# Second-order Conic Programming Approach for Wasserstein Distributionally Robust Two-stage Linear Programs 

Zhuolin Wang, Keyou You, Shiji Song and Yuli Zhang


#### Abstract

This paper proposes a second-order conic programming (SOCP) approach to solve distributionally robust two-stage stochastic linear programs over 1-Wasserstein balls. We start from the case with distribution uncertainty only in the objective function and exactly reformulate it as an SOCP problem. Then, we study the case with distribution uncertainty only in constraints, and show that such a robust program is generally NP-hard as it involves a norm maximization problem over a polyhedron. However, it is reduced to an SOCP problem if the extreme points of the polyhedron are given as a prior. This motivates to design a constraint generation algorithm with provable convergence to approximately solve the NP-hard problem. In sharp contrast to the exiting literature, the distribution achieving the worst-case cost is given as an "empirical" distribution by simply perturbing each sample for both cases. Finally, experiments illustrate the advantages of the proposed model in terms of the out-of-sample performance and the computational complexity.


Note to Practitioners-The two-stage program with the distribution uncertainty is a fundamental optimization problem with broad applications. This paper proposes a novel datadriven model over 1-Wasserstein balls to handle the distribution uncertainty. Moreover, it applies to the situation where the true distribution is slowly time-varying. An SOCP-based algorithm with provable convergence has been presented to solve the twostage program over 1-Wasserstein balls. The proposed model is particularly effective for applications with uncertainty only in the objective functions or constraints. This paper focuses on the applications in two-stage portfolio programs and two-stage material order programs. The good out-of-sample performance and computational complexity of the model are validated by the experiments on these applications. This model can be easily applied to other applications, such as the two-stage schedule problems and recourse allocation problems.

Index Terms-two-stage linear program, distribution uncertainty, data-driven robust, uncertainty modelling, Wasserstein ball

## I. INTRODUCTION

The two-stage program is one of the most fundamental optimization problems and has broad applications, see e.g., [1], [2]. It is observed that its coefficients are usually uncertain and ignoring their uncertainties may lead to poor decisions

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Zhuolin Wang, Keyou You and Shiji Song are with the Department of Automation and BNRist, Tsinghua University, Beijing 100084, China (e-mail: wangzl17@mails.tsinghua.edu.cn, youky@tsinghua.edu.cn, shijis@tsinghua.edu.cn.)

Yuli Zhang is with School of Management and Economics, Beijing Institute of Technology, Beijing 100081, China. (e-mail:zhangyuli@bit.edu.cn)
[3], [4]. In the literature, the classical robust optimization (RO) has been proposed to handle the uncertainty in the twostage program by restricting them to some given sets and then minimizes the worst-case cost over all possible realizations [5]. However, it ignores the distribution information of stochastic uncertainty and may return a conservative solution [6]. To this end, the stochastic program ( SP ) is adopted to address the uncertainty via a distribution function [7], and in practice is solved by using an empirical distribution in the sampleaverage approximation (SAA) method [8]. The SAA method is effective only when adequate and high-quality samples are obtained cheaply [8]. If samples are of low quality, the empirical distribution may significantly deviate from the true distribution and the SAA method exhibits poor performance.

An alternative approach is to apply the distributionally robust (DR) optimization technique to address stochastic uncertainty by assuming that the true distribution belongs to an ambiguity set of probability distributions [9], [10]. This method overcomes inherent drawbacks of the SP and RO as it does not require an exact distribution and can exploit the sample information. In fact, numerous evidence implies that the DR method can yield high-quality solutions within a reasonable computation cost [11]-[13]. Thus, our exposition concentrates on DR two-stage linear programs over an ambiguity set of distributions.

The ambiguity set is essential in the DR programs. It should be large enough to include the true distribution with a high probability but cannot be too "large" to avoid very conservative decisions. [14]-[16] adopt the moment-based ambiguity set, which includes distributions with specified moment constraints. The DR two-stage linear program over the set of distributions with exactly known first- and second-order moments are reformulated either as a semidefinite program [14] or the mixed-integer linear program of a polynomial size [15] under different settings. Observe that the moment mismatch is unavoidable, [16] further considers the moment uncertainty, which results in an intractable model.

In this work, we study a data-driven DR two-stage linear program over a ball centered at the empirical distribution of a finite sample dataset, and the ball radius reflects our confidence in the empirical distribution. Particularly, the lower the confidence, the larger the ball radius. The sample dataset can be utilized in a flexible way to handle the distribution uncertainty, e.g., the degree of conservatism can be controlled by tuning the radius. Moreover, our model applies to the situation where the true distribution is slowly time-varying.

Note that the empirical distribution is discrete and the true distribution is usually continuous. We adopt the 1 -Wasserstein metric to measure the distance between distributions, which is different from the Kullback-Leibler divergence in [17] and $L^{1}$-norm in [18]. Then, we obtain the DR two-stage linear program over 1-Wasserstein balls and develop a second-order conic programming (SOCP) approach to solve it. Since the Wasserstein ball contains the true distribution with a high probability [19], the proposed DR problem is expected to exhibit good out-of-sample performance. Moreover, the Wasserstein ball can asymptotically degenerate to the true distribution as the sample size increases to infinity [19].

This work considers the distribution uncertainty either in the objective function or constraints of two-stage linear programs. Specifically, we first study the case with distribution uncertainty only in the objective function and exactly reformulate it as an SOCP problem, which covers all the results of the conference version of this work [20]. Then we proceed to the case with the distribution uncertainty only in constraints and show that such a program is generally NP-hard as it requires to solve a norm maximization problem over a polyhedron. The good news is that the resulting program can be reduced to an SOCP problem if the extreme points of the polyhedron are given as a prior. Motivated by this and also inspired by [21], [22], we design a novel constraint generation algorithm with provable convergence to approximately solve it.

It should be noted that [23] and [24] study the DR two-stage linear programs with the 2 -Wasserstein and $\infty$-Wasserstein metrics, respectively. In [23], the distribution uncertainty arises simultaneously in the objective function and constraints, which renders their model NP-hard, and the co-positive programs are utilized to approximately solve it. [24] reformulates the DR model as a computational demanding mixed-integer problem. In comparison, we exactly reformulate our model with distribution uncertainty only in the objective as an SOCP problem and design an SOCP approach to approximately solve the NPhard problem with uncertainty only in constraints. Moreover, we explicitly derive the distribution achieving the worst-case cost by simply perturbing each sample, based on which we can further assess the quality of an optimal decision. This is clearly in sharp contrast to [25], [23] and [24]. Overall, we summarized our contributions as follows:

- We propose a novel SOCP approach to solve the datadriven DR two-stage linear programs over 1-Wasserstein balls.
- We exactly reformulate the model with uncertainty only in the objective as a solvable SOCP problem.
- The model with uncertainty only in the constraints is shown to be NP-hard. To approximately solve it, we develop an SOCP-based constraint generation algorithm with provable convergence.
- The good out-of-sample performance and the computational complexity of our model are validated by experiments.
The rest of this paper is organized as follows. Section II proposes the DR two-stage linear program over the 1-Wasserstein ball. Section 【II reformulates the model with
the distribution uncertainty only in the objective function as a tractable SOCP problem. Section IV studies the model with uncertainty only in constraints and presents an SOCPbased constraint generation algorithm. Section $V$ derives the distribution achieving the worst-case cost. Section VI reports numerical results to illustrate the performance of the proposed model and the paper is concluded in Section VII

Notation: We denote the set of real positive real numbers by $\mathbb{R}$ and $\mathbb{R}_{+}$. The boldface lowercase letter denotes a vector, e.g., $\boldsymbol{x}=\left[x_{1}, \ldots, x_{n}\right]^{T} \in \mathbb{R}^{n}$. Special vectors include the zero vector $\mathbf{0}$ and the all one vector $\boldsymbol{e} .\|\cdot\|_{p}$ denotes the $l_{p}$-norm. Let $[N]=\{1,2, \ldots, N\}$ and $|\mathcal{E}|$ denotes the cardinality of $\mathcal{E}$. The letters s.t. are an abbreviation of the phrase "subject to ". $\operatorname{Diag}(\cdot)$ denotes a diagonal matrix with vector $(\cdot)$ being diagonal elements.

## II. Problem Formulation

## A. The Two-stage Stochastic Linear Optimization

Consider the classical two-stage stochastic linear program [26]

$$
\begin{equation*}
\underset{\boldsymbol{x} \in \mathcal{X}}{\operatorname{minimize}} \quad \boldsymbol{c}^{T} \boldsymbol{x}+\mathbb{E}_{\mathbb{F}}[Q(\boldsymbol{x}, \boldsymbol{\xi})] \tag{1}
\end{equation*}
$$

where $\boldsymbol{x} \in \mathbb{R}^{n}$ is the first-stage decision vector from a compact set $\mathcal{X}$ and is decided before the realization of a random vector $\boldsymbol{\xi} \in \mathbb{R}^{m}$ with the distribution $\mathbb{F}$.

The second-stage cost is evaluated based on the expectation of the following recourse problem

$$
\begin{align*}
Q(\boldsymbol{x}, \boldsymbol{\xi})=\min & \boldsymbol{z}(\boldsymbol{\xi})^{T} \boldsymbol{y} \\
\text { s.t. } & A(\boldsymbol{\xi}) \boldsymbol{x}+B \boldsymbol{y} \geq \boldsymbol{b}(\boldsymbol{\xi})  \tag{2}\\
& \boldsymbol{y} \in \mathbb{R}_{+}^{m},
\end{align*}
$$

where $B \in \mathbb{R}^{k \times m}$ is the recourse matrix and $\boldsymbol{z}(\boldsymbol{\xi}) \in$ $\mathbb{R}^{m}, A(\boldsymbol{\xi}) \in \mathbb{R}^{k \times n}$ and $\boldsymbol{b}(\boldsymbol{\xi}) \in \mathbb{R}^{k}$ depend on the random vector $\boldsymbol{\xi}$.

In the sequel, we study models with uncertainty only in the objective function or constraints, each of which is motivated by two notable examples, see also [14], [16], [25].

Example 1. ( [16]) Consider a portfolio program with $n$ assets which investors can invest in two stages. Generally the return for assets in the second stage is random, hence a stochastic two-stage portfolio program is designed for a maximum return

$$
\begin{equation*}
\underset{\boldsymbol{e}^{T} \boldsymbol{x}=1, \boldsymbol{x} \geq \mathbf{0}}{\operatorname{minimize}}-(\boldsymbol{e}+\boldsymbol{c})^{T} \boldsymbol{x}+\mathbb{E}_{\mathbb{F}}[Q(\boldsymbol{x}, \boldsymbol{\xi})] \tag{3}
\end{equation*}
$$

where $\boldsymbol{x}, \boldsymbol{c} \in \mathbb{R}^{n}$ are vectors of the invested dollar and the return for the $n$ assets in the first stage, $Q(\boldsymbol{x}, \boldsymbol{\xi})$ is given by

$$
\begin{align*}
Q(\boldsymbol{x}, \boldsymbol{\xi})=\min & -(\boldsymbol{e}+\boldsymbol{\xi})^{T} \boldsymbol{y}  \tag{4}\\
\text { s.t. } & \boldsymbol{y} \geq \mathbf{0}, \boldsymbol{\Delta}^{s} \geq 0, \boldsymbol{\Delta}^{b} \geq 0 \\
& A \boldsymbol{x}+(1-\theta) \boldsymbol{\Delta}^{b}-(1+\theta) \boldsymbol{\Delta}^{s}=\boldsymbol{y}
\end{align*}
$$

where $\boldsymbol{y}, \boldsymbol{\xi} \in \mathbb{R}^{n}$ are vectors of the invested dollar and the random return for the assets in the second stage. The matrix $A=\operatorname{Diag}(\boldsymbol{e}+\boldsymbol{c}), \Delta^{s}$ and $\Delta^{b}$ are the vectors of the dollar
for selling and buying the assets, and $\theta$ is the transaction cost.

Example 2. ([27]) Consider a material order problem with $n$ raw materials and $m$ desired products. Let $\boldsymbol{b} \in \mathbb{R}^{m}$ denote the market demand vector for products. Let $a_{i j}$ be the amount of product $i$ produced by per unit of material $j$ and $A=\left[a_{i j}\right]_{m \times n}$ be the matrix of the production amount for all materials.

The market demand is usually time-varying and the uncertainty in the production amount is generally inevitable due to the quality of raw materials. Hence, it is unavoidable to introduce uncertainty $\boldsymbol{\xi}$ to the demand vector $\boldsymbol{b}$ and the matrix $A$, then the order problem is formulated as

$$
\begin{equation*}
\underset{e^{T} \boldsymbol{x} \leq u, \boldsymbol{x} \geq \mathbf{0}}{\operatorname{minimize}}\left\{\boldsymbol{c}^{T} \boldsymbol{x}+\mathbb{E}_{\mathbb{F}}[Q(\boldsymbol{x}, \boldsymbol{\xi})]\right\} \tag{5}
\end{equation*}
$$

where $u$ is the capacity of $n$ materials, $c \in \mathbb{R}^{n}$ is the cost vector of $n$ materials, and $Q(\boldsymbol{x}, \boldsymbol{\xi})$ is given as

$$
\begin{align*}
Q(\boldsymbol{x}, \boldsymbol{\xi})=\min & \boldsymbol{z}^{T} \boldsymbol{y} \\
\text { s.t. } & A(\boldsymbol{\xi}) \boldsymbol{x}+\boldsymbol{y} \geq \boldsymbol{b}(\boldsymbol{\xi})  \tag{6}\\
& \boldsymbol{y} \in \mathbb{R}_{+}^{m}
\end{align*}
$$

where $\boldsymbol{z} \in \mathbb{R}^{m}$ is the penalty vector for per unit of undeliverable products and $\boldsymbol{y} \in \mathbb{R}_{+}^{m}$ is the corresponding shortage amount vector.

Motivated by above examples, we consider that $\boldsymbol{z}(\boldsymbol{\xi}), A(\boldsymbol{\xi})$ and $\boldsymbol{b}(\boldsymbol{\xi})$ in 11 depend affinely on $\boldsymbol{\xi}$, i.e.,

$$
\begin{align*}
& \boldsymbol{z}(\boldsymbol{\xi})=\boldsymbol{z}_{0}+\sum_{i=1}^{m} \xi_{i} \boldsymbol{z}_{i}, \quad A(\boldsymbol{\xi})=A_{0}+\sum_{i=1}^{m} \xi_{i} A_{i} \\
& \boldsymbol{b}(\boldsymbol{\xi})=\boldsymbol{b}_{0}+\sum_{i=1}^{m} \xi_{i} \boldsymbol{b}_{i} \tag{7}
\end{align*}
$$

where $\boldsymbol{z}_{0}, \boldsymbol{z}_{1}, \ldots, \boldsymbol{z}_{m} \in \mathbb{R}^{m}, \boldsymbol{b}_{0}, \boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{m} \in \mathbb{R}^{k}$ and $A_{0}, A_{1}, \ldots, A_{m} \in \mathbb{R}^{k \times n}$ are given as prior. In fact, the affine uncertainty has also been adopt in [16], [25].

The following condition guarantees the feasibility of the second-stage problem in (2) and is satisfied by many problems, e.g., the production planning problem, the newsvendor problem and its variants [26].

Assumption 1. The second-stage problem in (2) is always feasible for any $\boldsymbol{x} \in \mathcal{X}$ and $\boldsymbol{\xi}$.

## B. Distributionally Robust Two-stage Problems

The program in (1) generally requires an exact distribution $\mathbb{F}$ of $\boldsymbol{\xi}$. In practice, $\mathbb{F}$ can only be estimated through a finite sample dataset $\left\{\widehat{\boldsymbol{\xi}}^{i}\right\}_{i=1}^{N}$ and a common idea is to adopt the SAA method, where $\mathbb{F}$ is approximated by an empirical distribution $\mathbb{F}_{N}$ over the sample dataset, i.e.,

$$
\mathbb{F}_{N}(\boldsymbol{\xi})=\frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\left\{\widehat{\boldsymbol{\xi}}^{i} \leq \boldsymbol{\xi}\right\}}
$$

where $1_{A}$ is the indicator of event $A$. Then the stochastic linear problem in (1) is approximated by

$$
\begin{equation*}
\underset{\boldsymbol{x} \in \mathcal{X}}{\operatorname{minimize}}\left\{\boldsymbol{c}^{T} \boldsymbol{x}+\frac{1}{N} \sum_{i=1}^{N} Q\left(\boldsymbol{x}, \widehat{\boldsymbol{\xi}}^{i}\right)\right\} \tag{8}
\end{equation*}
$$

By Glivenko-Cantelli theorem [28], the distribution $\mathbb{F}_{N}$ weakly converges to the true distribution $\mathbb{F}$ as $N$ increases to infinity. This implies the asymptotic convergence of (8) to the stochastic model (1). Hence, the SAA method is sensible only when $\mathbb{F}_{N}$ well approximates the true distribution $\mathbb{F}$.

However, insufficient and/or low-quality samples may lead to an empirical distribution $\mathbb{F}_{N}$ far from the true distribution F. Thus, the SAA model (8) may be not reliable with poor out-of-sample performance.

As in [19], a data-driven approach is adopted to address the distribution uncertainty in this work. We assume that $\mathbb{F}$ belongs to an ambiguity set $\mathcal{F}_{N}$ including all distributions within $\epsilon_{N}$-distance from the empirical distribution $\mathbb{F}_{N}$. Here $\epsilon_{N}$ indicates the confidence on $\mathbb{F}_{N}$, e.g., the larger the $\epsilon_{N}$, the lower the confidence.

Since the true distribution $\mathbb{F}$ is generally continuous and the empirical distribution $\mathbb{F}_{N}$ is discrete, the 1-Wasserstein metric [29] is adopted to measure their distance and consequently a 1-Wasserstein ball $\mathcal{F}_{N}$ is obtained. Then we are interested in the worst-case second-stage cost over $\mathcal{F}_{N}$, i.e.,

$$
\begin{equation*}
\beta(\boldsymbol{x})=\sup _{\mathbb{F} \in \mathcal{F}_{N}} \mathbb{E}_{\mathbb{F}}[Q(\boldsymbol{x}, \boldsymbol{\xi})], \tag{9}
\end{equation*}
$$

and the DR two-stage linear program is formulated as

$$
\begin{equation*}
\underset{\boldsymbol{x} \in \mathcal{X}}{\operatorname{minimize}} \boldsymbol{c}^{T} \boldsymbol{x}+\beta(\boldsymbol{x}) \tag{10}
\end{equation*}
$$

To evaluate an optimal solution, we also derive the worst-case distribution $\mathbb{F}^{*}$ that achieves the worst-case second-stage cost, i.e.,

$$
\begin{equation*}
\beta(\boldsymbol{x})=\sup _{\mathbb{F} \in \mathcal{F}_{N}} \mathbb{E}_{\mathbb{F}}[Q(\boldsymbol{x}, \boldsymbol{\xi})]=\mathbb{E}_{\mathbb{F}^{*}}[Q(\boldsymbol{x}, \boldsymbol{\xi})] \tag{11}
\end{equation*}
$$

## C. Ambiguity Set via the 1-Wasserstein Metric

We introduce the $r$-Wasserstein metric below.
Definition 1. ( [29]) Let $d\left(\boldsymbol{\xi}^{1}, \boldsymbol{\xi}^{2}\right)=\left\|\boldsymbol{\xi}^{1}-\boldsymbol{\xi}^{2}\right\|_{p}$ be the $l_{p^{-}}$ norm of $\boldsymbol{\xi}^{1}-\boldsymbol{\xi}^{2}$ on $\mathbb{R}^{n}$ and $(\Xi, d)$ be a Polish metric space. Given a pair of distributions $\mathbb{F}_{1} \in \mathcal{M}(\Xi)$ and $\mathbb{F}_{2} \in \mathcal{M}(\Xi)$ where $\mathcal{M}(\Xi)$ is a set containing all distributions supported on $\Xi$, the $r$-Wasserstein metric $W^{r}: \mathcal{M}(\Xi) \times \mathcal{M}(\Xi) \rightarrow \mathbb{R}_{+}$is defined as

$$
\begin{gather*}
W^{r}\left(\mathbb{F}_{1}, \mathbb{F}_{2}\right)=\inf \left\{\left(\int_{\Xi^{2}} d\left(\boldsymbol{\xi}^{1}, \boldsymbol{\xi}^{2}\right)^{r} K\left(\mathrm{~d} \boldsymbol{\xi}^{1}, \mathrm{~d} \boldsymbol{\xi}^{2}\right)\right)^{1 / r}:\right. \\
\left.\int_{\Xi} K\left(\boldsymbol{\xi}^{1}, \mathrm{~d} \boldsymbol{\xi}^{2}\right)=\mathbb{F}_{1}\left(\boldsymbol{\xi}^{1}\right), \int_{\Xi} K\left(\mathrm{~d} \boldsymbol{\xi}^{1}, \boldsymbol{\xi}^{2}\right)=\mathbb{F}_{2}\left(\boldsymbol{\xi}^{2}\right)\right\} \tag{12}
\end{gather*}
$$

where $r \geq 1$ and $K$ is a joint distribution with its marginal distributions being $\mathbb{F}_{1}$ and $\mathbb{F}_{2}$.

Without scarifying much modeling power and to obtain a real metric [29], we need the following requirement on the set $\mathcal{M}(\Xi)$.
Assumption 2. For any distribution $\mathbb{F} \in \mathcal{M}(\Xi)$, it holds

$$
\int_{\Xi}\|\boldsymbol{\xi}\|_{p}^{r} \mathbb{F}(\mathrm{~d} \boldsymbol{\xi})<\infty
$$

Different from [23] and [24], we adopt the 1-Wasserstein metric and $l_{2}$-norm, i.e., $r=1$ and $p=2$ in 12 to construct the ambiguity ball $\mathcal{F}_{N}$,

$$
\begin{equation*}
\mathcal{F}_{N}=\left\{\mathbb{F} \in \mathcal{M}(\Xi): W^{1}\left(\mathbb{F}_{N}, \mathbb{F}\right) \leq \epsilon_{N}\right\}, \tag{13}
\end{equation*}
$$

where $\epsilon_{N}>0$ is the ball radius, i.e., $\mathcal{F}_{N}$ is the set of distributions within $\epsilon_{N}$-distance from $\mathbb{F}_{N}$.

## D. Comparisons with the state-of-the-art methods

In [25], the ambiguity set of the DR two-stage linear programs is defined as a set of distributions with specified first- and second-order moment constraints.
[23] considers DR two-stage linear programs of the form (10) with 2 -Wasserstein balls, i.e., $r=p=2$ in (12), and $Q(\boldsymbol{x}, \boldsymbol{\xi})$ is defined as

$$
Q(\boldsymbol{x}, \boldsymbol{\xi})=\begin{array}{ll}
\min & (Q \boldsymbol{\xi}+\boldsymbol{q})^{T} \boldsymbol{y} \\
& \text { s.t. } T(\boldsymbol{x}) \boldsymbol{\xi}+h(\boldsymbol{x}) \leq B \boldsymbol{y} \tag{14}
\end{array}
$$

where $T(\cdot)$ and $h(\cdot)$ are two affine functions.
In [24], the DR two-stage program is defined via the $\infty$ Wasserstein metric, i.e, $r=\infty$ and $p=1, \infty$ in 12 with the uncertainty only in the objective function or constraints separately, i.e., $Q$ or $T(\boldsymbol{x})$ in (14) is set to 0 respectively.

Comparisons with those state-of-art models are summarized as follows:

- Model differences: Clearly, $Q(\boldsymbol{x}, \boldsymbol{\xi})$ in (2) of this work and [25] is different from (14] in [23] and [24]. Our model is motivated from a wide range of real applications, see e.g. Examples 1.2. Note that this "minor" difference may require a completely different solution approach.
- Solution approaches: [23] derives co-positive programs to approximate their NP-hard DR two-stage model. [24] reformulates the model as a computational demanding mixed-integer problem. [25] approximate their model by linear decision rule techniques.
In this work, we equivalently reformulate our model with distribution uncertainty only in the objective as an SOCP problem and design an SOCP-based constraint generation algorithm for the problem with distribution uncertainty only in constraints.
- Approximation gaps: There is no approximation gap in [23] and [25], under the condition that for any $t \in \mathbb{R}^{k}$, there exists a solution $\boldsymbol{y}$ to solve the inequality $B \boldsymbol{y} \geq \boldsymbol{t}$ (aka complete recourse). In this work, the zero-gap condition in Assumption 1 (aka relatively complete recourse) is weaker and satisfied by numerous real application models [26].
As explicitly stated in [25], "there are also problems that would generally not satisfy complete recourse, such as a production planning problem where a manager determines a production plan today to satisfy all uncertain demands for tomorrow instead of incurring penalty", see Example 2 which satisfies relatively complete recourse.
- The worst-case distribution: In sharp contrast to those state-of-art models, this work derives the distribution attaining the worst-case second-stage cost with distribution uncertainty either in the objective function or constraints, respectively.


## III. Uncertainty in the Objective Function

We first consider the distribution uncertainty only in the objective function of (2) via the following form

$$
\begin{align*}
Q(\boldsymbol{x}, \boldsymbol{\xi})=\min & \boldsymbol{z}(\boldsymbol{\xi})^{T} \boldsymbol{y} \\
\text { s.t. } & A \boldsymbol{x}+B \boldsymbol{y} \geq \boldsymbol{b}  \tag{15}\\
& \boldsymbol{y} \in \mathbb{R}_{+}^{m}
\end{align*}
$$

where $\boldsymbol{z}(\boldsymbol{\xi})$ is defined as (7) in Section II-A.
We convert the problem in 10 with $Q(\boldsymbol{x}, \boldsymbol{\xi})$ given by 15 over the 1 -Wasserstein ball $\mathcal{F}_{N}$ to an SOCP problem which can be solved efficiently by general-purpose commercial-grade solvers such as CPLEX.

Theorem 1. Under Assumptions 1,2 the worst-case $\beta(\boldsymbol{x})$ with $Q(\boldsymbol{x}, \boldsymbol{\xi})$ in 15 , over the 1-Wasserstein ball $\mathcal{F}_{N}$ is equivalent to the optimal value of an SOCP problem

$$
\begin{align*}
\beta(\boldsymbol{x})=\inf & \left\{\lambda \epsilon_{N}+\frac{1}{N} \sum_{i=1}^{N} s_{i}\right\} \\
\text { s.t. } & \lambda \geq\|Z \boldsymbol{y}\|_{2}  \tag{16}\\
& s_{i} \geq \boldsymbol{z}_{0}^{T} \boldsymbol{y}+\boldsymbol{y}^{T} Z^{T} \widehat{\boldsymbol{\xi}}^{i}, \forall i \in[N] \\
& A \boldsymbol{x}+B \boldsymbol{y} \geq \boldsymbol{b}, \quad \boldsymbol{y} \geq \mathbf{0}
\end{align*}
$$

where $Z^{T}=\left[\boldsymbol{z}_{1}, \ldots, \boldsymbol{z}_{m}\right]$.
Moreover, the associated DR problem (10) is equivalent to the following SOCP problem

$$
\begin{array}{ll}
\underset{\boldsymbol{x} \in \mathcal{X}}{\operatorname{minimize}} & \left\{\boldsymbol{c}^{T} \boldsymbol{x}+\lambda \epsilon_{N}+\frac{1}{N} \sum_{i=1}^{N} s_{i}\right\} \\
\text { subject to } & \lambda \geq\|Z \boldsymbol{y}\|_{2}  \tag{17}\\
& s_{i} \geq \boldsymbol{z}_{0}^{T} \boldsymbol{y}+\boldsymbol{y}^{T} Z^{T} \widehat{\boldsymbol{\xi}}^{i}, \forall i \in[N] \\
& A \boldsymbol{x}+B \boldsymbol{y} \geq \boldsymbol{b}, \boldsymbol{y} \geq \mathbf{0} .
\end{array}
$$

Proof. For any feasible first-stage decision vector $\boldsymbol{x}, \beta(\boldsymbol{x})$ over the 1 -Wasserstein ball can be obtained by solving a conic linear program

$$
\begin{align*}
\beta(\boldsymbol{x})=\sup & \sum_{i=1}^{N} \int_{\Xi} Q(\boldsymbol{x}, \boldsymbol{\xi}) K\left(\mathrm{~d} \boldsymbol{\xi}, \widehat{\boldsymbol{\xi}}^{i}\right) \\
\text { s.t. } & \int_{\Xi} K\left(\mathrm{~d} \boldsymbol{\xi}, \widehat{\boldsymbol{\xi}}^{i}\right)=\frac{1}{N}, \forall i \in[N]  \tag{18}\\
& \int_{\Xi} \sum_{i=1}^{N} d\left(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}}^{i}\right) K\left(\mathrm{~d} \boldsymbol{\xi}, \widehat{\boldsymbol{\xi}}^{i}\right) \leq \epsilon_{N} .
\end{align*}
$$

The Lagrange dual function for (18) is represented as

$$
\begin{aligned}
& g(\lambda, \boldsymbol{s}) \\
&= \sup _{\boldsymbol{\xi} \in \Xi}\left\{\int_{\Xi} \sum_{i=1}^{N}\left(Q(\boldsymbol{x}, \boldsymbol{\xi})-s_{i}-\lambda d\left(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}}^{i}\right)\right) K\left(\mathrm{~d} \boldsymbol{\xi}, \widehat{\boldsymbol{\xi}}^{i}\right)\right\} \\
&+\frac{1}{N} \sum_{i=1}^{N} s_{i}+\lambda \epsilon_{N} .
\end{aligned}
$$

Consequently, the dual problem of 18 is given as

$$
\begin{equation*}
\beta(\boldsymbol{x})=\inf \quad \lambda \epsilon_{N}+\frac{1}{N} \sum_{i=1}^{N} s_{i} \tag{19}
\end{equation*}
$$

$$
\begin{align*}
& \text { s.t. } \lambda \geq 0 \\
& \qquad Q(\boldsymbol{x}, \boldsymbol{\xi})-\lambda d\left(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}}^{i}\right) \leq s_{i}, \forall i \in[N], \boldsymbol{\xi} \in \Xi . \tag{20}
\end{align*}
$$

Since $\epsilon_{N}>0$, then $K=\mathbb{F}_{N} \times \mathbb{F}_{N}$ is a strictly feasible solution to (18), the Slater condition for the strong duality of primal problem (18) and its dual problem (19) is satisfied [30].

The constraints in 20 require a feasible second-stage solution $\widehat{\boldsymbol{y}}$ to guarantee the feasibility of the following inequality

$$
\boldsymbol{z}(\boldsymbol{\xi})^{T} \widehat{\boldsymbol{y}}-\lambda d\left(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}}^{i}\right) \leq s_{i}, \forall i \in[N], \boldsymbol{\xi} \in \Xi
$$

Note that Assumption 1 ensures the existence of such a $\widehat{\boldsymbol{y}}$. Hence, (20) can be expressed as

$$
\begin{equation*}
s_{i} \geq \boldsymbol{z}(\boldsymbol{\xi})^{T} \widehat{\boldsymbol{y}}-\lambda d\left(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}}^{i}\right), \quad \forall i \in[N], \boldsymbol{\xi} \in \Xi \tag{21}
\end{equation*}
$$

Since

$$
\boldsymbol{z}(\boldsymbol{\xi})^{T} \widehat{\boldsymbol{y}}=\left(\boldsymbol{z}_{0}+\sum_{i=1}^{m} \xi_{i} \boldsymbol{z}_{i}\right)^{T} \widehat{\boldsymbol{y}}=\boldsymbol{z}_{0}^{T} \widehat{\boldsymbol{y}}+\boldsymbol{\xi}^{T} Z \widehat{\boldsymbol{y}}
$$

it implies that

$$
\begin{aligned}
& \sup _{\boldsymbol{\xi}}\left\{\boldsymbol{z}(\boldsymbol{\xi})^{T} \widehat{\boldsymbol{y}}-\lambda\left\|\boldsymbol{\xi}-\widehat{\boldsymbol{\xi}}^{i}\right\|_{2}\right\} \\
& =\sup _{\boldsymbol{\xi}}\left\{\boldsymbol{z}_{0}^{T} \widehat{\boldsymbol{y}}+\boldsymbol{\xi}^{T} Z \widehat{\boldsymbol{y}}-\lambda\left\|\boldsymbol{\xi}-\widehat{\boldsymbol{\xi}}^{i}\right\|_{2}\right\} \\
& = \begin{cases}\boldsymbol{z}_{0}^{T} \widehat{\boldsymbol{y}}+\widehat{\boldsymbol{y}}^{T} Z^{T} \boldsymbol{\xi}^{i}, & \text { if }\|Z \widehat{\boldsymbol{y}}\|_{2} \leq \lambda \\
+\infty, & \text { if }\|Z \widehat{\boldsymbol{y}}\|_{2}>\lambda\end{cases}
\end{aligned}
$$

where the last equality follows from Lemma 1 in [31].
Consequently, 20) admits an equivalent form

$$
\left\{\begin{array}{l}
s_{i} \geq \boldsymbol{z}_{0}^{T} \widehat{\boldsymbol{y}}+\widehat{\boldsymbol{y}}^{T} Z^{T} \widehat{\boldsymbol{\xi}}^{i}, \forall i \in[N] \\
\lambda \geq\|Z \widehat{\boldsymbol{y}}\|_{2}
\end{array}\right.
$$

Inserting the above to (20) leads to the equivalence of (16) and (9). Hence, the two-stage problem (10) can be equivalently reformulated as the SOCP problem (17).

Theorem 1 shows that the optimization program 10 can be reformulated as a tractable SOCP problem. Furthermore, different $l_{p}$-norms in (12) lead to different equivalent forms of the DR two-stage problem, see Table $\square$ for details, where LP represents the linear programming.

TABLE I
EQUIVALENT PROBLEMS OF THE OUR DR PROBLEM, WHERE $p$ REPRESENTS THE $l_{p}$-NORM IN 12 .

| Norm | $p=1$ | $p=2$ | $p=\infty$ | Otherwise |
| :---: | :---: | :---: | :---: | :---: |
| Problem | LP | SOCP | LP | Convex Program |

## IV. Uncertainty in the Constraints

In this section we consider the distribution uncertainty only in constraints of (2), i.e.,

$$
\begin{align*}
Q(\boldsymbol{x}, \boldsymbol{\xi})=\min & \boldsymbol{z}^{T} \boldsymbol{y} \\
\text { s.t. } & A(\boldsymbol{\xi}) \boldsymbol{x}+B \boldsymbol{y} \geq \boldsymbol{b}(\boldsymbol{\xi})  \tag{22}\\
& \boldsymbol{y} \in \mathbb{R}_{+}^{m}
\end{align*}
$$

where $A(\boldsymbol{\xi})$ and $\boldsymbol{b}(\boldsymbol{\xi})$ are defined in (7) of Section II-A.

## A. Reformulation of the DR Problem

We first prove the NP-hardness of the problem with $Q(\boldsymbol{x}, \boldsymbol{\xi})$ given in 22.

Theorem 2. Under Assumptions 11|2 the worst-case $\beta(\boldsymbol{x})$ with $Q(\boldsymbol{x}, \boldsymbol{\xi})$ in 22 over the 1-Wasserstein ball $\mathcal{F}_{N}$ can be computed by an NP-hard problem

$$
\begin{align*}
\beta(\boldsymbol{x})=\inf & \left\{\lambda \epsilon_{N}+\frac{1}{N} \sum_{i=1}^{N} s_{i}\right\}  \tag{23}\\
\text { s.t. } & s_{i} \geq(C \boldsymbol{p})^{T} \widehat{\boldsymbol{\xi}}^{i}+\boldsymbol{p}^{T}\left(\boldsymbol{b}_{0}-A_{0} \boldsymbol{x}\right)  \tag{24}\\
& \lambda \geq\|C \boldsymbol{p}\|_{2}, \forall i \in[N], \boldsymbol{p} \in \mathcal{P} \tag{25}
\end{align*}
$$

where

$$
\begin{equation*}
C=\left[\boldsymbol{b}_{1}-A_{1} \boldsymbol{x}, \ldots, \boldsymbol{b}_{m}-A_{m} \boldsymbol{x}\right]^{T} \tag{26}
\end{equation*}
$$

and $\mathcal{P}$ is a polyhedron given by

$$
\begin{equation*}
\mathcal{P}=\left\{\boldsymbol{p} \in \mathbb{R}_{+}^{k}: B^{T} \boldsymbol{p} \leq \boldsymbol{d}\right\} \tag{27}
\end{equation*}
$$

Proof. The strong duality stills holds for $\beta(\boldsymbol{x})$, which is rewritten as

$$
\begin{align*}
& \beta(\boldsymbol{x})= \inf \quad \lambda \epsilon_{N}+\frac{1}{N} \sum_{i=1}^{N} s_{i}  \tag{28}\\
& \text { s.t. } \lambda \geq 0 \\
& \qquad Q(\boldsymbol{x}, \boldsymbol{\xi})-\lambda d\left(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}}^{i}\right) \leq s_{i}, \forall i \in[N], \boldsymbol{\xi} \in \Xi \tag{29}
\end{align*}
$$

Under the strong duality of the LP problem, $Q(\boldsymbol{x}, \boldsymbol{\xi})$ in 22) is equivalent to

$$
\begin{align*}
Q(\boldsymbol{x}, \boldsymbol{\xi})=\max & \boldsymbol{p}^{T}(\boldsymbol{b}(\boldsymbol{\xi})-A(\boldsymbol{\xi}) \boldsymbol{x}) \\
\text { s. t. } & \boldsymbol{z \geq B ^ { T } \boldsymbol { p }}  \tag{30}\\
& \boldsymbol{p} \geq 0
\end{align*}
$$

Then the constraints in 29) can be expressed as

$$
\begin{equation*}
s_{i} \geq \boldsymbol{p}^{T}(\boldsymbol{b}(\boldsymbol{\xi})-A(\boldsymbol{\xi}) \boldsymbol{x})-\lambda d\left(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}}^{i}\right), \forall \boldsymbol{\xi} \in \Xi, \boldsymbol{p} \in \mathcal{P} \tag{31}
\end{equation*}
$$

Furthermore, the right-hand side of (31) is expressed as

$$
\begin{aligned}
& \sup _{\boldsymbol{\xi}}\left\{\boldsymbol{p}^{T}(\boldsymbol{b}(\boldsymbol{\xi})-A(\boldsymbol{\xi}) \boldsymbol{x})-\lambda d\left(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}}^{i}\right)\right\} \\
& =\sup _{\boldsymbol{\xi}}\left\{(C \boldsymbol{p})^{T} \boldsymbol{\xi}+\boldsymbol{p}^{T}\left(\boldsymbol{b}_{0}-A_{0} \boldsymbol{x}\right)-\lambda d\left(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}}^{i}\right)\right\} \\
& = \begin{cases}(C \boldsymbol{p})^{T} \widehat{\boldsymbol{\xi}}^{i}-\boldsymbol{p}^{T}\left(\boldsymbol{b}_{0}-A_{0} \boldsymbol{x}\right), & \text { if }\|C \boldsymbol{p}\|_{2} \leq \lambda \\
+\infty, & \text { if }\|C \boldsymbol{p}\|_{2}>\lambda,\end{cases}
\end{aligned}
$$

where $C$ is defined in 26) and the second equality follows from Lemma 1 in [31].

Consequently, 29) is equivalent to

$$
\begin{cases}s_{i} \geq(C \boldsymbol{p})^{T} \widehat{\boldsymbol{\xi}}^{i}-\boldsymbol{p}^{T}\left(\boldsymbol{b}^{0}-A^{0} \boldsymbol{x}\right), & \forall i \in[N], \boldsymbol{p} \in \mathcal{P} \\ \lambda \geq\|C \boldsymbol{p}\|_{2}, & \forall \boldsymbol{p} \in \mathcal{P} .\end{cases}
$$

Thus, $\beta(\boldsymbol{x})$ in (9) is reformulated as 23).
The constraint 25 in 23 can be expressed as

$$
\lambda \geq \max _{\boldsymbol{p} \in \mathcal{P}}\|C \boldsymbol{p}\|_{2}
$$

Thus, the norm maximization problem over the polyhedron is NP-complete [32] and checking the feasibility of constraint (25) is NP-hard. This completes the proof.

Theorem 2 immediately implies the NP-hardness of the problem in 10 . If the extreme point set $\mathcal{E}$ of the polyhedron $\mathcal{P}$ is explicitly known, the problem 10 can be reformulated as a solvable SOCP problem.
Corollary 1. Suppose that Assumptions $1 \| 2$ hold and the extreme point set $\mathcal{E}$ of the polyhedron $\mathcal{P}$ in (27) is known, the 1 -Wasserstein problem (10) with $Q(\boldsymbol{x}, \boldsymbol{\xi})$ in (22) is equivalent to an SOCP problem

$$
\begin{array}{ll}
\underset{\boldsymbol{x} \in \mathcal{X}}{\operatorname{minimize}} & \left\{\boldsymbol{c}^{T} \boldsymbol{x}+\lambda \epsilon_{N}+\frac{1}{N} \sum_{i=1}^{N} s_{i}\right\} \\
\text { subject to } & s_{i} \geq(\boldsymbol{C p})^{T} \widehat{\boldsymbol{\xi}}^{i}+\boldsymbol{p}^{T}\left(\boldsymbol{b}^{0}-A^{0} \boldsymbol{x}\right),  \tag{32}\\
& \lambda \geq\|\boldsymbol{C} \boldsymbol{p}\|_{2}, \quad \forall i \in[N], \boldsymbol{p} \in \mathcal{E}
\end{array}
$$

Proof. Since the LP problem (30) attains its optimal value at an extreme point of its feasible set $\mathcal{P}$, it holds that

$$
Q(\boldsymbol{x}, \boldsymbol{\xi})=\max _{\boldsymbol{p} \in \mathcal{E}} \boldsymbol{p}^{T}(\boldsymbol{b}(\boldsymbol{\xi})-A(\boldsymbol{\xi}) \boldsymbol{x})
$$

Then the constraints in (25) and 24) can be explicitly expressed as

$$
\begin{cases}s_{i} \geq(C \boldsymbol{p})^{T} \widehat{\boldsymbol{\xi}}^{i}-\boldsymbol{p}^{T}\left(\boldsymbol{b}^{0}-A^{0} \boldsymbol{x}\right), & \forall i \in[N], \boldsymbol{p} \in \mathcal{E} \\ \lambda \geq\|C \boldsymbol{p}\|_{2}, & \forall \boldsymbol{p} \in \mathcal{E}\end{cases}
$$

which leads to the equivalence of (32) and 10. This completes the proof.

Corollary 1 shows that we can solve the DR two-stage problem by explicitly enumerating the extreme points of the polyhedron $\mathcal{P}$. Motivated by this, we design an algorithm to approximately solve the NP-hard DR two-stage problem via a constraint generation approach.

## B. Approximately Solving the DR Two-stage Problem with Uncertainty in Constraints

In this subsection, we propose a constraint generation algorithm to solve (10). Inspired by Corollary 1, the DR problem can be efficiently solved given all extreme points of $\mathcal{P}$. While the direct enumeration of all extreme points is computational demanding, we gradually select sets of "good" extreme points by solving a sequence of second-stage problems $\beta(\boldsymbol{x})$. Particularly, we utilize a master-subproblem framework to approximately solve 10 .

In the master problem (MP), we find an optimal solution under a selected subset of extreme points. Then a subproblem ( SuP ) is solved to obtain a better subset of extreme points. We add these points to the subset in MP as feasible cuts. Note that the optimal values of the MP and SuP are the lower and upper bounds for 10 respectively. Both the lower and upper bounds will converge and a good solution to (10) can be obtained. The algorithm based on such an MP-SuP framework is given in the sequel.

By Corollary 1, the MP is an SOCP problem given as

$$
\begin{align*}
& \underset{\boldsymbol{x} \in \mathcal{X}}{\operatorname{minimize}}\left\{\boldsymbol{c}^{T} \boldsymbol{x}+\lambda \epsilon_{N}+\frac{1}{N} \sum_{i=1}^{N} s_{i}\right\}  \tag{33}\\
& \text { subject to } s_{i} \geq(C \boldsymbol{p})^{T} \widehat{\boldsymbol{\xi}^{i}}+\boldsymbol{p}^{T}\left(\boldsymbol{b}^{0}-A^{0} \boldsymbol{x}\right)
\end{align*}
$$

$$
\lambda \geq\|C \boldsymbol{p}\|_{2}, \quad \forall i \in[N], \boldsymbol{p} \in \mathcal{E}_{s}
$$

where $\mathcal{E}_{s}$ is a given subset of extreme points of $\mathcal{P}$.
After obtaining an optimal solution $\boldsymbol{x}^{m}$ of the MP, an SuP is derived as follows

$$
\begin{align*}
& \beta\left(\boldsymbol{x}^{m}\right)= \min _{\lambda^{s}, s_{i}^{s}}  \tag{34}\\
&\left\{\lambda^{s} \epsilon_{N}+\frac{1}{N} \sum_{i=1}^{N} s_{i}^{s}\right\} \\
& \text { s.t. } \quad s_{i}^{s} \geq(C \boldsymbol{p})^{T} \widehat{\boldsymbol{\xi}^{i}}+\boldsymbol{p}^{T}\left(\boldsymbol{b}^{0}-A^{0} \boldsymbol{x}^{m}\right) \\
& \lambda^{s} \geq\|C \boldsymbol{p}\|_{2}, \quad \forall i \in[N] \boldsymbol{p} \in \mathcal{P} .
\end{align*}
$$

A weak condition is needed to obtain an good solution of the SuP.

Assumption 3. The polyhedron $\mathcal{P}=\left\{\boldsymbol{p} \in \mathbb{R}_{+}^{k}: B^{T} \boldsymbol{p} \leq \boldsymbol{z}\right\}$ is nonempty and bounded.

The decision variables $\lambda^{s}$ and $s^{s}$ in (34) are completely decoupled and hence we can find their optimal solutions separately. To achieve it, we have the following steps.

1) An optimal solution $s^{s}$ to SuP is obtained by solving a group of linear programs, i.e.,

$$
\begin{align*}
s_{i}^{s}=\max & (C \boldsymbol{p})^{T} \widehat{\boldsymbol{\xi}}^{i}+\boldsymbol{p}^{T}\left(\boldsymbol{b}^{0}-A^{0} \boldsymbol{x}^{m}\right)  \tag{35}\\
\text { s. t. } & \boldsymbol{p} \in \mathcal{P} .
\end{align*}
$$

2) An optimal $\lambda^{s}$ is obtained by solving a norm maximization problem, i.e.,

$$
\begin{gather*}
\lambda^{s}=\max \quad\|C \boldsymbol{p}\|_{2}  \tag{36}\\
\text { s.t. } \quad \boldsymbol{p} \in \mathcal{P}
\end{gather*}
$$

A sequence of optimal solutions $\left\{\boldsymbol{p}_{i}^{*}\right\}_{i=1}^{N}$ to (35) can be added to the extreme point subset $\mathcal{E}_{s}$ in MP, since the LP problem (35) obtains its optimal value at extreme points of the feasible region $\mathcal{P}$.

To solve the non-convex norm maximization problem, we adopt the consensus alternating direction method of multipliers (ADMM) method [33]. Particularly, 36] is reformulated as a consensus form via $m$ auxiliary variables $\left\{\boldsymbol{g}_{1}, \ldots, \boldsymbol{g}_{m}\right\}$, i.e.,

$$
\begin{align*}
\lambda^{s}=\min & -\boldsymbol{p}^{T} C^{T} C \boldsymbol{p} \\
\text { s.t. } & \boldsymbol{b}_{i}^{T} \boldsymbol{g}_{i} \leq z_{i}, \boldsymbol{g}_{i} \geq \mathbf{0}  \tag{37}\\
& \boldsymbol{g}_{i}=\boldsymbol{p}, \forall i \in[m],
\end{align*}
$$

where $\boldsymbol{b}_{i}$ is the $i$-th column of $B$. Algorithm 1 provides the detailed consensus-ADMM algorithm. We omit its convergence proof for brevity, which can be found in [33].

By Assumption 3, a solution $\boldsymbol{p}^{*}$ to (36) as an extreme point of polyhedron $\mathcal{P}$ is ensured to exist and then is added to the subset $\mathcal{E}_{s}$ [32].

We provide the MP-SuP based algorithm in Algorithm 2 Theorem 3 shows that Algorithm 2 terminates in a finite number of iterations.
Theorem 3. Under Assumption 3] Algorithm 2 generates an optimal solution of $\langle\overline{10}$ in $O(|\mathcal{E}|)$ iterations.
Proof. Let $\left\{\boldsymbol{x}_{k}, \lambda_{k}, \boldsymbol{s}_{k}\right\}$ be an optimal solution of MP in the $k$-th iteration and $\left\{\lambda_{k}^{s}, s_{k}^{s}\right\}$ be an optimal solution of SuP with $\left\{\boldsymbol{p}_{k}^{i}\right\}_{i=1}^{N} \cup\left\{\boldsymbol{p}_{k}\right\}$ being the extreme points of SuP. We show that

```
Algorithm 1 The consensus-ADMM for (37)
Input: Matrix \(B, C\), vector \(\boldsymbol{z}, \boldsymbol{g}_{i}\) and \(\boldsymbol{u}_{i}\), tolerance \(\tau\)
Output: Optimal solution \(\boldsymbol{p}^{*}\) and optimal value \(\lambda^{s}\)
    Initialize \(\boldsymbol{g}_{i}\) and \(\boldsymbol{u}_{i}\)
    repeat
        \(\boldsymbol{p} \leftarrow\left(\frac{-C^{T} C}{\rho}+m I\right)\left(\sum_{i=1}^{m}\left(\boldsymbol{g}_{i}+\boldsymbol{u}_{i}\right)\right)\)
        for each \(i \in[m]\) do
            \(\boldsymbol{g}_{i} \leftarrow \arg \min _{\boldsymbol{z}_{i}}\left\|\boldsymbol{g}_{i}-\boldsymbol{p}+\boldsymbol{u}_{i}\right\|^{2}\)
                subject to \(\boldsymbol{b}_{i}^{T} \boldsymbol{g}_{i} \leq z_{i}, \boldsymbol{g}_{i} \geq \mathbf{0}\)
            \(\boldsymbol{u}_{i} \leftarrow \boldsymbol{g}_{i}+\boldsymbol{u}_{i}-\boldsymbol{p}\)
        end for
    until The successive difference of \(\boldsymbol{p}\) is smaller than \(\tau\)
    Return \(\boldsymbol{p}^{*} \leftarrow \boldsymbol{p}\) and \(\lambda^{s} \leftarrow\left\|C \boldsymbol{p}^{*}\right\|_{2}\)
```

```
Algorithm 2 Solve the robust program
Input: A set of extreme points, \(U B=+\infty, L B=-\infty\),
        \(k=0\)
Output: Optimal solution \(\boldsymbol{x}^{*}\)
    repeat
        Add extreme points to \(\mathcal{E}_{s}\) in (33) and set \(k=k+1\)
        Solve (33) to obtain an optimal solution \(\left\{\boldsymbol{x}_{k}, \boldsymbol{s}_{k}, \lambda_{k}\right\}\)
        and set
\[
L B=\boldsymbol{c}^{T} \boldsymbol{x}_{k}+\lambda_{k} \epsilon_{N}+\frac{1}{N} \sum_{i=1}^{N} s_{k i}
\]
Solve (34) to obtain an optimal solution \(\left\{s_{k}^{s}, \lambda_{k}^{s}\right\}\) and extreme points \(\left\{\boldsymbol{p}_{k}^{i}\right\}_{i=1}^{N} \cup\left\{\boldsymbol{p}_{k}\right\}\) and set
\[
U B=\min \left\{U B, \boldsymbol{c}^{T} \boldsymbol{x}_{k}+\lambda_{k}^{s} \epsilon_{N}+\frac{1}{N} \sum_{i=1}^{N} s_{k i}^{s}\right\}
\]
until \(U B-L B \leq \varepsilon\)
Return \(\boldsymbol{x}^{*} \leftarrow \boldsymbol{x}_{k}\)
```

$\left\{\boldsymbol{p}_{k}^{i}\right\}_{i=1}^{N} \cup\left\{\boldsymbol{p}_{k}\right\} \subseteq \mathcal{E}_{s}$ implies the convergence of Algorithm 2) i.e., $L B=U B$.

Step 4 in Algorithm 2 implies that

$$
U B \leq \boldsymbol{c}^{T} \boldsymbol{x}_{k}+\frac{1}{N} \sum_{i=1}^{N} s_{k i}^{s}+\epsilon_{N} \lambda_{k}^{s} .
$$

Since $\left\{\boldsymbol{p}_{k}^{i}\right\}_{i=1}^{N} \cup\left\{\boldsymbol{p}_{k}\right\} \subseteq \mathcal{E}_{s}$, then MP in the $k$-th iteration is identical to that in the $(k-1)$-th iteration. Thus, $\boldsymbol{x}_{k}$ is an optimal solution to the ( $k-1$ )-th MP as well. By the Step 3 in Algorithm 2 we find that $L B \geq \boldsymbol{c}^{T} \boldsymbol{x}_{k}+\epsilon_{N} \lambda_{k}+\sum_{i=1}^{N} \frac{s_{k i}}{N} \geq$ $\boldsymbol{c}^{T} \boldsymbol{x}_{k}+\epsilon_{N} \lambda_{k}^{s}+\sum_{i=1}^{N} \frac{s_{k i}^{s}}{N}$, where the last inequality holds due to the fact that $\left\{\boldsymbol{p}_{k}^{i}\right\}_{i=1}^{N} \cup\left\{\boldsymbol{p}_{k}\right\} \subseteq \mathcal{E}_{s}$ and hence the related constraints are added to MP before the $(k-1)$-th iteration. Consequently, we have $U B=L B$.
The conclusion of the convergence in $O(|\mathcal{E}|)$ iterations follows immediately from the finite number of extreme points for the polyhedron $\mathcal{P}$.

## V. The Worst-case Distribution and the Asymptotic Consistency

## A. The Worst-case Distribution

In this subsection we derive the distribution achieving the worst-case $\beta(\boldsymbol{x})$ in (9) of Section II-B for any feasible vector $\boldsymbol{x} \in \mathcal{X}$.

Lemma 1. For any feasible first-stage decision vector $\boldsymbol{x}$, then

$$
\begin{equation*}
\beta(\boldsymbol{x})=\sup _{\tilde{\boldsymbol{\xi}} \in \mathcal{B}}\left\{\frac{1}{N} \sum_{i=1}^{N} Q\left(\boldsymbol{x}, \boldsymbol{\xi}^{(i)}\right)\right\} \tag{38}
\end{equation*}
$$

where

$$
\mathcal{B}=\left\{\left(\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(N)}\right) \left\lvert\, \frac{1}{N} \sum_{i=1}^{N} d\left(\boldsymbol{\xi}^{(i)}, \widehat{\boldsymbol{\xi}}^{i}\right) \leq \epsilon_{N}\right., \boldsymbol{\xi}^{(i)} \in \Xi\right\}
$$

Proof. Given a feasible solution $\boldsymbol{x}$, it follows that

$$
\begin{equation*}
\sup _{\tilde{\boldsymbol{\xi}} \in \mathcal{B}}\left\{\frac{1}{N} \sum_{i=1}^{N} Q\left(\boldsymbol{x}, \boldsymbol{\xi}^{(i)}\right)\right\} \leq \sup _{F \in \mathcal{F}_{N}} \mathbb{E}_{\mathbb{F}}\{Q(\boldsymbol{x}, \boldsymbol{\xi})\}, \tag{39}
\end{equation*}
$$

by Lemma 2 in [31].
By the equivalence between $\beta(\boldsymbol{x})$ and (19), then for any $\varepsilon \geq 0$, there exists $\left\{\tilde{\boldsymbol{\xi}}^{(i)}\right\}_{i \in[N]} \subseteq \Xi$ such that

$$
\begin{aligned}
& \sup _{F \in \mathcal{F}_{N}} \mathbb{E}_{\mathbb{F}}\{Q(\boldsymbol{x}, \boldsymbol{\xi})\}-\varepsilon \\
& <\inf _{\lambda \geq 0}\left\{\lambda \epsilon_{N}+\frac{1}{N} \sum_{i=1}^{N}\left\{Q\left(\boldsymbol{x}, \tilde{\boldsymbol{\xi}}^{(i)}\right)-\lambda d\left(\tilde{\boldsymbol{\xi}}^{(i)}, \widehat{\boldsymbol{\xi}}^{i}\right)\right\}\right\}
\end{aligned}
$$

If $\left(\tilde{\boldsymbol{\xi}}^{(1)}, \ldots, \tilde{\boldsymbol{\xi}}^{(N)}\right) \notin \mathcal{B}$ and let $\lambda>0$, it follows that

$$
\lambda\left\{\epsilon_{N}-\frac{1}{N} \sum_{i=1}^{N} d\left(\tilde{\boldsymbol{\xi}}^{(i)}, \widehat{\boldsymbol{\xi}}^{i}\right)\right\}<0 .
$$

Increasing $\lambda$ to $+\infty$ in 40 enforces $\sup _{\mathbb{F} \in \mathcal{F}_{N}} \mathbb{E}_{\mathbb{F}}\{Q(\boldsymbol{x}, \boldsymbol{\xi})\}$ to $-\infty$, which contradicts with the fact that

$$
\sup _{\mathbb{F} \in \mathcal{F}_{N}} \mathbb{E}_{\mathbb{F}}\{Q(\boldsymbol{x}, \boldsymbol{\xi})\} \geq \mathbb{E}_{\mathbb{F}_{N}}\{Q(\boldsymbol{x}, \boldsymbol{\xi})\}>-\infty
$$

where the second inequality follows from Assumption 1. Thus, $\left(\tilde{\boldsymbol{\xi}}^{(1)}, \ldots, \tilde{\boldsymbol{\xi}}^{(N)}\right) \in \mathcal{B}$.

By Lemma 2 in [31], it holds that

$$
\sup _{\mathbb{F} \in \mathcal{F}_{N}} \mathbb{E}_{\mathbb{F}}\{Q(\boldsymbol{x}, \boldsymbol{\xi})\}-\varepsilon<\sup _{\tilde{\boldsymbol{\xi}} \in \mathcal{B}}\left\{\frac{1}{N} \sum_{i=1}^{N}\left\{Q\left(\boldsymbol{x}, \boldsymbol{\xi}^{(i)}\right)\right\}\right\}
$$

Letting $\varepsilon$ to zero, it holds that

$$
\sup _{F \in \mathcal{F}_{N}} \mathbb{E}_{F}\left\{Q(\boldsymbol{x}, \boldsymbol{\xi}) \leq \sup _{\tilde{\boldsymbol{\xi}} \in \mathcal{B}}\left\{\frac{1}{N} \sum_{i=1}^{N} Q\left(\boldsymbol{x}, \boldsymbol{\xi}^{(i)}\right)\right\}\right.
$$

Jointly with 39), then 38 holds.
Since $Q(\boldsymbol{x}, \boldsymbol{\xi})$ is concave with respect to $\boldsymbol{\xi}$ and $\mathcal{B}$ is a compact set, 38) allows for an optimal solution, Then a worstcase distribution is explicitly derived below.

Theorem 4. For any solution $\boldsymbol{x} \in \mathcal{X}$ and let $\boldsymbol{\xi}_{\boldsymbol{x}}=$ $\left(\boldsymbol{\xi}_{\boldsymbol{x}}^{(1)}, \ldots, \boldsymbol{\xi}_{\boldsymbol{x}}^{(N)}\right)$ be an optimal solution to (38). The following distribution

$$
\mathbb{F}_{\boldsymbol{x}}^{*}=\frac{1}{N} \sum_{i=1}^{N} \delta_{\boldsymbol{\xi}_{\boldsymbol{x}}^{(i)}}
$$

is the distribution achieving the worst-case second-stage cost , i.e.,

$$
\sup _{\mathbb{F} \in \mathcal{F}_{N}} \mathbb{E}_{\mathbb{F}}\{Q(\boldsymbol{x}, \boldsymbol{\xi})\}=\mathbb{E}_{\mathbb{F}_{\boldsymbol{x}}^{*}}\{Q(\boldsymbol{x}, \boldsymbol{\xi})\}
$$

Proof. Obviously, the following distribution

$$
\Pi_{\boldsymbol{x}}=\frac{1}{N} \sum_{i=1}^{N} \delta_{\left(\boldsymbol{\xi}_{\boldsymbol{x}}^{(i)}, \widehat{\boldsymbol{\xi}}^{i}\right)}
$$

is a joint distribution of $F_{N}$ and $\mathbb{F}_{\boldsymbol{x}}^{*}$. Then it holds that

$$
\begin{aligned}
W\left(\mathbb{F}_{N}, \mathbb{F}_{\boldsymbol{x}}^{*}\right) & \leq \int\left\|\boldsymbol{\xi}-\boldsymbol{\xi}^{\prime}\right\|_{p} \Pi_{\boldsymbol{x}}\left(\mathrm{d} \boldsymbol{\xi}, \mathrm{~d} \boldsymbol{\xi}^{\prime}\right) \\
& =\frac{1}{N} \sum_{i=1}^{N}\left\|\boldsymbol{\xi}_{\boldsymbol{x}}^{(i)}-\widehat{\boldsymbol{\xi}}^{i}\right\|_{p} \leq \epsilon_{N}
\end{aligned}
$$

where the first inequality follows directly from the definition of the 1-Wasserstein metric and the last inequality follows from the fact that $\left(\boldsymbol{\xi}_{\boldsymbol{x}}^{(1)}, \ldots, \boldsymbol{\xi}_{\boldsymbol{x}}^{(N)}\right) \in \mathcal{B}$. Hence, $\mathcal{F}_{N}$ includes the distribution $F_{x}^{*}$. Thus, it yields that

$$
\begin{aligned}
\sup _{\mathbb{F} \in \mathcal{F}_{N}} \mathbb{E}_{\mathbb{F}}\{Q(\boldsymbol{x}, \boldsymbol{\xi})\} & \geq \mathbb{E}_{\mathbb{F}_{\boldsymbol{x}}^{*}}\{Q(\boldsymbol{x}, \boldsymbol{\xi})\}=\frac{1}{N} \sum_{i=1}^{N} Q\left(\boldsymbol{x}, \boldsymbol{\xi}_{\boldsymbol{x}}^{(i)}\right) \\
& =\sup _{\mathbb{F} \in \mathcal{F}_{N}} \mathbb{E}_{\mathbb{F}}\{Q(\boldsymbol{x}, \boldsymbol{\xi})\}
\end{aligned}
$$

where the last equality follows from Lemma 1 Hence, $\mathbb{F}_{\boldsymbol{x}}^{*}$ is the desired worst-case distribution.

## B. The Asymptotic Consistency

This subsection studies the asymptotic consistency of the DR problem 10 under a mild assumption.

Assumption 4. There exists a positive constant $c$ such that

$$
\int_{\Xi} \exp \left(\|\boldsymbol{\xi}\|_{2}^{c}\right) \mathbb{F}(\mathrm{d} \boldsymbol{\xi})<\infty
$$

for the true distribution $\mathbb{F}$.
Under Assumptions 14.4, we formalize the asymptotic consistency of the proposed DR problem below.
Theorem 5. Under Assumptions 1.4 and select $\beta_{N} \in(0,1)$ such that $\sum_{N=1}^{\infty} \beta_{N} \leq \infty$. Let the 1 -Wasserstein ball radius be
$\epsilon_{N}\left(\beta_{N}\right)= \begin{cases}\left(\frac{\log \left(c_{1} \beta_{N}^{-1}\right)}{c_{2} N}\right)^{1 / \max \{n, 2\}}, & \text { if } \mathrm{N} \geq \frac{\log \left(\mathrm{c}_{1} \beta_{\mathrm{N}}^{-1}\right)}{\mathrm{c}_{2}} \\ \left(\frac{\log \left(c_{1} \beta_{N}^{-1}\right)}{c_{2} N}\right)^{1 / c}, & \text { if } \mathrm{N}<\frac{\log \left(\mathrm{c}_{1} \beta_{\mathrm{N}}^{-1}\right)}{\mathrm{c}_{2}}\end{cases}$
where $c_{1}$ and $c_{2}$ are positive constants related to the constant $c$ in Assumption 4 . Then the DR problem (10) asymptotically converges to the stochastic problem (1) almost surely when the sample number increases to infinity.

Proof. For the problem with distribution uncertainty only in the objective function, the relatively complete recourse implies that $Q(\boldsymbol{x}, \boldsymbol{\xi})$ is feasible and finite. Then there exists a finite $\boldsymbol{y}$ such that $|Q(\boldsymbol{x}, \boldsymbol{\xi})|=\left|(Z \boldsymbol{y})^{T} \boldsymbol{\xi}\right| \leq\|Z \boldsymbol{y}\|_{2}\|\boldsymbol{\xi}\|_{2} \leq L(1+$ $\|\boldsymbol{\xi}\|_{2}$ ) for any $\boldsymbol{x} \in \mathcal{X}$ and $\boldsymbol{\xi} \in \Xi$, where $L \geq 0$ is a constant.

For the case of the distribution uncertainty only in constraints, the strong duality of LP problem shows that $Q(\boldsymbol{x}, \boldsymbol{\xi})=(C \tilde{\boldsymbol{p}})^{T} \boldsymbol{\xi}$, where $C$ is given in 26) of Section IV-A and $\tilde{\boldsymbol{p}}$ is the extreme point of polyhedron $\mathcal{P}$. Assumption 3 implies that $\|\tilde{\boldsymbol{p}}\|$ is bounded and hence there exists a positive constant $L$ such that $|Q(\boldsymbol{x}, \boldsymbol{\xi})| \leq\|C \tilde{\boldsymbol{p}}\|_{2}\|\boldsymbol{\xi}\|_{2} \leq L\left(1+\|\boldsymbol{\xi}\|_{2}\right)$ for $\boldsymbol{x} \in \mathcal{X}$ and $\boldsymbol{\xi} \in \Xi$.

Finally the asymptotic consistency of our model follows from Theorem 3.6 in [19].

## VI. Simulation

This section conducts experiments to evaluate the performance of the proposed model and the constraint generation algorithm. All experiments are performed on a 64 bit PC with an Intel Core $15-7500 \mathrm{CPU}$ at 3.4 GHz and 8 GB RAM. The Cplex 12.6 optimizer is used to solve the optimization programs.

## A. The Two-stage Portfolio Program

This subsection is devoted to the application in two-stage portfolio program with uncertainty only in the objective function as stated in Example 1, see [16] for details.

1) Problem Specification: Consider a portfolio of four assets: (1) Dow Jones Industrial Average Index, (2) Dow Jones Transportation Average Index, (3) Dow Jones Composite Average Index and (4) Dow Jones Utility Average. The daily returns of above assets over seven years from January 02th, 2011 to December 31th, 2018 are collected from the RESSET database (http://www.resset.cn).

Since the first-stage return $c$ is unknown in our simulation, we select the data from January 02th, 2011 to December 31th, 2016 to approximate it by the SAA method, i.e., $c=\sum_{i=1}^{N} \widehat{\boldsymbol{\xi}_{i}^{1}}$, where $\widehat{\boldsymbol{\xi}}_{i}^{1}$ is the $i$ th sample of the first-stage return.
2) Impact of the 1 -Wasserstein Radius and the Sample Size: Experiments are conducted to test the impact of the 1Wasserstein radius $\epsilon_{N}$ and the sample size $N$ on the out-ofsample performance of our model in this subsection. The out-of-sample performance is measured by the loss of the proposed model on new samples, i.e.,

$$
\begin{equation*}
\boldsymbol{c}^{T} \boldsymbol{x}+\mathbb{E}_{\mathbb{F}}\{Q(\boldsymbol{x}, \boldsymbol{\xi})\} \tag{41}
\end{equation*}
$$

We are unable to exactly calculate 41) due to the unknown true distribution $\mathbb{F}$. Instead, we randomly choose 300 test samples from the dataset to approximate it, i.e.,

$$
\boldsymbol{c}^{T} \boldsymbol{x}+\frac{1}{N_{T}} \sum_{i=1}^{N_{T}} Q\left(\boldsymbol{x}, \widehat{\boldsymbol{\xi}}_{T}^{i}\right)
$$

where $\widehat{\boldsymbol{\xi}}_{T}^{i}$ is the $i$-th test sample and $N_{T}$ is the number of test samples.

We first test the impact of the 1-Wasserstein radius $\epsilon_{N}$ on our model. We conduct 200 independent experiments and the
averaged out-of-sample performance is illustrated in Figure 1 . Experimental results show that the out-of-sample performance improves as the 1 -Wasserstein radius increases and decreases if the radius is greater than a specific value.

Experiments on different sample sizes are performed as well. The out-of-sample performance averaged over 200 independent experiments is presented in Figure 2. Theorem 5 is confirmed by the out-of-sample performance improvement with the growing sample size.
3) Comparisons with the State-of-the-art Methods: In this subsection, we compare the proposed 1-Wasserstein DR model (denoted as DRW) with the SAA method and the DR model with the moment-based ambiguity set (denoted as DRM), where the first- and second-order uncertainty are borrowed from [16]. Let $N=\{20,30,50,100,200,300\}$. Due to the dependence of the radius $\epsilon_{N}$ on the sample dataset size, we tune it to ensure a good out-of-sample performance.

We adopt the percentage difference

$$
\left(\frac{\mathrm{DR}}{\mathrm{SAA}}-1\right) \times 100 \%
$$

to compare the out-of-sample performance of those models, where DR denotes the out-of-sample performance of the DR two-stage problem and SAA denotes that of the SAA method.

TABLE II
PERCENTAGE DIFFERENCES OF OUT-OF-SAMPLE PERFORMANCE(IN \%) between the DR models and the SAA

| $N$ | 20 | 30 | 50 | 100 | 200 | 300 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| DRW | 1.1 | 1.6 | 1.7 | 2.1 | 4.1 | 4.8 |
| DRM | -1.3 | -0.7 | 0.7 | 1.5 | 3.6 | 3.5 |

TABLE III
AVERAGED COMPUTATION TIME (SECOND) OF DIFFERENT METHODS

| $N$ | 20 | 30 | 50 | 100 | 200 | 300 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| DRW | 0.14 | 0.15 | 0.15 | 0.17 | 0.16 | 0.19 |
| DRM | 0.12 | 0.14 | 0.14 | 0.16 | 0.15 | 0.16 |
| SAA | 0.13 | 0.15 | 0.16 | 0.17 | 0.16 | 0.16 |

Comparisons in terms of the out-of-sample performance and computation time are presented in Table $\Pi$ and Table III respectively. A positive value in Table $\Pi$ implies a better performance of the DR method than the SAA. Table II indicates the best out-of-sample performance of our proposed method among all models. Importantly, it can also be solved in an acceptable time even under a large sample dataset.

## B. The Two-stage Material Order Problem

Algorithm 2 is applied to solve the DR two-stage ordering problem in Example 2 We omit the comparison with the moment-based model since there is no effective method to solve it [16].

1) Problem Specification: Consider the crude oil order problem for the gasoline and fuel oil supply stated in [27]). The oil is from two countries and can be viewed as different
materials. Then the coefficients of the material order problem in Example 2 is set as
$\boldsymbol{c}=[2,3]^{T}, \boldsymbol{d}=[7,12]^{T}, u=100$,
$A(\boldsymbol{\xi})=\left[\begin{array}{cc}2+\xi_{1} & 3 \\ 6 & 3.4+\xi_{2}\end{array}\right], \boldsymbol{b}(\boldsymbol{\xi})=\left[\begin{array}{c}180+\xi_{3} \\ 162+\xi_{4}\end{array}\right]$,
where $\boldsymbol{\xi} \in \mathbb{R}^{4}$ is a random vector with an unknown distribution and the recourse matrix $B$ is the identity matrix. We assume that $\boldsymbol{\xi}$ follows a Gaussian distribution $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu}=$ $[0,0,0,0]^{T}$ and $\boldsymbol{\Sigma}=\operatorname{Diag}\left([9,12,0.21,0.16]^{T}\right)$, and generate $N$ samples to construct the 1 -Wasserstein ball $\mathcal{F}_{N}$.
2) Test the Tightness of Bounds: We test the tightness of the proposed bounds in MP and SuP for an optimal function value (O.F.V) and the first-stage cost over the 1-Wasserstein ball with different radii $\epsilon_{N}$. Obviously, the extreme points of the set $\mathcal{P}=\{\boldsymbol{p} \geq 0: \boldsymbol{p} \leq \boldsymbol{d}\}=\left\{\boldsymbol{p} \in \mathbb{R}_{+}^{2}: p_{1} \leq 7, p_{2} \leq 12\right\}$ are $[0,0]^{T},[0,12]^{T},[7,0]^{T}$ and $[7,12]^{T}$. Hence, we can solve (10) directly with explicitly known extreme points and compare with Algorithm 2 Let $\left(x_{1}^{d}, x_{2}^{d}\right)$ denote the solution obtained via solving (10) directly and $\left(x_{1}^{a}, x_{2}^{a}\right)$ obtained by Algorithm 2. Table IV indicates that the two methods under different 1-Wasserstein radius obtain identical results.

The O.F.V. and the first-stage cost compared to that of the method with known extreme points under 500 samples is shown in Fig 3(a) and Fig 3(b) We observe that both the lower bound and upper bound are tight, regardless of the radius of the 1-Wasserstein ball. Thus, these bounds can be viewed as a good reference to verify the performance of our algorithm.

Fig 4 shows the tendency of the upper and lower bound for the proposed two-stage program in a single experiment. We record the averaged number of the extreme points and iterations in Algorithm 2 under different sample sizes over 100 independent experiments in Table V and Table VI both of which validate the effectiveness of Algorithm 2.
3) The Test for High Dimension: A direct enumeration of all extreme points of the polyhedron $\mathcal{P}=\left\{\boldsymbol{p} \in \mathbb{R}_{+}^{M}: B^{T} \boldsymbol{p} \leq\right.$ $\boldsymbol{d}\}$ with a large $M$ is computational demanding [34]. In this subsection, we consider a high dimension problem to verify the efficiency of Algorithm 2, i.e.,

$$
\begin{aligned}
& u=1000, \boldsymbol{x} \in \mathbb{R}^{20}, A(\boldsymbol{\xi}) \in \mathbb{R}^{20 \times 20}, \boldsymbol{b}(\boldsymbol{\xi}) \in \mathbb{R}^{20} \\
& \boldsymbol{c}=[2,3,1,4,5,2,4,3,4,2,5,4,4,2,6,2,4,3,1,2]^{T} \\
& \boldsymbol{d}=[7,9,4,6,8,5,6,8,10,7,12,10,6,7,9,5,11,10,5,8]^{T}
\end{aligned}
$$

where $A(\boldsymbol{\xi})$ and $\boldsymbol{b}(\boldsymbol{\xi})$ are affinely dependent on the random vector $\boldsymbol{\xi}$ and $B$ is the identity matrix.
$\mathrm{Fig} 5(\mathrm{a})$ and $\mathrm{Fig} 5(\mathrm{~b})$ report the averaged performance of our proposed bounds for the O.F.V and the first-stage cost under different 1 -Wasserstein radii $\epsilon_{N}$ when the sample size $N=500$. As previous subsection, these proposed bounds are tight as well.

We record the averaged computation time, the number of extreme points and iterations in Algorithm 2 over 100 independent simulations as sample size $N$ varies from 10 to 1000 in Table VII, Table VIII and Table IX respectively. The convergence of the proposed algorithm in a single experiment is also illustrated in Fig 6.


Fig. 1. The averaged out-of-sample performance under sample dataset of different sizes as a function for 1-Wasserstein radius estimated by 200 independent simulation runs. (a) $N=20$, (b) $N=100$, (c) $N=200$.

TABLE IV
THE OPTIMAL SOLUTIONS UNDER DIFFERENT METHODS WITH DIFFERENT 1 -WASSERSTEIN BALL RADII $\epsilon_{N}$ WHEN SAMPLE SIZE $N=500$

| $\epsilon_{N}$ | 0.01 | 0.21 | 0.41 | 0.61 | 0.81 | 1 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\left(x_{1}^{d}, x_{2}^{d}\right)$ | $(42.7,57.2)$ | $(41.2,50.8)$ | $(38.7,41.5)$ | $(36.2,32.4)$ | $(34.7,26.4)$ | $(33.4,22.5)$ |
| $\left(x_{1}^{a}, x_{2}^{a}\right)$ | $(42.7,57.2)$ | $(41.2,50.8)$ | $(38.7,41.5)$ | $(36.2,32.4)$ | $(34.7,26.4)$ | $(33.4,22.5)$ |



Fig. 2. The averaged out-of-sample performance as a function of sample size $N$ for 200 independent experiments.

TABLE V
THE AVERAGED NUMBER OF EXTREME POINTS UNDER DIFFERENT SAMPLE SIZES

| $N$ | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 500 | 1000 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Num | 3.68 | 3.74 | 3.98 | 3.96 | 4 | 4 | 4 | 4 | 4 |

TABLE VI
THE AVERAGED NUMBER OF ITERATIONS UNDER DIFFERENT SAMPLE SIZES

| $N$ | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 500 | 1000 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Ite | 3.78 | 3.84 | 3.94 | 3.94 | 4 | 4 | 4 | 4 | 4 |

Results show that Algorithm 2 converges in a reasonable time even for the problem in a high dimension under a large sample dataset. The number of extreme points required in our algorithm is far smaller than the total number of extreme points.

## VII. CONCLUSION

We have proposed a novel SOCP approach to solve the data-driven DR two-stage linear programs over 1-Wasserstein balls. The model with distribution uncertainty in the objective

(a)

(b)

Fig. 3. The averaged performance of the proposed bounds for O.F.V. and the first-stage cost under the 1-Wasserstein ball with different radii. (a) O.F.V (b) the first-stage cost
function is reformulated as a solvable SOCP problem. While the DR model over the moment-based ambiguity set is generally unsolvable, we propose a constraint generation algorithm with provable convergence to approximately solve the NPhard model with distribution uncertainty only in constraints.


Fig. 4. The convergence of the O.F.V for the two-stage program with 500 samples.


Fig. 5. The averaged performance of the proposed bounds for O.F.V. and first-stage cost under the 1-Wasserstein ball with different radii. (a) O.F.V (b) first-stage cost


Fig. 6. The convergence of the O.F.V for the two-stage program with 500 samples.

We explicitly derive a distribution achieving the worst-case cost. Numerical results validate the good out-of-sample performance for our model and the high efficiency of the proposed

TABLE VII
THE AVERAGED COMPUTATION TIME (SECOND) UNDER DIFFERENT SAMPLE SIZES

| $N$ | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 500 | 1000 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Time | 10.9 | 11.6 | 11.6 | 11.6 | 12.3 | 13.9 | 17.2 | 23.3 | 36.2 |

TABLE VIII
THE AVERAGED NUMBER OF EXTREME POINTS UNDER DIFFERENT SAMPLE SIZES

| $N$ | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 500 | 1000 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Num | 35.2 | 46.5 | 49.2 | 60.3 | 72.4 | 100.1 | 123.7 | 156.4 | 181.1 |

TABLE IX
THE AVERAGED NUMBER OF ITERATIONS UNDER DIFFERENT SAMPLE SIZES

| $N$ | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 500 | 1000 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Ite | 10.28 | 9.58 | 9.32 | 8.92 | 8.80 | 8.54 | 8.58 | 8.16 | 8.46 |

## algorithm.

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Zhuolin Wang received the B.S. degree from the School of Information Computer Science and Technology, Beijing Normal University, Beijing, China, in 2017. She is currently pursuing the Ph.D. degree at the Department of Automation, Tsinghua University, Beijing, China. Her research interests include distributionally robust optimization, robust control, and their applications.


Keyou You (SM'17) received the B.S. degree in Statistical Science from Sun Yat-sen University, Guangzhou, China, in 2007 and the Ph.D. degree in Electrical and Electronic Engineering from Nanyang Technological University (NTU), Singapore, in 2012. After briefly working as a Research Fellow at NTU, he joined Tsinghua University in Beijing, China where he is now a tenured Associate Professor in the Department of Automation. He held visiting positions at Politecnico di Torino, The Hong Kong University of Science and Technology, The University of Melbourne and etc. His current research interests include networked control systems, distributed optimization and learning, and their applications.
Dr. You received the Guan Zhaozhi award at the 29th Chinese Control Conference in 2010, the CSC-IBM China Faculty Award in 2014 and the ACA (Asian Control Association) Temasek Young Educator Award in 2019. He was selected to the National 1000-Youth Talent Program of China in 2014 and received the National Science Fund for Excellent Young Scholars in 2017.


Shiji Song received the Ph.D. degree in the Department of Mathematics from Harbin Institute of Technology in 1996. He is a professor in the Department of Automation, Tsinghua University. His research interests include system modeling, control and optimization, computational intelligence and pattern recognition.


Yuli Zhang received the Ph.D. degree from Tsinghua University, Beijing, China, in 2014. He is an Associate Professor in the School of Management and Economics, Beijing Institute of Technology. His research interest includes robust optimization, intelligent transportation systems, and intelligent power grids.

