# Structured decomposition for reversible Boolean functions 

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#### Abstract

Reversible Boolean function is a one-to-one function which maps $n$-bit input to $n$-bit output. Reversible logic synthesis has been widely studied due to its relationship with low-energy computation as well as quantum computation. In this work, we give a structured decomposition for even reversible Boolean functions (RBF). Specifically, for $n \geq 6$, any even $n$-bit RBF can be decomposed to 7 blocks of $(n-1)$-bit RBF, where 7 is a constant independent of $n$; and the positions of those blocks have large degree of freedom. Moreover, if the $(n-1)$-bit RBFs are required to be even as well, we show for $n \geq 10$, $n$-bit RBF can be decomposed to 10 even $(n-1)$-bit RBFs. For simplicity, we say our decomposition has block depth 7 and even block depth 10 .

Our result improves Selinger's work in block depth model, by reducing the constant from 9 to 7 ; and from 13 to 10 when the blocks are limited to be even. We emphasize that our setting is a bit different from Selinger's. In Selinger's constructive proof, each block is one of two specific positions and thus the decomposition has an alternating structure. We relax this restriction and allow each block to act on arbitrary $(n-1)$ bits. This relaxation keeps the block structure and provides more candidates when choosing positions of blocks.


## Index Terms

Reversible computation, reversible logic, synthesis method, quantum computation, logic gates, integrated circuits.

## I. Introduction

REVERSIBLE Boolean function is a one-to-one function which maps $n$-bit input to $n$-bit output. Combinatorially, it represents a permutation over $\{0,1\}^{n}$. One historical motivation of studying reversible computation is to reduce the energy consumption caused by computation [1]-[3]. According to Landauer's principle [4], irreversible computation leads to energy dissipation of the order of $K T$ per bit, where $K$ refers to the Boltzmann constant and $T$ is the temperature of the environment. In contrast, if the computing process is reversible, we can in principle use no energy. A classic example of realization of reversible Boolean function - the billiard ball computer where computation costs no energy - can be found in Nielsen and Chuang's book [5]. In addition, reversible Boolean functions are widely used in the quantum circuit such as in the modular exponentiation part of Shor's factoring algorithm [6], or oracles in Grover's search algorithm [7], [8]. Any quantum circuit involving a Boolean function, which is generally irreversible and can not be implemented in quantum circuit directly, such as quantum arithmetic circuit [9], [10], may benefit from the study of reversible Boolean function.

When implementing an $n$-bit reversible Boolean function, the intuition is to use induction and divide the problem into smaller cases. That is, we try to decompose an $n$-bit reversible Boolean function into a product of several ( $n-1$ )-bit reversible Boolean functions. This decomposition is generally impossible, since if the $n$-bit reversible Boolean function represents an odd permutation over $\{0,1\}^{n}$, it can not be implemented by $(n-1)$-bit reversible Boolean functions, which are even when regarded as a permutation on $n$ bits. However, in 2017, Selinger [11] found the decomposition does exist for even $n$-bit reversible Boolean functions and remarkably, the number of required $(n-1)$-bit functions is a constant independent of $n$. More precisely, he proved that an arbitrary even $n$-bit reversible Boolean function can be represented by $9(n-1)$-bit reversible Boolean functions with an alternating structure shown in Figure 1. He also proved that, if we limit the $(n-1)$-bit functions to be even as well, then the number of $(n-1)$-bit functions is at most 13 . For simplicity, in the following we use block to refer to the ( $n-1$ )-bit reversible Boolean function, and even block to refer to the even $(n-1)$-bit reversible Boolean function.

Our main contributions are: we improve the constant from 9 to 7 for $n \geq 6$ and 13 to 10 for $n \geq 9$ when limiting the blocks to be even. To be concise, our decomposition has block depth 7 and even block depth 10 . We should

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Fig. 1. Alternating structure in [11]
emphasize that our setting is a bit different from Selinger's. In Selinger's work, the decomposition is restricted to an alternating structure. Instead of fixing two specific positions, we allow blocks to act on arbitrary $(n-1)$ bits. This relaxation keeps the block structure and provides more candidates when choosing the position of blocks. We believe this relaxation makes the model more flexible in application.

For convenience, we abbreviate reversible Boolean function as RBF. We further say a RBF is controlled RBF if it keeps a certain bit invariant (formal definition is in Section II). Our construction consists of two steps. In the first stage, we prove that an arbitrary even $n$-bit RBF can be transformed into an even controlled RBF by $3(n-1)$-bit blocks and the positions of those low-level blocks have a lot of freedom. It is worth mentioning that the number 3 is also essentially tight. Then we prove that an arbitrary even controlled RBF can be substituted with 5 blocks, where the third and fourth blocks have many choices as well. While putting it together, we can literally merge the last block in the first step with the first block in the second step, thus providing a 7 -depth full decomposition. As a partial result during the construction, we show that two different $(n-1)$-bit blocks are sufficient to formulate the cycle pattern of any even $n$-bit permutation free of $3 / 5$-cycle. We believe this result has some individual interest. Here, cycle pattern is the list $\left\{c_{k}\right\}$, where $c_{k}$ is the number of cycles of length $k$; and free of $3 / 5$-cycle means $c_{3}=c_{5}=0$. The limitation that cycle pattern is free of $3 / 5$-cycle is indeed inevitable since we can also prove two ( $n-1$ )-bit blocks can not compose a single $3 / 5$-cycle. The proof of even block depth 10 is similar. Since all the proofs in this paper are constructive in essence, our decomposition can be programmed as an efficient algorithm.

In 2003, Shende et al. [8] proved that any even reversible Boolean function can be decomposed into NOT gates, CNOT gates and Toffoli gates without using temporary storage. Besides, In 2010, Saeedi et al. [12] gave an algorithm which synthesizes a given permutation by 7 building blocks. These works focus on decomposing RBFs into smaller pieces, however, their constructions can not be merged into $7(n-1)$-bit blocks, thus they are different from our work. There are also some related works about decomposing $n$-bit unitary operator to smaller ones. In 2010, Saeedi et al. [13] showed how to decompose an arbitrary $n$-bit unitary operator down into $\ell$-bit unitary operators ( $\ell<n$ ) using quantum Shannon decomposition [14].

The structured decomposition may have some potential applications. Though not directly improving results in circuit synthesis, the structure of this decomposition implies some interesting results. For instance, in Selinger's construction in Figure 1, long-distance CNOT, i.e., CNOT between the first and the last bit prohibited by today's quantum devices [15], [16], shall be avoided. Although a similar effect can be realized with SWAP gates [17], this result actually indicates that such gate-costing alternatives will not happen frequently in a proper structure. In our setting, the positions of blocks have certain freedom to choose, which makes the construction even more flexible for different potential physical devices [18], [19].

Organization of the paper In Section II we give formal definitions of the key elements required in expressing problem and formulating proof. Then in Section III, we list our main results and give a proof sketch. In Section IV and Section V we give detailed proofs to the result of block depth 7. Specifically, in Section IV we transform an even $n$-bit RBF to an even controlled RBF by $3(n-1)$-bit blocks. In Section V, we show how to recover an even controlled RBF by 5 blocks. In addition, an explicit example of our algorithm is put in Section VI In Section VII, we give a proof sketch of the result of even block depth. This proof is similar to the proof of block depth but involves a much more sophisticated analysis. At last, the paper is concluded in Section VIII. Due to the page limit, the omitted proofs are deferred into the appendix.

## II. Preliminary

In general, our work aims to implement an even $n$-bit reversible Boolean function using ( $n-1$ )-bit reversible Boolean function. In order to state our problems and theorems properly, formal definitions are required.

Denote $[n]$ as $\{1,2, \cdots, n\}$ and $\{0,1\}^{n}$ as the set of $n$-bit binary strings. Define $S_{\{0,1\}^{n}}$ as the group of permutations over $\{0,1\}^{n}$; and $A_{\{0,1\}^{n}}$ as the group of even permutations over $\{0,1\}^{n}$. For any $\sigma \in S_{\{0,1\}^{n}}$ and $\boldsymbol{x}, \boldsymbol{y} \in\{0,1\}^{n}$, define

$$
\operatorname{dist}^{\sigma}(\boldsymbol{x}, \boldsymbol{y})=\min \left\{k \in \mathbb{N} \mid \sigma^{k}(\boldsymbol{x})=\boldsymbol{y}\right\}
$$



Fig. 2. Process of the algorithm for Theorem 1
(if $\boldsymbol{y}$ is not reachable from $\boldsymbol{x}$ under $\sigma$, $\left.\operatorname{dist}^{\sigma}(\boldsymbol{x}, \boldsymbol{y})=+\infty\right)$ and $\operatorname{dist}_{\text {min }}^{\sigma}(\boldsymbol{x}, \boldsymbol{y})=\min \left\{\operatorname{dist}^{\sigma}(\boldsymbol{x}, \boldsymbol{y}), \operatorname{dist}^{\sigma}(\boldsymbol{y}, \boldsymbol{x})\right\}$. We also define the support of $\sigma$ as $\operatorname{Supp}(\sigma)=\{\boldsymbol{x} \mid \sigma(\boldsymbol{x}) \neq \boldsymbol{x}\}$.

Recall that every permutation has a unique cycle decomposition. We say $\sigma$ has a $k$-cycle if there is a cycle of length $k$ in the cycle decomposition. We say $\boldsymbol{x} \in\{0,1\}^{n}$ is a fix-point if $\sigma(\boldsymbol{x})=\boldsymbol{x}$ and a fix-point is a 1-cycle as well. If $\sigma$ consists of $k_{1}$-cycle, $\ldots, k_{t}$-cycle, we say $\sigma$ is exactly $k_{1}, \ldots, k_{t}$-cycle. We may omit $k_{i}$ if $k_{i}=1$. For example, we may abbreviate $1,3,4$-cycle as 3,4 -cycle. In addition, we say $\sigma$ is free of $l_{1} / l_{2} / \ldots / l_{s}$-cycle if for any $i \in[s], j \in[t], l_{i} \neq k_{j}$.

For simplicity, we abbreviate reversible Boolean function as $R B F$ and permutation over $\{0,1\}^{n}$ as $n$-bit permutation. Since any $n$-bit RBF can be viewed as a permutation over $\{0,1\}^{n}$, thus the set of all $n$-bit RBFs is isomorphic to $S_{\{0,1\}^{n}}$. Moreover, we say an $n$-bit RBF is even if its corresponding permutation is even.

Given $\boldsymbol{x} \in\{0,1\}^{n}$, write $\boldsymbol{x}_{i}$ for the value of its $i$-th bit; and $\boldsymbol{x}^{\oplus i}:=\boldsymbol{x}_{1} \cdots \boldsymbol{x}_{i-1}\left(1-\boldsymbol{x}_{i}\right) \boldsymbol{x}_{i+1} \cdots \boldsymbol{x}_{n}$, i.e., $\boldsymbol{x}^{\oplus i}$ is $\boldsymbol{x}$ flipped the $i$-th bit. Furthermore, define $\boldsymbol{x}^{\oplus i_{1}, i_{2}, \ldots, i_{k}}$ recursively as $\left(\boldsymbol{x}^{\oplus i_{1}}\right)^{\oplus i_{2}, \cdots, i_{k}}$.
Definition 1 (Controlled RBF (CRBF)). Given $n>0$ and $i \in[n]$, we say $\pi$ is an $n$-bit i-CRBF if $\pi \in S_{\{0,1\}^{n}}^{(i)}$, where

$$
S_{\{0,1\}^{n}}^{(i)}:=\left\{\sigma \in S_{\{0,1\}^{n}} \mid \forall \boldsymbol{x} \in\{0,1\}^{n}, \sigma(\boldsymbol{x})_{i}=\boldsymbol{x}_{i}\right\}
$$

We also define

$$
A_{\{0,1\}^{n}}^{(i)}:=\left\{\sigma \in A_{\{0,1\}^{n}} \mid \forall \boldsymbol{x} \in\{0,1\}^{n}, \sigma(\boldsymbol{x})_{i}=\boldsymbol{x}_{i}\right\} .
$$

An $i$-CRBF keeps the $i$-th bit of any input invariant. For example, if $i=1$, then there exist $f_{0}, f_{1} \in S_{\{0,1\}^{n-1}}$ such that $\pi(0 \boldsymbol{y})=0 f_{0}(\boldsymbol{y}), \pi(1 \boldsymbol{y})=1 f_{1}(\boldsymbol{y})$ for any $\boldsymbol{y} \in\{0,1\}^{n-1}$. Moreover, we say $\pi$ is a concurrent controlled RBF (CCRBF) if $f_{0}=f_{1}$. Further, when $f_{0}$ is even, we say $\pi$ is concurrently even; and concurrently odd when $f_{0}$ is odd. The formal definitions are shown below.

Definition 2 (Concurrent Controlled RBF (CCRBF)). Given $n>0$ and $i \in[n]$, we say $\pi$ is an n-bit $i$-CCRBF if $\pi \in S C_{\{0,1\}^{n}}^{(i)}$, where

$$
\begin{aligned}
S C_{\{0,1\}^{n}}^{(i)} & :=\left\{\sigma \in S_{\{0,1\}^{n}}^{(i)} \mid \forall \boldsymbol{x} \in\{0,1\}^{n}\right. \\
& \left.\forall k \in[n] \backslash\{i\}, \sigma(\boldsymbol{x})_{k}=\sigma\left(\boldsymbol{x}^{\oplus i}\right)_{k}\right\} .
\end{aligned}
$$

Definition 3 (Concurrently Even/Odd). An n-bit i-CCRBF $\pi$ can be regarded as an $(n-1)$-bit RBF $\left.\sigma\right|_{-i}$ on bits $[n] /\{i\}$. We say that $\sigma$ is $i$-concurrently even/odd if $\left.\sigma\right|_{-i}$ is even/odd. Define $A C_{\{0,1\}^{n}}^{(i)}$ as the set of n-bit concurrently even $i$-CRBF.

When dimension $i$ is clear in the context, we simply use concurrently even/odd. Note that no matter whether $\left.\sigma\right|_{-i} \in S_{\{0,1\}^{n-1}}$ is odd or even, CCRBF $\sigma \in S_{\{0,1\}^{n}}$ itself is always even.
Definition 4 (Block depth and even block depth). Given $n \geq 2$ and $\sigma \in S_{\{0,1\}^{n}}$, we say $\sigma$ has block depth $d$ if there exist $\sigma_{1}, \sigma_{2}, \ldots, \sigma_{d} \in \bigcup_{j=1}^{n} S C_{\{0,1\}^{n}}^{(j)}$ such that $\sigma=\sigma_{1} \sigma_{2} \cdots \sigma_{d}$.

Similarly, we say $\sigma$ has even block depth d if those $\sigma_{i} \in \bigcup_{j=1}^{n} A C_{\{0,1\}^{n}}^{(j)}$.

Notice that the decomposition problem considered here is a bit different from Selinger's work [11]. In Selinger's work, any $\sigma_{i}$ is in one of two specific positions, thus the decomposition forms an alternating structure as Figure 1 . Here we relax the restriction and allow blocks acting on arbitrary $(n-1)$ bits. Thus we consider the block depth instead of alternation depth used in [11].

## III. Main results and proof sketch

In the previous work, Selinger [11] proved that an arbitrary even $n$-bit RBF has alternation depth 9 and even alternation depth 13 . Our main contribution is to improve the constant 9 to 7 in block depth model and 13 to 10 in even block depth model. The main theorems are stated as follows.

Theorem 1. For $n \geq 6$, any $\sigma \in A_{\{0,1\}^{n}}$ has block depth 7 .
Theorem 2. For $n \geq 10$, any $\sigma \in A_{\{0,1\}^{n}}$ has even block depth 10 .
Proof sketch of Theorem 1. To prove Theorem 1, we first turn $\sigma$ into an even CRBF by Proposition 1, then further break the even CRBF down into identity by Proposition 2. We achieve these two steps with 3 and 5 blocks respectively. By a finer analysis, the last block of the first step and the first block of the second step can be merged. Thus a 7-block implementation is obtained. The sketch of the whole process is depicted in Figure 2.

The proof of Theorem 2 is similar. Before Section VII, we only focus on the proof of block depth 7. Proposition 1 states that we can transform an even $n$-bit RBF to an even CRBF by 3 CCRBFs with many choices.

Proposition 1. For $n \geq 4, r_{1} \in[n]$ and $\sigma \in A_{\{0,1\}^{n}}$, there exist at leasts $(n-2)$ different $r_{2} \in[n] \backslash\left\{r_{1}\right\}$ such that $\sigma \pi_{1} \sigma_{1} \pi_{2} \in A_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ holds for some $\sigma_{1} \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}, \pi_{1}, \pi_{2} \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$.

In addition, we also show the tightness of Proposition 1 by Lemma 5 in Section IV It is also worth noting that the proof works for $\sigma \in S_{\{0,1\}^{n}}$ (with $\sigma \pi_{1} \sigma_{1} \pi_{2} \in S_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ ) as well. For our purpose, it is more convenient to state it as Proposition 1 .

Proposition 2 states that we can recover any even $n$-bit CRBF by 5 CCRBFs.
Proposition 2. For $n \geq 6, r_{1} \in[n], r_{2}, r_{3}, r_{4} \in[n] \backslash\left\{r_{1}\right\}, r_{3} \neq r_{4}$ and $\sigma \in A_{\{0,1\}^{n}}^{\left(r_{1}\right)}$, there exist $\pi_{1} \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$, $\sigma_{1}, \sigma_{2} \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}, \tau_{1} \in S C_{\{0,1\}^{n}}^{\left(r_{3}\right)}, \tau_{2} \in S C_{\{0,1\}^{n}}^{\left(r_{4}\right)}$ such that $\sigma \pi_{1} \sigma_{1} \tau_{1} \tau_{2} \sigma_{2}=$ id.

The key to the proof of Proposition 2 is the following proposition, which states two $n$-bit CCRBFs can formulate the cycle pattern of any even $n$-bit permutation free of $3 / 5$-cycle. We believe this proposition has some individual interest.
Proposition 3. For $n \geq 4$, distinct $r_{1}, r_{2} \in[n]$ and $\sigma \in A_{\{0,1\}^{n}}$ free of 3/5-cycle, there exist $\pi \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}, \tau \in$ $S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$ such that $\pi \tau$ and $\sigma$ have the same cycle pattern, which is equivalent to that $h \sigma h^{-1}=\pi \tau$ holds for some $h \in \dot{S}_{\{0,1\}^{n}}$.

The proof of Proposition 1 is in Section IV and the proof of Proposition 2 and Proposition 3 are in Section V.

## IV. Transforming even $n$-bit RBF to controlled RBF

In this section, we give proof of Proposition 1. That is, we transform an even $n$-bit RBF $\sigma$ to an even CRBF using 3 CCRBFs. $\sigma$ may involve $2^{n}$ elements and have a complicated pattern. However, to transform $\sigma$ to a controlled RBF, which keeps one bit invariant, the key point is whether the $i$-th bit of $\sigma(\boldsymbol{x})$ equals the $i$-th bit of $\boldsymbol{x}$. So we simplify the representation of a RBF by constructing a black-white cuboid, where the color indicates whether $\sigma(\boldsymbol{x})_{i}=\boldsymbol{x}_{i}$. Then proving Proposition 1 is equivalent to transforming the colored cuboid to white. An explicit example of the whole process of Proposition 1 can be seen in Section VI

Recall that $n$-bit RBF is in fact a permutation on $\{0,1\}^{n}$. Specifically, we visualize the permutation on a $2 \times 2 \times 2^{n-2} 3$-d cuboid. In Section IV-A, we give the construction for the black-white 3-d cuboid corresponding to $\sigma$. After that, in Section IV-B, we give a constructive proof to transform the colored cuboid to a white cuboid.

Proof sketch of Proposition 1] First we choose arbitrary two different $r_{1}, r_{2} \in[n]$ and construct a black-white cuboid. Then we transform the colored cuboid to a canonical form by $S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$ using Lemma 2 We also prove in most cases, by Lemma 1 the canonical form can be transformed to a white cuboid by $S C_{\{0,1\}^{n}}^{\left(r_{2}\right)} S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$,
$S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$. Finally, if the canonical form falls into a bad case, we prove for any $r_{3} \in[n] \backslash\left\{r_{1}, r_{2}\right\}$, by checking the new canonical form based on $r_{1}, r_{3}$, this case can be tackled with $S C_{\{0,1\}^{n}}^{\left(r_{3}\right)}, S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}, S C_{\{0,1\}^{n}}^{\left(r_{3}\right)}$ using Lemma 3,

## A. Visualizing a permutation on a 3-d cuboid

Given permutation $\sigma \in S_{\{0,1\}^{n}}$, in this section we construct a 3 -d black-white cuboid for $\sigma$ and discuss the effect of transformation, that is the new colored cuboid for $\sigma \tau, \tau \in S_{\{0,1\}^{n}}$.

Recall that $\sigma$ is a permutation over $2^{n}$ elements. Fixing $r_{1}, r_{2} \in[n]$ and compressing the other $(n-2)$ dimensions, we get a 3 -d cuboid. For example, if $n=4, r_{1}=1, r_{2}=2$, then we compress the remaining two dimensions into one by letting the coordinates to be $00,01,10,11$. We visualize $\sigma$ in Figure 3 , where

$$
\begin{aligned}
\sigma:= & (1001,1100,0101)(1110,0110,0111, \\
& 1111)(1010,0010,0011,1011) .
\end{aligned}
$$

As an example, 1100 is labelled on $(1,1,00)$, where 00 represents the third coordinate. The arrows in the figure stand for permutation $\sigma$. In this case, $\sigma(1100)=0101$, so we draw an arrow from 1100 to 0101 .


Fig. 3. Visualize $\sigma$ on a 3-d cuboid
The graph reflects both pattern and structure of the permutation. If we exert a CCRBF

$$
\begin{aligned}
\tau= & (1010,1011,0011,0010)(1110, \\
& 1111,0111,0110) \in S C_{\{0,1\}^{4}}^{(2)}
\end{aligned}
$$

on $\sigma$, it will have the same effect on the front and back face of the cuboid, eliminating the two 4 -cycles. That is, the 3 -d cuboid corresponding to $\sigma \tau$ will only have a 3 -cycle.

Back to Proposition 1, here we aim to eliminate cycles which have overlap with both top and bottom face. To further simplify the notation, we transform the cuboid with arrow pattern into a cuboid with black-white colored nodes. That is, we paint coordinate $\boldsymbol{x} \in\{0,1\}^{n}$ black if $\sigma(\boldsymbol{x})_{r_{1}} \neq \boldsymbol{x}_{r_{1}}$ as shown in Figure 4 Intuitively, the black node means that $\sigma(\boldsymbol{x})$ is in a wrong face.


Fig. 4. Visualize $\sigma$ on a colored cuboid
Now we consider the cuboid of $\sigma \pi$ with some permutation $\pi$. For example, if $\pi$ pushes $\boldsymbol{x}$ to the opposite face, the color of $\boldsymbol{x}$ in cuboid for $\sigma \pi$ will be the opposite of original $\boldsymbol{\pi}(x)$ 's in cuboid for $\sigma$. That is, assuming $\pi(\boldsymbol{x})=\boldsymbol{x}^{\prime}$ and $\boldsymbol{x}_{r_{1}} \neq \boldsymbol{x}_{r_{1}}^{\prime}$, if $\sigma\left(\boldsymbol{x}^{\prime}\right)_{r_{1}} \neq \boldsymbol{x}_{r_{1}}^{\prime}$, then $\sigma\left(\boldsymbol{x}^{\prime}\right)_{r_{1}}=\boldsymbol{x}_{r_{1}}$ (i.e., $\sigma(\pi(\boldsymbol{x}))_{r_{1}}=\boldsymbol{x}_{r_{1}}$ ), vice versa. An example is in Figure 5 and Figure 6 for $\pi=(1100,0101)(1000,0001) \in S C_{\{0,1\}^{4}}^{(2)}$.

Using colored cuboid, for some $\pi^{\prime}$, the cuboid for $\sigma \pi^{\prime}$ is white if and only if $\sigma \pi^{\prime} \in S_{\{0,1\}^{n}}^{\left(r_{1}\right)}$. To prove Proposition 1, it suffices to show that we can transform any black-white cuboid into a white cuboid, using CCRBFs.


Fig. 5. Colored cuboid for $\sigma$. Arrows refer to $\pi$.


Fig. 6. Colored cuboid for $\sigma \pi$

For simplicity, as shown in Figure 6, we use a double line to connect $\boldsymbol{x}$ and $\boldsymbol{x}^{\oplus r_{2}}$ for all $\boldsymbol{x}$ with $\boldsymbol{x}_{r_{1}}=1$; and zigzag line to connect $\boldsymbol{x}$ and $\boldsymbol{x}^{\oplus r_{2}}$ for all $\boldsymbol{x}$ with $\boldsymbol{x}_{r_{1}}=0$. Let $a_{1}, a_{2}, a_{3}, a_{4}$ be the number of , and $b_{1}, b_{2}, b_{3}, b_{4}$ be the number of ,

## B. Transforming $\sigma$ to controlled permutation

In this section, we transform the given permutation to CRBF. Following previous section, we construct a colored cuboid for $\sigma \in A_{\{0,1\}^{n}}$ and calculate corresponding $a_{i}$ 's, $b_{i}$ 's. According to $a_{i}$ 's, $b_{i}$ 's, we transform $\sigma$ to $A_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ using Lemma 1 or Lemma 3 We also show the tightness of 3 steps by Lemma 5

Firstly we prove Lemma 1 to show most cases are solvable by $S C_{\{0,1\}^{n}}^{\left(r_{2}\right)} S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}, S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$. Since the number of black nodes in lower and upper faces is the same, it is easy to see $a_{3}+a_{4}+b_{3}+b_{4}$ is even.

Lemma 1. There exist $\sigma_{1} \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ and $\pi_{1}, \pi_{2} \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$ such that $\sigma \pi_{1} \sigma_{1} \pi_{2} \in A_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ if

1. $a_{3}+a_{4}+b_{3}+b_{4}>2$ holds or,
2. $a_{3}+a_{4}+b_{3}+b_{4}=2$ and $\min \left\{b_{1}+a_{2}, a_{1}+b_{2}\right\}>0$ hold or,
3. $a_{3}+a_{4}+b_{3}+b_{4}=0$ and $b_{1}+a_{2}$ is even (equivalently $a_{1}+b_{2}$ is even) hold.

To give specific constructions, we first transform the colored cuboid to a canonical form by Lemma 2. Then we classify them into different cases and solve case by case.

A canonical form is a colored cuboid only containing 3 kinds of matching pairs ("cards") along the compressed dimensions, which are , We call them A-card, B-card, and C-card; and the numbers of these three kinds are $\alpha, \beta, \gamma$ respectively.

If $a_{2}+a_{3} \leq b_{2}+b_{3}$, we can use Lemma 2 to transform the colored cuboid to a canonical form.
Lemma 2. If $a_{2}+a_{3} \leq b_{2}+b_{3}$, there exists $\pi \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$ such that the colored cuboid for $\sigma \pi$ is of canonical form.

Proof. Recall that the color of a node $\boldsymbol{x}$ refers to whether $\sigma(\boldsymbol{x})$ is in the correct face. So if coordinate $\boldsymbol{x}^{\prime}$ is black and $\boldsymbol{x}_{r_{1}}^{\prime} \neq \boldsymbol{x}_{r_{1}}$, then coordinate $\boldsymbol{x}$ will be white after swapping $\boldsymbol{x}$ and $\boldsymbol{x}^{\prime}$, vice versa. See Figure 6 as an example.

We first apply $\tau \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$ such that the cuboid for $\sigma \tau$ satisfies $a_{4}^{\prime}=b_{4}^{\prime}, a_{2}^{\prime}=a_{3}^{\prime}=b_{3}^{\prime}=0$. Then we use $\tau^{\prime} \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$ to rearrange the nodes, such that the cuboid for $\sigma \tau \tau^{\prime}$ is a canonical form. $\tau$ is achieved by the following algorithm.


The correctness comes from the following observation. Since the number of black nodes is the same in the top and bottom face, if there is a black node in one face, the opposite face has one as well. Therefore, in line 3 the number of and is no fewer than ; in line 6 the number of $\sigma^{\circ}$ and is no fewer than . Since
$a_{2}+a_{3} \leq b_{2}+b_{3}$, it can be verified when algorithm executes in line 8 , the number of $O$ is no more than $\sigma^{\circ}$. After performing this algorithm, we have $a_{2}^{\prime}=a_{3}^{\prime}=b_{3}^{\prime}=0$ and $a_{4}^{\prime}=b_{4}^{\prime}$.

Then we rearrange the nodes to form A-, B-, C-cards. Since $a_{4}^{\prime}=b_{4}^{\prime}$, by some permutation $\tau^{\prime} \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$, we can assure that the colored cuboid corresponding to $\sigma \tau \tau^{\prime}$ only has these three kind of cards. Thus let $\pi=\tau \tau^{\prime}$, then the colored cuboid for $\sigma \pi$ is of canonical form.

Since the number of and is invariant in Algorithm 1, as well as and $\mathcal{O}^{\circ}$, we have $\alpha=\frac{1}{2}\left(a_{1}-\right.$ $\left.a_{2}+b_{2}-b_{1}\right), \beta=b_{1}+a_{2}, \gamma=\frac{1}{2}\left(a_{3}+a_{4}+b_{3}+b_{4}\right)$.

Now we give the proof of Lemma 1
Proof of Lemma 1 W.l.o.g, assume $a_{2}+a_{3} \leq b_{2}+b_{3}$. Using Lemma 2, we transform the colored cuboid to a canonical form with $\pi^{\prime} \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$. Record the number of the 3 kind of cards, i.e., $\alpha, \beta, \gamma$.


First notice that if we pair two A-cards or two B-cards, the paired A-cards and B-cards can be transformed to C-cards by the following permutations where $\tau_{1} \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}, \tau_{2} \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}, \tau_{3} \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$ :

This approach solves the $3^{r d}$ case directly and reduces the $1^{\text {st }}$ case to the following 3 subcases. Since these card groups can be tackled in parallel, in final construction, $\pi_{1}=\pi^{\prime} \tau_{1}, \sigma_{1}=\tau_{2}$, and $\pi_{2}=\tau_{3}$.

- $\alpha=1, \beta=1, \gamma \geq 2$ : This graph shows how to tackle 1 A -card and 1 B -card with 2 C -cards.

- $\alpha=1, \beta=0, \gamma \geq 2$ : This graph shows how to tackle 1 A -card with 2 C -cards.


- $\alpha=0, \beta=1, \gamma \geq 2$ : This graph shows how to tackle 1 B -card with 2 C -card.


For the $2^{\text {rd }}$ case, we reduce it to the following.

- $\alpha=2, \beta=1, \gamma \geq 1$ : This graph shows how to tackle 2 A -cards and 1 B -card with 1 C -card.

- $\alpha=0, \beta=3, \gamma \geq 1$ : This graph shows how to tackle 3 B -cards with 1 C -card.

- $\alpha=1, \beta=2, \gamma \geq 1$ : This graph shows how to tackle 1 A-card and 2 B -cards with 1 C -card.


For the other cases, which can not be solved by Lemma 1, can in turn be dealt with Lemma 3
Lemma 3. For any $r_{3} \in[n] \backslash\left\{r_{1}, r_{2}\right\}$, there exist $\sigma_{1} \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ and $\pi_{1}, \pi_{2} \in S C_{\{0,1\}^{n}}^{\left(r_{3}\right)}$ such that $\sigma \pi_{1} \sigma_{1} \pi_{2} \in$ $A_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ if

1) $a_{3}+a_{4}+b_{3}+b_{4}=2$ and $\min \left\{b_{1}+a_{2}, a_{1}+b_{2}\right\}=0$ hold or
2) $a_{3}+a_{4}+b_{3}+b_{4}=0$ and $b_{1}+a_{2}$ is odd (equivalently $a_{1}+b_{2}$ is odd) hold.

Fixing $r_{1}$, if for some $r_{2}$, the corresponding canonical form falls into Lemma 3. Then for any $r_{3} \in[n] \backslash\left\{r_{1}, r_{2}\right\}$, the canonical form corresponding with $r_{1}, r_{3}$ will fall into 3 -step solvable cases, that is, it can be solved by Lemma 1 with $r_{2}^{\prime}=r_{3}$.

Before the proof, we show how to switch dimensions. We visualize the permutation on a black-white 4-d cuboid as two 3 -d cuboids. When $r_{1}, r_{2}$ are fixed, pick $r_{3} \in[n] \backslash\left\{r_{1}, r_{2}\right\}$ and compress all the other $(n-3)$ dimensions. As before, paint $\boldsymbol{x}$ black if $\sigma(\boldsymbol{x})_{r_{1}} \neq \boldsymbol{x}_{r_{1}}$ for all $\boldsymbol{x} \in\{0,1\}^{n}$. An example of $n=4, r_{1}=1, r_{2}=2, r_{3}=4$ is Figure 7. The left and right 3 -d cuboids corresponding to $r_{3}=0$ and $r_{3}=1$.


Fig. 7. 4-d cuboid for $n=4, r_{1}=1, r_{2}=2, r_{3}=4$.
 respectively. When we switching dimension $r_{2}$ and $r_{3}$, Figure 7 changes to Figure 8 . Similarly, in Figure 8, denote $\hat{a}_{1}, \hat{a}_{2}, \hat{a}_{3}, \hat{a}_{4}$ to be the number of and $\hat{b}_{1}, \hat{b}_{2}, \hat{b}_{3}, \hat{b}_{4}$ to be the number of , , or, or respectively.


Fig. 8. Switching from $r_{2}$ to $r_{3}$

Proof of Lemma 3 For the $1^{\text {st }}$ case in Lemma 3, w.l.o.g, assume $b_{1}+a_{2}=0$. And we have the following 4 cases.

- $a_{3}+a_{4}=2$ : Thus all $\boldsymbol{x} \in\{0,1\}^{n}, \boldsymbol{x}_{r_{1}}=\boldsymbol{x}_{r_{2}}=0$ are black; and all $\boldsymbol{x} \in\{0,1\}^{n}, \boldsymbol{x}_{r_{1}}=0, \boldsymbol{x}_{r_{2}}=1$ are white. Therefore, $\hat{b}_{3}=\hat{b}_{4}=2^{n-3}$, which is 3 -step solvable in the $1^{\text {st }}$ case of Lemma 1
- $b_{3}+b_{4}=2$ : Similar with case $a_{3}+a_{4}=2$.
- $b_{3}=1$ : Thus all $\boldsymbol{x} \in\{0,1\}^{n}, \boldsymbol{x}_{r_{1}}=\boldsymbol{x}_{r_{2}}=0$ are black; and all $\boldsymbol{x} \in\{0,1\}^{n}, \boldsymbol{x}_{r_{1}}=0, \boldsymbol{x}_{r_{2}}=1$ are white except one. Therefore, $\hat{b}_{3}=2^{n-3}, \hat{b}_{4}=2^{n-3}-1$, which is 3 -step solvable in the $1^{\text {st }}$ case of Lemma 1
- $b_{4}=1$ : Similar with case $b_{3}=1$.

For the $2^{\text {nd }}$ case in Lemma 3, since $a_{3}+a_{4}+b_{3}+b_{4}=0$, then for any $\boldsymbol{x} \in\{0,1\}^{n}$ the color of $\boldsymbol{x}$ is different from the color of $\boldsymbol{x}^{\oplus r_{2}}$. Define

$$
\begin{aligned}
u_{b} & =\mid\left\{\text { black } \boldsymbol{x} \in\{0,1\}^{n} \mid \boldsymbol{x}_{r_{1}}=1, \boldsymbol{x}_{r_{2}}=\boldsymbol{x}_{r_{3}}=0\right\} \mid \\
u_{w} & =\mid\left\{\text { white } \boldsymbol{x} \in\{0,1\}^{n} \mid \boldsymbol{x}_{r_{1}}=1, \boldsymbol{x}_{r_{2}}=\boldsymbol{x}_{r_{3}}=0\right\} \mid \\
l_{b} & =\mid\left\{\text { black } \boldsymbol{x} \in\{0,1\}^{n} \mid \boldsymbol{x}_{r_{1}}=0, \boldsymbol{x}_{r_{2}}=\boldsymbol{x}_{r_{3}}=0\right\} \mid \\
l_{w} & =\mid\left\{\text { white } \boldsymbol{x} \in\{0,1\}^{n} \mid \boldsymbol{x}_{r_{1}}=0, \boldsymbol{x}_{r_{2}}=\boldsymbol{x}_{r_{3}}=0\right\} \mid
\end{aligned}
$$

and

$$
\begin{aligned}
u_{b}^{\prime} & =\mid\left\{\text { black } \boldsymbol{x} \in\{0,1\}^{n} \mid \boldsymbol{x}_{r_{2}}=0, \boldsymbol{x}_{r_{1}}=\boldsymbol{x}_{r_{3}}=1\right\} \mid \\
u_{w}^{\prime} & =\mid\left\{\text { white } \boldsymbol{x} \in\{0,1\}^{n} \mid \boldsymbol{x}_{r_{2}}=0, \boldsymbol{x}_{r_{1}}=\boldsymbol{x}_{r_{3}}=1\right\} \mid \\
l_{b}^{\prime} & =\mid\left\{\text { black } \boldsymbol{x} \in\{0,1\}^{n} \mid \boldsymbol{x}_{r_{1}}=\boldsymbol{x}_{r_{2}}=0, \boldsymbol{x}_{r_{3}}=1\right\} \mid \\
l_{w}^{\prime} & =\mid\left\{\text { white } \boldsymbol{x} \in\{0,1\}^{n} \mid \boldsymbol{x}_{r_{1}}=\boldsymbol{x}_{r_{2}}=0, \boldsymbol{x}_{r_{3}}=1\right\} \mid .
\end{aligned}
$$

By assumption, $a_{1}+b_{2}=u_{w}+u_{w}^{\prime}+l_{b}+l_{b}^{\prime}, b_{1}+a_{2}=u_{b}+u_{b}^{\prime}+l_{w}+l_{w}^{\prime}$. And $u_{w}+u_{b}=u_{w}^{\prime}+u_{b}^{\prime}=l_{w}+l_{b}=$ $l_{w}^{\prime}+l_{b}^{\prime}=2^{n-3}$. Thus

$$
\begin{aligned}
u_{b}+u_{b}^{\prime}+l_{b}+l_{b}^{\prime}= & \left(u_{b}+u_{b}^{\prime}+l_{w}+l_{w}^{\prime}\right)+\left(l_{w}+l_{b}\right) \\
& +\left(l_{w}^{\prime}+l_{b}^{\prime}\right)-2\left(l_{w}+l_{w}^{\prime}\right)
\end{aligned}
$$

is odd. On the other hand, $\left|\left\{\boldsymbol{x} \mid \boldsymbol{x}_{r_{2}}=0, \boldsymbol{x}_{r_{3}}=0\right\}\right|=\left|\left\{\boldsymbol{x} \mid \boldsymbol{x}_{r_{2}}=0, \boldsymbol{x}_{r_{3}}=1\right\}\right|=2^{n-2}$ is even. Therefore there exists $\boldsymbol{x} \in\{0,1\}^{n}, \boldsymbol{x}_{r_{2}}=0$ such that the color of $\boldsymbol{x}$ is the same with the color of $\boldsymbol{x}^{\oplus r_{3}}$. Thus, $\hat{a}_{3}+\hat{a}_{4}+\hat{b}_{3}+\hat{b}_{4}>0$.

- $\hat{a}_{3}+\hat{a}_{4}+\hat{b}_{3}+\hat{b}_{4}>2$ : It is 3-step solvable in the $1^{\text {st }}$ case of Lemma 1
- $\hat{a}_{3}+\hat{a}_{4}+\hat{b}_{3}+\hat{b}_{4}=2$ : Thus there exists $\boldsymbol{x} \in\{0,1\}^{n}, \boldsymbol{x}_{r_{1}}=0$, such that $\boldsymbol{x}$ is white; then $\boldsymbol{x}^{\oplus r_{3}}$ and $\boldsymbol{x}^{\oplus r_{2}}$ are all black; and $\boldsymbol{x}^{\oplus r_{2}, r_{3}}$ is white. Thus when $r_{2}$ is swapped with $r_{3}, \boldsymbol{x}$ with $\boldsymbol{x}^{\oplus r_{3}}$ and $\boldsymbol{x}^{\oplus r_{2}}$ with $\boldsymbol{x}^{\oplus r_{2}, r_{3}}$ form ? and $\widetilde{\sigma}^{\circ}$. Therefore $\hat{b}_{1}, \hat{b}_{2}>0$, which is 3-step solvable in the $2^{\text {nd }}$ case of Lemma 1

For completeness, in Lemma 4, we show that cases in Lemma 3 can not be solved in the order $r_{2}, r_{1}, r_{2}$. The proof is deferred into the appendix.
Lemma 4. For any $\sigma_{1} \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}, \pi_{1}, \pi_{2} \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}, \sigma \pi_{1} \sigma_{1} \pi_{2} \notin A_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ if

1) $a_{3}+a_{4}+b_{3}+b_{4}=2$ and $\min \left\{b_{1}+a_{2}, a_{1}+b_{2}\right\}=0$ hold or
2) $a_{3}+a_{4}+b_{3}+b_{4}=0$ and $b_{1}+a_{2}$ is odd (equivalently $a_{1}+b_{2}$ is odd) hold.

Lemma 5 shows that 3 steps is tight for transforming arbitrary permutation into a CRBF. The proof is put into the appendix.

Lemma 5. For all even number $n \geq 4$, there exists $\sigma \in A_{\{0,1\}^{n}}$ such that $\sigma \tau \pi \notin S_{\{0,1\}^{n}}^{\left(r_{3}\right)}$ for any $r_{1}, r_{2}, r_{3} \in$ $[n], \tau \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}, \pi \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$.

## V. Transforming CRBF to identity

In this section, we transform an even CRBF to id through 5 CCRBFs, where the first block can be merged with the last block of Proposition 1

Recall that given $\sigma \in S_{\{0,1\}^{n}}^{(1)}$, there exist $f, g \in S_{\{0,1\}^{n-1}}$ such that for all $\boldsymbol{y} \in\{0,1\}^{n-1}, \sigma(0 \boldsymbol{y})=$ $0 f(\boldsymbol{y}), \sigma(1 \boldsymbol{y})=1 g(\boldsymbol{y})$. We represent $\sigma$ by $2^{n} \times 2^{n}$ matrix and $f, g$ by $2^{n-1} \times 2^{n-1}$ matrix. For example, if $\tau=(00,01)(10,11) \in S C_{\{0,1\}^{2}}^{(1)}$, the basis is $00,01,10,11$, then

$$
\tau=\left[\begin{array}{llll}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{array}\right]
$$

where $f_{\tau}, g_{\tau}$ are

$$
f_{\tau}=g_{\tau}=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]=(0,1) \in S_{\{0,1\}^{1}}
$$

The proof in this section is based on the following two observations. The first observation is that, for any $h \in S_{\{0,1\}^{n-1}}$,

$$
\sigma=\left[\begin{array}{ll}
f & 0 \\
0 & g
\end{array}\right]=\left[\begin{array}{cc}
f h^{-1} & 0 \\
0 & f h^{-1}
\end{array}\right]\left[\begin{array}{cc}
\mathrm{id} & 0 \\
0 & h f^{-1} g h^{-1}
\end{array}\right]\left[\begin{array}{cc}
h & 0 \\
0 & h
\end{array}\right]
$$

The second observation is that, for $q \in S_{\{0,1\}^{n-2}}$, the following $\pi \in S_{\{0,1\}^{n}}^{(1)}$ is actually in $S C_{\{0,1\}^{n}}^{(2)}$

$$
\pi=\left[\begin{array}{cccc}
\text { id } & 0 & 0 & 0 \\
0 & \text { id } & 0 & 0 \\
0 & 0 & q & 0 \\
0 & 0 & 0 & q
\end{array}\right]
$$

Notice that $h f^{-1} g h^{-1}$ shares same cycle pattern with $f^{-1} g$. If we aim to prove $\sigma$ can be decomposed to identity in 4 steps, it suffices to show there exist $\sigma_{1} \in S C_{\{0,1\}^{n-1}}^{(j)}, \sigma_{2} \in S C_{\{0,1\}^{n-1}}^{(k)}$ such that $\sigma_{1} \sigma_{2}$ has same cycle pattern with $f^{-1} g \in S_{\{0,1\}^{n-1}}$.

However, Lemma 7 indicates $\sigma_{1} \sigma_{2}$ can not formulate a single $3 / 5$-cycle. In contrast, we show that $\sigma_{1} \sigma_{2}$ can indeed achieve any cycle pattern free of $3 / 5$-cycle by Proposition 3. To reduce $3 / 5$-cycles, we develop a cycle elimination algorithm as Lemma 6, which can be absorbed into the last block of Proposition 1 .
Lemma 6. For $n \geq 5, r_{1} \in[n]$ and $\sigma \in A_{\{0,1\}^{n}}$, there exists $\pi \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ such that $\sigma \pi$ is free of 3/5-cycles.
Proof. This $\pi$ is constructed in several rounds. In round- $i, \pi_{i} \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ is performed. Let $S_{i, c}$ be the set of $c$-cycles in $\sigma_{i-1}\left(\sigma_{0}=\sigma\right.$ and $\left.\sigma_{t}=\sigma \pi_{1} \pi_{2} \cdots \pi_{t}\right)$.

Denote $\zeta_{i}=\left|S_{i, 1}\right|+\left|S_{i, 2}\right|+\left|S_{i, 3}\right|+\left|S_{i, 4}\right|+\left|S_{i, 5}\right|$. If $S_{i-1,3} \cup S_{i-1,5} \neq \emptyset$, pick an arbitrary cycle $\mathscr{C}_{1}$ from it. Since $\mathscr{C}_{1}$ is an odd cycle, there exists $\boldsymbol{u} \in \mathscr{C}_{1}$ such that $\boldsymbol{v}:=\boldsymbol{u}^{\oplus r_{1}} \notin \mathscr{C}_{1}$. Let $\mathscr{C}_{2}$ be the cycle where $\boldsymbol{v}$ belongs. Define

$$
T=\mathscr{C}_{1} \cup\left\{\boldsymbol{w} \in \mathscr{C}_{2} \mid \text { dist }_{\min }^{\sigma_{i-1}}(\boldsymbol{v}, \boldsymbol{w}) \leq 5\right\}
$$

Note that $|T| \leq 5+11$. Since $n \geq 5$ and $2^{n-1}>|T|-1$, there must exist $t \notin T$ such that $\boldsymbol{u}_{r_{1}}=\boldsymbol{t}_{r_{1}}$ and $\boldsymbol{s}:=\boldsymbol{t}^{\oplus r_{1}} \notin T$. Then, let $\pi_{i}=(\boldsymbol{u}, \boldsymbol{t})(\boldsymbol{v}, \boldsymbol{s}) \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$. We will prove $\zeta_{\sigma_{i}}<\zeta_{\sigma_{i-1}}$, by checking the following cases.

- $\boldsymbol{t}, \boldsymbol{s} \notin \mathscr{C}_{2}$ : Swapping $\boldsymbol{u}, \boldsymbol{t}$ merges $\mathscr{C}_{1}$ with another cycle And similarly when swapping $\boldsymbol{v}, \boldsymbol{s}$.
- $\boldsymbol{t} \notin \mathscr{C}_{2}, s \in \mathscr{C}_{2}$ : Swapping $\boldsymbol{u}, \boldsymbol{t}$ merges $\mathscr{C}_{1}$ with another cycle. Then swapping $\boldsymbol{v}, \boldsymbol{s}$ splits new $\mathscr{C}_{2}$ into two cycles; and the length of neither is smaller than 6 , which will not increase the number of short cycles.
- $\boldsymbol{t} \in \mathscr{C}_{2}, \boldsymbol{s} \notin \mathscr{C}_{2}$ : Swapping $\boldsymbol{u}, \boldsymbol{t}$ merges $\mathscr{C}_{1}$ with $\mathscr{C}_{2}$. Then swapping $\boldsymbol{v}, s$ merges new $\mathscr{C}_{2}$ with another cycle.
- $\boldsymbol{t}, \boldsymbol{s} \in \mathscr{C}_{2}$ : Swapping $\boldsymbol{u}, \boldsymbol{t}$ merges $\mathscr{C}_{1}$ with $\mathscr{C}_{2}$. Then swapping $\boldsymbol{v}, \boldsymbol{s}$ splits new $\mathscr{C}_{2}$ into two cycles; and the length of neither is smaller than 6 , which will not increase the number of short cycles.
Repeat until $S_{i, 3} \cup S_{i, 5}=\emptyset$. Suppose this process has $k$ rounds, then the desired permutation $\pi$ is $\pi_{1} \pi_{2} \cdots \pi_{k}$.
Given $r_{1}, r_{2} \in[n]$, for any $\boldsymbol{x} \in\{0,1\}^{n}$, define $\boldsymbol{x}_{\text {out }}$ as the binary string of $\boldsymbol{x}$ throwing away the $r_{1}-$ and $r_{2}$-th bit; then for any $S \subseteq\{0,1\}^{n}$ and $a, b \in\{0,1\}$, define

$$
S_{a b}=\left\{\boldsymbol{x}_{\text {out }} \mid \boldsymbol{x} \in S, \boldsymbol{x}_{r_{1}}=a, \boldsymbol{x}_{r_{2}}=b\right\} .
$$

Now we present two algorithms (RPACK and TPACK) to generate desired cycle patterns. RPACK in Algorithm 2 performs two inplace concurrent permutations to obtain $a, b$-cycle. For example, Let $r_{1}=1, r_{2}=2$ and $a=4, b=6$,

$$
\begin{aligned}
S= & \{0000,0001,0010,0100,0101,0110 \\
& 1000,1001,1010,1100,1101,1110\}
\end{aligned}
$$

As in Figure 9, RPACK $\left(r_{1}, r_{2}, a, b, S\right)$ returns

$$
\begin{aligned}
\tau= & (1100,0100)(1000,0000) \\
\pi= & (1100,1101,1110,1010)(1000,1001) \\
& (0100,0101,0110,0010)(0000,0001)
\end{aligned}
$$



Fig. 9. An example of Algorithm 2

```
Algorithm 2: \(a, b\)-cycle in rectangles (RPACK)
    Input: \(r_{1}, r_{2}, a, b, S(0<a \leq b)\)
    Output: \(\pi \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}, \tau \in S C_{\{0,1\}}^{\left(r_{2}\right)}\)
    \(/ \star \pi \tau\) is \(a, b\)-cycle, \(\operatorname{supp}(\pi), \operatorname{supp}(\tau) \subseteq S\)
    if \((|S| \not \equiv 0 \bmod 4)\) or \((|S| \neq a+b)\) then
        return Error /* Invalid pattern */
    end
    if not ( \(S_{00}=S_{01}=S_{10}=S_{11}\) ) then
        return Error /* Invalid support */
    end
    \(k \leftarrow\lfloor a / 2\rfloor, l \leftarrow\lfloor b / 2\rfloor\)
    switch \(a, b\) do
        /* Fall into the first satisfied */
        case \(a=b\) do Top left case
        case \(a\) is even do Top right case
        case \(a=1, b \geq 7\) do Bottom left case
        case \(a\) is odd, \(\bar{a}, b \geq 5\) do Bottom right case
        otherwise do return Error
    end
    \(\pi \leftarrow\) solid arrows, \(\tau \leftarrow\) dashed arrows
    return \(\pi, \tau\)
    /* For the meaning of following figures, see Figure 3 and Example x */
                                \((k+l) / 2\)
    \(\underset{\mathrm{O} \rightarrow \mathrm{O} \rightarrow 0 \cdots \mathrm{O} \rightarrow 0}{\mathrm{O}} \mathrm{O}\)
                                \(\rightarrow \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0\)
```




```
        \((k-l+1) / 2\)
    \(\underbrace{\substack{0 \rightarrow 0}}_{(l+1) / 2}\)
```



The aim of TPACK in Algorithm 3 is to obtain $a, b, c, d$-cycle. It first divides the general rectangle shaped $S$ into two trapezoid shaped $X_{0}, X_{1}$, then performs two inplace concurrent permutations on $X_{0}, X_{1}$ to obtain $a, b$-cycle and $c, d$-cycle respectively. Since $a, b$-cycle and $c, d$-cycle are generated separately on $X_{0}, X_{1}$, these two parts can be performed simultaneously, thus can be combined together.

Now we give the proof of Proposition 3, which states two CCRBFs can compose most of the patterns.
Proof of Proposition 3 W.l.o.g, assume $r_{1}=1, r_{2}=2$. Let $c_{k}$ be the number of $k$-cycles in $\sigma$ and $c_{1}$ is the number of fix-points.

Now, we initialize $\pi=\tau=\mathrm{id}, T=\{0,1\}^{n-2}$ and construct them in two stages.
Stage I (Pairing). Initialize the set of pairs as $P=\emptyset$.

- Pick $i$ with $c_{i}>0$ and update $c_{i} \leftarrow c_{i}-1$.
- Pick $j$ with $c_{j}>0, i+j \equiv 0 \bmod 2$ and update $c_{j} \leftarrow c_{j}-1$.
- Swap $i, j$ if $i>j$. Then add $(i, j)$ to $P$.

Repeat the procedure until $c_{i}=0$ for any $i$.
Since $\sigma$ is even, we have $\sum_{i} c_{2 i} \equiv 0 \bmod 2$. Meanwhile, $\sum_{i} c_{2 i-1} \equiv \sum_{k} k c_{k} \equiv 2^{n} \equiv 0 \bmod 2$. Thus as long as the first step succeeds, the second step will not fail.
Stage II (Construct). Now we construct $\pi, \tau$.

- Pick $(a, b) \in P$ and remove it from $P$.

```
Algorithm 3: \(a, b, c, d\)-cycle in trapezoids (ТРАСК)
    Input: \(r_{1}, r_{2}, a, b, c, d, S(0<a \leq b, 0<c \leq d)\)
    Output: \(\pi \in S C_{\{0,1\}^{n}}^{\left(r_{1},\right.}, \tau \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}\)
    \(/ * \pi \tau\) is \(a, b, c, d\)-cycle, \(\operatorname{Supp}(\pi), \operatorname{Supp}(\tau) \subseteq S\) */
    if \((|S| \not \equiv 0 \bmod 4)\) or \((|S| \neq a+b+c+d)\) then
        return Error
                                    /* Invalid pattern */
    end
    if not ( \(S_{00}=S_{01}=S_{10}=S_{11}\) ) then
        return Error
    end
    if \(a+b \not \equiv 2 \bmod 4\) then
        return Error \(/ \star\) Invalid pattern */
    end
    Pick \(T \subseteq S_{00},|T|=\lfloor(a+b) / 4\rfloor\) and \(\boldsymbol{t} \in S_{00} \backslash T\)
    \(X_{0} \leftarrow\left\{\boldsymbol{x} \in S \mid\left(\boldsymbol{x}_{\text {out }} \in T_{0}\right) \vee\left(\boldsymbol{x}_{\text {out }}=\boldsymbol{t} \wedge \boldsymbol{x}_{r_{2}}=1\right)\right\}\)
    \(X_{1} \leftarrow S \backslash X_{0}\)
    \(\pi \leftarrow \mathrm{id}, \tau \leftarrow \mathrm{id}\)
    foreach \((u, v, i) \in\{(a, b, 0),(c, d, 1)\}\) do
        /* \(\operatorname{Supp}\left(\pi_{i}\right), \operatorname{Supp}\left(\tau_{i}\right) \subseteq X_{i}\)
        \(k \leftarrow\lfloor u / 2\rfloor, l \leftarrow\lfloor v / 2\rfloor\)
        switch \(u, v\) do
            /* Fall into the first satisfied
            case \(u=v\) do Top left case
            case \(u\) is even do Top right case
            case \(u=1, v \geq 7\) do Bottom left case
            case \(u\) is odd, \(u, v \geq 5\) do Bottom right case
            otherwise do return Error
        end
        \(\pi_{i} \leftarrow\) solid arrows, \(\tau_{i} \leftarrow\) dashed arrows
        \(\pi \leftarrow \pi \pi_{i}, \tau \leftarrow \tau \tau_{i}\)
    end
    return \(\pi, \tau\)
    /* For the meaning of following figures, see Figure 3 and Example x */
    \(\underset{0 \rightarrow 0 \rightarrow 0 \cdots 0 \rightarrow 0 \rightarrow 0}{ }\)
                                \(\overbrace{\substack{-\cdots+0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \\ \rightarrow 0 \cdots 0 \rightarrow 0}}^{(k+l-1) / 2}\)
```




- If $a+b \equiv 0 \bmod 4$, select $S \subseteq T,|S|=(a+b) / 4$. Let

$$
\pi^{\prime}, \tau^{\prime} \leftarrow \operatorname{RPACK}\left(r_{1}, r_{2}, a, b,\{0,1\}^{2} \times T\right)
$$

- If $a+b \equiv 2 \bmod 4$, pick $(c, d) \in P, c+d \equiv 2 \bmod 4$ and remove it from $P$. Select $S \subseteq T,|S|=$ $(a+b+c+d) / 4$. Let

$$
\pi^{\prime}, \tau^{\prime} \leftarrow \text { ТРАСК }\left(r_{1}, r_{2}, a, b, c, d,\{0,1\}^{2} \times T\right)
$$

- Update $T \leftarrow T \backslash S, \pi \leftarrow \pi \pi^{\prime}, \tau \leftarrow \tau \tau^{\prime}$.

Repeat the procedure until $P=\emptyset$.
Since $\sum_{(a, b) \in P} a+b=2^{n}$ and $n \geq 4$, if there is $a+b \equiv 2 \bmod 4$ then there must be another pair $c+d \equiv 2$ $\bmod 4$. Also, $\sigma$ is free of $3 / 5$-cycle, thus RPACK and TPACK will not err.

Since $\pi^{\prime}, \tau^{\prime}$ s are inplace and separate, $\pi, \tau$ is the desired permutation.
Combining these result, finally we are able to prove Proposition 2
Proof of Proposition 2 W.l.o.g, we assume $r_{1}=1, r_{2}=2$. Since $\sigma \in A_{\{0,1\}^{n}}^{\left(r_{1}\right)}$, there exist $f, g \in S_{\{0,1\}^{n-1}}$ such that

$$
\sigma=\left[\begin{array}{ll}
f & 0 \\
0 & g
\end{array}\right]
$$

Let $\pi_{1}=\left[\begin{array}{cc}\text { id } & 0 \\ 0 & g^{\prime}\end{array}\right]$, we have

$$
\begin{aligned}
\sigma \pi_{1} & =\left[\begin{array}{ll}
f & 0 \\
0 & g
\end{array}\right]\left[\begin{array}{cc}
\text { id } & 0 \\
0 & g^{\prime}
\end{array}\right] \\
& =\left[\begin{array}{cc}
f h^{-1} & 0 \\
0 & f h^{-1}
\end{array}\right]\left[\begin{array}{cc}
\text { id } & 0 \\
0 & h f^{-1} g g^{\prime} h^{-1}
\end{array}\right]\left[\begin{array}{cc}
h & 0 \\
0 & h
\end{array}\right]
\end{aligned}
$$

where $f, g^{\prime}, g, h \in S_{\{0,1\}^{n-1}}$ and $g^{\prime}, h$ shall be determined later.
Since $f^{-1} g$ is even, by Lemma 6, there exists $g^{\prime} \in S C_{\{0,1\}^{n-1}}^{\left(r_{2}\right)}$ such that $f^{-1} g g^{\prime}$ is free of $3 / 5$-cycle. Then by Proposition 3, there exist $\rho_{1} \in S C_{\{0,1\}^{n-1}}^{\left(r_{4}\right)}, \rho_{2} \in S C_{\{0,1\}^{n-1}}^{\left(r_{3}\right)}$ such that $\rho_{1} \rho_{2}$ has the same cycle pattern as $f^{-1} g g^{\prime}$. This condition is equal to that there exists $h \in S_{\{0,1\}^{n-1}}$ such that $h f^{-1} g g^{\prime} h^{-1}=\rho_{1} \rho_{2}$. Therefore

$$
\sigma \pi_{1}=\left[\begin{array}{cc}
f h^{-1} & 0 \\
0 & f h^{-1}
\end{array}\right]\left[\begin{array}{cc}
\text { id } & 0 \\
0 & \rho_{1}
\end{array}\right]\left[\begin{array}{cc}
\text { id } & 0 \\
0 & \rho_{2}
\end{array}\right]\left[\begin{array}{cc}
h & 0 \\
0 & h
\end{array}\right]
$$

Then setting

$$
\begin{gathered}
\pi_{1}=\left[\begin{array}{cc}
\text { id } & 0 \\
0 & g^{\prime}
\end{array}\right], \sigma_{1}=\left[\begin{array}{cc}
h^{-1} & 0 \\
0 & h^{-1}
\end{array}\right], \tau_{1}=\left[\begin{array}{cc}
\text { id } & 0 \\
0 & \rho_{2}^{-1}
\end{array}\right], \\
\tau_{2}=\left[\begin{array}{cc}
\text { id } & 0 \\
0 & \rho_{1}^{-1}
\end{array}\right], \sigma_{2}=\left[\begin{array}{cc}
h f^{-1} & 0 \\
0 & h f^{-1}
\end{array}\right]
\end{gathered}
$$

will do.
For completeness, we show in Lemma 7 that the restriction that the cycle pattern contains no $3 / 5$-cycle is inevitable. The proof is put into the appendix.
Lemma 7. For any $\sigma_{1} \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}, \sigma_{2} \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}, \sigma_{1} \sigma_{2}$ can not be a permutation that is merely a 3 -cycle or a 5-cycle.

## VI. AN EXPLICIT EXAMPLE OF OUR ALGORITHM

In this section, we decompose a specified $\sigma \in A_{\{0,1\}^{4}}$ to 7 blocks of 3-bit RBFs by our algorithm. Here

$$
\begin{aligned}
\sigma= & (1001,1100,0101)(1110,0110,0111,1111) \\
& (1010,0010,0011,1011)
\end{aligned}
$$

## A. Transform $\sigma$ to CRBF

Step 1. Choose $r_{1}=1, r_{2}=2$. Using method in Section IV-A we construct colored cube for $\sigma$ as Figure 10 Read the colored cube, we get $a_{1}=1, a_{2}=0, a_{3}=1, a_{4}=2 ; b_{1}=1, b_{2}=0, b_{3}=1, b_{4}=2$.
Step 2. Check Lemma 1 and Lemma 3, we find this case falls into Lemma 1 we can transform $\sigma$ to $S_{\{0,1\}^{4}}^{(1)}$ by $S C_{\{0,1\}^{4},}^{(2)} S C_{\{0,1\}^{4}}^{(1)} S C_{\{0,1\}^{4}}^{(2)}$ Lemma 1. Specific construction are as follows.
Step 2.1. Using Lemma 2 we transform $\sigma$ to canonical form by $\pi=\pi_{1} \pi_{2}$. Let

$$
\pi_{1}=(1110,0111)(1010,0011)
$$

which transforms the colored cube to a cube with $a_{3}=b_{3}=0, a_{2}=0$. Setting $\pi_{2}=(0100,0101)(0000,0001)$, it rearrange the cube to canonical form. The process is pictured as Figure 11, Figure 12,


Fig. 10. Visualize $\sigma$ on a colored cube


Fig. 11. Colored cube for $\sigma \pi_{1}$


Fig. 12. Colored cube for $\sigma \pi_{1} \pi_{2}$

Step 2.2. Using Lemma 1 we construct the following CCRBFs

$$
\begin{aligned}
\pi_{3}= & (0100,1110)(0000,1010)(1101, \\
& 0110)(1001,0010) \\
\pi_{4}= & (1000,1010)(0000,0010) \\
\pi_{5}= & (1100,0100)(1000,0000)(1101, \\
& 0110)(1001,0010)
\end{aligned}
$$

It's easy to verify $\pi_{1}, \pi_{2}, \pi_{3}, \pi_{5} \in S C_{\{0,1\}^{4}}^{(2)}, \pi_{4} \in S C_{\{0,1\}^{4}}^{(1)}$ and

$$
\begin{aligned}
\pi_{1} \pi_{2} \pi_{3}= & (0000,0011,1010,0001)(0100,0111 \\
& 1110,0101)(0010,1001)(0110,1101)
\end{aligned}
$$

And finally we transform the colored cube for $\sigma$ to a white cube by verifying

$$
\begin{aligned}
\sigma^{(1)}= & \sigma\left(\pi_{1} \pi_{2} \pi_{3}\right) \pi_{4} \pi_{5} \\
= & (0000,0001)(0010,0011)(0100,0101) \\
& (0110,0111)(1000,1100,1111,1110 \\
& 1001,1011,1010)
\end{aligned}
$$

B. Transform $\sigma^{(1)}$ to identity

We can use two 3-bit RBFs to represent $\sigma^{(1)}$. That is

$$
\begin{aligned}
& f=(000,001)(010,011)(100,101)(110,111) \\
& g=(000,100,111,110,001,011,010)
\end{aligned}
$$

such that

$$
\sigma^{(1)}:(0, \boldsymbol{x}) \rightarrow(0, f(\boldsymbol{x})),(1, \boldsymbol{x}) \rightarrow(1, g(\boldsymbol{x}))
$$

Step 1. Determine whether $f^{-1} g$ has $3 / 5$-cycle. By directly calculating $f^{-1} g$ we know the answer is no. So we can jump the process for eliminating $3 / 5$-cycles.

$$
f^{-1} g=(000,101,100,110)(001,010)
$$

Step 2. First, we construct a $\sigma_{1} \in S C_{\{0,1\}^{3}}^{(1)}$, a $\sigma_{2} \in S C_{\{0,1\}^{3}}^{(2)}$ to generate a 2,4-cycle pattern like $f^{-1} g$. Based on Algorithm TPACK

$$
\begin{aligned}
& \sigma_{1}=(000,011)(100,111) \\
& \sigma_{2}=(010,110)(000,100)
\end{aligned}
$$

Step 3. Find $h \in S_{\{0,1\}^{3}}$ such that $h\left(f^{-1} g\right) h^{-1}=\sigma_{1} \sigma_{2}$. By group theory we know that if $\tau=\left(i_{1}, i_{2}, \ldots, i_{k}\right)$, then $h \tau h^{-1}=\left(h\left(i_{1}\right), h\left(i_{2}\right), \ldots, h\left(i_{k}\right)\right)$. So we can construct

$$
h=(101,111)(001,010,110,011) .
$$

Step 4. Now we verify $h\left(f^{-1} g\right) h^{-1}=\sigma_{1} \sigma_{2}$. Thus

$$
\begin{aligned}
\sigma^{(1)} & =\left[\begin{array}{ll}
f & \\
& g
\end{array}\right] \\
& =\left[\begin{array}{ll}
f h^{-1} & \\
& f h^{-1}
\end{array}\right]\left[\begin{array}{ll}
\text { id } & \\
& \sigma_{1}
\end{array}\right]\left[\begin{array}{ll}
\text { id } & \\
& \sigma_{2}
\end{array}\right]\left[\begin{array}{ll}
h & \\
& h
\end{array}\right] \\
& \triangleq \pi_{6} \pi_{7} \pi_{8} \pi_{9} .
\end{aligned}
$$

Written in the form of permutation cycle pattern,

$$
\begin{aligned}
\pi_{6}= & (0000,0001,0010)(0011,0111,0100,0101,0110) \\
& (1000,1001,1010)(1011,1111,1100,1101,1110) \\
\pi_{7}= & (1000,1011)(1100,1111) \\
\pi_{8}= & (1010,1110)(1000,1100) \\
\pi_{9}= & (0101,0111)(0001,0010,0110,0011) \\
& (1101,1111)(1001,1010,1110,1011) .
\end{aligned}
$$

## C. Summary

In a word, $\sigma=\pi_{6} \pi_{7} \pi_{8} \pi_{9} \pi_{5}^{-1} \pi_{4}^{-1}\left(\pi_{1} \pi_{2} \pi_{3}\right)^{-1}$, where

$$
\begin{aligned}
& \pi_{6}, \pi_{9}, \pi_{4}^{-1} \in S C_{\{0,1\}^{4}}^{(1)} \\
& \pi_{7}, \pi_{5}^{-1},\left(\pi_{1} \pi_{2} \pi_{3}\right)^{-1} \in S C_{\{0,1\}^{4}}^{(2)} \\
& \pi_{8} \in S C_{\{0,1\}^{4}}^{(3)}
\end{aligned}
$$

## VII. Even block Depth

In previous sections, we prove for any $\sigma \in A_{\{0,1\}^{n}}, n \geq 6, \sigma$ has block depth 7 . However, the block itself may be an odd permutation which resists further decomposition. In this section, we address this concern and show that any $\sigma \in A_{\{0,1\}^{n}}$, with $n \geq 10$, has even block depth 10 , which is stated as Theorem 2 This is proven by some modification of the framework in previous sections. The idea is similar, but the analysis is much more complicated. Here we only sketch the proof and leave the detail in the appendix.

We prove Theorem 2 by the modified versions of Proposition 1 and Proposition 2. Specifically, we prove that arbitrary even $n$-bit permutation can be transformed to even CRBF by 3 even blocks; arbitrary even CRBF can be transformed to identity by 8 even blocks. Choosing carefully, we can merge some of them and finally decompose even $n$-bit permutation to identity using 10 even blocks. The results are summarized as the following two propositions.
Proposition 4. For $n \geq 4, \sigma \in A_{\{0,1\}^{n}}$ and $r_{1} \in[n]$, there exist at least $(n-2)$ different $r_{2} \in[n] \backslash\left\{r_{1}\right\}$ such that there exist $\sigma_{1} \in A C_{\{0,1\}^{n}}^{\left(r_{1}\right)}, \pi_{1}, \pi_{2} \in A C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$ satisfying $\sigma \pi_{1} \sigma_{1} \pi_{2} \in A_{\{0,1\}^{n}}^{\left(r_{1}\right)}$.

Here we only give the intuition. The key observation in the proof of Lemma 1 is that we can always swap some nodes without changing color in cuboid. For example, if we swap two nodes who has the the same color and lie in the same face, then the corresponding colored cuboid will not change. This observation can be used to modify the permutation to be concurrently even.

For example, we can transform two B-cards to white cube by the following two methods.


Fig. 13. Transform two B-cards to identity where $\tau_{1}$ is concurrently even, $\tau_{2}, \tau_{3}$ are concurrently odd.


Fig. 14. Transform two B-cards to identity where $\tau_{1}^{\prime}$ is concurrently odd, $\tau_{2}^{\prime}, \tau_{3}^{\prime}$ are concurrently even.
Proposition 5 states that we can recover any even $n$-bit CRBF by 8 concurrently even CCRBFs.
Proposition 5. For $n \geq 10, r_{1} \in[n], \sigma \in A_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ and distinct $r_{2}, r_{3}, r_{4} \in[n] /\left\{r_{1}\right\}$. There exist $\sigma_{1}, \sigma_{4}, \sigma_{7} \in$ $A C_{\{0,1\}^{n}}^{\left(r_{1}\right)}, \sigma_{6}, \sigma_{8} \in A C_{\{0,1\}^{n}}^{\left(r_{2}\right)}, \sigma_{2}, \sigma_{5} \in A C_{\{0,1\}^{n}}^{\left(r_{3}\right)}, \sigma_{3} \in A C_{\{0,1\}^{n}}^{\left(r_{4}\right)}$ such that $\sigma=\sigma_{1} \circ \cdots \circ \sigma_{8}$.

Similar to the proof of Proposition 2, here we first construct a concurrently even CCRBF $\pi$ such that $\sigma \pi$ is free of $3 / 5$-cycle and $\sigma \pi$ has an even cycle. Then we use concurrently even CCRBFs to formulate cycles. Besides, we need to solve some special cases. Those proofs are similar to the corresponding ones and are put into the appendix. Here is the new lemma for eliminating cycles.
Lemma 8. For $n \geq 8, r_{1} \in[n]$ and $\sigma \in A_{\{0,1\}^{n}}$, there exists $\pi \in A C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ such that $\sigma \pi$ is free of $3 / 5-c y c l e$, and $\sigma \pi$ has at least an even cycle.

The additional demand for an even cycle comes from the following lemma.
Lemma 9. For $\sigma, \pi \in S_{\{0,1\}^{n}} . \sigma, \pi$ have the same cycle pattern and $\sigma$ has an even cycle. Then there exists $h \in A_{\{0,1\}^{n}}$ such that $h \sigma h^{-1}=\pi$.

These following 2 lemmas ensure that cycle pattern can be constructed by 2 concurrently even CCRBFs on different dimensions under some restrictions.

Lemma 10. For $\sigma \in A_{\{0,1\}^{n}}$ which is free of $3 / 5$-cycle and contains at least 12 cycles with the length of at least 2, there exist $\pi \in A C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ and $\tau \in A C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$ such that $\pi \tau$ has the same cycle pattern with $\sigma$.

Lemma 11. For $\sigma \in A_{\{0,1\}^{n}}$ which is free of $3 / 5$-cycle and contains a cycle with the length of at least 12 , there exist $\pi \in A C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ and $\tau \in A C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$ such that $\pi \tau$ has the same cycle pattern with $\sigma$.

The last preparation is to construct a concurrently odd CCRBF by 4 concurrently even CCRBFs.
Lemma 12. For $n \geq 3$, distinct $r_{1}, r_{2}, r_{3} \in[n]$, there exists concurrently odd $\pi \in S C_{\{0,1\} n}^{\left(r_{1}\right)}$, such that $\pi=\tau_{1} \tau_{2} \tau_{3} \tau_{4}$, where $\tau_{1} \in A C_{\{0,1\}^{n}}^{\left(r_{3}\right)}, \tau_{2}, \tau_{4} \in A C_{\{0,1\}^{n}}^{\left(r_{2}\right)}, \tau_{3} \in A C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$.

Finally we give proof of Proposition 5.
Proof of Proposition 5. W.l.o.g, assume $r_{1}=1, r_{2}=2$. Similar to the proof of Proposition 2, since $\sigma \in A_{\{0,1\}^{n}}^{\left(r_{1}\right)}$, there exist $f, g \in S_{\{0,1\}^{n-1}}$ such that $\sigma=\left[\begin{array}{ll}f & \\ & g\end{array}\right]$. Observe that for any $g^{\prime}, s, h \in S_{\{0,1\}^{n-1}}$, let $\pi_{9}=\left[\begin{array}{ll}\text { id } & \\ & g^{\prime}\end{array}\right]$, we have

$$
\begin{aligned}
\sigma \pi_{9}= & {\left[\begin{array}{ll}
f s h & \\
& f s h
\end{array}\right]\left[\begin{array}{ll}
\text { id } & \\
& h^{-1}(f s)^{-1}\left(g g^{\prime} s\right) h
\end{array}\right] } \\
& {\left[\begin{array}{ll}
h^{-1} & \\
& h^{-1}
\end{array}\right]\left[\begin{array}{ll}
s^{-1} & \\
& s^{-1}
\end{array}\right] . }
\end{aligned}
$$

We first use Lemma 8 to choose $g^{\prime} \in A C_{\{0,1\}^{n-1}}^{\left(r_{2}\right)}$, such that $f^{-1} g g^{\prime}$ is free of $3 / 5$-cycle and has an even cycle. For convenience, we perform another pre-processing. Technically, if $f^{-1} g g^{\prime}$ has a cycle of length $\geq 12$ or has at least 12 cycles, we do nothing. Otherwise, there are at least 13 fix-point pairs $\left(\boldsymbol{x}_{1}, \boldsymbol{y}_{1}\right), \ldots,\left(\boldsymbol{x}_{13}, \boldsymbol{y}_{13}\right)$ in $f^{-1} g g^{\prime}$ satisfying $\left(\boldsymbol{x}_{i}\right)_{r_{1}}=\left(\boldsymbol{y}_{i}\right)_{r_{1}}=1$ and $\boldsymbol{x}_{i}=\boldsymbol{y}_{i}^{\oplus r_{2}}$ for all $i \in[13]$ since $n \geq 10$. Thus, we can perform $g^{\prime \prime} \in A C_{\{0,1\}^{n-1}}^{\left(r_{2}\right)}$ to add two 13 -cycles without affecting other cycle in $f^{-1} g g^{\prime}$. For simplicity, we update $g^{\prime}$ as $g^{\prime} g^{\prime \prime}$.

Since $\sigma, \pi_{9}$ are even, $f, g g^{\prime}$ are either both even or both odd. If $f, g g^{\prime}$ are both even, we choose $s=$ id. If otherwise, using Lemma 12, we choose concurrently odd $\left[\begin{array}{ll}s^{-1} & \\ & s^{-1}\end{array}\right] \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ where $s^{-1}$ is odd, and construct it with 4 even blocks in order $r_{3}, r_{2}, r_{1}, r_{2}$ (i.e., $\pi_{5}, \pi_{6}, \pi_{7}, \pi_{8}$ ). Then $f s, g g^{\prime} s$ will be both even.

Next we synthesize $\left[\begin{array}{cc}\text { id } & \\ & h^{-1}(f s)^{-1}\left(g g^{\prime} s\right) h\end{array}\right]$. Note that $f^{-1} g g^{\prime}$ either contains at least 12 cycles, or contains a long cycle of length at least 12. According to Lemma 10 and Lemma 11 there exist $\tau_{1} \in A C_{\{0,1\}^{n-1}}^{\left(r_{3}\right)}$ and $\tau_{2} \in A C_{\{0,1\}^{n-1}}^{\left(r_{4}\right)}$ such that $\tau_{1} \tau_{2}$ has the same cycle pattern with $f^{-1} g g^{\prime}$ and $(f s)^{-1} g g^{\prime} s$. Furthermore, since $f^{-1} g g^{\prime}$ has an even cycle, by Lemma 9, there exists $h \in A_{\{0,1\}^{n-1}}$ such that $\tau_{1} \tau_{2}=h^{-1}(f s)^{-1} g g^{\prime} s h$.

To sum up, let $\pi_{1}=\left[\begin{array}{cc}(f s) h & \\ & (f s) h\end{array}\right], \pi_{2}=\left[\begin{array}{cc}i d & \\ & \tau_{1}\end{array}\right], \pi_{3}=\left[\begin{array}{cc}i d & \\ & \tau_{2}\end{array}\right], \pi_{3}=\left[\begin{array}{cc}h^{-1} & \\ & h^{-1}\end{array}\right]$. Then $\pi_{1}, \pi_{4}, \pi_{7} \in$ $A C_{\{0,1\}^{n}}^{\left(r_{1}\right)}, \pi_{6}, \pi_{8}, \pi_{9} \in A C_{\{0,1\}^{n}}^{\left(r_{2}\right)}, \pi_{2}, \pi_{5} \in A C_{\{0,1\}^{n}}^{\left(r_{3}\right)}, \pi_{3} \in A C_{\{0,1\}^{n}}^{\left(r_{4}\right)}$, and

$$
\sigma=\pi_{1} \pi_{2} \pi_{3} \pi_{4} \pi_{5} \pi_{6} \pi_{7}\left(\pi_{8} \pi_{9}^{-1}\right)
$$

## VIII. Conclusion and open questions

In our work, we offer a method to decompose arbitrary even $n$-bit reversible Boolean function (RBF) into 7 blocks of ( $n-1$ )-bit RBFs for $n \geq 6$, or into 10 blocks of even $(n-1)$-bit RBFs for $n \geq 10$, where the blocks have certain freedom to choose. Technically, we transform even RBF to an even controlled reversible Boolean function (CRBF) by 3 blocks. Then we transform the even CRBF to identity by 5 blocks. In addition, the last block of the first step can be merged with the first block of the second step, thus providing a 7 -depth decomposition. The road map of even block depth is similar but much more complicated.

One direct open question is whether the constant 7 (and 10) can be further improved and what is the optimal constant. Besides, one may try to relax the conditions that $n \geq 6$ and $n \geq 10$. Another interesting question is, given an even $n$-bit RBF, if we are allowed to use general unitary blocks to synthesize it, can we use strictly fewer blocks than only using RBF blocks?

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## IX. Appendix

Proof of Lemma 4 Assume for contradiction there exist $\sigma_{1} \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}, \pi_{1}, \pi_{2} \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$, such that $\sigma \pi_{1} \sigma_{1} \pi_{2} \in$ $S_{\{0,1\}^{n}}^{\left(r_{1}\right)}$. Construct the black-white cuboid for $\sigma$.

For the $1^{\text {st }}$ case, define $\eta$ as the number of and ${ }^{\circ}$. It is easy to check $\eta \equiv 2 \bmod 4$ at the beginning and any permutation $\pi_{1} \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$ does not changes the value of $\eta \bmod 4$. Note that any permutation $\sigma_{1} \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ does not change $\eta$. Thus, $\pi_{2} \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$ can not transform all node to white, since it requires $\eta \equiv 0$ mod 4, which is a contradiction.

For the $2^{\text {nd }}$ case, define $\xi$ as the number of $\bullet$. It is easy to check $\xi$ is odd at the beginning and any permutation $\pi_{1} \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$ does not change its parity. Note that any permutation $\sigma_{1} \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ does not change $\xi$. Thus, $\pi_{2} \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$ can not transform all node to white, since it requires $\xi$ is even, which is a contradiction.
Proof of Lemma 5 Let

$$
\sigma_{3}=(000,001)(101,111)(010,110) \in S_{\{0,1\}^{3}}
$$

then define $\sigma_{k+1}$ recursively based on $\sigma_{k}$ and let $\sigma=\sigma_{n}$. Assume $\boldsymbol{u} \in\{0,1\}^{k}$ is a fix-point under $\tau_{k}$, then

$$
\sigma_{k+1}(x)= \begin{cases}0 \sigma_{k}(\boldsymbol{v}), & x=0 \boldsymbol{v}, \boldsymbol{v} \neq \sigma_{k}(v) \\ 1 \boldsymbol{u}, & \boldsymbol{x}=0 \boldsymbol{u} \\ 0 \boldsymbol{u}, & \boldsymbol{x}=1 \boldsymbol{u} \\ \boldsymbol{x}, & \text { otherwise }\end{cases}
$$

Thus $\sigma_{k} \in S_{\{0,1\}^{k}}$ is the composition of $k$ disjoint swaps.
We paint $\boldsymbol{x} \in\{0,1\}^{n}$ black if $\sigma(\boldsymbol{x})_{r_{3}} \neq \boldsymbol{x}_{r_{3}}$. Therefore, only two $\boldsymbol{x}$ 's will be black and their coordinates are distinct only in $r_{3}$-th. Thus, w.l.o.g, assume $r_{1}, r_{2}, r_{3}$ are distinct. Following the same notation $a_{i}$ 's, $b_{i}$ 's in Section IV-A, we have $a_{2}=b_{2}=1, a_{1}=a_{3}=a_{4}=b_{1}=b_{3}=b_{4}=0$. Thus after $\tau \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}, a_{1}+b_{2}=$ $b_{1}+a_{2}=1$. Since $\pi \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$ will have to eliminate all black nodes, the pattern in the $r_{2}=0$ part should be the same with the $r_{2}=1$ part. Thus a contradiction.

Proof of Lemma 7 W.l.o.g, assume $r_{1}=1, r_{2}=2$. Suppose $\sigma=\sigma_{1} \sigma_{2}$ is a 3-cycle.

- If $\sigma \in S_{\{0,1\}^{n}}^{(1)}$, then $\sigma_{2}=\sigma_{1}^{-1} \sigma$ must belong to $S_{\{0,1\}^{n}}^{(1)} \cap S C_{\{0,1\}^{n} n}^{(2)}$, thus there exist $\tau_{0}, \tau_{1} \in S_{\{0,1\}^{n-2}}$ that for any $\boldsymbol{x} \in\{0,1\}^{n-2}, \sigma_{2}(0 a \boldsymbol{x})=0 a \tau_{0}(\boldsymbol{x}), \sigma_{2}(1 a \boldsymbol{x})=1 a \tau_{1}(\boldsymbol{x})$, for $a=0,1$.
For $\sigma_{1} \in S C_{\{0,1\}^{n}}^{(1)}$, there exists $g \in S_{\{0,1\}^{n-1}}$ such that for any $\boldsymbol{y} \in\{0,1\}^{n-1}, \sigma_{1}(a \boldsymbol{y})=a g(\boldsymbol{y})$. Then $\sigma(a b \boldsymbol{x})=a g\left(b \tau_{a}(\boldsymbol{x})\right)$. Thus, if $\sigma$ is 3-cycle, then w.l.o.g, we can assume $\sigma(0 b \boldsymbol{x})=0 b \boldsymbol{x}$, then $g\left(b \tau_{0}(\boldsymbol{x})\right)=b \boldsymbol{x}$ and $\sigma(1 b \boldsymbol{x})=1 g\left(b \tau_{1}(\boldsymbol{x})\right)=1 b \tau_{0}^{-1} \tau_{1}(\boldsymbol{x})$. Patterns in $\{10\} \times\{0,1\}^{n-2}$ should be the same with patterns in $\{11\} \times\{0,1\}^{n-2}$. Thus patterns in the whole space can not be only a cycle., which means $\sigma$ can not be a 3 -cycle.
- If $\sigma \in S_{\{0,1\}^{n}}^{(2)}$, the analysis is similar as $\sigma^{-1}=\sigma_{2}^{-1} \sigma_{1}^{-1}$.
- If $\sigma \notin S_{\{0,1\}^{n}}^{(1)} \cup S_{\{0,1\}^{n}}^{(2)}$. We prove $\sigma \sigma_{2}^{-1} \sigma_{1}^{-1}$ does not belong to $S_{\{0,1\}^{n}}^{(1)}$ thus it can not be id. Towards this, we construct a colored cuboid described in Section IV. Then the cuboid will have 2 black nodes.
Notice that $\sigma_{1}^{-1}$ does not change the number of black nodes. Thus the colored cuboid for $\sigma \sigma_{2}^{-1}$ is white. If we use $\eta$ to denote the number of black nodes. Then $\eta$ in the colored cuboid for $\sigma$ must satisfy $\eta \equiv 0 \bmod 4$, thus a contradiction.
On the other hand, suppose $\sigma=\sigma_{1} \sigma_{2}$ is a 5 -cycle.
- If $\sigma \in S_{\{0,1\}^{n}}^{(1)} \cup S_{\{0,1\}^{n}}^{(2)}$, the analysis is similar.
- If $\sigma \notin S_{\{0,1\}^{n}}^{(1)} \cup S_{\{0,1\}^{n}}^{(2)}$. Construct a colored cuboid and use $\eta$ to denote the number of black nodes in the cuboid. According to the definition, $\eta$ must be even. If $\eta=2$, the analysis is similar.
Now we assume $\eta=4$. Since $\sigma \sigma_{2}^{-1} \sigma_{1}^{-1}=$ id and $\sigma_{1}^{-1}$ does not change number of black nodes, we conclude colored cuboid for $\sigma \sigma_{2}^{-1}$ is white and the 4 black nodes for $\sigma$ must be $\boldsymbol{x}, \boldsymbol{x}^{\oplus 2}, \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}}^{\oplus 2}$ for some $\boldsymbol{x}, \tilde{\boldsymbol{x}}, \boldsymbol{x}_{1} \neq$ $\tilde{\boldsymbol{x}}_{1}, \boldsymbol{x}_{2}=\tilde{\boldsymbol{x}_{2}}$. W.l.o.g, we assume the fifth element in the 5 -cycle to be $\boldsymbol{z}$ where $\boldsymbol{z}_{1}=\boldsymbol{x}_{1}$.

Also, we can assume the relative position of the black nodes in the cycle is $\boldsymbol{x}, \tilde{\boldsymbol{x}}, \boldsymbol{x}^{\oplus 2}, \tilde{\boldsymbol{x}}^{\oplus 2}$ or $\boldsymbol{x}, \tilde{\boldsymbol{x}}^{\oplus 2}, \boldsymbol{x}^{\oplus 2}, \tilde{\boldsymbol{x}}$. Let $\pi=(\boldsymbol{x}, \tilde{\boldsymbol{x}})\left(\boldsymbol{x}^{\oplus 2}, \tilde{\boldsymbol{x}}^{\oplus 2}\right) \in S C_{\{0,1\}^{n}}^{(2)}$. By checking all possible arrangement of $\boldsymbol{z}$, we have the following cases:

- $\sigma=\left(\boldsymbol{x}, \tilde{\boldsymbol{x}}, \boldsymbol{x}^{\oplus 2}, \tilde{\boldsymbol{x}}^{\oplus 2}, \boldsymbol{z}\right)$. Then $\sigma \pi=\left(\boldsymbol{x}, \boldsymbol{x}^{\oplus 2}, \boldsymbol{z}\right)=\sigma_{1}\left(\sigma_{2} \pi\right)$, which is impossible.
- $\sigma=\left(\boldsymbol{x}, \tilde{\boldsymbol{x}}^{\oplus 2}, \boldsymbol{x}^{\oplus 2}, \tilde{\boldsymbol{x}}, \boldsymbol{z}\right)$. Then $\sigma \pi=(\boldsymbol{x}, \boldsymbol{z})\left(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}}^{\oplus 2}\right)=\sigma_{1}\left(\sigma_{2} \pi\right)$. Construct the colored cuboid for $\sigma \pi$
with $r_{1}, r_{2}$ swapped and let $\xi$ be the number of $\bullet$. Then $\xi=1$. Since $\left(\sigma_{2} \pi\right)^{-1}$ does not change $\xi$, $(\sigma \pi)\left(\sigma_{2} \pi\right)^{-1} \sigma_{1}^{-1}=$ id requires $\xi \equiv 0 \bmod 2$, thus a contradiction.

Proof of Proposition 4. In the following, we transformed paired cards to identity by CCRBFs where $\tau_{1}$ has the different concurrently parity of the original construction in Lemma $1 \tau_{2}, \tau_{3}$ are concurrently even. Whether we use concurrently odd or even $\tau_{1}$ depends on the concurrently parity of $\pi^{\prime}$, which is constructed for modifying cycles in Lemma 8 .

Notice that, whether we use the even or concurrently odd construction of $\tau_{1}$ does not change the resulted cuboid, thus does not influence the following modification of $\tau_{2}, \tau_{3}$.

The constructions are as below. First, we give the new construction which transforms two A-cards or two B-cards into white cube.


The left cases can be modified to

- $\alpha=1, \beta=1, \gamma \geq 2$ : This graph shows how to tackle 1 A-card and 1 B -card with 2 C -cards.

- $\alpha=1, \beta=0, \gamma \geq 2$ : This graph shows how to tackle 1 A-card with 2 C -cards.

- $\alpha=0, \beta=1, \gamma \geq 2$ : This graph shows how to tackle 1 B -card with 2 C -card.

- $\alpha=2, \beta=1, \gamma \geq 1$ : This graph shows how to tackle 2 A -cards and 1 B -card with 1 C -card.


- $\alpha=0, \beta=3, \gamma \geq 1$ : This graph shows how to tackle 3 B-cards with 1 C -card.

- $\alpha=1, \beta=2, \gamma \geq 1$ : This graph shows how to tackle 1 A -card and 2 B -cards with 1 C -card.


Proof of Lemma 8 To ease the presentation, we say $\boldsymbol{u}, \boldsymbol{v}$ (or $\{\boldsymbol{u}, \boldsymbol{v}\}$ ) is a concurrent pair, if $\boldsymbol{u}=\boldsymbol{v}^{\oplus r_{1}}$.
The cycle transforming process is divided into following 4 stages:
Stage I. In the first stage, we attempt to construct $\pi_{0} \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ such that $\sigma \pi_{0}$ contains an even cycle $\mathscr{C}_{0}$ of length no more than 4.
Case 0. Suppose there exists a 2-cycle in $\sigma$ already, then simply let $\pi_{0}:=$ id.
Case 1. Suppose there exist $\boldsymbol{u}, \boldsymbol{v}$ such that $\boldsymbol{u}_{r_{1}}=\boldsymbol{v}_{r_{1}}=\sigma(\boldsymbol{u})_{r_{1}}=\sigma(\boldsymbol{v})_{r_{1}}$. If $\sigma(\boldsymbol{u})=\boldsymbol{v}$ (or $\left.\sigma(\boldsymbol{v})=\boldsymbol{u}\right)$, perform

$$
\pi_{0}:=(\boldsymbol{u}, \sigma(\boldsymbol{v}))\left(\boldsymbol{u}^{\oplus r_{1}}, \sigma(\boldsymbol{v})^{\oplus r_{1}}\right)
$$

and a 2 -cycle $\mathscr{C}_{0}=(\boldsymbol{v}, \sigma(\boldsymbol{v}))$ will appear. Otherwise, perform

$$
\pi_{0}^{\prime}:=\begin{aligned}
& (\boldsymbol{u}, \sigma(\boldsymbol{u}))\left(\boldsymbol{u}^{\oplus r_{1}}, \sigma(\boldsymbol{u})^{\oplus r_{1}}\right) \\
& (\boldsymbol{v}, \sigma(\boldsymbol{v}))\left(\boldsymbol{v}^{\oplus r_{1}}, \sigma(\boldsymbol{v})^{\oplus r_{1}}\right)
\end{aligned}
$$

and 2 fix-points $\sigma(\boldsymbol{u}), \sigma(\boldsymbol{v})$ will appear. Thus,

$$
\pi_{0}:=\pi_{0}^{\prime} \circ(\sigma(\boldsymbol{u}), \sigma(\boldsymbol{v}))\left(\sigma(\boldsymbol{u})^{\oplus r_{1}}, \sigma(\boldsymbol{v})^{\oplus r_{1}}\right)
$$

is as required such that $\sigma \pi_{0}$ contains a 2-cycle $\mathscr{C}_{0}=(\sigma(\boldsymbol{u}), \sigma(\boldsymbol{v}))$.
Case 2. Suppose there exists $\boldsymbol{u}$ such that $\boldsymbol{u}_{r_{1}}=\sigma^{2}(\boldsymbol{u})_{r_{1}}, \boldsymbol{u}_{r_{1}} \neq \sigma(\boldsymbol{u})_{r_{1}}$, and $\boldsymbol{u}, \sigma(\boldsymbol{u})^{\oplus r_{1}}, \sigma^{2}(\boldsymbol{u})$ are distinct. Let

$$
\pi_{0}:=\left(\boldsymbol{u}, \sigma^{2}(\boldsymbol{u})\right)\left(\boldsymbol{u}^{\oplus r_{1}}, \sigma^{2}(\boldsymbol{u})^{\oplus r_{1}}\right)
$$

Thus, $\sigma \pi_{0}$ will contain a 2-cycle $\mathscr{C}_{0}=\left(\sigma(\boldsymbol{u}), \sigma^{2}(\boldsymbol{u})\right)$.
Case 3. Suppose there exist fix-points $\boldsymbol{u}, \boldsymbol{v}$ such that $\boldsymbol{u}_{r_{1}}=\boldsymbol{v}_{r_{1}}$. Let

$$
\pi_{0}:=(\boldsymbol{u}, \boldsymbol{v})\left(\boldsymbol{u}^{\oplus r_{1}}, \boldsymbol{v}^{\oplus r_{1}}\right)
$$

Thus, $\sigma \pi_{0}$ will contain a 2 -cycle $\mathscr{C}_{0}=(\boldsymbol{u}, \boldsymbol{v})$.
Case 4. If none of the previous 3 cases holds, either there exists a 4 -cycle containing two concurrent pairs, or there exist distinct $\boldsymbol{u}_{1}, \ldots, \boldsymbol{u}_{6}$ such that $\left(\ldots, \boldsymbol{u}_{1}, \boldsymbol{u}_{2}, \boldsymbol{u}_{3}, \boldsymbol{u}_{4}, \boldsymbol{u}_{5}, \boldsymbol{u}_{6}, \ldots\right)$ is in $\sigma,\left(\boldsymbol{u}_{1}\right)_{r_{1}}=\left(\boldsymbol{u}_{3}\right)_{r_{1}}=$ $\left(\boldsymbol{u}_{5}\right)_{r_{1}}$ and $\left(\boldsymbol{u}_{1}, \boldsymbol{u}_{2}\right),\left(\boldsymbol{u}_{3}, \boldsymbol{u}_{4}\right),\left(\boldsymbol{u}_{5}, \boldsymbol{u}_{6}\right)$ are concurrent pairs. Then let $\pi_{0}=$ id for the first one; and $\pi_{0}=$ $\left(\boldsymbol{u}_{1}, \boldsymbol{u}_{3}, \boldsymbol{u}_{5}\right)\left(\boldsymbol{u}_{2}, \boldsymbol{u}_{4}, \boldsymbol{u}_{6}\right)$ for the second.


Fig. 15. One of the structures can be found in Case 4

Stage II. In this stage, several concurrent swaps will be performed to eliminate most of the $3 / 5$-cycles and keep $\mathscr{C}_{0}$ invariant. The following operation will be iterated in several rounds. In round- $i, \pi_{i} \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ is performed. Let $S_{i, c}$ be the set of all $c$-cycle on which each vertex $\boldsymbol{v}$ satisfies $\boldsymbol{v}, \boldsymbol{v}{ }^{\oplus r_{1}} \notin \mathscr{C} 0$ in $\sigma_{i-1}\left(\sigma_{t}:=\sigma \pi_{0} \pi_{1} \ldots \pi_{t}\right.$ in the following).

Denote $\zeta_{i}:=\left|S_{i, 1}\right|+\left|S_{i, 2}\right|+\left|S_{i, 3}\right|+\left|S_{i, 4}\right|+\left|S_{i, 5}\right|$. If $S_{i-1,3} \cup S_{i-1,5} \neq \emptyset$, k an arbitrary cycle $\mathscr{C}_{1}$ from it. Since $\mathscr{C}_{1}$ is an odd cycle, there exists $\boldsymbol{u} \in \mathscr{C}_{1}$ such that $\boldsymbol{v}:=\boldsymbol{u}^{\oplus r_{1}} \notin \mathscr{C}_{1}$. Let $\mathscr{C}_{2}$ be the cycle where $\boldsymbol{v}$ belongs (by choice of $\mathscr{C}_{1}$, here $\mathscr{C}_{2} \neq \mathscr{C}_{0}$ ). Define

$$
T:=\mathscr{C}_{0} \cup \mathscr{C}_{1} \cup\left\{\boldsymbol{w} \in \mathscr{C}_{2} \mid \operatorname{dist}_{\min }^{\sigma_{i-1}}(\boldsymbol{v}, \boldsymbol{w}) \leq 5\right\}
$$

Note that $|T| \leq 4+5+11$. Since $n \geq 8$, we can always find a concurrent pair $(\boldsymbol{s}, \boldsymbol{t})$ that $\boldsymbol{s}, \boldsymbol{t} \notin T$. Then, let

$$
\pi_{i}:=(\boldsymbol{u}, \boldsymbol{t})(\boldsymbol{v}, \boldsymbol{s}) \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)} .
$$

We will prove $\zeta_{i}<\zeta_{i-1}$, by checking the following cases:
Case 1. $\boldsymbol{t}, \boldsymbol{s} \notin \mathscr{C}_{2}$ : Swapping $\boldsymbol{u}, \boldsymbol{t}$ merges $\mathscr{C}_{1}$ with another cycle. And similarly when swapping $\boldsymbol{v}, \boldsymbol{s}$.
Case 2. $\boldsymbol{t} \notin \mathscr{C}_{2}, s \in \mathscr{C}_{2}$ : Swapping $\boldsymbol{u}, \boldsymbol{t}$ merges $\mathscr{C}_{1}$ with another cycle. Then swapping $\boldsymbol{v}, \boldsymbol{s}$ splits $\mathscr{C}_{2}$ into two cycles; and the length of neither is smaller than 6 , which does not increase the number of short cycles.
Case 3. $\boldsymbol{t} \in \mathscr{C}_{2}, \boldsymbol{s} \notin \mathscr{C}_{2}$ : Swapping $\boldsymbol{u}, \boldsymbol{t}$ merges $\mathscr{C}_{1}$ with $\mathscr{C}_{2}$. Then swapping $\boldsymbol{v}, \boldsymbol{s}$ merges new $\mathscr{C}_{2}$ with another cycle.
Case 4. $\boldsymbol{t}, \boldsymbol{s} \in \mathscr{C}_{2}$ : Swapping $\boldsymbol{u}, \boldsymbol{t}$ merges $\mathscr{C}_{1}$ with $\mathscr{C}_{2}$. Then swapping $\boldsymbol{v}, \boldsymbol{s}$ splits new $\mathscr{C}_{2}$ into two cycles; and the length of neither is smaller than 6 , which does not increase the number of short cycles.

Repeat until $S_{i, 3} \cup S_{i, 5}=\emptyset$. Suppose this process has $k$ rounds, then the permutation after Stage II is $\sigma_{k}=$ $\sigma \pi_{0} \pi_{1} \ldots \pi_{k}$.

Stage III. This stage is designed to remove remaining $3 / 5$-cycles by a permutation $\pi_{k+1}$. Notice that after Stage I if $\mathscr{C}_{0}$ is a 4 -cycle it must consist of 2 concurrent pairs, and in Stage II we exclude the cycles containing a vertex in $\left\{w, w^{\oplus r_{1}} \mid w \in \mathscr{C}_{0}\right\}$. Thus there are at most two $3 / 5$-cycles in $\sigma_{k}$.

Case 1. If there is no $3 / 5$-cycle, simply let $\pi_{k+1}:=$ id. Note that if $\left|\mathscr{C}_{0}\right|=4$, it must be in Case 1 .
Case 2. If there are two $3 / 5$-cycles $\mathscr{C}_{3}, \mathscr{C}_{4}$, we can always find $\boldsymbol{v}_{3} \in \mathscr{C}_{3}, \boldsymbol{v}_{4} \in \mathscr{C}_{4}$ such that $\boldsymbol{v}_{3}^{\oplus r_{1}}, \boldsymbol{v}_{4}^{\oplus r_{1}}$ are in $\mathscr{C}_{0}$. Perform $\pi_{k+1}:=\left(\boldsymbol{v}_{3}, \boldsymbol{v}_{4}\right)\left(\boldsymbol{v}_{3}^{\oplus r_{1}}, \boldsymbol{v}_{4}^{\oplus r_{1}}\right)$. If $\left(\boldsymbol{v}_{3}\right)_{r_{1}}=\left(\boldsymbol{v}_{4}\right)_{r_{1}}, \mathscr{C}_{3}, \mathscr{C}_{4}$ are merged into an even cycle and $\mathscr{C}_{0}$ becomes two fix-points. Otherwise, $\mathscr{C}_{0}, \mathscr{C}_{3}, \mathscr{C}_{4}$ are merged into an even cycle of length at most 12 . Let the new even cycle be $\mathscr{C}_{0}$.
Case 3. Suppose there is a unique $3 / 5$-cycle $\mathscr{C}_{3}$.
Case 3.1. If $\mathscr{C}_{3}$ contains a vertex $\boldsymbol{u}^{\prime}$ such that $\boldsymbol{u}^{\prime \oplus r_{1}} \notin \mathscr{C}_{0}, \mathscr{C}_{3}$, perform another round of Stage II with $\mathscr{C}_{1}=\mathscr{C}_{3}, \boldsymbol{u}=\boldsymbol{u}^{\prime}$; and construct a swap $\pi_{k+1}$.
Case 3.2. Otherwise, $\mathscr{C}_{3}$ contains a concurrent pair $(\boldsymbol{u}, \boldsymbol{v})$. Attempt to find a concurrent pair $\boldsymbol{s}, \boldsymbol{t}$ where $\boldsymbol{s}, \boldsymbol{t} \notin$ $\mathscr{C}_{0}, \mathscr{C}_{3}$ are contained by different cycles and assume $\boldsymbol{u}_{r_{1}}=\boldsymbol{t}_{r_{1}}$.
Case 3.2.1. If such $\boldsymbol{s}, \boldsymbol{t}$ exist, perform $\pi_{k+1}^{\prime}:=(\boldsymbol{u}, \boldsymbol{t})(\boldsymbol{v}, \boldsymbol{s})$ which will merge 3 different cycles including $\mathscr{C}_{3}$ and leaves $\mathscr{C}_{0}$ invariant.
Case 3.2.2. Otherwise, let $s \in \mathscr{C}_{0}$ such that $s^{\oplus r_{1}}$, denoted by $t$, is not in $\mathscr{C}_{3}$. In this case, such $s$ must exist. Also, let the cycle containing $t$ be $\mathscr{C}_{4}$; then $\mathscr{C}_{4}$ is of odd length.

Assume $\boldsymbol{u}_{r_{1}}=\boldsymbol{t}_{r_{1}}$. Thus, $\pi_{k+1}^{\prime}:=(\boldsymbol{u}, \boldsymbol{t})(\boldsymbol{v}, \boldsymbol{s})$ merges $\mathscr{C}_{0}, \mathscr{C}_{3}, \mathscr{C}_{4}$ as an even cycle if $\mathscr{C}_{0}=\left\{\boldsymbol{v}_{1}, \boldsymbol{v}_{2}\right\}$ and $\left(\boldsymbol{v}_{1}\right)_{r_{1}} \neq\left(\boldsymbol{v}_{2}\right)_{r_{1}}$. Otherwise $\left(\boldsymbol{v}_{1}\right)_{r_{1}}=\left(\boldsymbol{v}_{2}\right)_{r_{2}}, \pi_{k+1}^{\prime}$ will merge $\mathscr{C}_{3}, \mathscr{C}_{4}$ as an even cycle and breaks $\mathscr{C}_{0}$ into two fix-points. Let the new even cycle be $\mathscr{C}_{0}$.

Note that it is also the only possible case where the length of the smallest even cycle can be larger than 12 . Define $W:=\left\{\boldsymbol{v}_{1}, \boldsymbol{v}_{1}^{\oplus r_{1}}, \boldsymbol{v}_{2}, \boldsymbol{v}_{2}^{\oplus r_{2}}\right\}$. In this case, every concurrent pair $(\boldsymbol{s}, \boldsymbol{t})$, where $\boldsymbol{s}, \boldsymbol{t} \notin W$, is contained by the same cycle in $\sigma_{k} \pi_{k+1}^{\prime}$.
If all $3 / 5$-cycles are eliminated, let $\pi_{k+1}=\pi_{k+1}^{\prime}$. But when the remaining $\mathscr{C}_{3}$ is a 3 -cycle, $\pi_{k+1}^{\prime}$ may give a 5-cycle. Consider the (only) two bad instances:
$\therefore \mathscr{C}_{3}$ is merged with a 2 -cycle in Case 3.1 ;
$* \mathscr{C}_{3}$ is merged with two fix-points in Case 3.2.1.
In either bad instance, $\mathscr{C}_{0}$ is unchanged, all 3 -cycles are eliminated and at most one 5 -cycle is left. Try another round of Stage III with $\sigma_{k} \pi_{k+1}^{\prime}$ and get $\pi_{k+1}^{\prime \prime}$. Then let $\pi_{k+1}=\pi_{k+1}^{\prime} \pi_{k+1}^{\prime \prime}$; and $\sigma_{k+1}:=\sigma_{k} \pi_{k+1}$ is $3 / 5$-cycle free.
Stage IV. After Stage III, $\sigma_{k+1}$ is $3 / 5$-cycle free, and contains an even cycle. If $\pi_{0} \pi_{1} \cdots \pi_{k+1} \in A C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$, simply let $\pi_{k+2}:=$ id. If otherwise, we construct

$$
\pi_{k+2} \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)} \backslash A C_{\{0,1\}^{n}}^{\left(r_{1}\right)}
$$

which preserves an even cycle but forbids $3 / 5$-cycle.
Case 1. If there exists a concurrent pair $\boldsymbol{u}, \boldsymbol{v} \notin \mathscr{C}_{0}$ contained by different cycles, $\left|\mathscr{C}_{0}\right|$ can not be greater than 12 due to the analysis in Case 3.2.2 of Stage III. Let $\mathscr{C}_{1}$ and $\mathscr{C}_{2}$ be cycles that $\boldsymbol{u} \in \mathscr{C}_{1}, \boldsymbol{v} \in \mathscr{C}_{2}$. Define

$$
\begin{aligned}
T:= & \mathscr{C}_{0} \cup\left\{\boldsymbol{w} \mid \text { dist }_{\min }^{\sigma_{k+1}}(\boldsymbol{u}, \boldsymbol{w}) \leq 5\right\} \\
& \cup\left\{\boldsymbol{w} \mid \operatorname{dist}_{\min }^{\sigma_{k+1}}(\boldsymbol{v}, \boldsymbol{w}) \leq 5\right\} .
\end{aligned}
$$

Note that $|T| \leq 34$. Since $n \geq 8$ and $2^{n} \geq 2|T|+1$, we can always find a concurrent pair $\boldsymbol{t}, \boldsymbol{s} \notin T$ where $\boldsymbol{t}_{r_{1}}=\boldsymbol{u}_{r_{1}}$. Let $\pi_{k+2}:=(\boldsymbol{u}, \boldsymbol{t})(\boldsymbol{v}, \boldsymbol{s})$. Thus, $\sigma_{k+2}$ still contains $\mathscr{C}_{0}$. With the same argument in Stage II, no new 3/5-cycle appears.
Case 2. Otherwise, consider the size of $\mathscr{C}_{0}$. If $\left|\mathscr{C}_{0}\right| \leq 12$, define $W=\left\{\boldsymbol{w}, \boldsymbol{w}^{\oplus r_{1}} \mid \boldsymbol{w} \in \mathscr{C}_{0}\right\}$. If $\left|\mathscr{C}_{0}\right|>12$, it must comes from Case 3.2.2 of Stage III; and we adopt the definition of $W$ from there. In either case, $|W| \leq 24$.

Now, each concurrent pair out of $W$ is contained in the same cycle. If there exist 3 concurrent pairs $\boldsymbol{u}_{i}, \boldsymbol{v}_{i} \notin$ $W, i \in[3]$ and $\boldsymbol{u}_{1}, \boldsymbol{u}_{2}, \boldsymbol{u}_{3}$ are contained in 3 distinct cycles. Let $\tau:=\left(\boldsymbol{u}_{1}, \boldsymbol{u}_{2}, \boldsymbol{u}_{3}\right)\left(\boldsymbol{v}_{1}, \boldsymbol{v}_{2}, \boldsymbol{v}_{3}\right)$ (assuming $\left(\boldsymbol{u}_{1}\right)_{r_{1}}=$ $\left.\left(\boldsymbol{u}_{2}\right)_{r_{1}}=\left(\boldsymbol{u}_{3}\right)_{r_{1}}\right)$. Then $\tau \in A C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ and merges the 3 cycles. Repeat such merging operation until a large even cycle $\mathscr{C}_{1}$ of length $\ell \geq 2 \times(21 \times 2+12+1)=110$ appears. Since $n \geq 8$ and $2^{n} \geq|W|+2 \ell$, this is inevitable. Let $\pi_{k+2}^{\prime} \in A C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ as the merging process.

Denote $\sigma_{k+1} \tilde{\pi}_{k+2}^{\prime}$ by $\sigma^{\prime}$ for convenience. Pick 3 distinct concurrent pairs $\boldsymbol{u}_{i}, \boldsymbol{v}_{i} \in \mathscr{C}_{1}, i \in[3]$ such that

$$
\begin{gathered}
\operatorname{dist}_{\min }^{\sigma^{\prime}}\left(\boldsymbol{u}_{i}, \boldsymbol{u}_{j}\right), \operatorname{dist}_{\min }^{\sigma^{\prime}}\left(\boldsymbol{v}_{i}, \boldsymbol{v}_{j}\right), \\
\operatorname{dist}_{\min }^{\sigma^{\prime}}\left(\boldsymbol{v}_{i}, \boldsymbol{u}_{j}\right), \operatorname{dist}_{\min }^{\sigma^{\prime}}\left(\boldsymbol{u}_{i}, \boldsymbol{v}_{j}\right) \geq 6
\end{gathered}
$$

and $\boldsymbol{u}_{i}, \boldsymbol{v}_{j} \notin W$ for all distinct $i, j \in[3]$. Let $\pi^{i, j}:=\left(\boldsymbol{u}_{i}, \boldsymbol{u}_{j}\right)\left(\boldsymbol{v}_{i}, \boldsymbol{v}_{j}\right)$. The cycle pattern after $\pi^{i, j}$ is related to the order of the 4 vertices.

Since

$$
\operatorname{dist}^{\sigma^{\prime}}\left(\boldsymbol{u}_{i}, \boldsymbol{v}_{i}\right)=\left|\mathscr{C}_{1}\right|-\operatorname{dist}^{\sigma^{\prime}}\left(\boldsymbol{v}_{i}, u_{i}\right), \forall i \in[3]
$$

and $\left|\mathscr{C}_{1}\right|=\ell \geq 110$, there exist distinct $\hat{i}, \hat{j} \in[3]$ such that

$$
\operatorname{dist}^{\sigma^{\prime}}\left(\boldsymbol{u}_{\hat{i}}, \boldsymbol{v}_{\hat{i}}\right)+\operatorname{dist}^{\sigma^{\prime}}\left(\boldsymbol{v}_{\hat{j}}, \boldsymbol{u}_{\hat{j}}\right) \geq 6
$$

and

$$
\operatorname{dist}^{\sigma^{\prime}}\left(\boldsymbol{v}_{\hat{i}}, \boldsymbol{u}_{\hat{i}}\right)+\operatorname{dist}^{\sigma^{\prime}}\left(\boldsymbol{u}_{\hat{j}}, \boldsymbol{v}_{\hat{j}}\right) \geq 6
$$

Define a notation $a \rightsquigarrow b \rightsquigarrow c \rightsquigarrow d$ to represent that $\sigma_{k+1}$ contains a cycle in Figure 16 .


Fig. 16. Pattern $a \rightsquigarrow b \rightsquigarrow c \rightsquigarrow d$

Here we list possible orders of the 4 vertices.
Order 1. $\boldsymbol{u}_{i} \rightsquigarrow \boldsymbol{u}_{j} \rightsquigarrow \boldsymbol{v}_{j} \rightsquigarrow \boldsymbol{v}_{i}$ : Break into 3 cycles with the length of dist ${ }^{\sigma^{\prime}}\left(\boldsymbol{u}_{i}, \boldsymbol{u}_{j}\right)$, $\operatorname{dist}^{\sigma^{\prime}}\left(\boldsymbol{v}_{j}, \boldsymbol{v}_{i}\right)$ and $\operatorname{dist} t^{\sigma^{\prime}}\left(\boldsymbol{u}_{j}, \boldsymbol{v}_{j}\right)+\operatorname{dist}^{\sigma^{\prime}}\left(\boldsymbol{v}_{i}, \boldsymbol{u}_{i}\right)$ respectively;
Order 2. $\boldsymbol{u}_{i} \rightsquigarrow \boldsymbol{u}_{j} \rightsquigarrow \boldsymbol{v}_{i} \rightsquigarrow \boldsymbol{v}_{j}$ : Break into 3 cycles with the length of dist ${ }^{\sigma^{\prime}}\left(\boldsymbol{u}_{i}, \boldsymbol{u}_{j}\right)$, dist ${ }^{\sigma^{\prime}}\left(\boldsymbol{v}_{j}, \boldsymbol{v}_{i}\right)$ and $\operatorname{dist}^{\sigma^{\prime}}\left(\boldsymbol{u}_{j}, \boldsymbol{v}_{i}\right)+\operatorname{dist}^{\sigma^{\prime}}\left(\boldsymbol{v}_{j}, \boldsymbol{u}_{i}\right)$ respectively;
Order 3. $\boldsymbol{u}_{i} \rightsquigarrow \boldsymbol{v}_{i} \rightsquigarrow \boldsymbol{u}_{j} \rightsquigarrow \boldsymbol{v}_{j}$ : Remain a cycle of the same length.
Due to symmetry, other orders are not essentially different from these. Then, let $\pi_{k+2}:=\pi_{k+2}^{\prime} \pi^{\hat{i}, \hat{j}}$; we have $\pi_{0} \pi_{1} \cdots \pi_{k+2} \in A C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ and $\sigma \pi_{0} \cdots \pi_{k+2}$ satisfies the desired properties.

Proof of Lemma 9 W.l.o.g, suppose $(1, \ldots, 2 k)$ is an even cycle in $\sigma$. Define $h_{0} \in S_{\{0,1\}^{n}}$ as

$$
h_{0}(i)=\left\{\begin{array}{cl}
i+1 & i \in[2 k-1] \\
1 & i=2 k \\
i & \text { otherwise }
\end{array}\right.
$$

It is easy to see that $h_{0}$ is odd and satisfies $h_{0} \sigma h_{0}^{-1}=\sigma$. Since $\sigma, \pi$ has the same cycle pattern, then there exists $h_{1} \in S_{\{0,1\}^{n}}$ such that $h_{1} \sigma h_{1}^{-1}=\pi$. If $h_{1}$ is odd, define $h:=h_{1} h_{0}$. Otherwise, define $h:=h_{1}$. Thus, $h$ is even and satisfies $h \sigma h^{-1}=\pi$, which finishes the proof.
proof of Lemma 10. W.l.o.g, assume $r_{1}=1$ and $r_{2}=2$. There are at least 12 cycles $\mathscr{C}_{1}, \mathscr{C}_{2}, \ldots, \mathscr{C}_{k}$ with $\left|\mathscr{C}_{i}\right| \geq 2$ for all $i \in[k]$ in $\sigma$, which implies that there are at least 5 pairs of cycles $\left\{\mathscr{C}_{1}^{(1)}, \mathscr{C}_{2}^{(1)}\right\}, \ldots,\left\{\mathscr{C}_{1}^{(5)}, \mathscr{C}_{2}^{(5)}\right\}$ with the length of $\left\{a_{1}, b_{1}\right\}, \ldots,\left\{a_{5}, b_{5}\right\}$ respectively, such that $a_{i}+b_{i}$ is even and $\left\{a_{i}, b_{i}\right\} \neq\{2,4\}$ for all $i \in[5]$. W.l.o.g, assume $a_{1}+b_{1}+a_{2}+b_{2} \equiv a_{3}+b_{3}+a_{4}+b_{4} \equiv 0 \bmod 4$ and the selected 8 cycles are $\mathscr{C}_{1}, \ldots, \mathscr{C}_{8}$. Let $\ell_{1}:=a_{1}+b_{1}+a_{2}+b_{2}, \ell_{2}:=a_{3}+b_{3}+a_{4}+b_{4}$ and $\ell:=\ell_{1}+\ell_{2}$. Choose arbitrary $S \subseteq\{0,1\}^{n-2}$ with size of $\ell / 4$ and define $T:=\{0,1\}^{n-2} \backslash S$. Due to the fact that $\sigma$ is free of $3 / 5$-cycle and a simple generalization of Proposition 3, there exist $\pi_{3} \in S_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ and $\tau_{3} \in S_{\{0,1\}^{n}}^{\left(r_{2}\right)}$ such that $\operatorname{Supp}\left(\pi_{3}\right), \operatorname{Supp}\left(\tau_{3}\right) \subseteq\{0,1\}^{2} \times T$ and $\pi_{3} \tau_{3}$ is a $\left|\mathscr{C}_{9}\right|, \ldots,\left|\mathscr{C}_{k}\right|$-cycle.

In the remaining part of the proof, we provide 4 schemata to construct a $\left|\mathscr{C}_{1}\right|, \ldots\left|\mathscr{C}_{8}\right|$-cycle locally with paritydistinct $\pi^{(1)}, \pi^{(2)} \in A C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ and $\tau^{(1)}, \tau^{(2)} \in A C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$. Thus, not so strictly speaking, we can adjust the parity of $\pi$ and $\tau$ as required and keep $\pi \tau$ being a $\left|\mathscr{C}_{1}\right|, \ldots,\left|\mathscr{C}_{k}\right|$-cycle.

Divide $S=S_{1} \sqcup S_{2}$ where $\left|S_{1}\right|=\ell_{1} / 4$. Let $S_{1,1}, S_{1,2}$ be disjoint subsets of $S_{1}$ where $\left|S_{1, i}\right|=\left\lfloor\left(a_{i}+b_{i}\right) / 4\right\rfloor$ for $i \in[2]$. Consider the value of $\left(a_{1}+b_{1}\right) \bmod 4$ :
Case 1. If $\left(a_{1}+b_{1}\right) \equiv 2 \bmod 4$, call

$$
\operatorname{TPACK}\left(r_{1}, r_{2}, a_{1}, b_{1}, a_{2}, b_{2},\{0,1\}^{2} \times S_{1}\right)
$$

with $\pi_{1}, \tau_{1}$ as the outputs.
Case 2. Otherwise, call

$$
\operatorname{RPACK}\left(a_{i}, b_{i}, r_{1}, r_{2},\{0,1\}^{2} \times S_{1, i}\right)
$$

with $\pi_{1, i}, \tau_{1, i}$ as the outputs for $i \in[2]$. Define $\pi_{1}=\pi_{1,1} \circ \pi_{1,2}$ and $\tau_{1}=\tau_{1,1} \circ \tau_{1,2}$.
Since $\sigma$ is free of $3 / 5$-cycle, $a_{i}, b_{i}$ is valid as inputs of TPACK and RPACK.
The proof is based on the following observations: If we swap two pairs of consecutive nodes as shown in Figure 17) then the resulted permutation will have the same cycle pattern with the original one, no matter the two pairs belong to the same cycle or not.


Fig. 17. Swap two pairs of consecutive nodes

Formally, the following equations hold

$$
\begin{aligned}
& \left(i_{1}, i_{2}, \ldots, i_{k}, i_{k+1}, \ldots\right)\left(i_{1}, i_{k}\right)\left(i_{2}, i_{k+1}\right) \\
= & \left(i_{1}, i_{k+1}, i_{3}, \ldots, i_{k}, i_{2}, i_{k+2}, \ldots\right) \\
& \left(i_{1}, . ., i_{k}\right)\left(j_{1}, \ldots, j_{l}\right)\left(i_{1}, j_{1}\right)\left(i_{2}, j_{2}\right) \\
= & \left(i_{1}, j_{2}, i_{3}, \ldots, i_{k}\right)\left(j_{1}, i_{2}, j_{3}, \ldots, j_{l}\right) .
\end{aligned}
$$

In order to change the concurrent parity of $\tau_{1}$, we simply perform a swap in proper position to the original construction. For example, when $a=b=2 k$, we can construct a $a, b$-cycle with $\pi^{\prime} \tau^{\prime}$ or $\pi^{\prime} \tau^{\prime \prime}$, where $\tau^{\prime}$ (id) is concurrently even while $\tau^{\prime \prime}$ (a swap) is concurrently odd, as pictured in Figure 18 ,


Fig. 18. $\tau^{\prime \prime}$ is concurrently odd (left); $\tau^{\prime}$ is concurrently even (right).
We can use the similar method to change the parity of $\tau_{1}$. One technique which need be emphasized is that, by arranging nodes in proper positions, we can ensure the existence of 2 proper consecutive node pairs such that the concurrent swap on them will not change the cycle pattern. As the result, $\tau_{1}^{\prime}$ can be constructed, such that $\pi_{1} \tau_{1}^{\prime}$ has the same cycle pattern with $\pi_{1} \tau_{1}$, but $\tau_{1}^{\prime}$ has different concurrent parity with $\tau_{1}$. Furthermore, define $\pi_{2}, \tau_{2}$ and $\tau_{2}^{\prime}$ for $a_{3}, b_{3}, a_{4}, b_{4}$ in the same way.

Another essential ingredient is to "rotate" the constructed permutations in some way. Formally, we exchange $r_{1}, r_{2}$ dimensions by a permutation $\rho$, i.e, define $\rho_{r_{1}, r_{2}} \in S_{\{0,1\}^{n}}$ for $i<j$ as

$$
\rho_{i, j}: s_{1} \ldots s_{i} \ldots s_{j} \ldots s_{n} \mapsto s_{1} \ldots s_{j} \ldots s_{i} \ldots s_{n}
$$

which maps $s$ to the string constructed by exchanging the $r_{1}$-th and $r_{2}$-th elements. Define switch $: S_{\{0,1\}^{n}} \rightarrow$ $S_{\{0,1\}^{n}}$ as

$$
\operatorname{switch}(\nu):=\rho_{r_{1}, r_{2}}^{-1} \circ \nu \circ \rho_{r_{1}, r_{2}}
$$

Define

$$
\begin{aligned}
\pi^{(1)} & :=\pi_{1} \circ \operatorname{switch}\left(\tau_{2}\right) \circ \pi_{3} \\
\pi^{(2)} & :=\pi_{1} \circ \operatorname{switch}\left(\tau_{2}^{\prime}\right) \circ \pi_{3} \\
\tau^{(1)} & :=\tau_{1} \circ \operatorname{switch}\left(\pi_{2}\right) \circ \tau_{3} \\
\tau^{(2)} & :=\tau_{1}^{\prime} \circ \operatorname{switch}\left(\pi_{2}\right) \circ \tau_{3}
\end{aligned}
$$

where $\pi^{(1)}, \pi^{(2)} \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ have different concurrent parity, as well as $\tau^{(1)}, \tau^{(2)} \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$. Note the following facts:

- $\pi_{1} \circ \tau_{1} \circ \pi_{2} \circ \tau_{2} \circ \pi_{3} \circ \tau_{3}$ has the same cycle pattern with $\sigma$;
- $\operatorname{switch}\left(\tau_{2}\right) \circ \operatorname{switch}\left(\pi_{2}\right)$ is conjugated with $\pi_{2} \circ \tau_{2}$;
- $\pi^{\prime} \circ \tau^{\prime}$ is conjugated with $\tau^{\prime} \circ \pi^{\prime}$ for any $\pi^{\prime}, \tau^{\prime}$;
- These permutations noted with different subscripts act on disjoint supports.

Thus, it can be shown that $\pi^{(i)} \circ \tau^{(j)}$ has the same cycle pattern with $\sigma$ for all $i, j \in[2]$, which finishes the proof.

Proof of Lemma 11 W.l.o.g, assume $r_{1}=1$ and $r_{2}=2$. Due the restriction of given $\sigma$, there exists cycles $\mathscr{C}_{1}, \mathscr{C}_{2}$ with the length of $a, b$ respectively, such that $a+b \equiv 0 \bmod 2$ and $a \geq 12$. Due to a similar argument to the one
used in the proof of Lemma 10 it suffices to prove there exist $\pi_{1}, \ldots, \pi_{4} \in S C_{\{0,1\}^{n}}^{\left(r_{1}\right)}$ and $\tau_{1}, \ldots, \tau_{4} \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$ such that

- $\pi_{1}, \pi_{2}, \tau_{1}, \tau_{3}$ are concurrently even;
- $\pi_{3}, \pi_{4}, \tau_{2}, \tau_{4}$ are concurrently odd;
- $\pi_{i} \tau_{i}$ is an $a, b$-cycle for all $i \in[4]$.

Next, we will construct $\pi^{\prime}, \pi^{\prime \prime}, \tau^{\prime}, \tau^{\prime \prime}$ for the following cases such that $\pi^{\prime} \tau^{\prime}, \pi^{\prime \prime} \tau^{\prime \prime}$ are $a, b$-cycles, and $\pi^{\prime}, \pi^{\prime \prime}$ have different concurrent parity. Let $k:=\lfloor a / 2\rfloor$ and $l:=\lfloor b / 2\rfloor$.

Case 1. $a, b$ are even and $a=b$ :


Case 2. $a, b$ are even and $a \neq b$ :


The constructions for the cases where $k$ or $l$ is not even are similar.
Case 3. $a, b$ are odd and $a=b$ :


Case 4. $a, b$ are odd, $b \geq 7$ and $a \neq b$ :


The construction for $k \equiv l \bmod 2$ is similar.
Case 5. $a, b$ are odd and $b=1$ :


The construction for even $l$ is similar.
Furthermore, recalling the analysis in the proof of Lemma 10, it is easy to verified that there exist concurrently odd $\rho^{\prime}, \rho^{\prime \prime} \in S C_{\{0,1\}^{n}}^{\left(r_{2}\right)}$ such that $\pi^{\prime} \tau^{\prime}$ has the same cycle pattern with $\pi^{\prime} \tau^{\prime} \rho^{\prime}$, as well as $\pi^{\prime \prime} \tau^{\prime \prime}$ and $\rho^{\prime \prime}$, which finishes the proof.

Proof of Lemma 12. We give a constructive proof when $n=3$, the construction can be easily embeded into higher dimension. For $n=3$, let

$$
\begin{aligned}
\pi & =(001,011)(101,111) \\
\tau_{1} & =(010,100,110)(011,101,111) \\
\tau_{2} & =(001,100,101)(011,110,111) \\
\tau_{3} & =(001,010,011)(101,110,111) \\
\tau_{4} & =(001,101,100)(011,111,110) .
\end{aligned}
$$

For $n=4$, we simply padding 0 to the string, that is, let $\pi=(0010,0110)(101,1110), \tau_{1}=$ $(0100,1000,1100)(0110,1010,1110)$ and ditto for $n>4$.


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