On the size of data structures used in symbolic model checking

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Abstract

Temporal Logic Model Checking is a verification method in which we describe a system, the model, and then we verify whether some properties, expressed in a temporal logic formula, hold in the system. It has many industrial applications. In order to improve performance, some tools allow preprocessing of the model, verifying on-line a set of properties reusing the same compiled model; we prove that the complexity of the Model Checking problem, without any preprocessing or preprocessing the model or the formula in a polynomial data structure, is the same. As a result preprocessing does not always exponentially improve performance.

Symbolic Model Checking algorithms work by manipulating sets of states, and these sets are often represented by BDDs. It has been observed that the size of BDDs may grow exponentially as the model and formula increase in size. As a side result, we formally prove that a superpolynomial increase of the size of these BDDs is unavoidable in the worst case. While this exponential growth has been empirically observed, to the best of our knowledge it has never been proved so far in general terms. This result not only holds for all types of BDDs regardless of the variable ordering, but also for more powerful data structures, such as BEDs, RBCs, MTBDDs, and ADDs.

1 Introduction

Temporal Logic Model Checking [20] is a verification method for discrete systems. In a nutshell, the system, often called the model, is described by the possible transitions of its components, while the properties to verify are encoded in a temporal modal logic. It is used, for example, for the verification of protocols and hardware circuits [5]. Many tools, called model checkers, have been developed to this aim. The most famous ones are SPIN [35] and SMV [43] (with its many incarnations: NuSMV [17], RuleBase [6]), VIS [8], and FormalCheck [34].

There are many languages to express the model; the most widespread ones are Promela and SMV. Two temporal logics are mainly used to define the specification: CTL [20] and LTL [45]. In this paper we focus on the latter. In many cases, the two inputs of the model checking problem (the model and the formula) can be processed in a different way. If we want to verify several properties of the same system, it makes sense to spend more time on the model alone, if the verification of the properties becomes faster. Many tools allow to build the model separately from checking the formula [16, 53, 36]. This way, one can reuse the same model, compiled into a data structure, in order to check several formulae.

In the same way, we may wish to verify the same property on different systems: the property is this time the part we can spend more time on. Many tools allow populating a property database [16, 53, 36], i.e., a collection of temporal formulae which will be checked on the models. We imagine a situation in which we early establish the requirements that our system must satisfy, even before the system is actually designed. As a result, and we can fill a database of temporal formulae, but we do not yet describe the system. While the design/modeling of the system goes on, we can preprocess the formulae (without knowledge of the model, which is not yet known). Whenever the system is specified, we can then use the result of this preprocessing step to check the model against the formulae.

In this paper, we analyze whether preprocessing a part of the model checking problem instances improve the performances. The technical tool we use is the compilability theory [15, 39]. This theory characterizes the complexity of problems when the problem instances can be divided into two parts (the fixed and the varying part), and we can spend more time on the first part alone, provided that the result of this preprocessing step has polynomial size respect the fixed part. We show that the Model Checking problem remains PSPACE-hard even if we can preprocess either the model or the formula, if this preprocessing step is constrained to have a polynomial size. These theorems hold for all model checkers.

Finally, we answer to a long-time standing question in Symbolic Model Checking [43, 11]. It has been observed that the BDDs that are used by SMV and other Symbolic Model Checking systems become exponentially large in some cases. However, it has not yet been established whether this size increase is due to the choice of variable ordering, or to the kind of BDDs employed, or it is intrinsic of the problem. We show that, if PSPACE $\not\subseteq \Pi_2^p \cap \Sigma_2^p$, such a growth is, in the worst case, unavoidable. This result is independent from the particular class of BDDs and from the variable order of the BDDs. It also holds for all decision diagrams representing integer-value functions whose evaluation problem is in the polynomial hierarchy, such as BEDs [54], BMD and *BMD [9], RBCs [1], MTBDDs [18], and ADDs [2].

2 Preliminaries

2.1 Model Checking

In this section, we briefly recall the basic definitions about model checking that are needed in the rest of the paper. We follow the notation of [47, 46]. LTL (Linear Temporal Logic) is a modal logic aimed at encoding how states evolve over time. It has three unary modal operators (X, G, and F) and one binary modal operator (U). Their meaning is: $X\phi$ is true in particular state if and only if the formula ϕ is true in the next state; $G\phi$ is true if and only ϕ is true from now on; $F\phi$ is true if ϕ will become true at some time in the future; $\phi U\psi$ is true if ψ will eventually become true and ϕ stays true until then. We indicate with $L(O_1, \ldots, O_n)$ the LTL fragment in which the only temporal operators allowed are O_1, \ldots, O_n ; for instance, L(F, X) is the fragment of LTL in which only Fand X are allowed.

The semantics of LTL is based on Kripke models. In the following, for an 'atomic proposition' we mean a Boolean variable. Given a set of atomic proposition, a Kripke structure for LTL is a tuple $\langle Q, R, \ell, I \rangle$, where Q is a set of states, R is a binary relation over states (the transition relation), ℓ is a function from states to atomic propositions (it labels every state with the atomic propositions that are true in that state), I is a set of initial states. A run of a Kripke structure is a Kripke model. A Kripke model for LTL is an infinite sequence of states, where the transition relation links each state with the one immediately following it in the sequence. The semantics of the modal operators is defined in the intuitive way: for example, $F\phi$ is true in a state of a Kripke model if ϕ is true in some following state.

The main problem of interest in practice is to verify whether all runs of a Kripke structure (all of its Kripke models) satisfy the formula; this is the Universal Model Checking problem. The Existential Model Checking one is to verify whether there is a run of the Kripke structure that satisfies the formula. In formal verification, we encode the behavior of a system as a Kripke structure, and the property we want to check as an LTL formula. Checking the structure against the formula tells whether the system satisfies the property. Since the Kripke structure is usually called a "model" (which is in fact very different from a Kripke model, which is only a possible run), this problem is called Model Checking.

In practice, all model checkers describe a system by the Kripke structure of its components. A Kripke structure can be seen as a transition system [20]. Thus the global system is obtained by parallel composition of the transition systems representing its components and sharing some variables [42, 20]; using this approach, we can give results valid for all model checkers.

2.2 Composition of Transition Systems

Each component of the global system is modeled using a transition system, which is a formal way to describe a possible transition a system can go through. Intuitively, all is needed is to specify the state variables, the possible initial states, and which transitions are possible, i.e., we have to say whether the transition from state s to state s' is possible for any pair of states s and s'. The formal definition is as follows [42, 20].

Definition 1 A finite-state transition system is a triple (V, I, ϱ) , where $V = \{x_1, \ldots, x_n\}$ is a set of Boolean variables, I is a formula over V, and $\varrho(V, V')$ is a formula over $V \cup V'$, where $V' = \{x'_1, \ldots, x'_n\}$ is a set of new variables in one to one relation with elements of V.

Intuitively, V is the set of state variables, I is a formula that is true on a truth assignment if and only if it represents a possible initial state, and ρ is true on a pair of truth assignments if they represent a possible transition of

the system. The set of variables V' is needed because ρ must refer to both the value of a variable in the current state (x_i) and in the next state (x'_i) . In other words, in this formula x_i means the value of x_i in the current state, while x'_i is the value of the same variable in the next state. For example, the fact that x_i remains true is encoded by $\rho = x_i \to x'_i$: if x_i is true now, then x'_i is true, i.e., x_i is true in the next state.

Formally, a state s is an assignment to the variables; a state s' is successor of a state s iff $\langle s, s' \rangle \models \varrho(V, V')$. A computation is an infinite sequence of states s_0, s_1, s_2, \ldots , satisfying the following requirements:

Initiality: s_0 is initial, i.e. $s_0 \models I$

Consecution: For each $j \ge 0$, the state s_{j+1} is a successor of the state s_j

For the sake of simplicity, without loss of any generality, we only consider Boolean variables and Boolean assertions.

In order to model a complex system, we assume that each of its parts can be modeled by a transition system. Clearly, there is usually some interaction between the parts; as a result, some variables may be *shared* between the transition systems. In the following, we consider k transition systems M_1, \ldots, M_k . Every M_i is described by $((V_i^L \cup V_i^S), I_i(V_i), \varrho_i(V_i, V_i'))$ for i $1 \le i \le k$ where V_i^L is the set variables local to M_i , V_i^S is the set of shared variables of M_i , and $V_i = V_i^L \cup V_i^S$. A group of transition systems can be composed in different ways: synchronous, interleaved asynchronous, and asynchronous. The third way is not frequently used in Model Checking, so we only define the first two ways of composition. In the following, a process is any of the transition systems M_i .

The synchronous parallel composition of k transition systems is obtained by assuming that the global transition is due to all processes M_i making a transition simultaneously. In other words, all processes must make a transition at any time step, and no process is allowed to "idle" at any time step.

Definition 2 The synchronous parallel composition of processes M_1, \ldots, M_k , denoted by $M_1 \| \ldots \| M_k$, is the transition system $M = (V, I, \varrho)$ described by:

$$V = \bigcup_{i=1}^{k} V_i \qquad I(V) = \bigwedge_{i=1}^{k} I_i(V_i)$$
$$\varrho(V, V') = \bigwedge_{i=1}^{k} \varrho_i(V_i, V'_i)$$

The basic idea of the interleaved asynchronous parallel composition is that only one process is active at the same time. As a result, a global transition can only result from the transition of a single process. The variables that are not changed by this process must maintain the same value.

Definition 3 The interleaved asynchronous parallel composition of M_1, \ldots, M_k is the transition system $M = (V, I, \varrho)$:, where V and I are as in the synchronous composition and ϱ is:

$$\varrho(V,V') = \bigvee_{i=1}^{k} \left[\varrho_i(V_i,V_i') \land \bigwedge_{\substack{j=1\\j\neq i}}^{k} V_i^L = V_i^{L'} \right]$$

The interleaved asynchronous parallel composition of M_1, \ldots, M_k , is denoted by $M_1 | \ldots | M_k$.

A model can be described as the composition of transition systems. As a result, we can define the model checking problem for concurrent transition systems as the problem of verifying whether the model described by the composition of the transition systems satisfies the given formula.

2.3 Complexity and Compilability

We assume that the reader knows the basic concepts of complexity theory [48, 31]. What we mainly use in this paper are the concepts of polynomial reduction and the class PSPACE.

The Model Checking problem is PSPACE-complete, and is thus intractable. On the other hand, as said in the Introduction, it makes sense to preprocess only one part of the problem (either the model or the formula), if this reduces the remaining running time. The analysis of how much can be gained by such preprocessing, however, cannot be done using the standard tools of the polynomial classes and reductions. The compilability classes [15] have to be used instead.

The way in which the complexity of the problem is identified in the theory of NP-completeness is that of giving a set of increasing classes of problems. If a problem is in a class C but is not in an inner class C', then we can say that this problem is more complex to solve that a problem in C'. A similar characterization, with similar classes, can be given when preprocessing is allowed. For example the class $|\!| \rightarrow P$ is the class of problems that can be solved in polynomial time after a preprocessing step. Crucial to this definition are two points:

- 1. which part of the problem instance can be preprocessed?
- 2. how expensive is the preprocessing part allowed to be?

The first point depends on the specific problem and on the specific settings: depending on the scenario, for example, we can preprocess either the model or the formula for the model checking problem. The second question instead allows for a somehow more general answer. First, we cannot limit this phase to take polynomial time, as otherwise there would be no gain in doing preprocessing from the point of view of computational complexity. Second, we cannot allow the final result of this part to be exponentially large, for practical reasons; we bound the result of the preprocessing phase only to take a polynomial amount of space.

In order to denote problems in which only one part can be preprocessed, we assume that their instances are composed of two parts, and that the part that can be preprocessed is the first one. As a result, the model checking problem written as $\langle M, \phi \rangle$ indicates that M can be preprocessed; written as $\langle \phi, M \rangle$ indicates that ϕ can be preprocessed.

The "complexity when preprocessing is allowed" is established by characterizing how hard a problem is *after* the preprocessing step. This is done by building over the usual complexity classes: if C is a "regular" complexity class such as NP, then a problem is in the (non-uniform) compilability class $\parallel \rightarrow C$ if the problem is in C after a preprocessing step whose result takes polynomial space. In other words, $\parallel \rightarrow C$ is "almost" C, but preprocessing is allowed and will not be counted in the cost of solving the problem. More details can be found in [15].

In order to identify how hard a problem is, we also need a concept of hardness. Since the regular polynomial reductions are not appropriate when preprocessing is allowed, ad-hoc reductions (called nu-comp reductions in [15]) have been defined.

In this paper, we do not show the hardness of problems directly, but rather use a sufficient condition called representative equivalence. For example, in order to prove that model checking is $\parallel \rightarrow PSPACE$ -hard, we first show a (regular) polynomial reduction from a PSPACE-hard problem to model checking and then show that this reduction satisfies the condition of representative equivalence.

Let us assume that we know that a given problem A is $\Vdash C$ -hard and we have a polynomial reduction from the problem A to the problem B. Can we use this reduction to prove the $\Vdash C$ -hardness of B? Liberatore [39] shows sufficient conditions that should hold on A as well as on the reduction. If all these conditions are verified, then there is a nucomp reduction from *A to B, where $*A = \{\langle x, y \rangle | y \in A\}$, thus proving the $\Vdash C$ -hardness of B.

Definition 4 (Classification Function) A classification function for a problem A is a polynomial function Class from instances of A to nonnegative integers, such that $Class(y) \leq ||y||$.

Definition 5 (Representative Function) A representative function for a problem A is a polynomial function Repr from nonnegative integers to instances of A, such that Class(Repr(n)) = n, and that ||Repr(n)|| is bounded by some polynomial in n.

Definition 6 (Extension Function) An extension function for a problem A is a polynomial function from instances of A and nonnegative integers to instances of A such that, for any y and $n \ge Class(y)$, the instance y' = Exte(y, n) satisfies the following conditions:

- 1. $y \in A$ if and only if $y' \in A$;
- 2. Class(y') = n.

Let us give some intuitions about these functions. Usually, an instance of a problem is composed of a set of objects combined in some way. For problems on boolean formulas, we have a set of variables combined to form a formula. For graph problems, we have a set of nodes, and the graph is indeed a set of edges, which are pairs of nodes. The classification function gives the number of objects in an instance. The representative function thus gives an instance with the given number of objects. This instance should be in some way "symmetric", in the sense that its elements should be interchangeable (this is because the representative function must be determined only from the number of objects). Possible results of the representative function can be the set of all clauses of three literals over a given alphabet, the complete graph over a set of nodes, the graph with no edges, etc. Let for example A be the problem of propositional satisfiability. We can take Class(F) as the number of variables in the formula F, while Repr(n) can be the set of all clauses of three literals over an alphabet of n variables. Finally, a possible extension function is obtained by adding tautological clauses to an instance. Note that these functions are related to the problem A only, and do not involve the specific problem B we want to prove hard, neither the specific reduction used. We now define a condition over the polytime reduction from A to B. Since B is a problem of pairs, we can define a reduction from A to B as a pair of polynomial functions $\langle r, h \rangle$ such that $x \in A$ if and only if $\langle r(x), h(x) \rangle \in B$.

Definition 7 (Representative Equivalence) Given a problem A (having the above three functions), a problem of pairs B, and a polynomial reduction $\langle r, h \rangle$ from A to B, the condition of representative equivalence holds if, for any instance y of A, it holds:

$$\langle r(y), h(y) \rangle \in B$$
 iff $\langle r(Repr(Class(y)), h(y) \rangle \in B$

The condition of representative equivalence can be proved to imply that the problem B is \rightarrow C-hard, if A is C-hard [39]. As an example, we show these three functions for the $PLANSAT_1^*$ problem. $PLANSAT_1^*$ is the following problem of planning: giving a STRIPS [29] instance $y = \langle P, O, I, G \rangle$ in which the operators have an arbitrary number of preconditions and only one postcondition, is there a plan for y? $PLANSAT_1^*$ is PSPACE-Complete [12]. Without loss of generality we consider $y = (P, O \cup o_0, I, G)$, where o_0 is a operator which is always usable (it has no preconditions) and does nothing (it has no postconditions). We use the following notation: $P = \{x_1, \ldots, x_n\}$, I is the set of conditions true in the initial state, $G = \langle \mathcal{M}, \mathcal{N} \rangle$. A state in STRIPS is a set of conditions. In the following we indicate with ϕ_i^h the *h*th positive precondition of the operator o_i , with ϕ_i all its the positive preconditions, with η_i^h its *h*th negative precondition, and with η_i all its negative preconditions; α_i is the positive postcondition of the operator o_i , β_i is the negative postcondition of the operator o_i . Since any operator has only one postcondition, for every operator i it holds that $\|\alpha_i \cup \beta_i\| =$ 1.

Since we shall use them in the following, we define a classification function, a representative function and a extension function for $PLANSAT_1^*$:

- Classification Function: Class(y) = ||P||. Clearly, it satisfies the condition $Class(y) \le ||y||$.
- Representative Function: $Repr(n) = \langle P_n, \emptyset, \emptyset, \emptyset \rangle$, where $P_n = \{x_1, \ldots, x_n\}$. Clearly, this function is polynomial and satisfies the following conditions: (i) Class(Repr(n))=n, (ii) $||Repr(n)|| \leq p(n)$ where p(n) is a polynomial.
- Extension Function: Let $y = \langle P, O, I, G \rangle$ and $y' = Exte(y, n) = \langle P_n, O, I, G \rangle$. Clearly for any y and n s.t. $n \ge Class(y)$ y' satisfies the following conditions: (i) $y \in A$ iff $y' \in A$, (ii) Class(y') = n.

Given the limitation of space we cannot give the full definitions for compilability, for which the reader should refer to [15] for an introduction, to [14, 13] for an application to the succinctness of some formalisms, to [39] for further applications and technical advances.

3 Results

The Model Checking problem for concurrent transition systems is PSPACEcomplete [37]. In Section 3.1, we prove that the following problems are \gg PSPACEhard, which implies that they remain PSPACE-hard even if preprocessing is allowed.

- 1. model checking on the synchronous and interleaved asynchronous composition of transition systems, where the transitions systems are the fixed part of the problem and the LTL formula is the varying part;
- 2. the same problem, where the LTL formula is the fixed part and the transition system is the varying part;
- 3. given a set of transition systems and a formula as the fixed part, a state as the varying part, checking whether the state is a legal initial state.

We can conclude that preprocessing the model or the formula does not lead to a polynomial algorithm for model checking. We recall that the fixed part is preprocessed off-line in a polynomial data structure during the preprocessing phase, and the varying part is given on-line.

The relevance of the first two problems is clear: in formal verification, it is often the case that many properties (formulae) have to be verified over the same system (the model, in this case modeled by the transition systems); on the other hand, it may also be that the same property has to be verified on different systems.

The result about the third problem is less interesting by itself. On the other hand, we use it to prove that the superpolynomial growth of the size of the data structures (e.g. OBDDs) currently used in model checkers based on the Symbolic Model Checking algorithms [43] (such as SMV and NuSMV) cannot be avoided in general. The result is independent from its variable ordering, and it holds for others data structures that can be employed. We show these results in Section 3.2.

We point out that most of Temporal Logic Model Checking algorithms [20] fall in one of three classes: Symbolic Model Checking algorithms, which work on symbolic representation of M; algorithms based on Bounded Model Checking [7] (i.e. based on reduction from Model Checking into SAT); algorithms that work on an explicit representation of M (e.g. [32]). Our results concerning the size of the BDD (or some other decision diagrams) are valid for all algorithms of the first class.

In the proofs of the following sections we consider Existential Model Checking problems, but the results are valid also for the Universal case; in fact PSPACE is closed under complementation also for compilability.

3.1 Preprocessing Model Checking

We now identify the complexity of the Model Checking problem when the preprocessing of the model (represented as the composition of transition systems) is allowed, both in the synchronous and in the interleaved case. **Theorem 1** The model checking problem for k synchronous concurrent process $MC_{syn} = \langle (M_1 || \dots || M_k), \varphi \rangle$ where $\varphi \in LTL$ is $\parallel \rightarrow PSPACE$ -hard, and remains $\parallel \rightarrow PSPACE$ -hard for $\varphi \in L(F, G, X)$.

Proof. It is similar to the proof of the Theorem 2. We carry out a reduction from the $PLANSAT_1^*$ problem, that satisfies the conditions of representative equivalence; the main difference is about the LTL formula.

We now consider the Model Checking problem for concurrent processes composed in a interleaved way when the model can be preprocessed.

Theorem 2 The model checking problem for k interleaved concurrent process $MC_{asyn} = \langle (M_1| \dots | M_k), \varphi \rangle$ where $\varphi \in LTL$ is $\parallel \rightarrow PSPACE$ -complete, and remains $\parallel \rightarrow PSPACE$ -hard for $\varphi \in L(F, G, X)$.

Proof. We show a reduction, that translates an instance $y \in PLANSAT_1^*$ into an instance $\langle r(y), h(y) \rangle \in M_{asyn}$, satisfying the condition of representative equivalence. Given $y = \langle P, O, I, G \rangle \in PLANSAT_1^*$

- r(y) defines a concurrent transition systems M_1, \ldots, M_n , where each M_i is obtained from a variable $x_i \in P$ and it is described by:
 - $V_{i} = \{x_{i}\}$ $I_{i}(V_{i}) = (x_{i}) \lor (\neg x_{i})$ $\varrho_{i}(V_{i}, V_{i}') = (x_{i} = 0 \land x_{i}' = 0) \lor (x_{i} = 0 \land x_{i}' = 1) \lor$ $(x_{i} = 1 \land x_{i}' = 0) \lor (x_{i} = 1 \land x_{i}' = 1)$

The process $M = M_1 \| \dots \| M_n$ represents all possible computations, starting from all possible initial assignments, over the variables x_1, \dots, x_n .

- $h(y) = h(I, G, O) = \neg(\phi_I \land \phi_G \land \phi_O)$ where:

$$\varphi_{I} = \bigwedge_{i \in I} x_{i} \wedge \bigwedge_{i \notin I} \neg x_{i}$$

$$\varphi_{G} = F(\bigwedge_{i \in \mathcal{M}} x_{i} \wedge \bigwedge_{i \in \mathcal{N}} \neg x_{i})$$

$$\varphi_{O} = G \bigvee_{i=0}^{m} [\bigwedge_{h=1}^{\|\phi_{i}\|} \phi_{i}^{h} \wedge \bigwedge_{h=1}^{\|\eta_{i}\|} \neg \eta_{i}^{h} \wedge X\gamma_{i} \wedge \bigwedge_{\substack{j \neq i \\ j=1}}^{n} (x_{j} \leftrightarrow Xx_{j})]$$

where

$$\gamma_i = \begin{cases} \alpha_i & \text{if } \alpha_i \neq \emptyset \\ \neg \beta_i & \text{if } \beta_i \neq \emptyset \end{cases}$$

 φ_I adds constraints about the initial states of y represented by I.

- φ_G adds constraints about the goal states of y represented by G: it tells that a goal state will be reached.
- φ_O describes the operators in O: globally (i.e. in every state) one of the operators must be used to go in the next state; φ_O also describes the nop operator o_0 .

Now, we prove that $y \in PLANSAT_1^*$ iff $\langle r(y), h(y) \rangle \in M_{asyn}$. Given $y = \langle P, O, I, G \rangle$, a solution for y is a plan which generates the following sequence of states: (s_1, \ldots, s_p) where s_1 is an initial state and s_p is a goal state. This sequence of states is obtained applying a sequence of operators $(o_{h_1}, \ldots, o_{h_p})$ chosen in $O = \{o_1, \ldots, o_m\}$ in the following way: for all i s.t. $1 \leq i \leq p$, preconditions for o_{h_i} are included in the state s_i , and the state s_{i+1} is obtained from the state s_i modifying the postcondition associated with o_{h_i} . We remark that a state in STRIPS is the set of conditions.

The model M = r(y) = r(P) represents all possible traces starting from all possible initial configurations, over the variables x_1, \ldots, x_n . Thus, in this case the Existential Model Checking problem $\langle M, \varphi \rangle$ reduces to the satisfiability problem for φ : we check whether ther exists a trace among all traces over the variables x_1, \ldots, x_n that satisfies the LTL formula φ . Therefore, we have to prove that $y \in A$ iff $\varphi = h(y)$ is satisfiable:

 \Rightarrow . Given a solution for $y \in A$, we identify a model for $\varphi = h(y)$; by construction such a model has:

- initial state s_1^M s.t. $\ell(s_1) = I \cup \{\neg x_i | x_i \notin I\}$
- a state s_p^M s.t. $\ell(s_p) \subseteq \mathcal{M} \cup \{\neg x_i | x_i \notin \mathcal{N}\}$
- given a state s_i^M , s_{i+1}^M is successor of s_i^M iff
 - $\ell(s_i^M) \subseteq Precond(o_{h_i})$, where $Precond(o_{h_i}) = \{x_j | x_j \in \phi_{h_i}\} \cup \{\neg x_j | x_j \in \eta_{h_i}\}$
 - $\ell(s_{i+1}^M) = \ell(s_i^M) \cup \alpha_i \beta_i$ where α_i is the positive postcondition of o_{h_i} and β_i is the negative postcondition of o_{h_i} .
- an infinite number of states: when the state s_p is reached this state is repeated for at least once or for ever (applying the nop operator o_0), or it is possible, it depends from y, to apply any operators whose preconditions are satisfied by $\ell(s_p^M)$.

 \Leftarrow . Let $(s_1^M, \ldots, s_p^M, \ldots)$ a model for φ , and let s_p the goal state, that the first state satisfying φ_G . We obtain the sequence of states visited by a plan which is a solution for y, by cutting the states after the goal state s_p and assigning $s_i = \ell(s_i^M)$; thus this sequence of states (s_1, \ldots, s_p) , associated with the plan, has by construction:

- initial state s_1 s.t. $s_1 = I \cup \{\neg x_i | x_i \notin I\}$
- a state s_p s.t. $s_p \subseteq \mathcal{M} \cup \{\neg x_i | x_i \notin \mathcal{N}\}$
- given a state s_i , s_{i+1} is successor of s_i iff
 - $s_i \subseteq Precond(o_{h_i})$
 - $s_{i+1} = s_i \cup \alpha_i \beta_i$ where α_i is the positive postcondition of o_{h_i} and β_i is the negative postcondition of o_{h_i} .

Now we show the complexity results, both in the synchronous and in the interleaved case, when the formula can be preprocessed.

Theorem 3 The model checking problem for k synchronous concurrent process $MC'_{syn} = \langle \varphi, (M_1 || \dots || M_k) \rangle$ where $\varphi \in LTL$ is $\parallel \rightarrow PSPACE$ -complete, and remains $\parallel \rightarrow PSPACE$ -hard for $\varphi \in L(F, G, X)$.

Proof. $PLANSAT_1^*$ is the following problem of planning: giving a STRIPS [29] instance $y = \langle P, O, I, G \rangle$ in which the operators have an arbitrary number of preconditions and only one postcondition, is there a plan for y? $PLANSAT_1^*$ is PSPACE-complete [12]. Without loss of generality we consider $y = (P, O \cup o_0, I, G)$, where o_0 is a operator which is always usable (it has no preconditions) and does nothing (it has no postconditions). We use the following notation: $P = \{x_1, \ldots, x_n\}$, I is the set of conditions true in the initial state, $G = \langle \mathcal{M}, \mathcal{N} \rangle$. A state in STRIPS is a set of conditions.

In the following we indicate with ϕ_i^h the *h*th positive precondition of the operator o_i , and with η_i^h the *h*th negative precondition of the operator o_i ; α_i is the positive postcondition of the operator o_i , β_i is the negative postcondition of the operator o_i . Since any operator has only one postcondition, for every operator *i* it hold that $\|\alpha_i \cup \beta_i\| = 1$.

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$$r(y) = r(P) = \neg \left\{ F(x_g) \land G \bigwedge_{i=0}^n \left[\neg (x_i \leftrightarrow Xx_i) \to \bigwedge_{\substack{j=1 \ j \neq i}}^n (x_j \leftrightarrow Xx_j) \right] \right\}$$

- h(y) defines the transition systems $M_1 \parallel \ldots \parallel M_k$. The generic M_i is obtained from the operators $o_{i_1}, \ldots, o_{i_{d_i}}$ whose postcondition involves the variable $x_i \in P$; d_i is the number of such operators. We add the variable x_g ; thus we have at most as many processes as variables: if k is the number of variables used as postcondition of operators plus one, we have $k \leq n+1$. Let M_k the process associated with the variable x_g ; this variable is 0 at the beginning and it becomes 1 only when the goal of the *PLANSAT* problem is reached. M_i , for i s.t. $1 \leq i < k$, is defined by:

$$\begin{split} V_i &= \bigcup_{q=1}^{d_i} \phi_{i_q} \cup \eta_{i_q} \cup \alpha_{i_q} \cup \beta_{i_q} \\ I_i(V_i) &= \bigwedge_{x_j \in I \cap V_i} x_j \wedge \bigwedge_{x_j \in \overline{I} \cup V_i} \neg x_j \\ \varrho_i(V_i, V_i') &= \bigvee_{k=1}^{d_i} \bigwedge_{h=1}^{\|\phi_{i_k}\|} \phi_{i_k}^h \wedge \bigwedge_{h=1}^{\|\eta_{i_k}\|} \neg \eta_{i_k}^h \wedge \neg (\bigwedge_{i \in \mathcal{M}} x_i \wedge \bigwedge_{i \in \mathcal{N}} \neg x_i) \wedge (x_i' \equiv b_{i_k}) \\ \text{where } b_{i_k} &= \begin{cases} 1 & \text{if } \alpha_{i_k} \neq \emptyset \\ 0 & \text{if } \beta_{i_k} \neq \emptyset \end{cases} \end{split}$$

The process M_k is defined by:

$$V_k = \{x_g\}$$

$$I_k(V) = (x_g = 0)$$

$$\varrho_k(V_k, V'_k) = \bigwedge_{i \in \mathcal{M}} x_i \wedge \bigwedge_{i \in \mathcal{N}} \neg x_i \wedge x'_q = 1$$

Now we prove that this reduction is correct, i.e. $y \in PLANSAT_1^*$ iff $\langle r(y), h(y) \rangle \in MC'_{SYN}.$

 \Rightarrow . Given a solution for $y \in PLANSAT_1^*$, we show a path of M which satisfies φ (r(y) defined above).

A solution for y is a plan which generates the following sequence of states: (s_1,\ldots,s_p) where s_1 is a initial state and s_p is a goal state. This sequence of states is obtained by applying a sequence of operators $(o_{h_1}, \ldots, o_{h_p})$. By construction M admits a path $(s_1^M, \ldots, s_p^M, s_{p+1}^M, \ldots)$ s.t.:

- $\ell(s_i^M) = s_i \cup \neg x_g$ for i $1 \le i \le p$ - $\ell(s_{p+1}^M) = s_p \cup x_q$

This path satisfies φ :

- φ does not constrain about the initial state, therefore every initial state of the model is legal;
- $x_g \subseteq \ell(s_{p+1}^M)$, therefore $F(x_g)$ is true;
- the path shown is s.t. only one variable change at a time, therefore the subformula under the Globally is true.

 \Leftarrow . Given a path of M which satisfies φ , we show a solution for $y \in$ $PLANSAT_1^*$.

The path is a sequence $(s_1^M, \ldots, s_p^M, s_{p+1}^M, \ldots)$. We can obtain the sequence of states visited by a plan for y in this way:

-
$$s_i = \ell(s_i^M) - \{\neg x_g\}$$
 for i $1 \le i \le p;$

- we ignore the rest of the path of M.

Theorem 4 The model checking problem for k interleaved concurrent process $MC'_{asyn} = \langle \varphi, (M_1 | \dots | M_k) \rangle$ where $\varphi \in LTL$ is $\parallel \rightarrow PSPACE$ -complete, and remains \gg PSPACE-hard for $\varphi \in L(F)$.

Proof. We carry out a reduction from the $PLANSAT_1^*$ problem, that satisfies the conditions of representative equivalence. The proof is similar to the proof of the Theorem 3.

Now we introduce the decision problem $MC_{s_0} = \langle [M, \varphi], s_0 \rangle$, where M is specified by the interleaved parallel composition of k transition systems M_1, \ldots, M_k , $\varphi \in L(F)$, and s_0 is a specific state. MC_{s_0} is true if the model checking problem for concurrent transition system $\langle M, \varphi \rangle$ has solution and s_0 is a legal initial state i.e., is an initial state belonging to M that satisfies φ .

Theorem 5 MC_{s_0} is $\parallel \rightarrow PSPACE$ -complete.

Proof. The hardness follows from a polynomial time reduction from the problem $\langle (P, O, G), I \rangle$, that can be easily shown $\parallel \rightarrow PSPACE$ -complete on the basis of the results in [40].

We sketch the reduction. We encode each operator in O into each process M_i , and the goal G into the formula φ . We encode the set of initial states I using s_0 .

3.2 The Size of BDDs

In this section we prove that the size of BDDs and others data structures increases superpolynomially with the size of the input data, in the worst case, when are used in a Symbolic Model Checking algorithm.

Let M a model specified by k concurrent transition systems M_1, \ldots, M_k , and let φ an LTL (or a CTL or CTL*) formula.

Theorem 6 If PSPACE $\not\subseteq \Pi_2^p \cap \Sigma_2^p$, then there is not always a BDD of any kind and with any variable order that is polynomially large and represents the set of initial states consistent with M and φ .

Proof. The evaluation problem for any kind of BDD, i.e. giving a BDD and an assignment of its variables evaluate the BDD, is in P. If there exists a poly-size BDD representing the set of initial states consistent with M and φ , then we can compile M and φ in the BDD and evaluate the assignment (representing a initial state) in polynomial time. This implies that MC_{s_0} is in $\| \rightarrow P$. We know from Theorem 5 that MC_{s_0} is $\| \rightarrow PSPACE$ -complete. Therefore if such a BDD exists, then $\| \rightarrow PSPACE = \| \rightarrow P$. Now, by applying Theorem 2.12 in [15], we conclude that there is no poly-size reduction from MC_{s_0} to the evaluation problem for a BDD, if PSPACE $\not\subseteq \Pi_2^p \cap \Sigma_2^p$.

Symbolic Model Checking algorithms work by building a representation of the set of the initial states of M that satisfy φ . In particular, this set is represented by BDDs. Therefore, the last theorem proves that these algorithms, in the worst case, end up with a BDD of superpolynomial size. This result does not depend on the kind of BDD used (free, ordered, etc.) and on the variable ordering. On the contrary, it holds also when the states are labeled with enumerative variable; in other words it holds not only for BDD but also for any decision diagram, provided that the evaluation problem over this representation of the states is in a class of the polynomial hierarchy. More formally, we consider an arbitrary representation of a set of states. The evaluation problem is that of determining whether a state belongs to a set.

Theorem 7 Given a method for representing a set of states whose evaluation problem is in a class Σ_i^p of the polynomial hierarchy, it is not always possible to represent in polynomial space the set of legal initial states of a model M and a formula φ , provided that $\Sigma_{i+1}^p \neq \prod_{i+1}^p$.

The proof of this theorem has the same structure of the proof of the Theorem 6.

Instances of such data structures, currently used in Symbolic Model Checking tools, are BDDs, Boolean Expression Diagrams (BEDs) [54] and Reduced Boolean Circuits (RBCs) [1]. Our results hold also for data structures used to represent integer-value functions, like Multi terminal binary decision diagrams (MTBDDs) [18], Algebraic Decision Diagrams (ADDs) [2]; see for details the survey [25].

On the other hand, it is also possible to prove that the above two theorems cannot be stated unconditionally: indeed if P = PSPACE, then there is a data structure of polynomial size allowing the representation of the set of initial states in such a way deciding whether a state is in this set can be decided in polynomial time. As a result, the non-conditioned version of the above two theorems implies a separation in the polynomial hierarchy.

4 Related Works

Some works in the literature are related to the results in this article:

- 1. the exponential growth of the BDD size respect to a particular problem (e.g. integer multiplication [10]); some results concern the size growth of other decision diagrams [25] respect to particular problems. While these results are not conditional to the collapse of the polynomial hierarchy as the ones reported in this paper, they are also more specific, as they concern only specific kinds of data structures (e.g. OBDDs) respect to particular problems (e.g. integer multiplication).
- 2. the complexity of model checking:
 - (a) the parametrized complexity [24] of a wide variety of model checking problems [23], analyzing the state explosion problem;
 - (b) it has been shown that [28]:
 - i. the complexity of model checking does not decrease under the ipotheses of some structural restrictions (e.g. treewidth) in the input.
 - ii. despite a CNF formula of bounded treewidth can be represented by an OBDD of polynomial size, the nice properties of treewidthbounded CNF formulas are not preserved under existential quantification or unrolling, that is a basic operation of model checking algorithms.
 - (c) the compilability of the model checking problem [27]: it remains PSPACE-complete even if a part of the input, either the implicit model or the formula, is preprocessed using any amount of time and storing the result of this preprocessing step in a polynomial-sized data structure.
- 3. the theoretical limitations of Symbolic Model Checking. The state explosion problem can be partially explained by complexity theoretic results [19]; in fact, problems (also whose inputs are graphs), usually increase their worst case complexity when the input is represented by BDD or other Boolean formalisms [26, 30, 44, 3, 41, 4, 49, 50, 51, 52, 33]. Moreover, a

classic information theoretic argument shows that only a small fraction of all finite Kripke structures can be exponentially compressed [38].

4. succinctness of languages; for instance [21], in which succinctness of language for preferences are discussed, and [22] that presents results on the succinctness of several formalism, including BDDs and CNF.

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