Optimization of Tree Modes for Parallel Hash Functions: A Case Study

Kevin Atighehchi¹ and Robert Rolland²

 Aix Marseille Univ, CNRS, LIF, Marseille, France kevin.atighehchi@univ-amu.fr
 Aix Marseille Univ, CNRS, I2M, Marseille, France robert.rolland@acrypta.fr

Abstract. This paper focuses on parallel hash functions based on tree modes of operation for an inner Variable-Input-Length function. This inner function can be either a single-block-length (SBL) and *prefix-free* MD hash function, or a sponge-based hash function. We discuss the various forms of optimality that can be obtained when designing parallel hash functions based on trees where all leaves have the same depth. The first result is a scheme which optimizes the tree topology in order to decrease the running time. Then, without affecting the optimal running time we show that we can slightly change the corresponding tree topology so as to minimize the number of required processors as well. Consequently, the resulting scheme decreases in the first place the running time and in the second place the number of required processors.

Keywords. Hash functions, Hash tree, Merkle tree, Parallel algorithms

1 Introduction

A mode of operation for hashing is an algorithm iterating (operating) an underlying function over parts of a message, under a particular composition method, in order to compute a digest. A hash function is obtained by applying such a mode to a concrete underlying function; we then say that the former is constructed on top of the latter. Usually, when the purpose is to process messages of arbitrary length, the underlying function may be a fixed-input-length (FIL) compression function, a block cipher or a permutation. However, there may also be an interest in using a sequential (or serial) hash function as underlying function, in order to add to it other features, like coarsegrained parallelism. The resulting hash function must satisfy the usual properties of pre-image resistance (given a digest value, it is hard to find any pre-image producing this digest value), second pre-image resistance (given a message m_1 , it is hard to find a second message m_2 which produces the same digest value), and collision resistance (it is hard to find two distinct messages which produce the same digest value). A sequential hash function can only use Instruction-Level Parallelism (ILP) and SIMD instructions [19,20]. A cryptographic hash function has numerous applications, the main one is its use in a signature algorithm to compress a message before signing it.

The most well known sequential hashing mode is the Merkle-Damgård [14,25] construction which can only take advantage of the fine-grained parallelism of the operated compression function. If such a low-level "primitive" can benefit from the Instruction-Level Parallelism, by using also SIMD instructions, the outer algorithm iterating this building block could benefit from a coarse-grained parallelism. This parallelism can be employed in multithreaded implementations. Suppose that we have a collision-free compression function taking as input a fixed-size data, $f:\{0,1\}^{2N}\to\{0,1\}^N$. By using a balanced binary tree structure, Merkle and Damgård [14,24] show that we can extend the domain of this function so that the new outer function, denoted $H:\{0,1\}^*\to\{0,1\}^N$, has an arbitrary sized domain and is still collision-resistant. Note that if the function f is a sequential hash function, the purpose of this tree structure is merely the addition of coarse-grained parallelism.

A construction using a balanced binary tree allows simultaneous processing of multiple parts of data at a same level of the tree, reducing the running time to hash the message from $\mathcal{O}(n)$ to $\mathcal{O}(\log n)$ if we have $\mathcal{O}(n)$ processors [14,24]. If we want to further reduce the amount of resources involved, we can use one of the following rescheduling techniques:

- Each processor is assigned the processing of a subtree (in the data structure sense) having $\log n$ leaves. There are approximatively $n/\log n$ such subtrees. The processing of the remaining ancestor nodes, at each remaining level of the tree, is distributed as fairly as possible between the processors. An example is depicted in Figure 1.
- An alternative solution is, at each level of the tree, to distribute as fairly as possible the node computations among $O(n/\log n)$ processors.

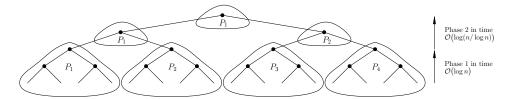


Fig. 1. Example of the computation of the root node in $\mathcal{O}(\log n)$ time using $\mathcal{O}(n/\log n)$ processors. The message to hash is of size n=16. In Phase 1, the computation of each hash subtree containing $4=\log_2 16$ leaves is assigned to each processor. The first subtree is assigned to processor P_1 , the second one to processor P_2 , the third one to processor P_3 and the last one to processor P_4 . A fine-grained allocation is then performed in Phase 2.

The number of processors is then reduced by a factor $\log n$ and the asymptotic running time is conserved (with, nevertheless, a multiplicative factor 2). In this paper we are not interested in tradeoffs between the amount of used resources and the running time but instead we study optimal algorithms in finite distance. More precisely, we de-

termine the hash tree structures which give the best concrete (parallel) time complexity for finite message lengths.

A tree structure is notably used in parallel hashing modes of Skein [16], BLAKE2 [4] or MD6 [29]. To give some examples, Skein uses a tree whose topology is controlled by the user thanks to three parameters: the arity of base level nodes which is a power of two; the arity of other inner nodes, which is also a power of two, and a last parameter limiting the height of the tree. MD6 uses a full (but not necessarily perfect) quaternary tree, in the sense that an inner node has always four children. Some fictive leaves or nodes padded with 0 are added so that a rightmost node has the correct number of children. Like Skein, MD6 offers a parameter which serves to limit the height of the tree.

Some proposals [30,31,26] consider that a tree covering all the message blocks is not a good thing, because the number of processors should not grow with the size of the message. For instance, the domain extension parallel algorithm from Sarkar et al. [30,31,26] uses a perfect binary tree of processors, of fixed size. This perfect binary tree of compression/hash function calls can be seen as a big compression function, sequentially iterated over large parts of the message. In other words only the nodes computations performed in the tree can be done in parallel. The number of usable processors is a system parameter chosen by the issuer of the cryptographic form when hashing the message. The value of this parameter has to be reused by the recipients, for instance when verifying a signature. Thus, this one limits the scalability and the potential speedup. In this paper we consider that the scalability and the potential speedup should be independent of the characteristics (configuration) of the transmitting computer.

Bertoni et al. [9,11] give sufficient conditions for a tree-based hash function to ensure its indifferentiability from a random oracle. They propose several tree hashing modes for different usages. For example we can make use of a tree of height 2, defined in the following way: we divide the message in as many parts (of roughly equal size) as there are processors so that each processor hashes each part, and then the concatenation of all the results is sequentially hashed by one processor. To divide the message in parts of roughly equal size, the algorithm needs to know in advance the size of the message. Bertoni et al. propose also a variant which still makes use of a tree having two levels and a fixed number of processors, but this one interleaves the blocks of the message. This interleaving offers a number of advantages, as it allows an efficient parallel hashing of a streamed message, a fairly equal distribution of the data processed by each processor in the first level of tree (without prior knowledge of the message size), and a correct alignment of the data in the processors' registers. This kind of solution is suitable for multithreaded and SIMD implementations [18]. In this paper we study theoretically optimal speedups, and, as a consequence, the message to hash is supposed to be already available.

Our concern in this paper is with hash tree modes using an underlying variable-input-length (VIL) function that needs l invocations of a lower level primitive to process a message of l blocks, where a block and the hash output have the same size. To make such a complexity concrete, we choose to use a single-block-length (SBL) hash function as underlying function and to focus on the *prefix-free* Merkle-Damgård construction from Coron *et al.* [13]. We make this choice for two reasons: first, we need a

hash function whose mode of operation is proven indifferentiable from a random oracle (when its underlying primitive is assumed to be ideal). Second, assuming that we have applied the *prefix-free* encoding [13] and another encoding [9,11] to identify the type of input in the tree, it is possible to precompute a constant number of hash states, making the aforementioned complexity possible. Note that, even if we take this construction as an example, the possible use of a sponge-based function will be discussed. In this work, we aim to show that we can improve the performance of a hash tree mode of operation by reworking the tree-structured circuit topology. While we focus on the case of trees having all their leaves at the same depth, we are interested in minimizing the depth (parallel time) of the circuit and the width (number of processors involved). This kind of work has been done for parallel exponentiation in finite fields [32,34,1,22,36] where the multiplication operator is both associative and commutative. In the case of parallel hashing, the considered operator can be a FIL (Fixed-Input Length) compression function. This is quite different since we do not have these two properties and we need to cope with other problems (the space consumption of a padding rule, a length encoding, or other information bits). To the best of our knowledge, it is the first time that the problem of optimizing hash trees is addressed. The main interest of this paper is the methodology provided. The results can be presented as follows:

- The first result is an algorithm which optimizes the tree topology in order to decrease the depth. We first show that a node arity greater than 5 is not possible and then we prove that we can construct such an optimal tree using exclusively levels of arity 2 and 3.
- Without affecting this optimal depth, we show that we can change the corresponding tree topology in order to decrease the width as much as possible. In particular, we show that for some message lengths l, the width can be decreased to $\lceil l/5 \rceil$.
- We also provide an algorithm which optimizes the number of processors at each step of the hash computation. We prove that eleven tree topologies are possible.
- With the assumption that the message size is Pareto-distributed, we estimate the relative frequency of each tree topology using the Monte Carlo method.
- Finally, we show that by using a SBL hash function as underlying function and by assuming a constant number of precomputed values, these optimisations can be applied safely.

Suppose that the processing of one block of the message by the underlying function costs one unit of time. A binary tree is not necessarily the structure which gives the best running time. Figure 2 shows two different tree topologies for hashing a 6-block message. The binary tree depicted in (2a) gives a (parallel) running time of 6 units while the rightmost one with a different arity at each level, depicted in (2b), gives a running time of 5 units. Furthermore, one may note that for messages of length less that 5 blocks, the use of the topology (2a) has no utility compared to a purely sequential mode (*i.e.* a completely degenerated binary tree).

In what follows, we suppose the use of variable-input-length (VIL) compression functions (or hash functions) having a domain space $\{0,1\}^{xN}$ for $x \ge 2$ and a fixed length range space $\{0,1\}^N$. We also assume that such a function has an ideal computational cost of x units when compressing x blocks of size N bits. In other words, if we consider a tree of calls of this function, the computation of a node having k children

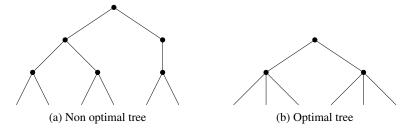


Fig. 2. Tree hashing with a 6-block message. The hash tree on the left requires 2 units of time to process each level, while the one on the right requires 3 units of time to process the base level and 2 units of time to process the root node.

(i.e. k blocks) has a cost of k units. Such a computational cost is realist. For instance, the UBI transformation function³ used in the hash function family Skein [16] performs x calls to the tweakable block cipher Threefish to compress a data of length x blocks. Assuming a hash tree of height k and k the arity of level k (for k = 1 . . . k), we define the parallel running time to obtain the root node value as being k to k the arity of level k the arity of level

The paper is organized in the following way. In Section 2 we give background information and definitions. In Section 3, we first describe the approach to minimize the running time of a hash function. Then, we give an algorithm to construct a hash tree topology which achieves the same optimal running time while requiring an optimal number of processors. We also show that we can optimize the number of processors at each step of the hash computation. This leads to eleven possible tree topologies, whose probability distribution is empirically analyzed in Section 4. We propose in Section 5 a concrete tree-based hash function that safely implements these optimizations. Finally, in Section 5, we conclude the paper.

2 Preliminaries

2.1 Tree structures

Throughout this paper, we use the convention⁴ that a node is the result of a function called on a data composed of the node's children. A node value (or chaining value) then corresponds to an image by such a function and a child of this node can be either an other image or a message block. We call a base level node a node at level 1 pointing to the leaves representing message data blocks. The leaves (or leaf nodes) are then at level 0. Then, a tree of nodes of height h has h+1 levels. We define the arity of a level in the tree as being the greatest node arity in this level.

A k-ary tree is a tree where the nodes are of arity at most k. For instance, a tree with only one node of arity k is said to be a k-ary tree. A full k-ary tree is a tree where all

³ UBI stands for Unique Block Iteration, the sequential operating mode used in Skein. The UBI transformation function refers to the application of this mode to the underlying tweakable compression function, itself based on the tweakable block cipher Threefish.

⁴ This corresponds to the convention used to describe Merkle trees.

nodes have exactly k children. A perfect k-ary tree is a full k-ary tree where all leaves have the same depth.

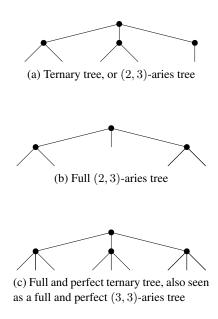


Fig. 3. Examples of trees

We also define other "refined" types of tree. We say that a tree is of arities $\{k_1,k_2,\ldots,k_n\}$ (we can call it a $\{k_1,k_2,\ldots,k_n\}$ -aries tree) if it has n levels (not counting level 0) whose nodes at the first level are of arity at most k_1 , nodes at level 2 are of arity at most k_2 , and so on. We say that such a tree is full if all nodes at the first level have exactly k_1 children, all nodes at level 2 have exactly k_2 children, and so on. As before, we say that such a tree is perfect if it is full and if all the leaves are at the same depth. Some examples are depicted in Figure 3.

A tree of nodes has a corresponding tree of f-inputs having one less level. For instance, the simple tree of nodes of height 1 and arity 3 has 2 levels and 4 nodes: one root node and 3 children which are the message blocks. Its corresponding tree of f-inputs has a single level containing a single f-input. This f-input consists of the concatenation of the message blocks and some meta-information bits. This representation is further defined in the following subsection 2.2.

2.2 Security of tree-based hash functions

In this section (and in Subsection 5.2), we represent a hash tree as a tree of f-input, assuming that a single inner function f is operated in the outer hash function, denoted H. This is the representation adopted in [9,10] to prove the desired security properties. We recall that a tree of f-inputs has one less level compared to a tree of nodes.

A f-input is a finite sequence of bits from the following elements: message bits, chaining value bits (*i.e.* bits coming from a f-image), and frame bits (bits which are fully determined by the hash algorithm and the message size). In a tree of f-inputs, there are pointers from children to their corresponding parent. When a chaining value is present in a formatted f-input, it is pointed by another f-input which is considered as its children. Each f-input has an associated index which locates it in the tree. In addition, we need to define, for a tree T of f-inputs, its corresponding tree template Z which has the same topology, where the corresponding f-inputs have the same lengths and the frame bits match the corresponding bits in T, but where message and chaining value bits are not valuated. Thus, a tree template is fully determined by the message size and the parameters of the tree mode algorithm. This tree template is used by the tree hash mode to instantiate the tree of f-inputs, by valuating progressively message bits and chaining value bits.

A tree T of f-inputs is said to be *compliant* with a tree hash mode τ if the latter can produce a tree of f-inputs whose corresponding tree template is compatible with it (its topology, the frame bits and the sizes of its f-inputs match those of T). A tree T of f-inputs is said to be *final-subtree-compliant* with τ if the latter can produce a tree of f-inputs whose proper subtree (*i.e.* containing at least the root f-input) has a corresponding tree template with which T is compatible.

Bertoni *et al.* [9,10] give some guidelines to design correctly a tree hash mode τ operating an inner hash (or compression) function f. They define three sufficient conditions which ensure that the constructed hash function τ_f , which makes use of an ideal hash (or compression) function f, is indifferentiable from an ideal hash function. Besides, they propose to use particular frame bits in order to meet these conditions. We refer to [9,10] for the detailed definitions, and we give here a short description for each of them:

- message-completeness: suppose we have a tree of f-inputs produced by the tree hash mode. There is an algorithm $A_{message}$ which, among the bits in the tree, uniquely determines the message. This requires that each message bit is processed at least once by f. The message can be reconstructed correctly if, given the sequence of bits of a f-input, we can identify those which are message bits, and we are able to say what their positions in the message are. Generally, only the end of the message is problematic. To cope with this, dedicated frame bits can be used such as a reversible padding f for the message or a coding of the message length. The running time of $A_{message}$ should be linear in the total number of bits in the tree.

⁵ Since a hash function processes an entire number of blocks (whose size N depends on the underlying primitive), a reversible padding is an efficient way of revealing the end of the message. This consists in applying to the message M, whatever its length, a function pad which returns a bit-string of length a multiple of N. Such a padding has to be reversible, *i.e.* there is a function unpad such that unpad(pad(M)) = M for all messages M. A well-known technique consists in appending the bit "1" to the end of the message, followed by the minimum number of bits "0", so that the total bit-length of the padded message is a multiple of N.

- tree-decodability: intuitively, given a tree T of f-inputs generated by τ , it is impossible to extract a final proper subtree T' of T which could have legitimately been generated by τ . In other words, given such a subset of f-inputs, we are able to say whether there is a missing input or not. More formally, this property is satisfied by τ if there are no trees of f-inputs which are both compliant and final-subtree-compliant with it, and there is a decoding algorithm A_{decode} that can parse the tree progressively on subtrees, starting from the root f-input, to retrieve frame bits, chaining value bits and message bits unambiguously. Also, when terminating, A_{decode} must decide if the tree is compliant, final-subtree-compliant, or incompliant with τ . The running time of A_{decode} should be linear in the total number of bits in the tree.
- final-f-input separability: whatever the tree T of f-inputs generated by τ , we can distinguish between a root f-input and any other f-input. Such a property is useful to prevent length extension attacks. One straightforward way to fulfil this property is by means of domain separation between this final (root) f-input and other f-inputs, e.g. by augmenting them with a frame bit identifying them as such.

These conditions ensure that no weaknesses are introduced on top of the risk of collisions in the inner function. For instance, with *tree-decodability*, an inner collision in the tree is impossible without a collision for the inner function. Andreeva *et al.* have shown in [2,3] that a hash function indifferentiable from a random oracle satisfies the usual security notions, up to a certain degree, such as pre-image and second pre-image resistance, collision resistance and multicollisions resistance.

2.3 Definition of the inner hash function

For our inner function, the hash functions based on the Merkle-Damgård construction, such as MD5, SHA-1 or SHA-2, have to be discarded. First, these functions cannot emulate a random oracle and we need this property to construct a tree-based hash function, constructed on top of it, which is still indifferentiable from a random oracle [10]. Second, for efficiency purposes, we want the inner function to have a running time linear in the number of blocks of the message. When MD-strengthened padding is applied, the size of the message is appended to its end. This makes it difficult to obtain a running time perfectly linear in the number of blocks. As we will see, the *prefix-free* Merkle-Damgård construction from Coron *et al.* [13] is a solution to both these problems.

Our inner hash function, based on the *prefix-free* MD construction, iterates a SBL (Single-Block-Length) compression function, *i.e.* whose output length is the same as the message block length. We use the compression function $c:\{0,1\}^{2N} \to \{0,1\}^N$ based on a N-bit block cipher with a N-bit key, such as the Davies-Meyer compression function: $c(x,y) = E_y(x) \oplus x$ where x is the previous hash state and y is a block of the message. Such a hash function [13], denoted f, consists in applying the plain MD construction to a prefix-free encoding of the message input.

The considered inner function f.

INPUT: message m. OUTPUT: hash value.

- 1. The message m is padded with 10^r where r is the minimum number (possibly zero) of bits 0 such that its bitlength is a multiple of N.
- 2. The *N*-bits encoding of the number of blocks is prepended to the message (prefix-free encoding step).
- 3. The message is parsed into k blocks $m_1, m_2, ..., m_k$ of size N bits.
- 4. The plain MD mode is applied on these k blocks.

```
Let y_0 = 0^N (or a fixed IV value).
For i = 1 to k do y_i \leftarrow c(y_{i-1}, m_i).
Return y_k \in \{0, 1\}^N.
```

At first sight, due to the padding and the prefix-free encoding, this hash function requires i + 2 calls to the underlying compression function to process a message of jN bits. In fact, the node arities of our tree topologies can be upper-bounded by a constant. Thus, the first block of the prefix-free encoding of an input involved in our trees has a constant number of possible values, and their possible corresponding hash states y_1 can be precomputed. Assuming a constant number of precomputed hash states, the running time of this hash function is then reduced to j+1 calls to the underlying primitive. Hence, it is explained in the subsection above that before applying the function f, domain separation bits have to be added to the input. Bertoni et al. [10] have stated that 2 bits is sufficient. The number of distinct domain separation codes can then be considered small. For domain separation purposes, we choose to prepend a code of N-1 bits to the message so that there is no extra bits 0 due to the padding operation. With this second "large" encoding, the first bit of the message is at the end of the second block. Then, the running time is still j + 1 when the possible values of y_1 are precomputed. Since the number of distinct domain separation codes is small and that only one bit of the message is in the second block, we can precompute the possible values of y_2 (instead of y_1) so as to reduce the running time of f to j units of time. Further details will be provided in Section 5.

A hash function that performs one call to a SBL compression function to process one block of the (padded) message is also called a SBL hash function.

On the use of other SBL compression functions. Chang et al. [12] have extended the work of Coron et al. by checking if other hash functions using this prefix-free MD construction are still indifferentiable from an RO. It turns out that sixteen of the PGV (Preneel Govaerts Vandewalle [27]) compression functions yield sound hash functions. Again, these hash functions divide the message in blocks of the same size than the digest.

On the use of a sponge-based hash function as inner function. The function f can be a sponge-based hash function [8] if the message block and output sizes are equal. This constraint can be fulfilled by setting the rate (also denoted N) of this sponge function to

the output size. Let us suppose that we use a function having the same padding rule than Keccak [7]. This padding rule consists in adding to the end of the message the bitstring 10^*1 where 0^* is the minimum number of bits 0 such that the padded message has a size a multiple of N. Hence, we choose to use N-2 bits for the prepended domain separation codes so that the padded message ends with the two bits 11. With this large encoding, the first two bits of the message are at the end of the first block. Taking into account these overheads, the running time of f is j+1 to process a message of j blocks. Some precomputations can be done to reduce the running time of f to j units of time. Further details will be provided in Section 5.

About the padding overhead in other designs. The UBI transformation function of Skein [16] is collision resistant and requires exactly j calls to the tweakable block-cipher Threefish to process a message of j blocks. Indeed, when the message size is already a multiple of N, a flag in the tweak indicates whether or not the message is padded. Such a design choice is also present in the BLAKE2 [4] hash function.

2.4 Parallel computation model

We make the assumption that the number of processors is equal to the number of nodes at the first level of the tree. Once the nodes have been computed at a given level, the processors are reused to process the next (upper) level.

We use the classic PRAM (Parallel Random Access Machine) model of computation [17], assuming the strategy EREW (Exclusive Read Exclusive Write). When dealing with hash trees, this model can indeed be restricted to this strategy: we do not need that two processors write simultaneously into the same memory location, nor that a same data can be read simultaneously by two or more processors. In the context of parallel hashing, it serves *a priori* no purpose to process twice a same message block or chaining value.

Let us denote by L_i the list of nodes at level i. Given the definitions of a level arity and of our inner hash function, the parallel running time to process a hash tree of height h is equal to

$$\sum_{i=1}^{h} \max_{\text{Node} \in L_i} \text{arity(Node)},$$

where the function arity() returns the arity of a node. The *total work* to process a hash tree is equal to

$$\sum_{i=1}^{h} \sum_{\text{Node} \in L_i} \text{arity(Node)}.$$

In other words, this quantity corresponds to the running time when it is executed sequentially (*i.e.* by a single processor).

3 Optimization of hash trees for parallel computing

3.1 Minimizing the running time

In order to optimize the running time of a tree mode, we make a certain degree of flexibility on the choices of node arities. We can note immediately that allowing different node arities in a same level of the tree provides no efficiency gains. Worse, the running time may be less interesting since a tree level processing running time is bounded by the running time to process the node having the highest arity. This observation suggests that, in order to hope a reduction of the tree processing running time, node arities at the same level need to be set to the same value⁶ while allowing arities to vary from one level to another. Therefore our strategy allows a different arity at each level of the tree.

Let us denote l the block-length of a message. The problem is to find a tree height h and integer arities $x_1, x_2, ..., x_h$ such that $\sum_{i=1}^h x_i$ is minimized. When constructing a hash tree having its leaves at the same depth, we seek integers $x_1, x_2, ..., x_h$ such that $\lceil \lceil \cdots \lceil \lceil l/x_1 \rceil / x_2 \rceil \cdots \rceil / x_h \rceil = 1$. Since we have

$$\lceil \lceil \cdots \lceil \lceil l/x_1 \rceil / x_2 \rceil \cdots \rceil / x_h \rceil = \lceil l/(x_1 x_2 \cdots x_h) \rceil$$

for (strictly) positive integers $(x_j)_{j=1...h}$, any solution to the problem must necessarily satisfy the following constraints:

$$\prod_{i=1}^{h} x_i \ge l \text{ and } \left(\prod_{i=1}^{h} x_i\right) / x_j < l \quad \forall j \in [[1, h]].$$
 (1)

Note that if a solution (x_1, x_2, \ldots, x_h) does not satisfy the second contraint in (1), this means that a better solution exists. A solution to this problem is a multiset of arities. First, we show that, in a non-asymptotic setting, a perfect ternary tree comes closer to optimality than a perfect binary tree. Then we examine the case of trees having different arities at each level.

First of all, we can start by considering the h and x_i (for $i=1\ldots h$) as real numbers. Thus, we have to minimize the summation of x_i subject to the constraint that their product is l. We know that the minimum is reached when the x_i are equal to the same number, which we will denote x. So we have $x^h=l$, that is $x=l^{\frac{1}{h}}$. We must now determine h so that $hl^{\frac{1}{h}}$ is minimized. The calculation of a derivative shows that this minimum is reached for $h=\ln(l)$, which implies x=e. Consequently, we can wonder what the best solution is between a perfect binary tree and a perfect ternary tree. The comparison of these two cases is done in Appendix A and shows that beyond a certain message size l ($l=2^{28}$), a perfect ternary tree gives a better running time than a perfect binary tree. In fact, as the present general study shows, a tree having different level arities can give better results.

Let us remind that node arities are not allowed to vary in a same level (same stage) of the tree. A level of the tree is said to be of arity a when all nodes at this level are of arity at most a. Given an optimal tree (in the sense of the running time) for hashing, we can ask what the possible arities are for its levels. We have the following Theorem:

⁶ Except maybe the rightmost node which may be of smaller arity.

Theorem 1. For a hash tree whose running time is optimal, the following statements hold:

- It can be comprised of levels of arity 2, 3, 4, or 5. Higher arities are not possible.
- It can be constructed using only levels of arity 2 and 3.

Proof. We prove these two assertions separately:

- We first show that levels of arity a with $a \ge 7$ lead to trees having a suboptimal running time. Indeed, any node of arity $a \ge 7$ can be replaced by a tree of arity 2 having a better running time. We simply have to note that $2\lceil \log_2 a \rceil < a$ for all a > 6, meaning that a a-ary tree of height 1 can be advantageously replaced by a binary tree of height $\lceil \log_2 a \rceil$. In contrast, for all nodes of arity a with $a \in [3, 6]$ and for all $i \in [2, 5]$ we have $i \lceil \log_i a \rceil \ge a$. Finally, a node of arity 6 can be replaced by a $\{3,2\}$ -aries tree, since $2 \cdot 3 = 6$, thereby reducing the running time to 2+3=5 units.
- As regards the second assertion, a node of arity 5 can be replaced by a tree of arities $\{3,2\}$, since $2\cdot 3=6>5$. This transformation does not change the running time since 2+3=5. Finally, a node of arity 4 can be replaced by a binary tree of height 2 for a running time which is still unchanged.

An optimal tree has not necessarily a single topology, Firstly, a solution satisfying constraints (1) can be defined as a multiset of arities since we can permute them. For instance, suppose a tree has three levels with the first level of arity 3, the second one of arity 2 and the last one (that is, the root node) of arity 3. We can permute these arities so that the first level is of arity 2 and the latter two levels of arity 3. If this new tree has the same running time, its topology has however changed. Secondly, we can find examples where different multisets of arities lead to trees of optimal running time. For instance, if we consider a 7-block message, the multisets of arities $\{2,2,2\}$, $\{3,3\}$ and $\{4,2\}$ allow the construction of trees having the optimal running time (see Figure 4). We can, however, construct optimal trees by restricting the set of possible arities. We have the following theorem:

Theorem 2. Let a message of length l blocks and let i be the lowest integer such that $3^i \ge l$. Let us note $x \in [0,2]$ the value which minimizes the product $3^{i-x}2^x$ under the constraint $3^{i-x}2^x \ge l$. There exists an optimal tree (in the sense of optimal running time) which has i - x levels of arity 3 and x levels of arity 2. More precisely, we can state the followings:

- If $l \le 3^i < \frac{3l}{2}$ then a ternary hash tree is optimal for a running time of 3i.
- If $\frac{3l}{2} \leq 3^i < \frac{9l}{4}$ then an optimal hash tree has i-1 levels of arity 3 and one level of arity 2, for a running time of 2 + 3(i - 1).
- Otherwise $\frac{9l}{4} \leq 3^i < 3l$, and then an optimal hash tree has i-2 levels of arity 3and 2 levels of arity 2, for a running time of 4 + 3(i - 2).

Such an optimal tree maximizes the number of levels of arity 3.

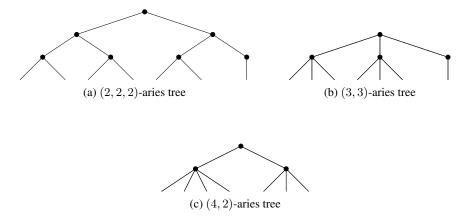


Fig. 4. Different topologies for a 7-block message

Proof. According to Theorem 1, a hash tree whose running time is optimal can be constructed using only levels of arity 2 and levels of arity 3. We still need to find out their numbers. If we have at least 3 levels of arity 2 then we can replace these 3 levels by 2 levels of arity 3 ($3^2 = 9 > 2^3 = 8$). The running time to process 3 levels of arity 2 or 2 levels of arity 3 is 6. Therefore, it is always possible to construct optimal trees with maximum 2 levels of arity 2. Let i be such that $3^i \geq l$. From the parallel running time standpoint, it is preferable to trade a level of arity 3 for a level of arity 2. This means that the sought solution corresponds to the highest value $x \in [0, 2]$ such that $3^{i-x}2^x \geq l$. The three assertions follow immediately.

To determine the level arities of an optimal tree, we apply the following algorithm:

Algorithm 1.

INPUT: a message length l.

OUTPUT: a multiset of arities minimizing the running time.

- We first compute $i = \lceil \log l / \log 3 \rceil$ and then $x = \lfloor \log(l/3^i) / \log(2/3) \rfloor$.
- We return a multiset which consists of i-x first levels of arity 3 and x last levels of arity 2.

Examples. For messages of lengths l=4,5 and 10 blocks respectively, Algorithm 1 returns the multisets of arities $\{2,2\}$, $\{3,2\}$ and $\{3,2,2\}$ respectively. The number of processors is not optimized here. This aspect is addressed in the following section.

The result can be either a *perfect* tree where the number of leaves is greater than the message length (the tree is said to be *perfect* since, on the one hand, nodes at a same level are all of same arity, and, on the other hand, all the leaves are at the same depth), or a *truncated* tree since it is possible to prune some right branches to remove this surplus of leaves. In the rest of the paper, we refer to a truncated (x_1, x_2, \ldots, x_h) -aries tree to speak about a tree having a number of leaves equal to the message length and where the

nodes of the base level are of arity at most x_1 , nodes at the second level are of arity at most x_2 and so on.

As a last remark, since the hash function must be deterministic, the multiset of arities must also be chosen deterministically as a function of the message size. For instance, we can arrange in descending order the elements of the multiset of arities. The solution to the problem of minimizing the running time is then uniquely determined as an ordered multiset.

Performance improvements. We have seen that for a message of 6 blocks (see Figure 2), the performance gain of an optimal tree compared to a binary tree is 20%. Figure 5a shows the running times of an optimal tree and a binary tree as functions of the message size varying from 1 to 10^5 blocks. Figure 5b shows the speed gain obtained with an optimal tree. The gain in time (or speedup gain) is computed as $100(T_b/T_o-1)$ where T_b is the running time of a binary tree and T_o the running time of an optimal tree. As we can see, the gain differs from one message size to another. The gain can be greater than 30% for very short messages but decreases quickly, to cap at 10%. As regards the message size, although the diagram does not cover a sufficiently long range, one can note a slight downward slope.

3.2 Minimizing the number of processors

In this section we look into how to minimize the number of required processors to obtain the optimal running time. We have two cases to study, the trees having all leaves at the same depth and the others. We fully treat the first case and we make a few observations regarding the second type of tree, which we intuitively sense to further reduce the number of required processors.

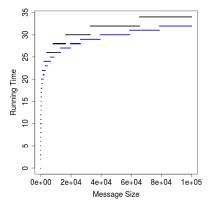
At the outset, one may be interested in the maximum possible number of levels of arity 5 or 4. We have the following Lemma:

Lemma 1. In a tree having an optimal running time the following statements hold:

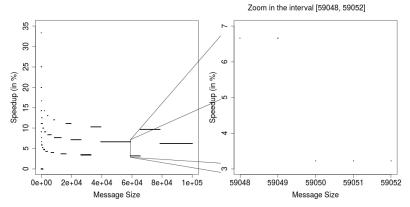
- There can be up to 1 level of arity 5;
- There can be up to 6 levels of arity 4.

Proof. We prove these two assertions separately:

- Suppose that the tree has 2 levels of arity 5. We replace these 2 levels by 3 levels of arity 3 since $3^3 = 27 > 5^2 = 25$. The running time is improved since $3 \cdot 3 = 9 < 2 \cdot 5 = 10$. We can then state that 2 levels of arity 5 lead to a tree having a suboptimal running time.
- Now, let us look for a pair of minimum integers (i,j) satisfying $3^i \ge 4^j$ and $3 \cdot i < 4 \cdot j$. The first pair which satisfies these constraints is i=9 and j=7. We can then replace 7 levels of arity 4 by 9 levels of arity 3 in order to decrease the running time.



(a) Running time of an optimal tree (shown in blue) compared to a binary tree (in black)



(b) Speed gain of an optimal tree compared to a binary tree

Fig. 5. Performance comparison between an optimal tree and a binary tree

We have seen that it is possible to construct a tree optimizing the running time by using only levels of arity 2 and 3. In what follows, we show how to deduce an optimal tree minimizing the number of involved processors. Let us suppose that level arities $x_1, x_2 ..., x_h$ are noted in (no strictly) decreasing order so that x_1 is the arity of the base level and x_h the arity of the last level, *i.e.* the arity of the root node. The trees optimizing the running time, defined above, are not necessarily full in the sense that a rightmost node at a given level can be of arity strictly lower than the arity of this level. First, we note that for the trees constructed with Algorithm 1, the number of required processors is equal to $\lceil l/3 \rceil$ in the best case, and equal to $\lceil l/2 \rceil$ when there are only levels of arity 2. Moreover, according to Theorem 1 we know that a level arity cannot be greater than 5. This means that in the best case, after optimization, the number of

required processors could be reduced to $\lceil l/5 \rceil$. Thus, we could in the best case decrease the number of processors by a factor of about 5/2.

Given an optimal tree for the running time, the intent is to increase the arity of the first level (base level) while decreasing arities of the following levels so that the sum of the level arities remains constant and their product remains greater than or equal to l. To solve this problem we propose in Appendix B two solutions (Algorithm 2a or 2b). However, as will be discussed below, we can further optimize hash trees.

According to Theorem 1, a level arity of a tree minimizing the running time cannot exceed 5. Thus, Algorithm 2a (or Algorithm 2b) of Appendix B allows us to substitute any sub-multiset A for another one, denoted A', where the sum of arities remains the same, and by trying to increase the arity of the base level up to 5. Consider, for instance, a message of size l=95 blocks. With such a message size, Algorithm 1 returns the multiset of arities $A_0=\{3,3,3,2,2\}$ which defines a tree structure involving 32 processors. By applying Algorithm 2, we obtain the multiset $A_1=\{4,3,3,3\}$ which reduces the number of involved processors to 24 while leaving the running time unchanged.

What if the arity of each level is increased? As much as possible? We just saw that we can increase the arity of the first level. It would also be preferable to increase the arity of each level of the tree in order to free up the highest number of processors at each step of the computation. An example is depicted in Figure 6.

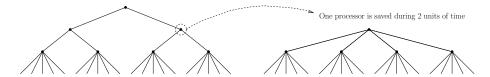


Fig. 6. Two trees compressing a 20-block message, optimized both for the running time and the number of involved processors. Both trees require 4 processors. Nevertheless, we note that the right tree is the best choice. Indeed, the one on the left needs 4 processors during 5 units of time, then 2 processors during 2 units of time, and finally one processor during 2 units of time. The one on the right needs 4 processors during 5 units of time and then one processor during 4 units of time.

While we propose an iterative algorithm in Appendix B to construct an optimal tree maximizing the arity of each level, we also enumerate all possible cases in the following Theorem:

Theorem 3. For any integer $l \ge 32$ there is an unique ordered multiset A of h_5 arities 5, h_4 arities 4, h_3 arities 3 and h_2 arities 2 such that the corresponding tree covers a message size l, has a minimal running time and has first h_5 as large as possible, then h_4 as large as possible, and then h_3 as large as possible. More precisely, if i is the lowest

integer such that $l \leq 3^i < 3l$, this ordered multiset is such that:

$$\begin{cases} |A|=i, h_5=0, h_4=0, h_3=i, h_2=0 & \text{if} \quad l \leq 3^i < \frac{9l}{8}, \\ |A|=i, h_5=0, h_4=1, h_3=i-2, h_2=1 & \text{if} \quad \frac{9l}{8} \leq 3^i < \frac{81l}{64}, \\ |A|=i-1, h_5=0, h_4=3, h_3=i-4, h_2=0 & \text{if} \quad \frac{9l}{64} \leq 3^i < \frac{27l}{20}, \\ |A|=i-1, h_5=1, h_4=1, h_3=i-3, h_2=0 & \text{if} \quad \frac{27l}{20} \leq 3^i < \frac{3}{2}, \\ |A|=i, h_5=0, h_4=0, h_3=i-1, h_2=1 & \text{if} \quad \frac{3l}{2} \leq 3^i < \frac{27l}{16}, \\ |A|=i-1, h_5=0, h_4=2, h_3=i-3, h_2=0 & \text{if} \quad \frac{27l}{16} \leq 3^i < \frac{9l}{5}, \\ |A|=i-1, h_5=1, h_4=0, h_3=i-2, h_2=0 & \text{if} \quad \frac{9l}{16} \leq 3^i < \frac{8ll}{40}, \\ |A|=i-1, h_5=1, h_4=1, h_3=i-4, h_2=1 & \text{if} \quad \frac{8ll}{40} \leq 3^i < \frac{9l}{4}, \\ |A|=i-1, h_5=0, h_4=1, h_3=i-2, h_2=0 & \text{if} \quad \frac{9l}{4} \leq 3^i < \frac{8ll}{32}, \\ |A|=i-1, h_5=0, h_4=2, h_3=i-4, h_2=1 & \text{if} \quad \frac{8ll}{32} \leq 3^i < \frac{27l}{10}, \\ |A|=i-1, h_5=0, h_4=2, h_3=i-4, h_2=1 & \text{if} \quad \frac{8ll}{32} \leq 3^i < \frac{27l}{10}, \\ |A|=i-1, h_5=1, h_4=0, h_3=i-3, h_2=1 & \text{if} \quad \frac{27l}{10} \leq 3^i < 3l, \end{cases}$$

where the number h_3 is at least 1 in the first case and can be 0 in the other cases.

Proof. Let us start from the 3 cases of Theorem 2 which maximize the number of levels of arity 3. For a given message length l, we consider the corresponding optimal tree (in the sense of the running time). We denote by a the initial number of levels of arity 2 and by i-a the initial (maximized) number of levels of arity 3. We want to transform this tree in order to increase the arity of each level as much as possible, while leaving the running time unchanged. According to Lemma 1, there can be one level of arity 5 and up to six levels of arity 4. Since we want to maximize the number of levels of arity 4 after having maximized the number of levels of arity 5, there cannot be more than one level of arity 2. Thus, $h_5 \in [0,1]$, $h_4 \in [0,6]$ and $h_2 \in [0,1]$, meaning there shall be at most 28 cases. Note that among these 28 cases, many may not be valid solutions. The aim is to transform the initial product $2^a 3^{i-a}$ into a product $2^w 3^{i-a-b} 4^v 5^u$ where a and b are respectively the number of levels of arity 2 and the number of levels of arity 3 that we have transformed, and u, v, w the number of levels of arity 5, 4, 2 respectively. For each triple $(h_5 = u, h_4 = v, h_2 = w)$ with $u \in [0, 1], v \in [0, 6]$ and $w \in [0, 1],$ we can verify that there is a solution (a, b) with a an integer in [0, 2] and b a positive integer such that 3b + 2a = 5u + 4v + 2w. We remark that 3b + 2a can be rewritten 3b if a=0, 3(b+1)+1 if a=2, and 3b+2 if a=1. Thus, all but one integers are produced. We can also verify that this solution is unique. Indeed, let us suppose a second solution (a', b'). Since 3b + 2a = 3b' + 2a', we have 3(b' - b) = 2(a' - a), meaning that 3 divides (a'-a). This is impossible, unless a'=a. Such a solution must satisfy $3^{i-a-b}2^w4^v5^u \ge l$, that is

$$3^i \ge \frac{3^{a+b}l}{2^w 4^v 5^u}. (2)$$

According to Theorem 2, we have $(3/2)^a l \leq 3^i < \min(3l, (3/2)^{a+1}l)$. Consequently, if we have

$$\frac{3^{a+b}l}{2^w4^v5^u} \geq \min\left(3l, \left(\frac{3}{2}\right)^{a+1}l\right),$$
 where
$$\min\left(3l, \left(\frac{3}{2}\right)^{a+1}l\right) = \begin{cases}3l & \text{if } a=2\\ (3/2)^{a+1}l & \text{if } a=0,1\end{cases},$$

this solution does not meet the constraint (2), and then cannot exist. Among the 28 cases, we observe that 13 of them are not valid solutions. Thus, we have 15 solutions, denoted (u,v,w,a,b), for which we compute and sort the values $3^{a+b}l/(2^w4^v5^u)$ so that we can establish their domains of validity. We then obtain the fifteen following cases:

$$\text{(II)} \begin{cases} |A| = i, h_5 = 0, h_4 = 0, h_3 = i, h_2 = 0 & \text{if} \quad l \leq 3^i < \frac{3l}{2}, \\ |A| = i, h_5 = 0, h_4 = 1, h_3 = i - 2, h_2 = 1 & \text{if} \quad \frac{9l}{8} \leq 3^i < \frac{3l}{2}, \\ |A| = i - 1, h_5 = 0, h_4 = 3, h_3 = i - 4, h_2 = 0 & \text{if} \quad \frac{81l}{64} \leq 3^i < \frac{3l}{2}, \\ |A| = i - 1, h_5 = 1, h_4 = 1, h_3 = i - 3, h_2 = 0 & \text{if} \quad \frac{27l}{20} \leq 3^i < \frac{3l}{2}, \\ |A| = i - 1, h_5 = 0, h_4 = 4, h_3 = i - 6, h_2 = 1 & \text{if} \quad \frac{729l}{512} \leq 3^i < \frac{3l}{2}, \\ |A| = i - 1, h_5 = 0, h_4 = 2, h_3 = i - 3, h_2 = 0 & \text{if} \quad \frac{27l}{16} \leq 3^i < \frac{9l}{4}, \\ |A| = i - 1, h_5 = 0, h_4 = 2, h_3 = i - 3, h_2 = 0 & \text{if} \quad \frac{27l}{16} \leq 3^i < \frac{9l}{4}, \\ |A| = i - 1, h_5 = 1, h_4 = 0, h_3 = i - 2, h_2 = 0 & \text{if} \quad \frac{9l}{5} \leq 3^i < \frac{9l}{4}, \\ |A| = i - 1, h_5 = 0, h_4 = 3, h_3 = i - 5, h_2 = 1 & \text{if} \quad \frac{243l}{40} \leq 3^i < \frac{9l}{4}, \\ |A| = i - 1, h_5 = 1, h_4 = 1, h_3 = i - 4, h_2 = 1 & \text{if} \quad \frac{8ll}{40} \leq 3^i < \frac{9l}{4}, \\ |A| = i - 2, h_5 = 0, h_4 = 5, h_3 = i - 7, h_2 = 0 & \text{if} \quad \frac{9l}{4} \leq 3^i < 3l, \\ |A| = i - 1, h_5 = 0, h_4 = 2, h_3 = i - 4, h_2 = 1 & \text{if} \quad \frac{8ll}{32} \leq 3^i < 3l, \\ |A| = i - 1, h_5 = 0, h_4 = 2, h_3 = i - 4, h_2 = 1 & \text{if} \quad \frac{8ll}{32} \leq 3^i < 3l, \\ |A| = i - 1, h_5 = 0, h_4 = 2, h_3 = i - 4, h_2 = 1 & \text{if} \quad \frac{8ll}{32} \leq 3^i < 3l, \\ |A| = i - 1, h_5 = 0, h_4 = 2, h_3 = i - 3, h_2 = 1 & \text{if} \quad \frac{27l}{10} \leq 3^i < 3l, \\ |A| = i - 1, h_5 = 0, h_4 = 4, h_3 = i - 6, h_2 = 0 & \text{if} \quad \frac{729l}{10} \leq 3^i < 3l, \\ |A| = i - 1, h_5 = 0, h_4 = 4, h_3 = i - 6, h_2 = 0 & \text{if} \quad \frac{729l}{10} \leq 3^i < 3l, \\ |A| = i - 1, h_5 = 0, h_4 = 4, h_3 = i - 6, h_2 = 0 & \text{if} \quad \frac{729l}{10} \leq 3^i < 3l, \\ |A| = i - 1, h_5 = 0, h_4 = 4, h_3 = i - 6, h_2 = 0 & \text{if} \quad \frac{729l}{10} \leq 3^i < 3l, \\ |A| = i - 1, h_5 = 0, h_4 = 4, h_3 = i - 6, h_2 = 0 & \text{if} \quad \frac{729l}{100} \leq 3^i < 3l, \\ |A| = i - 1, h_5 = 0, h_4 = 4, h_3 = i - 6, h_2 = 0 & \text{if} \quad \frac{729l}{100} \leq 3^i < 3l, \\ |A| = i - 1, h_5 = 0, h_4 = 4, h_3 = i - 6, h_2 = 0 & \text{if} \quad \frac{729l}{1000} \leq 3^i < 3l, \\ |A| = i - 1, h_5 = 0, h_4 = 4, h_3 = i - 6, h_2 = 0 & \text{if} \quad \frac{729l}{10$$

In accordance with Theorem 2, we have this grouping:

- Group I consists of the five cases ensuring a running time of 3i;
- Group II consists of the six cases ensuring a running time of 3i 1;
- Group III consists of the four cases ensuring a running time of 3i-2.

Now, we have to optimize the arities. In the first group, we delete the last case since it decreases h_5 compared to the immediately preceding case. For the same reasons, we delete in the second group the fourth and sixth cases. Again, in the last group, we need to delete the last case. Overall, 11 cases are deduced by intersecting the intervals of validity.

Remark 1. The number of cases is lower when l < 32. Their determinations are let to the reader.

Remark 2. Let us consider a tree which is optimal in the sense of the theorem 3. If we extract a final subtree by deleting one or several lower levels (at the bottom), the resulting tree is still optimal. Indeed, let us suppose that the original tree has height h and has l leaves (for l message blocks). If we delete the j lower levels, the resulting tree has $l' = \lceil l/(x_1x_2\cdots x_j) \rceil$ leaves and is already optimal for a number l' of blocks. If this form of local optimality does not exist, we can further optimize the original tree. Indeed, let us suppose that a $(x'_{j+1}, x'_{j+2}, \ldots, x'_{h'})$ -aries tree covers the number of blocks l' and improves either the running time or the number of processors (in the sense of Theorem 3), compared to the $(x_{j+1}, x_{j+2}, \ldots, x_h)$ -aries tree. This means that a $(x_1, \ldots, x_j, x'_{j+1}, x'_{j+2}, \ldots, x'_{h'})$ -aries tree is a better choice to process l leaves.

Let us now consider trees having a number of leaves equal to the message length. Having a multiset of arities arranged in descending order, that we denote $A = \{x_1, x_2, \dots, x_{|A|}\}$, the number of nodes of level i is $\lceil l/(x_1x_2\dots x_i) \rceil$. One important thing is the number of nodes of the base level. We have the following Corollary:

Corollary 1. Let the message size be $l \ge 2$ and let i be the lowest integer such that $3^i \ge l$. The number of processors required to process such a message is:

```
 \begin{array}{l} - \lceil l/3 \rceil \ if \ 3^i \in [l, \frac{9l}{8} \lfloor \cup \lfloor \frac{3l}{2}, \frac{27l}{16} \rfloor, \\ - \lceil l/4 \rceil \ if \ 3^i \in [\frac{9l}{8}, \frac{27l}{20} \lfloor \cup \lfloor \frac{27l}{16}, \frac{9l}{5} \lfloor \cup \lfloor \frac{9l}{4}, \frac{27l}{10} \rfloor, \\ - \lceil l/5 \rceil \ if \ 3^i \in [\frac{27l}{20}, \frac{3l}{2} \lfloor \cup \lfloor \frac{9l}{5}, \frac{9l}{4} \lfloor \cup \lfloor \frac{27l}{10}, 3l \rfloor. \end{array}
```

Proof. These results follow immediately from Theorem 3.

An other important thing is the minimization of the *total work* done by the hash tree algorithm for the processing of a single message. Since we are interested in hash trees having an optimal running time for a given message size l, we apply Theorem 2 or 3 to retrieve a topology. For a perfect (x_1, x_2, \ldots, x_h) -aries tree constructed thanks to this theorem, the *total work* is:

$$W_l = x_h + x_h x_{h-1} + x_h x_{h-1} x_{h-2} + \dots + x_h x_{h-1} \dots x_2 x_1.$$

We notice that $\lceil \lceil \cdots \lceil \lceil l/x_1 \rceil / x_2 \rceil \cdots \rceil / x_i \rceil = \lceil l/(x_1 x_2 \cdots x_i) \rceil$ for (strictly) positive integers $(x_j)_{j=1...i}$. Consequently, for a (x_1, x_2, \ldots, x_h) -aries truncated tree constructed thanks to this theorem, the *total work* is:

$$W_{tr,l} = l + \lceil l/x_1 \rceil + \lceil l/(x_1x_2) \rceil + \dots + \lceil l/(x_1x_2 \dots x_{h-1}) \rceil.$$

This quantity is necessarily greater than or equal to l. Regarding truncated trees minimizing the running time, Theorem 3 indicates a topology which minimizes the *total* work, by first choosing x_1 as large as possible, then choosing x_2 as large as possible, and so on.

Remark 3. Decreasing the total work consists in decreasing as much as possible the number of nodes (apart from the number of leaves). We have to check if the multisets provided by Theorem 3 are preferable to others. Let us suppose that, for a given message size l, we have a (non-ordered) multiset minimizing the running time. Let us also suppose that at least one order of this multiset minimizes the total work. We can show that, among all the possible orders, this is the one represented in decreasing order which minimizes the total work. We denote such a solution by (x_1, x_2, \ldots, x_h) with $x_i \geq x_{i+1}$. Indeed, for a given message size l and for any random permutation π of the indices $[\![1,h]\!]$, we have $x_1x_2\ldots x_i \geq x_{\pi(1)}x_{\pi(2)}\ldots x_{\pi(i)}$ for all $i \leq h$. Thus, summing left sides and right sides from the h inequalities, we have $\sum_{j=1}^i \lceil l/(x_1x_2\ldots x_j) \rceil \leq \sum_{j=1}^i \lceil l/(x_{\pi(1)}x_{\pi(2)}\ldots x_{\pi(j)}) \rceil$. When this ordered multiset cannot be derived from Theorem 3, we show that the transformations performed in its proof can further decrease the total work. We recall that the composition of five types of transformation

can lead to the eleven cases of this theorem. These transformations change the following pairs of arities (3,3), (2,2), (3,2), (4,3) and (4,4) into (4,2), (4), (5), (5,2) and (5, 3) respectively. It is sufficient to show that each of these transformations can reduce the total work:

- Case $(2,2) \rightarrow (4)$: We indeed have $\lceil l/4 \rceil \leq \lceil l/2 \rceil + \lceil l/(2 \cdot 2) \rceil$. Case $(3,2) \rightarrow (5)$: We indeed have $\lceil l/5 \rceil \leq \lceil l/3 \rceil + \lceil l/(3 \cdot 2) \rceil$.
- Case $(3,3) \to (4,2)$: Since $\lceil l/4 \rceil + \lceil l/8 \rceil \le l/4 + l/8 + 2$ and $l/3 + l/9 \le l/4 + l/8 + 2$ $\lceil l/3 \rceil + \lceil l/(3 \cdot 3) \rceil$, we can show that $\lceil l/4 \rceil + \lceil l/(4 \cdot 2) \rceil \leq \lceil l/3 \rceil + \lceil l/(3 \cdot 3) \rceil$ for $l \geq 29$. As regards the other values of l, it appears that this inequality does not hold for l = 9, but we remark that this message length is not concerned by such a transformation.
- Case $(4,3) \to (5,2)$: By the same reasoning, we can show that $\lceil l/5 \rceil + \lceil l/(5 \cdot 2) \rceil \le$ $\lceil l/4 \rceil + \lceil l/(4 \cdot 3) \rceil$ for $l \neq 11$ and $l \neq 12$. We then remark that a message of length l = 11 or 12 blocks cannot be covered by a (5, 2)-aries tree.
- Case $(4,4) \rightarrow (5,3)$: Again, we can show that $\lceil l/5 \rceil + \lceil l/(5\cdot 3) \rceil \leq \lceil l/4 \rceil + \lceil l/(5\cdot 3) \rceil$ $\lceil l/(4\cdot 4) \rceil$ for $l \neq 16$. We remark that a length of l=16 cannot be covered by a (5,3)-aries tree.

About the distribution of cases

For the purpose of minimizing the number of processors at each step of the computation, we apply Theorem 3. There are 11 possible cases and we would like to estimate their distribution. We then perform an empirical estimation with a Monte Carlo simulation. Numerous papers have analysed the distribution of file sizes, and more particularly the sizes of files transferred accross a network. It is shown that the average size of transferred files is about 10KB [33,35,6], and we observe that the Pareto distribution is the predominantly used model [6,15,35,21] to fit the datasets, with a *shape* parameter ρ generally estimated between 1 and 2, and a location parameter ν estimated by hand or as a function of the average file size (using the mean formula). For this simulation, we use the Pareto model with $\rho = 1.5$ and $\nu = 10^4 \cdot (\rho - 1)/\rho$ bytes in order to generate a sample of 10^6 byte sizes. We initialize 11 counters $cpt_i = 0$ for $i = 1, \dots, 11$. With the assumption of a 32-byte block size, we perform the following operations for each generated byte-size:

- 1. We compute $v = 3^{\lceil \log(\lceil \text{byte-size}/32 \rceil)/\log 3 \rceil}$;
- 7. We compute v=5 . We check which one of the following cases is satisfied: Case 1: $l \leq v < \frac{9l}{8}$; Case 2: $\frac{9l}{8} \leq v < \frac{81l}{64}$; Case 3: $\frac{81l}{64} \leq v < \frac{27l}{20}$; Case 4: $\frac{27l}{20} \leq v < \frac{3l}{2}$; Case 5: $\frac{3l}{2} \leq v < \frac{27l}{16}$; Case 6: $\frac{27l}{16} \leq v < \frac{9l}{5}$; Case 7: $\frac{9l}{5} \leq v < \frac{81l}{40}$; Case 8: $\frac{81l}{40} \leq v < \frac{9l}{4}$; Case 9: $\frac{9l}{4} \leq v < \frac{81l}{32}$; Case 10: $\frac{81l}{32} \leq v < \frac{27l}{10}$; Case 11: $\frac{27l}{10} \leq v < 3l$; The index of the satisfied case is denoted j;
- 3. We increment cpt_i .

The relative frequency of the *i*-th case is then $cpt_i/10^6$ for $i=1,\ldots,11$. These estimations are made using R software [28] with the VGAM library [37]. According to Corollary 1, it follows that the number of required processors is $\lceil l/3 \rceil$ with probability of about 19.1%, $\lceil l/4 \rceil$ with probability of about 33.1%, and $\lceil l/5 \rceil$ with probability of about 47.8%. The relative proportions of each individual case are depicted in Figure 7.

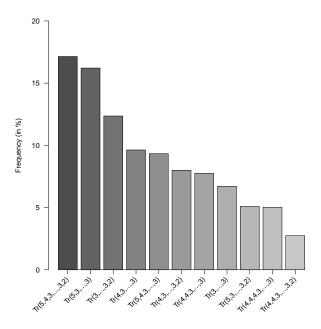


Fig. 7. Proportions for the eleven cases of Theorem 3. The bars are drawn in decreasing order of frequency. The notation $Tr(a_1,a_2,...,a_h)$ stands for a tree having arities $a_1, a_2, ..., a_h$ from the base level to the root node.

5 Applying our optimizations safely

Suppose that we have 4 different inner functions f_{BL} , f_I , f_F and f_{SN} with the following properties:

- $f_t: \{0,1\}^{qN} \to \{0,1\}^N$ for $q \ge 2$ and $t \in \{BL, I, F, SN\}$.
- Without precomputation, they have the same running time of q+2 units of time when compressing q message blocks of size N bits.
- With a constant number of precomputations, they have the same running time of q
 units of time when compressing q message blocks.
- They behave like independent random oracles.

In the hash tree construction we propose, we use f_{BL} to compute base level nodes, f_I to compute inner nodes, f_F to compute the root node. If the tree is of height one, there is only one node computed using f_{SN} . In order to simulate four independent functions, we use the same inner function f but with domain separation. Indeed, since f behaves like a random oracle, by construction the functions $f_{BL}(x) = f(100^{N-3} || x)$, $f_I(x) = f(000^{N-3} || x)$, $f_F(x) = f(010^{N-3} || x)$ and $f_{SN}(x) = f(110^{N-3} || x)$ behave like independent random oracles.

Since f is based on the *prefix-free* MD construction, the first block encodes the number of blocks of the message, comprised between 2 and 5. Due to the domain separation codings, the second block contains the bitstring $b_1b_20^{N-3}m_{11}$ where the 1-bit

values b_1 and b_2 depend on the type of processed input, and m_{11} is the first bit of the input. Overall, we have to precompute 16 hash states (resulting from the processing of these two blocks) in order to obtain an inner function having a running time equals to the node arity.

On the precomputations costs induced by the use of a sponge function. The first block merely consists of the bitstring $b_1b_20^{N-4}m_{11}m_{12}$ where the 1-bit values b_1 and b_2 are as above, and m_{11} and m_{22} are the first two bits of the message input. Again, we have to precompute 32 hash states (resulting from the processing of only one block) in order to obtain an inner function having a running time equal to the node arity.

5.1 An example of hash function

Given a message M, a hash tree mode could be the following:

- 1. Whatever the message bit-length is, we append to M a bit "1" and the minimum number of bits "0" so that the total bit-length is a multiple of N. The new message is denoted M_0 and its total number of blocks of size N bits is $l = |M_0|_2/N$, where $|M_0|_2$ is the bitlength of M_0 .
- 2. We apply Theorem 3 on l to retrieve the height h of the tree and an ordered multiset A of arities $a_1, a_2, ..., a_h$ (arranged in decreasing order). This step requires the computation of $\log_3 l$, which can be considered free.
- 3. If h = 1, we compute and return the hash value $f_{SN}(M_0)$. Otherwise, we go to the following step.
- 4. We first split M_0 into blocks $M_{0,1}$, $M_{0,2}$, ..., M_{0,l_1} where: (i) $l_1 = \lceil l/a_1 \rceil$; (ii) all blocks but the last one are a_1N bits long and the last block is between N and a_1N bits long. Then, we compute the message

$$M_{1}:=\prod_{j=1}^{l_{1}}f_{BL}\left(M_{0,j}\right) .$$

- 5. If h=2 we go to step 6. Otherwise, for k=2 to h-1, we perform the following operations:
 - (a) We split M_{k-1} into blocks $M_{k-1,1}$, $M_{k-1,2}$, ..., M_{k-1,l_k} where: (i) $l_k = \left\lceil l/\left(\prod_{j=1}^k a_j\right)\right\rceil$; (ii) all blocks but the last one are a_kN bits long and the last block is between N and a_kN bits long.
 - (b) We compute the message

$$M_{k} := \prod_{j=1}^{l_{k}} f_{I}\left(M_{k-1,j}\right).$$

6. We compute and return the hash value $f_F(M_{h-1})$.

5.2 Security

Since our mode can be rewritten as if it was using only f, it suffices to check whether the three conditions (seen in Section 2.2) are satisfied to prove soundness.

We first need to describe some rules regarding our mode:

Rule 0. The root f-input has a prepended code 010^{N-3} or 110^{N-3} .

Rule 1. A f-input with a prepended code 010^{N-3} has children having a prepended code 0^{N-1} or 10^{N-2} .

Rule 2. A f-input with a prepended code 10^{N-2} or 110^{N-3} has no children.

Rule 3. A f-input with a prepended code 00 has children having a prepended code 10^{N-2} or 0^{N-1} .

Rule A. A f-input must be (2 + kN)-bit long with an integer $k \ge 1$.

Rule B. At a same level of the tree, the number of chaining values is the same for all the *f*-inputs, except for the righmost one where this number may be smaller.

Rule C. At a same level of the tree, prepended codes are the same for all the f-inputs.

Note that the satisfaction of the rules 0, 1, 2, 3 and C imply that the leaves are at the same depth. So, we do not need to define a rule to express this.

By construction our mode is final-f-input separable. Our mode is trivially messagecomplete since it processes all message bits. Indeed, having the valuated tree of f-inputs produced by τ , the algorithm $A_{message}$ reaches directly the base level f-inputs and recovers message blocks by discarding the frame bits, whether they serve as padding purpose (in the rightmost f-input) or for identifying the type of f-input. This algorithm runs in linear time in the number of bits in the tree. Finally, our mode is also tree-decodable. Thanks to domain separation between base level f-inputs and other finputs, we cannot find a tree which is both *compliant* and *final-subtree-compliant*. Given only one f-input, the prepended coding allows its content to be recognized correctly. We can then construct a decoding algorithm A_{decode} which runs in 2 phases: Phase 1 starts from the root f-input and fully determines the tree structure by recursively decoding each f-input. The size of a f-input determines the number of its children. This phase terminates with the "correct" state C0 if all the visited f-inputs respect the rules defined above. This phase terminates with the "incorrect" state C1 if one of the rule 0, 2, A, B or C is not respected. If Rule 1 is not respected it terminates with the "incorrect" state C2. Otherwise, it terminates with the "incorrect" state C3. Phase 2 examines the properties of the decoded tree by taking into account the termination state of the first phase. The details of Phase 2 are the followings:

- 1. If the state is C0, it runs the $A_{message}$ algorithm in order to check the message size. If for the corresponding number l of blocks, Theorem 3 indicates a topology which differs from the one that Phase 1 has just decoded, then it returns "incompliant". Otherwise, it returns "compliant".
- 2. If the state is C1, it returns "incompliant".
- 3. If the state is C2, the following examinations are made:
 - (a) If the tree seems incomplete with a single f-input, it checks, after having discarded the prepended code, the number of blocks of size N bits. If there are 2, 3 or 4 blocks, then it returns "final-subtree-compliant". Otherwise, it returns "incompliant".
 - (b) Otherwise, the coding is incompatible with the mode, and it returns "incompliant".

- 4. If the state is C3, this means that only the rule 3 is not respected. The following examinations are then made:
 - (a) If there is a f-input with a prepended code 00 which has a child with a prepended code not equal to 10, then it returns "incompliant".
 - (b) Otherwise, there is at least one missing f-input. We note h the maximum number of f-inputs visited on a path in this proper final subtree. The algorithm has to check if its topology is consistent with Theorem⁷ 3. First, it establishes a system of contraints regarding the arity of each level. If there is only one f-input at a level which is not the root, and if it is the righmost one⁸ in the complete tree, then the number of (message or chaining) blocks it contains defines a lower bound for the arity of this level. Otherwise, the arity for this level has a single possible value and the constraint is an equality. Having these h constraints, it checks in Theorem 3 which cases (among the eleven) can satisfy these contraints. We denote by L the list of compatible cases and for each case L in L we denote by L the arity of the level L in L in L, the algorithm performs the following operations:
 - i. It completes this final subtree until its leaves are at the same depth h. It performs this by (virtually) creating the missing f-inputs with a maximal arity (*i.e.*, even if a missing f-input at level i is the rightmost one in the complete tree, it chooses its arity to be exactly a_i).
 - ii. It counts the number l of blocks covered by the completed subtree. If l is in the domain of validity of the case j, then it returns "final-subtree-compliant".

A this point, no cases in L are suitable. This phase 2 finally returns "incompliant".

The total running time of A_{decode} is linear in the number of bits in the tree.

Remark 4. If prepending 2 bits is sufficient [10] for soundness, we could have used a reduction to the Sakura coding [11] where meta-information bitstrings are longer. Using Sakura coding allows any tree-based hash function to be automatically indifferentiable from a random oracle, without the need of further proofs. According to the Sakura ABNF grammar [11], the number of chaining values (i.e. the number of children of a f-input) is also coded in the formatted input to f. In our context and using this coding, the number of ways to format the input to f (depending on its location in the tree topology) is at least 10. A Sakura coding bit expresses the fact that the input is or is not the last one (for the computation of the root hash). Another bit indicates whether the input contains a message block or a certain number of chaining values (whose number, 2, 3, 4 or 5, is also coded at the beginning of the input). Overall, this corresponds to 10 ways to format an input in our tree topology. This larger encoding and the fact to have meta-information bits at the beginning (for prepended bits) and at the end of the input (for appended bits) both complicates our construction and increases the number of precomputed hash states.

⁷ Meaning that it has to detect if this final subtree can be extracted from an optimal tree, in the sense of this theorem.

⁸ A *f*-input is the righmost one in the complete tree if we see that it does not have a right sibling, when looking at the retrieved topology by Phase 1.

Remark 5. Yet another solution is to use different IVs (Initial Values) instead of particular frame bits, as suggested in [10,23]. We could use a *free-IV* hash function, like the *suffix-free-prefix-free* hash function from Bagheri *et al.* [5]. The distinction between the f-inputs would be done by using 4 distinct IVs: BL_IV for base level f-inputs, I_IV for inner f-inputs, F_IV for the root f-input and SN_IV for a tree reduced to a single f-input.

6 Conclusion

In this paper, we focused on trees having their leaves at the same depth. We have shown, for a given message length, how to construct a hash tree minimizing the running time. In particular, we have shown how to minimize the number of processors allowing such a running time. The proposed construction makes use of a prepended coding for each input to the inner function in order to satisfy the three conditions of Bertoni *et al.* [10]. Besides, our tree topologies could also be used in substitution of the tree hash mode of Skein, provided that the tweaks to the UBI mode are carefully chosen for each node.

References

- Gordon B. Agnew, Ronald C. Mullin, and Scott A. Vanstone. Fast exponentiation in GF(2ⁿ).
 In Advances in Cryptology EUROCRYPT '88, Workshop on the Theory and Application of of Cryptographic Techniques, Davos, Switzerland, May 25-27, 1988, Proceedings, pages 251–255, 1988.
- Elena Andreeva, Bart Mennink, and Bart Preneel. Security reductions of the second round SHA-3 candidates. Cryptology ePrint Archive, Report 2010/381, 2010.
- Elena Andreeva, Bart Mennink, and Bart Preneel. Security reductions of the second round SHA-3 candidates. In *Information Security - 13th International Conference, ISC 2010, Boca Raton, FL, USA, October 25-28, 2010, Revised Selected Papers*, pages 39–53, 2010.
- Jean-Philippe Aumasson, Samuel Neves, Zooko Wilcox-O'Hearn, and Christian Winnerlein. BLAKE2: Simpler, smaller, fast as MD5. In *Proceedings of the 11th International Conference on Applied Cryptography and Network Security*, ACNS'13, pages 119–135, Berlin, Heidelberg, 2013. Springer-Verlag.
- Nasour Bagheri, Praveen Gauravaram, Lars R. Knudsen, and Erik Zenner. The suffix-freeprefix-free hash function construction and its indifferentiability security analysis. *Interna*tional Journal of Information Security, 11(6):419–434, 2012.
- Paul Barford and Mark Crovella. Generating representative web workloads for network and server performance evaluation. In *Proceedings of the 1998 ACM SIGMETRICS Joint International Conference on Measurement and Modeling of Computer Systems*, SIGMETRICS '98/PERFORMANCE '98, pages 151–160, New York, NY, USA, 1998. ACM.
- Guido Bertoni, Joan Daemen, Michaël Peeters, and Gilles Van Assche. Keccak. In Advances in Cryptology EUROCRYPT 2013, 32nd Annual International Conference on the Theory and Applications of Cryptographic Techniques, Athens, Greece, May 26-30, 2013. Proceedings, pages 313–314, 2013.
- Guido Bertoni, Joan Daemen, Michaël Peeters, and Gilles Van Assche. On the indifferentiability of the sponge construction. In Proceedings of the Theory and Applications of Cryptographic Techniques 27th Annual International Conference on Advances in Cryptology, EUROCRYPT'08, pages 181–197, Berlin, Heidelberg, 2008. Springer-Verlag.

- Guido Bertoni, Joan Daemen, Michael Peeters, and Gilles Van Assche. Sufficient conditions for sound tree and sequential hashing modes. Cryptology ePrint Archive, Report 2009/210, 2009
- Guido Bertoni, Joan Daemen, Michaël Peeters, and Gilles Van Assche. Sufficient conditions for sound tree and sequential hashing modes. *Int. J. Inf. Secur.*, 13(4):335–353, August 2014.
- 11. Guido Bertoni, Joan Daemen, Michaël Peeters, and Gilles Van Assche. Sakura: A flexible coding for tree hashing. 8479:217–234, 2014.
- 12. Donghoon Chang, Sangjin Lee, Mridul Nandi, and Moti Yung. Indifferentiable security analysis of popular hash functions with prefix-free padding. In *Proceedings of the 12th International Conference on Theory and Application of Cryptology and Information Security*, ASIACRYPT'06, pages 283–298, Berlin, Heidelberg, 2006. Springer-Verlag.
- Jean-Sébastien Coron, Yevgeniy Dodis, Cécile Malinaud, and Prashant Puniya. Merkle-damgård revisited: How to construct a hash function. In Advances in Cryptology CRYPTO 2005: 25th Annual International Cryptology Conference, Santa Barbara, California, USA, August 14-18, 2005, Proceedings, pages 430–448, 2005.
- Ivan Damgård. A design principle for hash functions. In CRYPTO '89: Proceedings of the 9th Annual International Cryptology Conference on Advances in Cryptology, pages 416– 427, London, UK, 1990. Springer-Verlag.
- Allen B. Downey. Lognormal and Pareto distributions in the Internet. Comput. Commun., 28(7):790–801, May 2005.
- Niels Ferguson, Stefan Lucks Bauhaus, Bruce Schneier, Doug Whiting, Mihir Bellare, Tadayoshi Kohno, Jon Callas, and Jesse Walker. The skein hash function family (version 1.2), 2009.
- Alan Gibbons and Wojciech Rytter. Efficient parallel algorithms. Cambridge University Press, 1988.
- Shay Gueron. Parallelized hashing via j-lanes and j-pointers tree modes, with applications to SHA-256. IACR Cryptology ePrint Archive, 2014:170, 2014.
- 19. Shay Gueron and Vlad Krasnov. Parallelizing message schedules to accelerate the computations of hash functions. *J. Cryptographic Engineering*, 2(4):241–253, 2012.
- 20. Shay Gueron and Vlad Krasnov. Simultaneous hashing of multiple messages. *J. Information Security*, 3(4):319–325, 2012.
- 21. Mor Harchol-Balter, Bianca Schroeder, Nikhil Bansal, and Mukesh Agrawal. Size-based scheduling to improve web performance. *ACM Trans. Comput. Syst.*, 21(2):207–233, May 2003
- 22. Mun-Kyu Lee, Yoonjeong Kim, Kunsoo Park, and Yookun Cho. Efficient parallel exponentiation in gf(qn) using normal basis representations. *J. Algorithms*, 54(2):205–221, 2005.
- 23. Stefan Lucks. Tree hashing: A simple generic tree hashing mode designed for SHA-2 and SHA-3, applicable to other hash functions. In *Early Symmetric Crypto (ESC)*, 2013.
- 24. Ralph C. Merkle. Protocols for public key cryptosystems. In *Proceedings of the 1980 IEEE Symposium on Security and Privacy*, pages 122–134, 1980.
- Ralph Charles Merkle. Secrecy, Authentication, and Public Key Systems. PhD thesis, Stanford, CA, USA, 1979.
- Pinakpani Pal and Palash Sarkar. PARSHA-256- A new parallelizable hash function and a multithreaded implementation. In Fast Software Encryption, 10th International Workshop, FSE 2003, Lund, Sweden, February 24-26, 2003, Revised Papers, pages 347–361, 2003.
- 27. Bart Preneel, René Govaerts, and Joos Vandewalle. Hash functions based on block ciphers: A synthetic approach. In *Proceedings of the 13th Annual International Cryptology Conference on Advances in Cryptology*, CRYPTO '93, pages 368–378, London, UK, UK, 1994. Springer-Verlag.
- 28. R Core Team. *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria, 2014.

- 29. Ronald L. Rivest, Benjamin Agre, Daniel V. Bailey, Christopher Crutchfield, Yevgeniy Dodis, Kermin Elliott, Fleming Asif Khan, Jayant Krishnamurthy, Yuncheng Lin, Leo Reyzin, Emily Shen, Jim Sukha, Drew Sutherland, Eran Tromer, and Yiqun Lisa Yin. The MD6 hash function: A proposal to nist for sha-3, 2008.
- Palash Sarkar and Paul J. Schellenberg. A parallel algorithm for extending cryptographic hash functions. In *Progress in Cryptology - INDOCRYPT 2001, Second International Con*ference on Cryptology in India, Chennai, India, December 16-20, 2001, Proceedings, pages 40–49, 2001.
- 31. Palash Sarkar and Paul J. Schellenberg. A parallelizable design principle for cryptographic hash functions. *IACR Cryptology ePrint Archive*, 2002:31, 2002.
- 32. Douglas R. Stinson. Some observations on parallel algorithms for fast exponentiation in $GF(2^n)$. SIAM J. Comput., 19(4):711–717, 1990.
- 33. K. Thompson, G. J. Miller, and R. Wilder. Wide-area internet traffic patterns and characteristics. *Netwrk. Mag. of Global Internetwkg.*, 11(6):10–23, November 1997.
- 34. Joachim von zur Gathen. Efficient exponentiation in finite fields (extended abstract). In 32nd Annual Symposium on Foundations of Computer Science, San Juan, Puerto Rico, 1-4 October 1991, pages 384–391, 1991.
- 35. Adepele Williams, Martin Arlitt, Carey Williamson, and Ken Barker. Web Workload Characterization: Ten Years Later, pages 3–21. Springer US, Boston, MA, 2005.
- Chia-Long Wu, Der-Chyuan Lou, Jui-Chang Lai, and Te-Jen Chang. Fast parallel exponentiation algorithm for RSA public-key cryptosystem. *Informatica, Lith. Acad. Sci.*, 17(3):445–462, 2006.
- 37. Thomas W. Yee. The VGAM package for categorical data analysis. *Journal of Statistical Software*, 32(10):1–34, 2010.

A Comparison between a perfect binary tree and a perfect ternary tree

Let $l \geq 2$ an integer. Let h_2 the lowest integer such that $2^{h_2} \geq l$ and h_3 the lowest integer such that $3^{h_3} \geq l$. We assume that we use a perfect binary (or ternary) tree as in the original Merkle (and Damgård) hash tree mode, *i.e.* the message is padded to obtain a message size which is a power of 2 (or 3). The problem is to compare $2h_2$ and $3h_3$.

Any l can be uniquely written

$$l = 2^k + u,$$

where u is an integer such that $0 \le u < 2^k$. Then

$$l = 2^k (1+a)$$
 where $a = \frac{u}{2^k}$.

If a = 0 then $h_2 = k$ else $h_2 = k + 1$.

A.1 The case a = 0

In this case

$$l = 2^k, h_2 = k, h_3 = \left\lceil \frac{k \log(2)}{\log(3)} \right\rceil.$$

Then

$$h_3 = \frac{k \log(2)}{\log(3)} + \alpha,$$

where $0 < \alpha < 1$. We must compare $3h_3$ with $2h_2$, namely

$$3\frac{k\log(2)}{\log(3)} + 3\alpha \text{ with } 2k,$$

or

$$3\frac{\log(2)}{\log(3)} + 3\frac{\alpha}{k} \text{ with } 2.$$

As α is bounded by 1 and $3\frac{\log(2)}{\log(3)} < 2$, for k sufficiently large we have $3h_3 < 2h_2$.

More precisely if $k \ge 28$ then $3h_3 < 2h_2$, meaning that a perfect ternary tree gives a better running time than a perfect binary tree. When $2 \le k \le 27$, we compute the 27 values

$$T = \frac{3}{k} \left\lceil \frac{k \log(2)}{\log(3)} \right\rceil - 2$$

and we look at the sign of the result:

- For k=3s $(s=1,\cdots,9)$, a perfect binary tree and a perfect ternary tree give the same result (T=0).
- For k = 11, 14, 17, 19, 20, 22, 23, 25, 26, a perfect ternary tree is better (T < 0).
- For k = 2, 4, 5, 7, 8, 10, 13, 16, a perfect binary tree is better (T > 0).

A.2 The case $a \neq 0$

In this case $h_2 = k + 1$ and

$$h_3 = \left\lceil \frac{k \log(2)}{\log(3)} + \frac{\log(1+a)}{\log(3)} \right\rceil.$$

We must compare $3h_3$ to $2h_2$. But:

$$\frac{3h_3}{k} \le \frac{3\log(2)}{\log(3)} + \frac{3\log(2)}{k\log(3)} + \frac{3}{k}$$

and

$$\frac{2h_2}{k} = 2 + \frac{2}{k}.$$

As $\frac{3\log(2)}{\log(3)} < 2$, for k sufficiently large we have $3h_3 < 2h_2$. More precisely for $k \geq 27$ then $3h_3 < 2h_2$, meaning that a perfect ternary tree gives a better running time than a perfect binary tree. For any $2 \leq k \leq 26$ and any u such that $1 \leq u < 2^k$ we must compute the sign of

$$R = 3 \left[\frac{k \log(2)}{\log(3)} + \frac{\log(1 + \frac{u}{2^k})}{\log(3)} \right] - 2k - 2.$$

As R is an increasing function of u, it is sufficient to determine for any k < 27 the value of u where the sign changes. This can be done by dichotomy. Results are in Table 1.

B Algorithms for reducing the number of processors

B.1 Reducing the number of processors at the base level

We propose two (different) algorithms to construct an optimal tree (in the sense of the running time) which covers exactly l blocks (the tree is not necessarily perfect) and increases as much as possible the arity of the base level. The first solution consists to check if there exists an optimal tree having a level of arity 5 or 4.

Algorithm 2a.

INPUTS: a message length l and a multiset of arities (arranged in descending order) minimizing the running time, denoted $A = \{x_1, x_2, ..., x_{|A|}\}$.

OUTPUT: a multiset of arities (still sorted in descending order) minimizing the number of processors while leaving unchanged the running time.

Let t_l the optimal running time for a message of size l, *i.e.* the sum of arities of A. The algorithm proceeds as follows:

- 1. Use Algorithm 1 to construct a tree for a message length $l' = \lceil l/5 \rceil$ and denote by A' the corresponding ordered multiset of arities. If $t_l = t_{l'} + 5$ then return the multiset $A'' = \{5, A'\}$, otherwise go to the following step.
- 2. Use Algorithm 1 to construct a tree for a message length $l' = \lceil l/4 \rceil$ and denote by A' the corresponding ordered multiset of arities. If $t_l = t_{l'} + 4$ then return the multiset $A'' = \{4, A'\}$, otherwise go to the following step.
- 3. Return A (which cannot be further optimized).

k=2	Sign = 0 for any u
k = 3	Sign < 0 for u = 1 and Sign > 0 for u > 1
k=4	$Sign < 0$ for $u \le 11$ and $Sign > 0$ for $u > 11$
k = 5	Sign = 0 for any u
k = 6	$Sign < 0$ for $u \le 17$ and $Sign > 0$ for $u > 17$
k = 7	$Sign < 0$ for $u \le 115$ and $Sign > 0$ for $u > 115$
k = 8	Sign = 0 for any u
k = 9	$Sign < 0$ for $u \le 217$ and $Sign > 0$ for $u > 217$
k = 10	Sign < 0 for any u
k = 11	$Sign < 0$ for $u \le 139$ and $Sign = 0$ for $u > 139$
k = 12	$Sign < 0$ for $u \le 2465$ and $Sign > 0$ for $u > 2465$
k = 13	Sign < 0 for any u
k = 14	$Sign < 0$ for $u \le 3299$ and $Sign = 0$ for $u > 3299$
k = 15	$Sign < 0$ for $u \le 26281$ and $Sign > 0$ for $u > 26281$
k = 16	Sign < 0 for any u
k = 17	$Sign < 0$ for $u \le 46075$ and $Sign = 0$ for $u > 46075$
k = 18	Sign < 0 for any u
k = 19	Sign < 0 for any u
k = 20	$Sign < 0$ for $u \le 545747$ and $Sign = 0$ for $u > 545747$
k = 21	Sign < 0 for any u
k = 22	Sign < 0 for any u
k = 23	$Sign < 0$ for $u \le 5960299$ and $Sign = 0$ for $u > 5960299$
k = 24	Sign < 0 for any u
k = 25	Sign < 0 for any u
k = 26	$Sign < 0 \text{ for } u \le 62031299 \text{ and } Sign = 0 \text{ for } u > 62031299$

Table 1. Comparison between a perfect binary tree and a perfect ternary tree. If Sign < 0 a perfect ternary tree has a better running time. If Sign = 0 the two trees give the same running time. Otherwise a perfect binary tree is better.

The second approach uses the following hints:

Hints. Let us note that if k > 0, then $a > b \iff (a - k)b > a(b - k)$. Moreover, if $b \le a$ then $(b - 1)(a + 1) \le ab$. This suggests that a product of several numbers, where the sum is constant, is maximized when these numbers are as close together as possible. In order to decrease the product of arities as slowly as possible we use the fact that if $c \ge b \ge a$ we have $(c + 1)(b - 1)a \ge (c + 1)b(a - 1)$.

Algorithm 2b.

INPUTS: a message length l and a multiset of arities (arranged in descending order) minimizing the running time, denoted $A = \{x_1, x_2, ..., x_{|A|}\}$.

OUTPUT: a multiset of arities (still sorted in descending order) minimizing the number of processors while leaving unchanged the running time.

The algorithm proceeds as follows:

- 1. We start by replacing in A each pair of arities 2 by an arity 4 (leaving possibly only one arity 2 in A). We sort A in descending order.
- 2. We repeat at most twice the following routine to determine the solution:
 - Case |A| = 1: we return A.
 - Case |A| = 2:
 - Case $x_1 = 5$: we return A.
 - Case $x_1 \ge 3, x_2 \ge 3$: if $(x_1 + 1)(x_2 1) \ge l$ then $A = \{x_1 + 1, x_2 1\}$, otherwise we return A.
 - Case $x_1 = 4, x_2 = 2$: we return A.
 - Case $x_1 = 3, x_2 = 2$: if $5 \ge l$ then $A = \{5\}$. We return A.
 - **-** Case |A| ≥ 3:
 - Case $x_1 = 5$: we return A.
 - Case $x_1 \geq 3, x_2 \geq 3, x_3 \geq 2$: if $(x_1 + 1)(x_2 1) \prod_{i=3}^{|A|} x_i \geq l$ then we perform the following operations: (i) we add 1 to x_1 and we subtract 1 to x_2 ; (ii) we replace a possible pair of arities 2 by an arity 4; (iii) we reorder A. If either the check fails or $x_1 = 5$ then we return A.

Reducing the number of processors at all the levels **B.2**

The following algorithm uses Algorithm 1 and 2 in order to compute a multiset of arities (sorted in descending order) minimizing the running time and the required number of processors at each step of the computation.

Algorithm 3.

INPUT: a message length l.

OUTPUT: an ordered multiset of arities minimizing the running time and the required number of processors at each step of the computation.

Let $A_0 = \{x_1, x_2, ..., x_{|A_0|}\}$ be the multiset of arities returned by Algorithm 1. We then use Algorithm 2 with a message of length l and the multiset A_0 to compute the multiset of arities $A_1 = \{x'_1, x'_2, ..., x'_{|A_1|}\}$. The rest of the algorithm proceeds iteratively as follows:

- 1. We apply Algorithm 2 on inputs $l' = \lceil l/x_1' \rceil$ and $A_1' = \{x_2', ..., x_{|A_1|}' \}$ to compute
- We apply Algorithm 2 on inputs t = |t/x_1| and X_1 = |x_2, ..., -|A_1|.
 the multiset A'_2 = {x''_2, ..., x''_{|A'_1|}}. We set n = 1.
 As long as one of the following termination conditions is not met, namely (i) A_{n+1}^{(n)} = A_n^{(n)}; (ii) the highest number of levels of arity 4 has been reached (see Lemma 1); or (iii) A_{n+1}^{(n)} = Ø, we set n = n+1 and apply Algorithm 2 with the inputs l(n) = [l(n-1)/x_n^{(n)}] and A_n^{(n)} = {x_{n+1}^{(n)}, ..., x_{|A_n^{(n-1)}|}^{(n)}} to compute the multiset $A_{n+1}^{(n)} = \{x_{n+1}^{(n+1)}, \dots, x_{|A_n^{(n)}|}^{(n+1)}\}.$

The resulting multiset of arities $A_r = \{x_1', x_2'', \dots, x_n^{(n)}, x_{n+1}^{(n+1)}, \dots, x_{|A_n^{(n)}|}^{(n+1)}\}$ minimizes the number of required processors at each step of the computation.