq i}^N a_{ij}(y_i(t) - y_j(t))$$
(4)

where $\bar{d}_{in}(i)$ is an *upper bound* of $d_{in}(i) = \sum_{j=1}^{N} a_{ij}$ for i = 1, ..., N. Next we write ζ_i as

$$\zeta_i(t) = \sum_{j=1, j \neq i}^N d_{ij} (y_i(t) - y_j(t)),$$
(5)

where $d_{ij} \ge 0$, and we choose $d_{ii} = 1 - \sum_{j=1, j \ne i}^{N} d_{ij}$ such that $\sum_{j=1}^{N} d_{ij} = 1$ with $i, j \in \{1, \dots, N\}$. Note that d_{ii} satisfies $d_{ii} > 0$. The weight matrix $D = [d_{ij}]$ is then a so-called, row stochastic matrix, where all eigenvalues of D satisfy $|\lambda_i| \le 1$ and 1 has one simple eigenvalue. Let $D_{in} = \text{diag}\{\overline{d}_{in}(i)\}$. Then the relationship between the row stochastic matrix D and the Laplacian matrix L is

$$(I + D_{\rm in})^{-1}L = I - D.$$
(6)

Our goal is to achieve state synchronization, i.e.,

$$\lim_{t \to \infty} (x_i(t) - x_j(t)) = 0 \tag{7}$$

for all $i, j \in \{1, ..., N\}$.

We need the following definition to explicitly state our problem formulation.

Definition 1 We define the following set. \mathbb{G}^N denotes the set of fixed, directed graphs of N agents which contains a directed spanning tree.

We formulate the scale-free state synchronization problem of a MAS without localized collaborative information exchange, i.e. non-collaborative protocol, as follows.

Problem 1 The scale-free state synchronization problem without localized collaborative information exchange for MAS (1) and communication network with (3) for continuous-time case or (5) for discrete-time case is to find, if possible, a fixed linear protocol of the form:

$$\begin{cases} x_{i,c}^{+} = A_{c} x_{i,c} + B_{c} \zeta_{i}, \\ u_{i} = F_{c} x_{i,c} + G_{c} \zeta_{i}, \end{cases}$$
(8)

where $x_{c,i}(t) \in \mathbb{R}^{n_c}$ is the state of protocol, and matrices A_c, B_c, F_c, G_c are predesigned parameters, such that the state synchronization (7) is achieved for any number of agents N, any fixed communication graph \mathcal{G} and all initial conditions of agents.

Remark 1 Note that the number of agents N and the weight a_{ij} are fixed in a control period.

In discrete-time MAS, we work with a row stochastic matrix which is a scaled version of the Laplacian matrix. This is completely in line with all papers in this area. For the scaling it should be noted that this only uses some local information about the graph, namely, $\bar{d}_{in}(i)$ as used in (4). One might ask whether scale-free state synchronization problem without localized collaborative information exchange is possible without using this scaled Laplacian. It can actually be shown that this latter problem is never solvable in discrete-time MAS.

3 Necessary conditions for solvability

The first important result that we provides necessary conditions for the solvability of Problem 1 for both continuous- and discrete-time MAS.

Theorem 1 (Continuous-time MAS) Consider a continuous-time, single-input, singleoutput MAS (1) with communication via (3). There exists a linear protocol of the form (8) which achieves scale-free state synchronization problem without localized collaborative information exchange only if:

- 1. Agent model is stabilizable and detectable.
- 2. Agent model is neutrally stable.
- 3. Agent model is weakly minimum phase.
- 4. Agent model has relative degree equal to 1.

Proof: The necessity of stabilizability and detectability is obvious. If we define

$$\tilde{A} = \begin{pmatrix} A & BF_c \\ 0 & A_c \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} BG_c \\ B_c \end{pmatrix}, \qquad \tilde{C} = \begin{pmatrix} C & 0 \end{pmatrix}$$
(9)

then [19, Chapter 2] has shown that we achieve synchronization if

$$\tilde{A} + \lambda_i \tilde{B} \tilde{C}$$

is asymptotically stable for all nonzero eigenvalues $\{\lambda_2, \ldots, \lambda_N\}$ of the Laplacian matrix *L*. To obtain a scale-free design we should therefore guarantee that

$$\tilde{A} + \lambda_i \tilde{B} \tilde{C} \tag{10}$$

is asymptotically stable for all $\lambda_i \in \mathbb{C}$ with $\operatorname{Re}(\lambda_i) > 0, i = 2, \dots, N$. We define

$$g(s) = C(sI - A)^{-1}B,$$
 $g_c(s) = F_c(sI - A_c)^{-1}B_c.$

Note that (10) asymptotically (Hurwitz) stable is equivalent to:

$$1 - \lambda_i g(s) g_c(s) \neq 0$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s \ge 0$ without unstable pole-zero cancellations in $g(s)g_c(s)$. Since this must be true for all $\lambda_i \in \mathbb{C}$ with $\operatorname{Re}(\lambda_i) > 0$, this yields the requirement that $g(s)g_c(s)$ is positive-real. From [6, Section 3.51] this requires that $g(s)g_c(s)$ satisfies:

- The poles of $g(s)g_c(s)$ are in the closed left half plane and the poles on the imaginary axis are simple.
- The zeros of $g(s)g_c(s)$ are in the closed left half plane and the zeros on the imaginary axis are simple.
- The relative degree of $g(s)g_c(s)$ is less than or equal to 1.

Since there are no unstable pole-zero cancellations in $g(s)g_c(s)$, the above conditions immediately yield that the agent model should be neutrally stable, weakly minimum-phase, and have relative degree 1.

Theorem 2 (Discrete-time MAS) Consider a discrete-time, single-input, singleoutput MAS (1) with communication via (5). There exists a linear protocol of the form (8) which achieves scale-free state synchronization problem without localized collaborative information exchange only if:

- 1. Agent model is stabilizable and detectable.
- 2. Agent model is neutrally stable.

Proof: The necessity of stabilizability and detectability is obvious, too. Using (9). we obtain from [19, Chapter 3] that we need

$$\tilde{A} + (1 - \lambda_i)\tilde{B}\tilde{C} \tag{11}$$

is asymptotically (Schur) stable for all $\lambda_i \in \mathbb{C}$ with $|\lambda_i| < 1$. Using similar arguments as in the continuous time, we obtain that we need that $g(s)g_c(s) + \frac{1}{2}$ has to be positive real. From [27] we obtain that this requires that the poles of $g(s)g_c(s)$ are in the closed unit disc and the poles on the unit circle are simple. Since there are no unstable pole-zero cancellations in $g(s)g_c(s)$, this immediately yields that the agent model should be neutrally stable.

4 Scale-free non-collaborative protocol design: Continuoustime case

We make the following assumption for agent models.

Assumption 1 Continuous-time agents (1) satisfy the following properties:

- 1. Agent model is stabilizable and detectable.
- 2. Agent model is neutrally stable.
- 3. Agent model is minimum phase.
- 4. Agent model must be uniform rank with order of infinite zero equal to one.

Remark 2 If we compare the above with the necessary conditions we obtained for SISO systems in Theorem 1 then we note that we only strengthened to condition of weakly minimum-phase to minimum-phase. The other conditions are the same.

We would like to emphasize that the agent model can be non-square and neither right nor left invertible. Also it is known that passive agents satisfy these Assumptions 1 and as such form a subset of the class of agents that we consider in this paper.



Figure 1: Architecture of the scalable non-collaborative linear protocol

We provide a scale-free non-collaborative linear protocol design in continuous via partial-state coupling. The design architecture is shown in Fig. 1. In other words, the design has two steps:

1. The first module designs a precompensator to make the agent model (1) left-invertible.

2. The second module designs a non-collaborate dynamical protocols for leftinvertible agents to achieve state synchronization.

4.1 Protocol design for partial-state coupling

The detailed design procedure is listed as follows.

Step I: Design of pre-compensator



Figure 2: The compensated agent with pre-compensator

In this step we design the following asymptotically stable pre-compensator such that the compensated agent shown at Fig. 2 is left-invertible and satisfies Assumption 1.

$$\begin{cases} \dot{p}_i = A_p p_i + B_p v_i \\ u_i = C_p p_i + D_p v_i \end{cases}$$
(12)

where $p_i \in \mathbb{R}^q$ and $v_i \in \mathbb{R}^{m_v}$ are state and input of pre-compensator. All eigenvalues of A_p are in open left-half plane.

The following lemma guarantees the existence of this pre-compensator.

Lemma 1 Consider a continuous-time agent of the form (1) which is stabilizable and detectable. In that case there exists an asymptotically stable pre-compensator (1), such that the interconnection of (1) and this pre-compensator which is given by,

$$\begin{cases} \dot{z}_i = \tilde{A} z_i + \tilde{B} v_i \\ y_i = \tilde{C} z_i \end{cases}$$
(13)

where

$$z_{i} = \begin{pmatrix} x_{i} \\ p_{i} \end{pmatrix}, \tilde{A} = \begin{pmatrix} A & BC_{p} \\ 0 & A_{p} \end{pmatrix}, \tilde{B} = \begin{pmatrix} BD_{p} \\ B_{p} \end{pmatrix}, \tilde{C} = \begin{pmatrix} C & 0 \end{pmatrix}$$

has the following properties:

- It is stabilizable and detectable,
- It is left-invertible,

- Its poles are the poles of the system (1) plus the stable poles of the precompensator (i.e., the eigenvalues of A_p),
- Its infinite zero structure is the same as the infinite zero structure of the system (1),
- Its invariant zeros are the invariant zeros of the system (1) and some additional invariant zeros that can be arbitrarily placed in the open left-half complex plane,

Proof: Obviously, we just need to prove the case where agent model is not left-invertible, i.e., right-invertible and neither left-invertible or right-invertible.

If the agent model is right-invertible, we can directly use the results in [18, Section III-B, and the dual results of Theorem 3.1 and Remark 3.3] or [20, Theorem 1-(2) and Remark 1].

If the agent model is neither left-invertible or right-invertible, we can design a pre-compensator only to make the compensated agent left-invertible, by using the results in [18, Section III-C, and the dual results of Theorem 3.1 and Remark 3.3] or [20, Theorem 1-(3) and Remark 1].

Step II: Design of a scalable non-collaborative linear protocol

Under Assumption 1, we can use the Special Coordinate Basis (SCB) [18] to achieve the following transformation for the compensated agents (13). In other words, there exists a non-singular state transformation matrix S with

$$\begin{pmatrix} \bar{z}_{1i} \\ \bar{z}_{2i} \end{pmatrix} = S z_i,$$

such that the dynamics of \bar{z}_{1i} and \bar{z}_{2i} are represented by

$$\begin{cases} \bar{z}_{1i} = A_{11}\bar{z}_{1i} + A_{12}\bar{z}_{2i}, \\ \bar{z}_{2i} = A_{21}\bar{z}_{1i} + A_{22}\bar{z}_{2i} + \bar{B}v_i, \\ y_i = \begin{pmatrix} y_{1i} \\ y_{2i} \end{pmatrix} = \begin{pmatrix} \bar{C}\bar{z}_{1i} \\ \bar{z}_{2i} \end{pmatrix}, \end{cases}$$
(14)

where $\bar{z}_{1i} \in \mathbb{R}^{n+q-\bar{n}}$ and $\bar{z}_{2i} \in \mathbb{R}^{\bar{n}}$, \bar{B} is a non-singular matrix, and (A_{11}, \bar{C}) is detectable while

$$S\tilde{A}S^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, S\tilde{B} = \begin{pmatrix} 0 \\ \bar{B} \end{pmatrix}, \tilde{C}S^{-1} = \begin{pmatrix} \bar{C} & 0 \\ 0 & I \end{pmatrix}.$$

Meanwhile, we have

$$\zeta_i = \begin{pmatrix} \zeta_{1i} \\ \zeta_{2i} \end{pmatrix} = \begin{pmatrix} \bar{C}\zeta_{si} \\ \zeta_{2i} \end{pmatrix}, \qquad \zeta_{si} = \sum_{j=1}^N \ell_{ij}\bar{z}_{1j}, \qquad \zeta_{2i} = \sum_{j=1}^N \ell_{ij}\bar{z}_{2j}.$$

Since the compensated agents are neutrally stable, we have the eigenvalues of A are on the imaginary axis, if any, are semi-simple. According to Lemma 1, the poles of compensated system (13) are on closed left half plane and therefore, there exists a positive definite matrix P such that

$$P\tilde{A} + \tilde{A}^{\mathrm{T}}P \leqslant 0 \tag{15}$$

Now we are ready to give our scale-free protocol design below.

$$\begin{cases} \dot{p}_{i} = A_{p}p_{i} - \rho B_{p}\tilde{B}^{\mathrm{T}}PS^{-1} \left[\begin{pmatrix} I_{n+q-\bar{n}} \\ 0 \end{pmatrix} \hat{z}_{1i} + \begin{pmatrix} 0 & 0 \\ 0 & I_{\bar{n}} \end{pmatrix} \zeta_{i} \right] \\ \dot{\bar{z}}_{1i} = (A_{11} - H\bar{C})\hat{z}_{1i} + (H & A_{12})\zeta_{i} \\ u_{i} = C_{p}p_{i} - \rho D_{p}\tilde{B}^{\mathrm{T}}PS^{-1} \left[\begin{pmatrix} I_{n+q-\bar{n}} \\ 0 \end{pmatrix} \hat{z}_{1i} + \begin{pmatrix} 0 & 0 \\ 0 & I_{\bar{n}} \end{pmatrix} \zeta_{i} \right] \end{cases}$$
(16)

where *H* is a matrix such that $A_{11} - H\overline{C}$ is Hurwitz stable, P > 0 satisfies (15), $\rho > 0$, and i = 1, ..., N.

Next, we have the following theorem to achieve state synchronization.

Theorem 3 Consider a continuous-time MAS described by (1) and (2). Assume Assumption 1 is satisfied. Let the set \mathbb{G}^N denote all graphs satisfy Definition 1.

Then, the scale-free non-collaborative state synchronization problem via linear protocol as stated in Problem 1 is solvable. More specifically, the protocol (16) achieve state synchronization for any fixed graph $\mathscr{G} \in \mathbb{G}^N$ with any size of the network N.

Proof: According to Lemma 1, we can know that there must exist pre-compensator (12) to make agent (1) left-invertible, and obtain the compensated system (13).

From (14) and (16), we have

$$\begin{aligned} \dot{\bar{z}}_{1i} &= A_{11}\bar{z}_{1i} + A_{12}\bar{z}_{2i} \\ \dot{\bar{z}}_{1i} &= A_{11}\bar{\bar{z}}_{1i} + A_{12}\zeta_{2i} + H(\zeta_{1i} - \bar{C}\bar{\bar{z}}_{1i}) \end{aligned}$$

By defining

$$\bar{z}_1 = \begin{pmatrix} \bar{z}_{11} \\ \vdots \\ \bar{z}_{1N} \end{pmatrix}, \qquad \bar{z}_2 = \begin{pmatrix} \bar{z}_{21} \\ \vdots \\ \bar{z}_{2N} \end{pmatrix}, \qquad \hat{z}_1 = \begin{pmatrix} \hat{\bar{z}}_{11} \\ \vdots \\ \hat{\bar{z}}_{1N} \end{pmatrix},$$

we obtain

$$\dot{\bar{z}}_1 = (I \otimes A_{11})\bar{z}_1 + (I \otimes A_{12})\bar{z}_2$$
$$\dot{\bar{z}}_1 = [I \otimes (A_{11} - H\bar{C})]\hat{\bar{z}}_1 + (L \otimes A_{12})\bar{z}_2 + (L \otimes H\bar{C})\bar{z}_1$$

Let $e = (L \otimes I)\overline{z}_1 - \hat{\overline{z}}_1$, then we have

$$\dot{e} = [I \otimes (A_{11} - H\bar{C})]e.$$

Since $A_{11} - H\bar{C}$ is Hurwitz stable, it is obvious that *e* is asymptotically stable, i.e.

$$\lim_{t\to\infty}\hat{\bar{z}}_{1i}\to\zeta_{si}=\sum_{j=1}^N\ell_{ij}\bar{z}_{1j}.$$

Meanwhile, we obtain

$$\begin{pmatrix} \hat{\bar{z}}_{1i} \\ \zeta_{2i} \end{pmatrix} \to S \begin{pmatrix} \sum_{j=1}^{N} \ell_{ij} z_j \end{pmatrix} \text{ as } t \to \infty.$$
(17)

On the other hand, from (12), (13), and (16) we have

$$v_i = -\rho \tilde{B}^{\mathrm{T}} P S^{-1} \begin{pmatrix} \hat{\bar{z}}_{1i} \\ \zeta_{2i} \end{pmatrix}$$

According to agent model (13) and result (17), we have

$$\dot{z}_i = \tilde{A}z_i - \rho \tilde{B}\tilde{B}^{\mathrm{T}}P \sum_{j=1} \ell_{ij}z_j$$

Then, by setting

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix},$$

we obtain

$$\dot{z} = (I \otimes \tilde{A} - \rho L \otimes \tilde{B}\tilde{B}^{\mathrm{T}}P)z.$$
⁽¹⁸⁾

By using the method from [11, Lemma 2], there exists a non-singular matrix T such that (18) can be transformed as

$$\begin{cases} \dot{\eta}_1 = \tilde{A}\eta_1, \\ \dot{\eta}_i = (\tilde{A} - \rho\lambda_i \tilde{B}\tilde{B}^{\mathrm{T}}P)\eta_i, \qquad i = 2, \dots, N, \end{cases}$$
(19)

where λ_i denotes all non-zero eigenvalues of *L*. Therefore, we need to prove the stability of (19) to obtain original MAS' state synchronization, i.e. the stability of $\tilde{A} - \rho \lambda_i \tilde{B} \tilde{B}^T P$ for i = 2, ..., N where we know that $\text{Re}(\lambda_i) > 0$, i.e. the real part of λ_i is positive.

Choosing P > 0 satisfying (15), then we have

$$P(\tilde{A} - \rho\lambda_{i}\tilde{B}\tilde{B}^{T}P) + (\tilde{A} - \rho\lambda_{i}\tilde{B}\tilde{B}^{T}P)^{*}P$$

= $P\tilde{A} + \tilde{A}^{T}P - 2\rho \operatorname{Re}(\lambda_{i})P\tilde{B}\tilde{B}^{T}P$
 $\leq -2\rho \operatorname{Re}(\lambda_{i})P\tilde{B}\tilde{B}^{T}P.$

Since (\tilde{A}, \tilde{B}) is stabilizable and $\operatorname{Re}(\lambda_i) > 0$, it follows from LaSalle's invariance principle that $\tilde{A} - \rho \lambda_i \tilde{B} \tilde{B}^{\mathrm{T}} P$ is Hurwitz stable and we obtain the required stability of (19).

Meanwhile, from [11, Lemma 2], we can obtain the state synchronization result

$$\lim_{t\to\infty} z_i - z_j \to 0.$$

Furthermore, it implies that

$$\lim_{t\to\infty} x_i - x_j \to 0.$$

Therefore, the synchronization result can be obtained for any graph $\mathscr{G} \in \mathbb{G}^N$ with any size of the network N.

4.2 Protocol design for full-state coupling, i.e. C = I

When C = I, we only need the assumption that the agents are stabilizable and neutrally stable, i.e., the other conditions in Assumption 1 are satisfied automatically. Moreover, since (3) can be rewritten as

$$\zeta_i = \sum_{j=1}^N \ell_{ij} x_j, \tag{20}$$

it means that we do need neither a pre-compensator nor use SCB to transform the compensated system (13). Thus, we can obtain a static protocol, i.e., the estimator (or observer) is not needed to achieve the synchronization. Of course, the protocol design in (16) can still be applied.

Firstly, since agent model (1) is neutrally stable, there still exists a positive definite matrix P such that

$$PA + A^{\mathrm{T}}P \leqslant 0. \tag{21}$$

The scale-free protocol design for continuous-time MAS with neutrally stable agent is listed as follows.

 $u_i = -\rho B^{\mathrm{T}} P \zeta_i,$

where P > 0 satisfies (21) and $\rho > 0$.

Then, we have the following theorem.

Theorem 4 Consider a continuous-time MAS consisting of neutrally stable agents described by (1) and (20) where (A, B) is stabilizable. Let the set \mathbb{G}^N denote all graphs satisfy Definition 1.

Then, the scale-free state synchronization problem via linear protocol as stated in Problem 1 is solvable. More specifically, then protocol (22) achieves state synchronization for any fixed graph $\mathscr{G} \in \mathbb{G}^N$ with any size of the network N.

Proof: Combining (1) and (22), we obtain

$$\dot{x}_i = Ax_i - \rho B^{\mathrm{T}} P \sum_{j=1}^N \ell_{ij} x_j$$
(23)

Then we have

$$\dot{x} = (I \otimes A - \rho L \otimes (BB^{\mathrm{T}}P))x \tag{24}$$

by defining

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}.$$

Similar to the proof of Theorem 3, we can obtain the following transformed system

$$\begin{cases} \dot{\phi}_1 = A\phi_1, \\ \dot{\phi}_i = (A - \rho\lambda_i BB^{\mathrm{T}} P)\phi_i, \qquad i = 2, \dots, N \end{cases}$$

by using a non-singular matrix T_f . According to [11, Lemma 2], we just prove the stability of $A - \rho \lambda_i B B^T P$ to obtain the state synchronization.

Since P > 0 satisfies (21), we have

$$P(A - \rho\lambda_i BB^{\mathrm{T}}P) + (A - \rho\lambda_i BB^{\mathrm{T}}P)^*P \leq -2\rho \operatorname{Re}(\lambda_i)PBB^{\mathrm{T}}P \leq 0$$

for $\rho > 0$.

Since (A, B) is stabilizable and $\operatorname{Re}(\lambda_i) > 0$, it follows from LaSalle's invariance principle that $A - \rho \lambda_i B B^{\mathsf{T}} P$ is Hurwitz stable. Thus, the synchronization result can be obtained for any graph $\mathscr{G} \in \mathbb{G}^N$ with any size of the network N.

(22)

5 Scale-free non-collaborative protocol design: Discretetime case

We make the following assumptions:

Assumption 2 Discrete-time agents (1) satisfy the following properties:

- 1. Agent model is stabilizable and detectable.
- 2. Agent model is neutrally stable.

Remark 3 These assumptions are equal to the necessary conditions we obtained for SISO systems in Theorem 2

Meanwhile, there still exists a positive definite matrix P such that

$$A^{\mathrm{T}}PA - P \leqslant 0. \tag{25}$$

Our design is intrinsically different from the continuous-time. We first start of with the partial-state coupling, which is going to use a stable observer with the so-called CSS architecture.

5.1 Protocol design for partial-state coupling

We have the following scale-free protocol design for discrete-time MAS with neutrally stable agents:

$$\chi_i(t+1) = (A - HC)\chi_i(t) + H\zeta_i(t)$$

$$u_i(t) = -\delta B^{\mathrm{T}} P A \chi_i(t)$$
(26)

where $\delta \in (0, \delta^*]$ and δ^* is obtained only from the knowledge of agent model (A, B, C). P > 0 satisfies (25) and H is a matrix such that A - HC is Schur stable. For computation of δ^* , see the proof of Theorem 5.

Then, we have the following theorem.

Theorem 5 Consider a discrete-time MAS described by (1) and (5). Assume Assumption 2 is satisfied. Let the set \mathbb{G}^N denote all graphs satisfy Definition 1.

Then, the scale-free state synchronization problem via non-collaborative linear protocol as stated in Problem 1 is solvable. More specifically, there exists $\delta^* > 0$ which is obtained only from agent model (A, B, C), such that for all $\delta \in (0, \delta^*]$, the protocol (26) achieves state synchronization for any fixed graph $\mathscr{G} \in \mathbb{G}^N$ with any size of the network N.

Proof: For agent model (1) and (26), we have

$$\begin{cases} x(t+1) = (I \otimes A)x(t) - [\delta(I-D) \otimes BB^{\mathsf{T}}PA]\chi(t), \\ \chi(t+1) = [I \otimes (A-HC)]\chi(t) + (I \otimes HC)x(t). \end{cases}$$
(27)

By using [12, Lemma 3], there exists a non-singular matrix T_f , we can transform (27) to

$$\begin{cases} \phi_i(t+1) = A\phi_i(t) - \delta(1-\lambda_i)BB^{\mathrm{T}}PA\psi_i(t),\\ \psi_i(t+1) = (A-HC)\psi_i(t) + HC\phi_i(t) \end{cases}$$
(28)

for i = 2, ..., N, where λ_i satisfies $|\lambda_i| < 1$. Thus, we only need to prove that the system (28) is asymptotically stable for all $|\lambda_i| < 1$.

Define $e_i(t) = \phi_i(t) - \psi_i(t)$. The system (28) can be rewritten in terms of $\phi_i(t)$ and $e_i(t)$ as

$$\begin{pmatrix}
\phi_i(t+1) = (A - (1 - \lambda_i)\delta BB^{\mathsf{T}}PA)\phi_i(t) + (1 - \lambda_i)\delta BB^{\mathsf{T}}PAe_i(t), \\
e_i(t+1) = (A - HC + (1 - \lambda_i)\delta BB^{\mathsf{T}}PA)e_i(t) \\
-(1 - \lambda_i)\delta BB^{\mathsf{T}}PA\phi_i(t).
\end{cases}$$
(29)

Let Q be the positive definite solution of the Lyapunov equation,

$$(A - HC)^{\mathrm{T}}Q(A - HC) - Q + 4I = 0.$$

There exists a δ_1 such that for all $\delta \in (0, \delta_1]$, we have

$$(A - HC + (1 - \lambda_i)\delta BB^{\mathsf{T}}PA)^*Q(A - HC + (1 - \lambda_i)\delta BB^{\mathsf{T}}PA) - Q + 3I \leq 0.$$

for all λ_i with $|\lambda_i| < 1$. Consider $V_1(t) = e_i(t)^* Q e_i(t)$ and let $\mu = \delta B^T P A \phi_i(t)$. We have

$$V_{1}(t+1) - V_{1}(t)$$

$$\leq -3\|e_{i}(t)\|^{2} + |1 - \lambda_{i}|^{2}\mu^{*}B^{T}QB\mu$$

$$+ 2\left|\left((1 - \lambda_{i})^{*}\mu^{*}B^{T}Q[A - HC + (1 - \lambda_{i})\delta BB^{T}PA]e_{i}(t)\right)\right|$$

$$\leq -3\|e_{i}(t)\|^{2} + |1 - \lambda_{i}|^{2}M_{2}\|\mu\|^{2}$$

$$+ (|1 - \lambda_{i}|M_{1} + |1 - \lambda_{i}|^{2}\delta M_{3})\|\mu\|\|e_{i}(t)\|,$$

where $M_1 = 2||B^TQ||||A - HC||$, $M_2 = ||B^TQB||$, and $M_3 = 2||B^TQ||||BB^TPA||$. It should be noted that M_1 , M_2 , and M_3 are independent of δ and λ . Consider $V_2(t) = \phi_i^*(t)P\phi_i(t)$. Note that

$$[A - (1 - \lambda_i)\delta BB^{\mathsf{T}}PA]^*P[A - (1 - \lambda_i)\delta BB^{\mathsf{T}}PA] - P$$

$$\leq -2\operatorname{Re}(1 - \lambda_i)\delta A^{\mathsf{T}}PBB^{\mathsf{T}}PA + |1 - \lambda_i|^2\delta^2 A^{\mathsf{T}}PBB^{\mathsf{T}}PBB^{\mathsf{T}}PA.$$

There exists a $\delta_2 < \delta_1$ such that, for all $\delta \in (0, \delta_2]$, we have $2\delta B^{\mathsf{T}}PB \leq I_m$. Since $|1 - \lambda_i|^2 \leq 2 \operatorname{Re}(1 - \lambda_i)$ for $|\lambda_i| < 1$, we get for all $\delta \in (0, \delta_2]$,

$$[A - (1 - \lambda_i)\delta BB^{\mathsf{T}}PA]^*P[A - (1 - \lambda_i)\delta BB^{\mathsf{T}}PA] - P$$

$$\leq -\frac{1}{2}|1 - \lambda_i|^2\delta A^{\mathsf{T}}PBB^{\mathsf{T}}PA.$$

Hence

$$V_{2}(t+1) - V_{2}(t)$$

$$\leq -\frac{1}{2\delta} |1 - \lambda_{i}|^{2} ||\mu||^{2} + |1 - \lambda|^{2} \delta^{2} e_{i}^{*}(t) A^{\mathsf{T}} P B B^{\mathsf{T}} P B B^{\mathsf{T}} P A e_{i}(t)$$

$$+ 2 \left| (1 - \lambda^{*}) e_{i}^{*}(t) A^{\mathsf{T}} P B \mu - |1 - \lambda|^{2} \delta e_{i}^{*}(t) A^{\mathsf{T}} P B B^{\mathsf{T}} P B \mu \right|$$

$$\leq -\frac{1}{2\delta} |1 - \lambda|^{2} ||\mu||^{2} + \theta_{1} |1 - \lambda| ||e_{i}(t)|| ||\mu||$$

$$+ \theta_{2} \delta^{2} ||e_{i}(t)||^{2} + \theta_{3} \delta |1 - \lambda|^{2} ||e_{i}(t)|| ||\mu||,$$

where $\theta_1 = 2||A^T P B||$, $\theta_2 = 4||A^T P B B^T P B B^T P A||$, and $\theta_3 = 2||A^T P B B^T P B||$. Define a Lyapunov candidate $V(t) = V_1(t) + \delta \kappa V_2(t)$ with $\kappa = 4 + 2M_2 + 2M_1^2$. We get that

$$V(t+1) - V(t) \leq - (3 - \delta^3 \theta_2 \kappa) ||e_i(t)||^2 - (2 + M_1^2) |1 - \lambda_i|^2 ||\mu||^2 + (M_1 + \delta \theta_1 \kappa) |1 - \lambda_i| ||\mu|| ||e_i(t)|| + (\delta M_3 + \delta^2 \theta_3 \kappa) |1 - \lambda_i|^2 ||\mu|| ||e(t)||.$$

There exists a $\delta^* < \delta_2$ such that for a $\delta \in (0, \delta^*]$, $3 - \delta^3 \theta_2 \kappa \ge 2.5$, $M_1 + \delta \theta_1 \kappa \le 2M_1$, and $\delta M_3 + \delta^2 \theta_3 \kappa \le 1$. This yields,

$$\begin{split} V(t+1) - V(t) \\ \leqslant &- 2.5 \|e_i(t)\|^2 - (2 + M_1^2) |1 - \lambda_i|^2 \|\mu\|^2 \\ &+ (2M_1 |1 - \lambda_i| + |1 - \lambda_i|^2) \|\mu\| \|e_i(t)\| \\ \leqslant &- 0.5 \|e_i(t)\|^2 - |1 - \lambda_i|^2 \|\mu\|^2 - (\|e_i(t)\| - M_1 |1 - \lambda_i| \|\mu\|)^2 \\ &- |1 - \lambda_i|^2 (\frac{1}{2} \|e_i(t)\| - \|\mu\|)^2 \\ \leqslant &- 0.5 \|e_i(t)\|^2 - |1 - \lambda_i|^2 \|\mu\|^2. \end{split}$$

Since (A, B) is controllable, it follows from LaSalle's invariance principle that the system (29) is globally asymptotically stable for $\delta \leq \delta^*$.

5.2 Protocol design for full-state coupling, i.e. C = I

Firstly, the information measurement (5) is rewritten as

$$\zeta_i(t) = \sum_{j=1, j \neq i}^N d_{ij}(x_i(t) - x_j(t)).$$
(30)

The scalable protocol for discrete-time MAS with neutrally stable agent via fullstate coupling is listed as follows.

$$u_i(t) = -\varepsilon B^{\mathrm{T}} P A \zeta_i(t), \qquad (31)$$

where $P > 0$ satisfies (25), and $\varepsilon \in (0, \varepsilon^*]$ with $\varepsilon^* = \|B^{\mathrm{T}} P B\|^{-1}$.

Then, we have the following theorem.

Theorem 6 Consider a discrete-time MAS consisting of neutrally stable agents described by (1) and (30) where (A, B) is stabilizable. Let the set \mathbb{G}^N denote all graphs satisfy Definition 1.

Then, the scale-free state synchronization problem via linear protocol as stated in Problem 1 is solvable. More specifically, for any given $\varepsilon \in (0, \varepsilon^*]$ with $\varepsilon^* = ||B^T P B||^{-1}$, the protocol (31) achieves state synchronization for any fixed graph $\mathscr{G} \in \mathbb{G}^N$ with any size of the network N.

Proof: For agent model (1) and (31), we have

$$x(t+1) = [I \otimes A - \varepsilon(I-D) \otimes BB^{\mathrm{T}}PA]x(t).$$
(32)

By using [12, Lemma 3], there exists a non-singular matrix T_f , we can transform (32) to

$$\phi_i(t+1) = (A - \varepsilon(1 - \lambda_i)BB^{\mathrm{T}}PA)\phi_i(t), \quad i = 2, \dots, N,$$
(33)

where λ_i satisfies $|\lambda_i| < 1$. Thus, we just need to prove the stability of $A - \varepsilon(1 - \lambda_i)BB^{T}PA$.

Since the matrix P > 0, we obtain the stability of (33).

$$[A - \varepsilon(1 - \lambda_i)BB^{\mathsf{T}}PA]^*P[A - \varepsilon(1 - \lambda_i)BB^{\mathsf{T}}PA] - P \leq -\varphi A^{\mathsf{T}}PBB^{\mathsf{T}}PA$$

with $\varphi = \varepsilon [2 \operatorname{Re}(1 - \lambda_i) - |1 - \lambda_i|^2]$. Note that $|\lambda_i| < 1$ implies

$$|1 - \lambda_i|^2 \leq 2\operatorname{Re}(1 - \lambda_i), \tag{34}$$

and therefore we have $\varphi > 0$. Since (A, B) is stabilizable, it then follows from LaSalle's invariance principle that the system (33) is globally asymptotically stable.

Note that ε^* depends only on agent's model, hence the synchronization result can be obtained for any graph $\mathscr{G} \in \mathbb{G}^N$ with any size of the network N.

Remark 4 The results in [11] and [12] are used in Theorems 3-6. Compared with this paper, [11] focused on continuous-time MAS with agents which are squared-down passive and passifiable. The linear protocol for squared-down passive agents is scalable and a subset of the design in this paper. In particular, the additional structure in [11] enabled the use of static protocols which is not possible for the more general class of agents in this paper. The nonlinear adaptive protocols are also scalable for the undirected communication network. [12] developed a linear protocol design for discrete-time MAS only with squared-down passifiable via input feedforward agents. The designs is not scale-free.

6 Numerical Examples

In this section, we will illustrate the effectiveness of our designs with two numerical examples for state synchronization of continuous- and discrete-time MAS with partial-state coupling. Meanwhile, we consider two communication networks with different topologies to show the scalability of our protocols.

Case *I*: We consider MAS with 4 agents N = 4, and directed communication topology shown in Figure 3.



Figure 3: Directed topology network with 4 nodes

Case *II*: In this case, we consider MAS with 60 agents i.e. N = 60, and directed communication topology with associated adjacency matrix \mathcal{A}_{II} being $a_{i+1,i} = a_{1,60} = 1$ and $i = 1, \dots, 59$.

Then, the continuous- and discrete-time MAS are studied respectively.

6.0.1 Continuous-time MAS

Consider continuous-time agent models (1) with the following parameter:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ B = I, \ C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We design pre-compensator (12) with the choice of

$$A_p = -2, B_p = 1, C_p = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}}, D_p = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^{\mathrm{T}}.$$

Then the other protocol parameters in (16) are as follows,

$$P = \begin{pmatrix} 1 & 0 & -1 & -0.6 \\ 0 & 1 & 1 & 0.2 \\ -1 & 1 & 3 & 1.3 \\ -0.6 & 0.2 & 1.3 & 2 \end{pmatrix}, S^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \tilde{B} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$
$$A_{11} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & -2 & 1 \\ 0 & -1 & 0 \end{pmatrix}, A_{12} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, H = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \bar{C} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

The simulation results for both Cases I and II are demonstrated in Figure 4 and 5. And the error states $x_{ij} - x_{i1}$ are shown in 6 to show the synchronization more clearly. The results show that the protocol design is independent of the communication graph and is scale free so that we can achieve synchronization with one-shot protocol design, for any graph with any number of agents.



Figure 4: State synchronization for continuous-time MAS with communication graph in Case *I*.

Compared with scale-free collaborative protocol design in [10], the synchronized time is deteriorating since no extra information exchange is employed. For example, the running time of the 60-node case is 21.6768s, but this time is 6.8072s under the same parameters using the scalable collaborative protocol by SIMULINK. However, non-collaborative protocol design does not need extra information exchange through communication network and is more likely applied in practical.



Figure 5: State synchronization for continuous-time MAS with communication graph in Case *II*.



Figure 6: Error state for continuous-time MAS with communication graph in Case *II*.

6.0.2 Discrete-time MAS

Consider discrete-time agent models (1) with the following parameters:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

We design protocol (26) with the following parameters

$$H = \begin{pmatrix} 0.5 \\ -0.5 \\ 0.4 \end{pmatrix}, P = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}, \delta = 0.1.$$

We let the information exchange ζ_i satisfy (4). Then the simulation results for both Cases I and II are demonstrated in Figure 7 and 8. And the error states $x_{ij} - x_{i1}$ are shown in 6 to show the synchronization more clearly. The results show that the protocol design is independent of the communication graph and is scale free so that we can achieve synchronization with one-shot protocol design, for any graph with any number of agents.



Figure 7: State synchronization for discrete-time MAS with communication graph in Case *I*.



Figure 8: State synchronization for discrete-time MAS with communication graph in Case *II*. In particular, we only show the synchronized trajectories for states x_{i1} and x_{i2} , since there are many lines to make the figures difficult illuminating.



Figure 9: Error state for discrete-time MAS with communication graph in Case II.

7 Conclusion

In this paper, we have proposed a scale-free non-collaborative protocol design to achieve state synchronization for homogeneous MAS with the agents satisfying Assumptions 1 and 2. Moreover, we have provided these assumptions (conditions) are very close to necessary. The non-collaborative protocols are designed for one

class of continuous- and discrete-time MAS, which are solely based on agent models without utilizing localized collaborative information exchange, and work for any number of agents and any fixed communication graph containing a spanning tree.

References

- H. Bai, M. Arcak, and J. Wen. *Cooperative control design: a systematic, passivity-based approach.* Communications and Control Engineering. Springer Verlag, 2011.
- [2] F. Bullo. Lectures on network systems. Kindle Direct Publishing, 2019.
- [3] N. Chopra. Output synchronization on strongly connected graphs. *IEEE Trans. Aut. Contr.*, 57(1):2896–2901, 2012.
- [4] D. Chowdhury and H. K. Khalil. Synchronization in networks of identical linear systems with reduced information. In *American Control Conference*, pages 5706–5711, Milwaukee, WI, 2018.
- [5] C. Godsil and G. Royle. *Algebraic graph theory*, volume 207 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.
- [6] P. Ioannou and G. Sun. *Robust adaptive control*. Prentice Hall, New Jersey, 1996.
- [7] L. Kocarev. Consensus and synchronization in complex networks. Springer, Berlin, 2013.
- [8] Z. Li, Z. Duan, G. Chen, and L. Huang. Consensus of multi-agent systems and synchronization of complex networks: A unified viewpoint. *IEEE Trans. Circ. & Syst.-I Regular papers*, 57(1):213–224, 2010.
- [9] Z. Liu, A. Saberi, A.A. Stoorvogel, and D. Nojavanzadeh. Global regulated state synchronization for homogeneous networks of non-introspective agents in presence of input saturation: Scale-free nonlinear and linear protocol designs. *Automatica*, 119:109041(1–8), 2020.
- [10] Z. Liu, A. Saberi, A.A. Stoorvogel, and D. Nojavanzadeh. H_{∞} almost state synchronization for homogeneous networks of non-introspective agents: A scale-free protocol design. *Automatica*, 122:109276 (1–7), 2020.
- [11] Z. Liu, M. Zhang, A. Saberi, and A. A. Stoorvogel. State synchronization of multi-agent systems via static or adaptive nonlinear dynamic protocols. *Automatica*, 95:316–327, 2018.

- [12] Z. Liu, M. Zhang, A. Saberi, and A.A. Stoorvogel. Passivity based state synchronization of homogeneous discrete-time multi-agent systems via static protocol in the presence of input delay. *European Journal of Control*, 41:16– 24, 2018.
- [13] M. Mesbahi and M. Egerstedt. *Graph theoretic methods in multiagent networks*. Princeton University Press, Princeton, 2010.
- [14] D. Nojavanzadeh, Z. Liu, A. Saberi, and A.A. Stoorvogel. Synchronization for homogeneous and heterogeneous discrete-time multi-agent systems: A scale-free protocol design. In *the 39th Chinese Control Conference*, pages 4736–4741, Shenyang, China, 2020.
- [15] R. Olfati-Saber and R.M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans. Aut. Contr.*, 49(9):1520–1533, 2004.
- [16] W. Ren and R.W. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Trans. Aut. Contr.*, 50(5):655–661, 2005.
- [17] W. Ren and Y.C. Cao. *Distributed coordination of multi-agent networks*. Communications and Control Engineering. Springer-Verlag, London, 2011.
- [18] A. Saberi and P. Sannuti. Squaring down by static and dynamic compensators. *IEEE Trans. Aut. Contr.*, 33(4):358–365, 1988.
- [19] A. Saberi, A. A. Stoorvogel, M. Zhang, and P. Sannuti. Synchronization of multi-agent systems in the presence of disturbances and delays. Birkhäuser, New York, 2022.
- [20] P. Sannuti, A. Saberi, and M. Zhang. Squaring down of general MIMO systems to invertible uniform rank systems via pre- and/or post-compensators. *Automatica*, 50(8):2136–2141, 2014.
- [21] A. A. Stoorvogel, A. Saberi, and M. Zhang. Solvability conditions and design for state synchronization of multi-agent systems. *Automatica*, 84:43–47, 2017.
- [22] S. Stüdli, M. M. Seron, and R. H. Middleton. Vehicular platoons in cyclic interconnections with constant inter-vehicle spacing. *IFAC-PapersOnLine*, 50(1):2511–2516, 2017.
- [23] E. Tegling, B. Bamieh, and H. Sandberg. Localized high-order consensus destabilizes large-scale networks. In *American Control Conference*, pages 760–765, Philadelphia, PA, 2019.

- [24] E. Tegling, B. Bamieh, and H. Sandberg. Scale fragilities in localized consensus dynamics. Available: arXiv:2203.11708, 2023.
- [25] E. Tegling, R. H. Middleton, and M. M. Seron. Scalability and fragility in bounded-degree consensus networks. In 8th IFAC Workshop on Distributed Estimation and Control in Networked Systems, volume 52(20), pages 85–90, Chicago, IL, 2019. IFAC-PapersOnLine, Elsevier.
- [26] C.W. Wu. Synchronization in complex networks of nonlinear dynamical systems. World Scientific Publishing Company, Singapore, 2007.
- [27] C. Xiao and D.J. Hill. Generalizations and new proof of the discrete-time positive real lemma and bounded real lemma. *IEEE Trans. Circ. & Syst.-I Fundamental theory and applications*, 46(6):740–743, 1999.