$\mathcal{H}_\infty$ Almost Output Synchronization for Heterogeneous Networks Without Exchange of Controller States

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***Abstract*—We consider the $\mathcal{H}_\infty$ almost output synchronization and regulated output synchronization problem for heterogeneous directed networks with external disturbances where agents are introspective (i.e., agents have access to a part of their own states). A decentralized protocol is designed for each agent, without exchanging internal controller states with neighboring agents, to achieve output synchronization and regulated output synchronization while making the $\mathcal{H}_\infty$ norm from the disturbances to the synchronization errors in the network arbitrarily small. Furthermore, the idea for output synchronization is applied to a formation problem in the presence of disturbances. The method is illustrated by simulation examples.**

***Index Terms*—Distributed processing, disturbances, linear feedback control systems, multi-agent systems, synchronization.**

I. Introduction

In THE past decade, a large amount of research has been conducted on the topic of *synchronization*, where the goal is to secure asymptotic agreement among a set of networked agents on a common state or output trajectory. Most of this work has focused on the problem of *state synchronization* in *homogeneous* networks, based on diffusive *full-state* or *partial-state coupling* (e.g., [6], [9], [12]–[15], [22]–[24]).

Recently, several authors have also addressed the synchronization problem for *heterogeneous* networks, where the agents are governed by non-identical dynamics (e.g. [1], [2], [5], [20], [26], [27]). In a heterogeneous network, the agents' internal states may not be directly comparable; thus, the goal is often to achieve *output synchronization* rather than state synchronization. In most work on heterogeneous networks, it is assumed that the agents are *introspective*, meaning that they have access to their own state or output independent of the information received via the network. In a recent paper, Grip, Yang, Saberi, and Stoorvogel [4] addressed the synchronization problem for a class of nonintrospective agents.

The vast majority of the research has focused on the idealized case where the agents are unaffected by external disturbances. Among the authors that have considered external disturbances, Lin, Duan, Chen, and Huang [6] considered the problem of minimizing the $\mathcal{H}_\infty$ norm from an external disturbance to the output of each agent, whereas Lin and Jia [7] and Li, Lin, and Jia [8] considered minimization of the $\mathcal{H}_\infty$ norm from a disturbance to the average of the states in a network of single or double integrators. In contrast, Peymani, Grip, Saberi, Wang, and Fossen [11] introduced the notion of $\mathcal{H}_\infty$ *almost synchronization* for a class of heterogeneous networks, where the goal is to reduce the $\mathcal{H}_\infty$ norm from an external disturbance to the synchronization *error* to any arbitrary desired level.

A. Contributions of This Paper

In this paper, we again consider the $\mathcal{H}_\infty$ almost synchronization problem for heterogeneous networks; however, we depart from the work of Peymani, Grip, Saberi, Wang, and Fossen [11] in the crucial aspect of *interagent communication*. Traditional formulations of the synchronization problem for single- and double-integrator dynamics assume that the only information available to each agent is a linear combination of the state or output of neighboring agents in the network [9], [13]. However, since the research has turned toward networks with more complex agent dynamics, it has become commonplace to assume availability of an *additional* communication channel, used to exchange information about an internal controller or observer states between neighboring agents (e.g., [6] and [11]). As shown in the early work of Li, Duan, Chen, and Huang [6], the availability of an additional communication channel permits the construction of a distributed observer giving each agent an asymptotic estimate of its own state; for this reason, the work of Peymani, Grip, Saberi, Wang, and Fossen [11] also assumed the availability of such a communication channel.

In this paper, we dispense with the assumption of an additional communication channel and show that, for a large class of heterogeneous networks, we can achieve almost synchronization in the presence of arbitrary external disturbances. The practical consequences of this development are significant. Considering, for example, a group of robots seeking synchronization or formation, one no longer requires an active

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communication link between the robots; instead, passive observation of neighboring robots is sufficient to ensure an arbitrary level of synchronization or formation accuracy, even in the presence of unknown disturbances.

In order to achieve these results, we build upon the work of Seo, Shim, and Back [21], who developed a *low-gain* design methodology for ensuring synchronization among identical nonexponentially unstable linear agents without additional communication; and Grip, Saberi, and Stoorvogel [3], who combined low-gain and *high-gain* techniques to achieve synchronization in networks of minimum-phase single-input-single-output (SISO) agents, including nonlinear and heterogeneous networks. In both of these cases, no external disturbances were considered.

We note that for the synchronization problem of networks with MIMO agents in the presence of external disturbances, to our knowledge, there are only two papers by Peymani *et al.* in [11] (heterogeneous, introspective) and [10] (homogeneous, nonintrospective). However, in [11], the additional communication channel mentioned from before has been used. In this paper, we abandon this additional communication channel in the design of the protocol. The only information agents have access to is the internal information and the relative output information from neighboring agents, that is, our protocol design is purely decentralized. We consider three distinct problems: $\mathcal{H}_\infty$ *almost output synchronization*, where the goal is to achieve asymptotic agreement with any desired accuracy; $\mathcal{H}_\infty$ *almost regulated output synchronization*, where the goal is to track an exogeneous reference system with any desired accuracy; and $\mathcal{H}_\infty$ *almost formation*, where the goal is to achieve formation with any desired accuracy.

B. Notation and Definitions

Given a matrix A, A' denotes its transpose and $\|A\|$ is the induced 2-norm. I_N depicts the N-dimensional identity matrix. We denote the Kronecker product between A and B by $A \otimes B$. Moreover, $\mathrm{col}\{x_i\}$ is a vector constructed by stacking vectors $x_1, x_2, \ldots, x_n$. We will denote by $\mathrm{blkdiag}\{A_i\}$, a block-diagonal matrix with A_i $(i = 1, \ldots, n)$ as the diagonal elements. Finally, the $\mathcal{H}_\infty$ norm of a transfer function T is indicated by $\|T\|_\infty$.

A graph $\mathcal{G}$ is defined by a pair $(\mathcal{N}, \mathcal{E})$ where $\mathcal{N} = \{1, \ldots, N\}$ is a node set and $\mathcal{E}$ is a set of pairs of vertices (i, j). Each pair in $\mathcal{E}$ is called an *edge*. A *directed path* from node i_1 to i_k is a sequence of vertices $\{i_1, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, \ldots, k-1$. A graph $\mathcal{G}$ contains a *directed spanning tree* if there is a node r such that a directed path exists between r and every other node.

The graph $\mathcal{G}$ is *weighted* if each edge $(i, j) \in \mathcal{E}$ is assigned a real number $a_{ij} > 0$. For $(i, j) \notin \mathcal{E}$, we have $a_{ij} = 0$. For a weighted graph $\mathcal{G}$, the matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^{N} a_{ik}, & i = j, \\ -a_{ij}, & i \neq j \end{cases}$$

is called the *Laplacian matrix* associated with graph $\mathcal{G}$. The matrix L has all of its eigenvalues in the closed right-half plane and at least one eigenvalue at zero associated with the right eigenvector $\mathbf{1}$ (see [17]). Here, $\mathbf{1}$ is a column vector with all elements equal to 1. If $\mathcal{G}$ has a directed spanning tree, L has a simple eigenvalue at zero and all of the other eigenvalues have strictly positive real parts (see [16]).

Definition 1: A linear time-invariant dynamics (A, B, C, D) is right-invertible if, given a smooth reference output y_r, there exists an initial condition $x(0)$ and an input u that ensures $y(t) = y_r(t)$ for all $t \geq 0$. For an SISO system, a system is right-invertible if and only if its transfer function is nonzero.

II. Heterogeneous Multiagent Systems

We consider a heterogeneous network consisting of N nonidentical linear time-invariant agents Σ_i with $i \in \nu := \{1, 2, \ldots, N\}$ described by

$$\Sigma_i : \begin{cases} \dot{\breve{x}}_i = A_i \breve{x}_i + B_i \bar{u}_i + G_i \bar{w}_i, \\ y_i = C_i \breve{x}_i \end{cases} \tag{1}$$

where $\breve{x}_i \in \mathbb{R}^{n_i}$, $\bar{u}_i \in \mathbb{R}^{m_i}$, and $y_i \in \mathbb{R}^p$ are the state, input, and output of agent i. Finally, $\bar{w}_i \in \mathbb{R}^{\bar{w}_i}$ is the external disturbance, the power of which is finite, i.e.,

$$\limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \bar{w}_i'(t)\bar{w}_i(t)\mathrm{d}t < \infty.$$

Remark 1: We have also looked at the case where statistics of the disturbances are available and have formulated H_2 almost synchronization which has been submitted [28]. In this formulation, the goal is to utilize the statistical knowledge (i.e., power spectrum density) of disturbances and design a protocol to reduce the stochastic *rms* of disagreement error dynamics.

The agents are introspective, that is, they have access to parts of their own states as given by

$$z_{m,i} = C_{m,i} \breve{x}_i \tag{2}$$

where $z_{m,i} \in \mathbb{R}^{p_{m,i}}$.

The network infrastructure provides each agent with a linear combination of its own output relative to that of other neighboring agents. In particular, each agent $i \in \nu$ has access to the quantity

$$\zeta_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j) \tag{3}$$

where $a_{ij} \geq 0$ and $a_{ii} = 0$ with $i, j \in \nu$. The communication topology of the network can be described by a weighted graph $\mathcal{G}$ with nodes corresponding to the agents in the network and the weight of edges given by the coefficient a_{ij}. In particular, $a_{ij} > 0$ indicates that there is an edge in the graph from agent j to agent i.

In terms of the coefficients of the Laplacian matrix L associated with this weighted graph, ζ_i can be rewritten as

$$\zeta_i = \sum_{j=1}^{N} \ell_{ij} y_j. \tag{4}$$

We make the following assumption on the agent dynamics.

Assumption 1: For each agent $i \in \nu$, we have:

- (A_i, B_i, C_i), which is right-invertible;
- (A_i, B_i), which is stabilizable, and (A_i, C_i) is detectable;
- $(A_i, C_{m,i})$, which is detectable.

III. $\mathcal{H}_\infty$ Almost Output Synchronization

Here, we consider the $\mathcal{H}_\infty$ almost synchronization problem for heterogeneous multiagents systems. We also recall the definition of $\mathcal{H}_\infty$ almost synchronization from [11].

In the following text, we will use the notation of $\bar{w} := \mathrm{col}\{\bar{w}_i\}$. Synchronization among agents is measured by mutual disagreement. For example, the disagreement between agent i and N is denoted by $\mathbf{e}_i := y_N - y_i$ for $i \in \{1, \ldots, N-1\}$. For the complete network, the disagreement is defined as $\mathbf{e} := \mathrm{col}\{\mathbf{e}_i\}$. The transfer function from the disturbance $\bar{w}$ to the mutual disagreement $\mathbf{e}$ is denoted by $T_{\bar{w}\mathbf{e}}$.

We can formulate the $\mathcal{H}_\infty$ almost output synchronization as follows.

Problem 1: Consider a multiagent system described by (1). Given a set of network graphs $\mathbb{G}$, the $\mathcal{H}_\infty$ almost synchronization problem is to find, for any $\gamma > 0$, if possible, a linear time-invariant dynamic protocol such that, for any $\mathcal{G} \in \mathbb{G}$, the following result holds:

- In the absence of disturbances, we have $\lim_{t\to\infty} \mathbf{e}(t) = 0$ for all possible initial conditions.
- The closed-loop transfer function from $\bar{w}$ to $\mathbf{e}$ satisfies $\|T_{\bar{w}\mathbf{e}}\|_\infty < \gamma$.

Before stating the results, we need the following definition for the set of network graphs.

Definition 2: For given $\alpha, \beta, \varphi > 0$, $\mathbb{G}^{\varphi,N}_{\alpha,\beta}$ is the set of directed graphs composed of N nodes such that for every $\mathcal{G} \in \mathbb{G}^{\varphi,N}_{\alpha,\beta}$, the graph has a directed spanning tree, the associated eigenvalues of its Laplacian L, denoted by $\lambda_1, \ldots, \lambda_N$, satisfy $Re\{\lambda_i\} > \beta$ and $|\lambda_i| < \alpha$ whenever $\lambda_i \neq 0$ and, finally, the condition number[1] of L is bounded by φ.

Remark 2: Note that for undirected graphs, the condition number of the Laplacian matrix is always equal to 1. Moreover, if we have a *finite* set of possible graphs, each of which has a directed spanning tree, then there always exists a set of the form $\mathbb{G}^{\varphi,N}_{\alpha,\beta}$ for suitable $\alpha, \beta, \varphi > 0$ and N containing these graphs. The only limitation is that we cannot find one protocol for a sequence of graphs converging to a graph without a spanning tree or whose Laplacian either diverges or approaches some ill-conditioned matrix.

The main result in this section is stated in the following theorem.

Theorem 1: Consider a multiagent system described by (1) and a set of network graphs $\mathbb{G}^{\varphi,N}_{\alpha,\beta}$. Under Assumption 1, the $\mathcal{H}_\infty$ almost output synchronization problem is solvable, that is, there exists a family of low-and-high gain linear time-invariant dynamic protocols, parameterized in terms of low and high gain parameters $\delta, \varepsilon \in (0, 1]$ of the form

$$\begin{cases} \dot{\chi}_i = \mathcal{A}_i(\delta,\varepsilon)\chi_i + \mathcal{B}_i(\delta,\varepsilon)\begin{pmatrix}\zeta_i \\ z_{m,i}\end{pmatrix}, \\ \bar{u}_i = \mathcal{C}_i(\delta,\varepsilon)\chi_i + \mathcal{D}_i(\delta,\varepsilon)\begin{pmatrix}\zeta_i \\ z_{m,i}\end{pmatrix} \end{cases} \tag{5}$$

where $\chi_i \in \mathbb{R}^{qi}$ and $i \in \nu$, such that:

1) in the absence of disturbances, for any given $\alpha, \beta, \varphi > 0$, there exists a $\delta^* \in (0, 1]$ such that, for each $\delta \in (0, \delta^*]$, there exists an $\varepsilon_1^*(\delta) \in (0, 1]$ such that for all $\varepsilon \in (0, \varepsilon_1^*(\delta)]$, the protocol (5) achieves output synchronization for any $\mathcal{G} \in \mathbb{G}^{\varphi,N}_{\alpha,\beta}$, that is, we have $\lim_{t\to\infty} \mathbf{e}(t) = 0$;
2) in the presence of disturbances, for any given $\gamma > 0$ and $\alpha, \beta, \varphi > 0$, there exits a $\delta^* \in (0, 1]$ such that, for each $\delta \in (0, \delta^*]$, there exists an $\varepsilon_2^*(\delta) \in (0, 1]$ such that for all $\varepsilon \in (0, \varepsilon_2^*(\delta)]$, the protocol (5) yields a closed-loop transfer function from $\bar{w}$ to $\mathbf{e}$, called $T_{\bar{w}\mathbf{e}}$, satisfying $\|T_{\bar{w}\mathbf{e}}\|_\infty < \gamma$ for any $\mathcal{G} \in \mathbb{G}^{\varphi,N}_{\alpha,\beta}$.

The proof of Theorem 1 is presented in the subsequent section in a constructive way.

A. Proof of Theorem 1

In this section, we first show that agents in the heterogeneous network of the form (1) can be shaped to agents of the form (11) by augmenting each original agent with a dynamic precompensator.

In the Appendix, it is shown that local dynamic precompensators exist for each agent $i \in \nu$ of the form

$$\begin{aligned} \dot{x}_{p,i} &= A_{p,i}x_{p,i} + B_{p1,i}u_i + B_{p2,i}z_{mi} \\ \bar{u}_i &= C_{p,i}x_{p,i} + D_{p1,i}u_i + D_{p2,i}z_{mi} \end{aligned} \tag{6}$$

such that the interconnection of (1) and (6) has the following form:

$$\begin{aligned} \dot{x}_i &= Ax_i + B(Mu_i + Rx_i) + E_{d,i}\bar{w}_i + \rho_i \\ y_i &= Cx_i \end{aligned} \tag{7}$$

where

$$A = \begin{pmatrix} 0 & I_{p(n_q-1)} \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ I_p \end{pmatrix}, \quad C = (\, I_p \quad 0 \,). \tag{8}$$

Moreover, an invertible M and R can be arbitrarily chosen. In the above text, ρ_i is a signal generated by the following stable system driven by $\bar{w}_i$:

$$\begin{aligned} \dot{\tilde{x}}_i &= H_i\tilde{x}_i + E_{o,i}\bar{w}_i, \\ \rho_i &= W_i\tilde{x}_i. \end{aligned} \tag{9}$$

The matrices H_i, $E_{o,i}$, and W_i follow directly from the explicit design in the Appendix. Next, we define

$$E_i = (\, E_{d,i} \quad I \,), \quad w_i = \begin{pmatrix} \bar{w}_i \\ \rho_i \end{pmatrix}.$$

[1] In this context, we mean by condition number the minimum of $\|U\|\|U\|^{-1}$ over all possible matrices U whose columns are the (generalized) eigenvectors of the matrix L.

Clearly, there exists ψ such that

$$\|\bar{w}_i\| \leq \|w_i\| \leq \psi\|\bar{w}_i\|. \tag{10}$$

Therefore, if we can achieve almost disturbance rejection from w_i to $\mathbf{e}_i$, then we are automatically guaranteed that we can also guarantee almost disturbance rejection from $\bar{w}_i$ to $\mathbf{e}_i$. Note that the above result is based on a feedforward precompensator which guarantees that each agent becomes a square system (using the idea of squaring down) with each agent having the same uniform rank n_q. Next, a feedback precompensator is used to obtain the identical dynamics except for some asymptotically stable dynamics which is incorporated through the signal ρ_i.

We can therefore guarantee that each agent $i \in \nu$ has the following form:

$$\begin{aligned}\dot{x}_i &= Ax_i + B(Mu_i + Rx_i) + E_i w_i,\\ y_i &= Cx_i.\end{aligned} \tag{11}$$

For agent $i \in \nu$, the measurement (4) is available. The local measurements (2) will no longer be used; they only play a role in bringing the system into the special form expressed before.

We will use a low-gain parameter $\delta \in (0,1]$ and a high-gain parameter $\varepsilon \in (0,1]$ to design the controller. First select K such that $A - KC$ is Hurwitz. Next, choose $F_\delta = -B'P_\delta$ where $P_\delta > 0$ is uniquely determined by the following algebraic Riccati equation:

$$P_\delta A + A'P_\delta - \tau P_\delta BB'P_\delta + \delta I = 0 \tag{12}$$

where $\tau > 0$ is the lower bound on the real parts of the nonzero eigenvalues of all Laplacian matrices L (i.e., $\tau \leq \beta$). Next, define $K_\varepsilon = \varepsilon^{-1}S_\varepsilon^{-1}K$, $F_{\delta\varepsilon} = \varepsilon^{-n_q}F_\delta S_\varepsilon$, and

$$S_\varepsilon = \operatorname{blkdiag}\{I_p, \varepsilon I_p, \ldots, \varepsilon^{n_q-1}I_p\}.$$

Then, we define the dynamic controller for each agent $i \in \nu$

$$\begin{aligned}\dot{\hat{x}}_i &= A\hat{x} + BR\hat{x}_i + K_\varepsilon(\zeta_i - C\hat{x}_i)\\ u_i &= M^{-1}F_{\delta\varepsilon}\hat{x}_i.\end{aligned} \tag{13}$$

The state $\hat{x}_i$ is not an estimate for x_i but is actually an estimate for

$$\sum_{j=1}^{N} a_{ij}(x_i - x_j) = \sum_{j=1}^{N} \ell_{ij}x_j$$

that is, we use an estimator for a linear combination of the other agents' relative state with the same weights as in the measurement ζ_i. Theorem 1 remains to be proved when all agents are of the form (11).

For each $i \in \{1, \ldots, N-1\}$, let $\bar{x}_i := x_N - x_i$ and $\hat{\bar{x}}_i := \hat{x}_N - \hat{x}_i$. Moreover, we define $\hat{w}_i = E_N w_N - E_i w_i$. Then, using (11), we can write

$$\begin{aligned}\dot{\bar{x}}_i &= A\bar{x}_i + B(F_{\delta\varepsilon}\hat{\bar{x}}_i + R\bar{x}_i) + \hat{w}_i\\ \mathbf{e}_i &= C\bar{x}_i.\end{aligned}$$

We define $\bar{g}_{ij} = \ell_{ij} - \ell_{Nj}$ for $i,j \in \{1, \ldots, N-1\}$. From (13), we obtain

$$\dot{\hat{\bar{x}}}_i = A\hat{\bar{x}}_i + BR\hat{\bar{x}}_i + \sum_{j=1}^{N-1}\bar{g}_{ij}K_\varepsilon C\bar{x}_j - K_\varepsilon C\hat{\bar{x}}_i. \tag{14}$$

Next, define $\xi_i = S_\varepsilon\bar{x}_i$ and $\hat{\xi}_i = S_\varepsilon\hat{\bar{x}}_i$. Then

$$\begin{aligned}\varepsilon\dot{\xi}_i &= A\xi_i + BF_\delta\hat{\xi}_i + V_{\varepsilon i}\xi_i + \varepsilon\bar{E}_{\varepsilon i}\hat{w}_i\\ \varepsilon\dot{\hat{\xi}}_i &= A\hat{\xi}_i + V_{\varepsilon i}\hat{\xi}_i + \sum_{j=1}^{N-1}\bar{g}_{i,j}KC\xi_j - KC\hat{\xi}_i\end{aligned} \tag{15}$$

where $V_{\varepsilon i} = \varepsilon^{n_q}BRS_\varepsilon^{-1}$ and $\bar{E}_{\varepsilon i} = S_\varepsilon$.

Define $\bar{G} = [\bar{g}_{ij}]$ with $i,j \in \{1, \ldots, N-1\}$. It follows from the proof of [29, Lemma 1] that the eigenvalue of $\bar{G}$ are the nonzero eigenvalues of L. Moreover, let

$$\xi = \operatorname{col}\{\xi_i\}, \quad \hat{\xi} = \operatorname{col}\{\hat{\xi}_i\} \text{ and } \hat{w} = \operatorname{col}\{\hat{w}_i\}.$$

Then, we have

$$\begin{aligned}\varepsilon\dot{\xi} &= (I_{N-1}\otimes A)\xi + (I_{N-1}\otimes BF_\delta)\hat{\xi} + V_\varepsilon\xi + \varepsilon\bar{E}_\varepsilon\hat{w}\\ \varepsilon\dot{\hat{\xi}} &= (I_{N-1}\otimes A)\hat{\xi} + V_\varepsilon\hat{\xi} + (\bar{G}\otimes KC)\xi - (I_{N-1}\otimes KC)\hat{\xi}\end{aligned} \tag{16}$$

where

$$V_\varepsilon = \operatorname{blkdiag}\{V_{\varepsilon i}\} \text{ and } \bar{E}_\varepsilon = \operatorname{blkdiag}\{\bar{E}_{\varepsilon i}\}.$$

Define $U^{-1}\bar{G}U = J$, where J is the Jordan form of $\bar{G}$. The fact that the condition number of L is bounded implies that the condition number of $\bar{G}$ is bounded. Moreover, $\bar{G}$ and L have the same nonzero eigenvalues. Since the condition number of $\bar{G}$ is bounded, we can guarantee uniform bounds on $\|U\|$ and $\|U^{-1}\|$. Next, let

$$v = \left(JU^{-1}\otimes I_{pn_q}\right)\xi, \quad \tilde{v} = v - \left(U^{-1}\otimes I_{pn_q}\right)\hat{\xi}.$$

Then

$$\begin{aligned}\varepsilon\dot{v} &= (I_{N-1}\otimes A)v + (J\otimes BF_\delta)(v-\tilde{v}) + W_\varepsilon v + \varepsilon\bar{E}\hat{w}\\ \varepsilon\dot{\tilde{v}} &= (I_{N-1}\otimes A)\tilde{v} + (J\otimes BF_\delta)(v-\tilde{v}) + W_\varepsilon\nu + \varepsilon\bar{E}\hat{w}\\ &\quad - \hat{W}_\varepsilon(v-\tilde{v}) - (I_{N-1}\otimes KC)\tilde{v}\end{aligned} \tag{17}$$

where

$$\bar{E} = \left(JU^{-1}\otimes I_{pn_q}\right)\bar{E}_\varepsilon$$

while we use

$$\begin{aligned}W_\varepsilon &= \left(JU^{-1}\otimes I_{pn_q}\right)V_\varepsilon\left(UJ^{-1}\otimes I_{pn_q}\right)\\ \hat{W}_\varepsilon &= \left(U^{-1}\otimes I_{pn_q}\right)V_\varepsilon\left(U\otimes I_{pn_q}\right).\end{aligned}$$

Finally, define Z such that

$$\eta := Z\begin{pmatrix} v \\ \tilde{v}\end{pmatrix} = \begin{pmatrix} v_1 \\ \tilde{v}_1 \\ \vdots \\ v_{N-1} \\ \tilde{v}_{N-1}\end{pmatrix} \text{ where } Z = \begin{pmatrix} e_1 & 0 \\ 0 & e_1 \\ \vdots & \vdots \\ e_{N-1} & 0 \\ 0 & e_{N-1}\end{pmatrix} \otimes I_{pn_q}$$

where $e_i \in \mathbb{R}^{N-1}$ is the ith standard basis vector whose elements are all zero except for the ith element which is equal to 1. Then (17) can be written as

$$\varepsilon\dot{\eta} = \tilde{A}_\delta \eta + \tilde{W}_\varepsilon \eta + \varepsilon \tilde{E}\hat{w} \tag{18}$$

where

$$\tilde{A}_\delta = I_{N-1} \otimes \begin{pmatrix} A & 0 \\ 0 & A-KC\end{pmatrix} + J \otimes \begin{pmatrix} BF_\delta & -BF_\delta \\ BF_\delta & -BF_\delta \end{pmatrix} \tag{19}$$

and

$$\tilde{W}_\varepsilon = Z\begin{pmatrix} W_\varepsilon & 0 \\ W_\varepsilon - \hat{W}_\varepsilon & \hat{W}_\varepsilon\end{pmatrix} Z^{-1}, \quad \tilde{E} = Z\begin{pmatrix}\bar{E} \\ \bar{E}\end{pmatrix}.$$

We recall the following result from [3] which is a consequence of earlier results in [21, Proof of Theor. 4].

Lemma 1: The matrix $\tilde{A}_\delta$ as defined in (19) has the property that for any α, we can choose δ small enough such that $\tilde{A}_\delta$ is asymptotically stable for all matrices J in Jordan form whose eigenvalues satisfy $|\lambda_i| < \alpha$.

The above lemma implies that there exists a small δ^*, such that for any $\delta \in (0, \delta^*]$, $\tilde{A}_\delta$ is Hurwitz stable since $\mathcal{G} \in \mathbb{G}_{\alpha,\beta}^{\varphi,N}$ guarantees that the eigenvalues of J satisfy $|\lambda_i| < \alpha$. Thus, we have $P_\delta = P_\delta' > 0$, satisfying

$$\tilde{A}_\delta' P_\delta + P_\delta' \tilde{A}_\delta = -3I_{2(N-1)pn_q}. \tag{20}$$

Then, define $V_p = \varepsilon\eta' P_\delta \eta$ as a Lyapunov function for (18). The derivative of V_p is bounded by

$$\begin{aligned}\dot{V}_p &= -3\|\eta\|^2 + 2\mathrm{Re}(\eta^* P_\delta \tilde{W}_\varepsilon \eta) + 2\varepsilon \mathrm{Re}(\eta^* P_\delta \tilde{E}\hat{w}) \\ &\le -3\|\eta\|^2 + 2\|P_\delta\|\|\tilde{W}_\varepsilon\|\|\eta\|^2 + 2\sigma\varepsilon\|\hat{w}\|\|\eta\| \\ &= -\left(2 - 2\|P_\delta\|\|\tilde{W}_\varepsilon\|\right)\|\eta\|^2 - (\|\eta\| - \sigma\varepsilon\|\hat{w}\|)^2 \\ &\quad + \varepsilon^2\sigma^2\|\hat{w}\|^2 \\ &\le -\left(2 - 2\|P_\delta\|\|\tilde{W}_\varepsilon\|\right)\|\eta\|^2 + \varepsilon^2\sigma^2\|\hat{w}\|^2\end{aligned}$$

where σ is an upper bound for $\|P_\delta \tilde{E}\|$. Moreover, there exists an ε^* such that, for any $\varepsilon \in (0, \varepsilon^*]$, we have that $2 - 2\|P_\delta\|\|\tilde{W}_\varepsilon\| > 1$. The

$$\begin{aligned}\dot{V}_p &\le -\left(2 - 2\|P_\delta\|\|\tilde{W}_\varepsilon\|\right)\|\eta\|^2 + \varepsilon^2\sigma^2\|\hat{w}\|^2 \\ &\le -\|\eta\|^2 + \varepsilon^2\sigma^2\|\hat{w}\|^2.\end{aligned}$$

So

$$\dot{V}_p + \|\eta\|^2 - \varepsilon^2\sigma^2\|\hat{w}\|^2 \le 0. \tag{21}$$

From the Kalman-Yakubovich-Popov Lemma (see, for example, [30]), we conclude that (21) implies $\|T_{\hat{w}\eta}\|_\infty \le \varepsilon\sigma$. Next, we will show that this implies that $\|T_{\bar{w}\mathbf{e}}\|_\infty \le \varepsilon\sigma'$ for a certain σ'.

Recall that $\hat{w}_i = E_N w_N - E_i w_i$ which implies that $\hat{w} = \Delta_w w$, where

$$\Delta_w = \begin{pmatrix} -E_1 & 0 & \cdots & 0 & E_N \\ 0 & -E_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & E_N \\ 0 & \cdots & 0 & -E_{N-1} & E_N\end{pmatrix}.$$

Following the proof above, we find that

$$\begin{aligned}\mathbf{e} &= (I_{N-1} \otimes C)\left(I_{N-1} \otimes S_\varepsilon^{-1}\right)\left(UJ^{-1} \otimes I_{pn_q}\right)(I_{N-1}\ 0)Z^{-1}\eta \\ &= \Theta\eta\end{aligned} \tag{22}$$

for a suitably chosen matrix Θ. The norm of Θ is obviously bounded for $CS_\varepsilon^{-1} = C$. Therefore

$$\begin{aligned}\|\mathbf{e}\| &= \|\Theta\eta\| \\ &\le \|\Theta\|\|T_{\hat{w}\eta}\|_\infty\|\hat{w}\| \\ &\le \|\Theta\|\|T_{\hat{w}\eta}\|_\infty\|\Delta_w\|\|w\| \\ &\le \varepsilon\sigma\|\Theta\|\|\Delta_w\|\|w\|.\end{aligned} \tag{23}$$

Recall that we have

$$w_i = \begin{pmatrix}\bar{w}_i \\ \rho_i\end{pmatrix}$$

with the norm bound (10). This implies that we have

$$\|T_{\bar{w}\mathbf{e}}\|_\infty \le \varepsilon\psi\sigma\|\Theta\|\|\Delta_w\| \tag{24}$$

which can be made arbitrarily small by an appropriate choice for ε.

IV. $\mathcal{H}_\infty$ Almost Regulated Output Synchronization

In this section, we consider the case where the agents try to asymptotically track a particular trajectory in the presence of external disturbances. In this paper, the reference trajectory is generated by an autonomous system of the form

$$\bar{\Sigma}_0 : \begin{cases}\dot{\bar{x}}_0 = \bar{A}_0\bar{x}_0, & \bar{x}_0(0) = \bar{x}_{00} \\ y_0 = \bar{C}_0\bar{x}_0\end{cases} \tag{25}$$

where $\bar{x}_0 \in \mathbb{R}^{n_0}$, $y_0 \in \mathbb{R}^p$. Moreover, we have that $(\bar{A}_0, \bar{C}_0)$ is observable, all eigenvalues of $\bar{A}_0$ are in the closed-right half complex plane and, finally, $\bar{C}_0$ has full-row rank.

Define $e_{i0} := y_i - y_0$ as the regulation error for agent $i \in \nu$ and $\mathbf{e}_0 = \mathrm{col}\{e_{i0}\}$. In a network graph $\mathcal{G}$, the set π is denoted the set of agents receiving information from the exosystem. When the network graph $\mathcal{G}$ consists of one connected component with a spanning tree, the set π may only contain one node which is the root of such a spanning tree.

In order to achieve regulated output synchronization for all agents, the following assumption is clearly necessary.

Assumption 2: Every node of the network graph $\mathcal{G}$ is a member of a directed tree with a root contained in the root set $\pi \subset \nu$.

To regulate all agents in the network to a reference trajectory, agents in the so-called root set π will receive the regulation error e_{i0} from the network. It implies that the network measurement is altered to

$$\tilde{\zeta}_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j) + \Psi_i(y_i - y_0) \tag{26}$$

where $\Psi_i = 1$ for $i \in \pi$; while $\Psi_i = 0$ otherwise.

Note that the reference system can be viewed as a new root node, denoted as node 0. This expanded network will be referred to as the augmented network and will be denoted by $\tilde{\mathcal{G}}$. Based on Assumption 2, this augmented network will contain a directed spanning tree with node 0 as its root [4]. The associated Laplacian matrix $\tilde{L} = [\tilde{l}_{ij}]$ is updated as

$$\tilde{L} = \begin{pmatrix} 0 & 0 \\ -\mathrm{col}\{\Psi_i\} & L + \mathrm{diag}\{\Psi_i\} \end{pmatrix}.$$

So (26) can be rewritten as

$$\tilde{\zeta}_i = \sum_{j=0}^{N} \tilde{l}_{ij} y_j.$$

We will define the $\mathcal{H}_\infty$ almost regulated output synchronization problem as follows.

Problem 2: Consider a multiagent system described by (1) and associated root set π. Given a set of network graphs $\mathbb{G}$ and any $\gamma > 0$, the $\mathcal{H}_\infty$ almost regulated output synchronization, with respect to a reference y_0 generated by (25), is to find, if possible, a linear time-invariant dynamic protocol such that, for any $\mathcal{G} \in \mathbb{G}$, the following holds:

- In the absence of disturbances, we have $\lim_{t\to\infty} \mathbf{e}_0(t) = 0$ for all possible initial conditions for the agents and for the exosystem.
- The closed-loop transfer function from $\bar{w}$ to $\mathbf{e}_0$ satisfies $\|T_{\bar{w}e_0}\|_\infty < \gamma$.

Clearly, the aforementioned problem will never be solvable for all possible sets of network graphs. The definition below defines the sets of network graphs that we will consider in this paper.

Definition 3: Consider a multiagent system described by (1) and associated root set π. For given $\alpha, \beta, \varphi > 0$, $\mathbb{G}^{\varphi,N}_{\alpha,\beta,\pi}$ is the set of directed graphs composed of N nodes such that every $\mathcal{G} \in \mathbb{G}^{\varphi,N}_{\alpha,\beta,\pi}$ satisfies Assumption 2 with the root set π, and the eigenvalues of the Laplacian matrix $\tilde{L}$ of the augmented graph $\tilde{\mathcal{G}}$ satisfy $Re\{\tilde{\lambda}_i\} > \beta$ and $|\tilde{\lambda}_i| < \alpha$ for nonzero eigenvalues where $i \in \{0, 1, 2, \ldots, N\}$ and the condition number of $\tilde{L}$ is bounded by φ.

Here is the main result in this section.

Theorem 2: Consider a multiagent system described by (1) and a set of network graphs $\mathbb{G}^{\varphi,N}_{\alpha,\beta,\pi}$. Under Assumption 1, the $\mathcal{H}_\infty$ almost regulated output synchronization problem is solvable. To be specific, there exists a family of low- and high-gain linear time-invariant dynamic protocols, parameterized in terms of low- and high-gain parameters $\delta, \varepsilon \in (0, 1]$ of the form

$$\begin{cases} \dot{\chi}_i = \mathcal{A}_i(\delta,\varepsilon)\chi_i + \mathcal{B}_i(\delta,\varepsilon)\begin{pmatrix} \tilde{\zeta}_i \\ z_{m,i} \end{pmatrix}, \\ \bar{u}_i = \mathcal{C}_i(\delta,\varepsilon)\chi_i + \mathcal{D}_i(\delta,\varepsilon)\begin{pmatrix} \tilde{\zeta}_i \\ z_{m,i} \end{pmatrix} \end{cases} \tag{27}$$

where $\chi_i \in \mathbb{R}^{qi}$ and $i \in \nu$, such that:

1) in the absence of disturbances, for any given $\alpha, \beta, \varphi > 0$, there exits a $\delta^* \in (0, 1]$ such that, for each $\delta \in (0, \delta^*]$, there exists an $\varepsilon_1^*(\delta) \in (0, 1]$ such that for all $\varepsilon \in (0, \varepsilon_1^*(\delta)]$, the protocol (27) achieves regulated output synchronization to the reference y_0 for any $\mathcal{G} \in \mathbb{G}^{\varphi,N}_{\alpha,\beta,\pi}$, that is, $\mathbf{e}_0(t) \to 0$ as $t \to \infty$.
2) In the presence of disturbances, for any given $\gamma > 0$ and $\alpha, \beta, \varphi > 0$, there exits a $\delta^* \in (0, 1]$ such that, for each $\delta \in (0, \delta^*]$, there exists an $\varepsilon_2^*(\delta) \in (0, 1]$ such that for all $\varepsilon \in (0, \varepsilon_2^*(\delta)]$, the protocol (27) yields a closed-loop transfer function from $\bar{w}$ to $\mathbf{e}_0$, called $T_{\bar{w}\mathbf{e}_0}$, satisfying $\|T_{\bar{w}\mathbf{e}_0}\|_\infty < \gamma$ for any $\mathcal{G} \in \mathbb{G}^{\varphi,N}_{\alpha,\beta,\pi}$.

The proof is again constructive and similar to the proof of Theorem 1, except that we first need to guarantee that the exosystem has a uniform rank.

In [25, Appendix] it has been established that a fictitious input u_0 can be added to the reference system (25), such that the expanded exosystem (A_0, B_0, C_0)

$$\Sigma_0 : \begin{cases} \dot{\tilde{x}}_0 = A_0\tilde{x}_0 + B_0 u_0, \quad \tilde{x}_0(0) = \tilde{x}_{00} \\ y_0 = C_0 x_0 \end{cases} \tag{28}$$

is invertible and without invariant zeros and is such that it has a uniform rank n_q, which is equal to the maximum relative degree of all agents. Moreover, for every initial condition $\bar{x}_{00}$ of the exosystem (25), we have that there exists $\tilde{x}_{00}$ such that (28) yields the same output for $u_0 = 0$. According to [11, Lemma 4], we can then, by a suitable basis transformation, bring this expanded exosystem in the form

$$\Sigma_0 : \begin{cases} \dot{x}_0 = Ax_0 + B(M_0u_0 + R_0x_0), \quad x_0(0) = x_{00} \\ y_0 = Cx_0 \end{cases} \tag{29}$$

where $x_0 \in \mathbb{R}^{pn_q}$ and $u_0 \in \mathbb{R}^p$ while A, B, and C are given by (8) while $M_0 \in \mathbb{R}^{p\times p}$ is an arbitrary nonsingular matrix. The exosystem is autonomous and, therefore, we choose $u_0 = 0$. Finally, $R_0 \in \mathbb{R}^{p\times pn_q}$ such that (29) can generate the same outputs as (25). By adding a term E_0w_0 with $E_0 = 0$, in (29), we obtain the same form for the exosystem as the agents in the network (11), with $i \in \{0, \nu\}$.

The protocol designed here is similar to the one (13) used in the synchronization case. The dynamic protocol used for each agent in the augmented network is given by

$$\begin{aligned} \dot{\hat{x}}_i &= A\hat{x} + BR_0\hat{x}_i + K_\varepsilon(\tilde{\zeta}_i - C\hat{x}_i), \\ u_i &= M_0^{-1}F_{\delta\varepsilon}\hat{x}_i \end{aligned} \tag{30}$$

where $K_\varepsilon = \varepsilon^{-1}S_\varepsilon^{-1}K$ and $F_{\delta\varepsilon} = \varepsilon^{-n_q}F_\delta S_\varepsilon$. K and F_δ are chosen in the same way as in Section III. Notice that setting $\hat{x}_0(0) = 0$ leads to $\hat{x}_0(t) = 0$ and $u_0(t) = 0$ for $\forall t \geq 0$.

The remaining proof of Theorem 2 is then the same as the proof of Theorem 1.

V. $\mathcal{H}_\infty$ Almost Formation

In this section, the synchronization method posted before is shown to also be applicable to the formation problem in the presence of disturbances. The objective of formation control is to keep the relative output of each agent in the network at a desired value.

Define a family of vectors $V_f := \mathrm{col}\{f_i\}$ with $f_i \in \mathbb{R}^p$ and let $y_{fi} = y_i - f_i$ be the relative output for agent $i \in \nu$. The network disagreement is defined as $\bar{e}_{fi} = y_{fN} - y_{fi}$ for $i \in \nu$. Define $\mathbf{e}_f := \mathrm{col}\{\bar{e}_{fi}\}$.

The $\mathcal{H}_\infty$ almost formation problem is defined as.

Problem 3: Consider a multiagent system described by (1) with a communication topology $\mathcal{G}$. Given a set of network graphs $\mathbb{G}$ and any $\gamma > 0$, the $\mathcal{H}_\infty$ almost formation problem with respect to a formation set V_f is to find, if possible, a linear time-invariant dynamic protocol where the following holds:

- In the absence of disturbances, we have $\lim_{t\to\infty} \mathbf{e}_f(t) = 0$ for all possible initial conditions.
- The closed-loop transfer function from $\bar{w}$ to $\mathbf{e}_f$ satisfies $\|T_{\bar{w}\mathbf{e}_f}\|_\infty < \gamma$.

In formation control, the network measurement is modified as

$$\zeta_{fi} = \sum_{j=1}^{N} \ell_{ij} y_{fi}. \tag{31}$$

Define $\bar{f}_i = \mathrm{col}\{f_i, 0\}$, $\bar{f}_i \in \mathbb{R}^{pn_q \times p}$, $i \in \nu$, such that $C\bar{f}_i = f_i$ and $(A + BR)\bar{f}_i = 0$. Finally, define $x_{fi} = x_i - \bar{f}_i$, and the network agents (11) can be rewritten as

$$\begin{aligned} \dot{x}_{fi} &= Ax_{fi} + B(Mu_i + Rx_{fi}) + E_i w_i \\ y_{fi} &= Cx_{fi}. \end{aligned} \tag{32}$$

Similar to (13), the controller for each agent in the formation problem is

$$\begin{aligned} \dot{\hat{x}}_{fi} &= A\hat{x}_{fi} + BR\hat{x}_{fi} + K_\varepsilon(\zeta_{fi} - C\hat{x}_{fi}), \\ u_i &= M^{-1} F_{\delta\varepsilon} \hat{x}_{fi}. \end{aligned} \tag{33}$$

Here, K_ε, $F_{\delta\varepsilon}$, ε, and S_ε have the same definition and structure as in (13).

The main result in this section is stated as follows.

Theorem 3: Consider a multiagent system described by (1) and a set of network graphs $\mathbb{G}^{\varphi,N}_{\alpha,\beta}$. Under assumption 1, the problem of $\mathcal{H}_\infty$ almost formation is solvable. In particular, there exists a family of low- and high-gain linear time-invariant dynamic protocols, parameterized in terms of low- and high-gain parameters $\delta, \varepsilon \in (0, 1]$ of the form

$$\begin{cases} \dot{\chi}_i = \mathcal{A}_i(\delta,\varepsilon)\chi_i + \mathcal{B}_i(\delta,\varepsilon)\begin{pmatrix} \zeta_{fi} \\ z_{m,i} \\ f_i \end{pmatrix}, \\ \bar{u}_i = \mathcal{C}_i(\delta,\varepsilon)\chi_i + \mathcal{D}_i(\delta,\varepsilon)\begin{pmatrix} \zeta_{fi} \\ z_{m,i} \\ f_i \end{pmatrix} \end{cases} \tag{34}$$

where $\chi_i \in \mathbb{R}^{q_i}$, $i \in \nu$, such that:

1) in the absence of disturbances, for any given $\alpha, \beta, \varphi > 0$, there exists a $\delta^* \in (0, 1]$ such that, for each $\delta \in (0, \delta^*]$, there exists an $\varepsilon_1^*(\delta) \in (0, 1]$ such that for all $\varepsilon \in (0, \varepsilon_1^*(\delta)]$, the protocol (34) achieves the desired formation for any $\mathcal{G} \in \mathbb{G}^{\varphi,N}_{\alpha,\beta}$; that is, $\mathbf{e}_f(t) \to 0$, as $t \to \infty$;
2) in the presence of disturbances, for any given $\gamma > 0$ and $\alpha, \beta, \varphi > 0$, there exists a $\delta^* \in (0, 1]$ such that, for each $\delta \in (0, \delta^*]$, there exists an $\varepsilon_2^*(\delta) \in (0, 1]$ such that for all $\varepsilon \in (0, \varepsilon_2^*(\delta)]$, the protocol (34) yields a closed-loop transfer function from $\bar{w}$ to $\mathbf{e}_f$, called $T_{\bar{w}\mathbf{e}_f}$, satisfying $\|T_{\bar{w}\mathbf{e}_f}\|_\infty < \gamma$ for any $\mathcal{G} \in \mathbb{G}^{\varphi,N}_{\alpha,\beta}$.

The proof follows a similar route as in Section III.

Remark 3: We should note that the given protocol imposes no restrictions on the agreement trajectories; that is, although agents establish the desired configuration, it is not clear where the entire system heads to. The formation problem can be combined with the regulation problem, giving rise to the problem of $\mathcal{H}_\infty$ almost formation with regulated output synchronization.

VI. Example

We illustrate the result on a network of ten agents, which are of the form (1) with

$$A_{i_1} = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}, \ B_{i_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ C'_{i_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$C'_{m,i_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ G_{i_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$A_{i_2} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \ B_{i_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$C'_{i_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ C'_{m,i_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ G_{i_2} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$A_{i_3} = \begin{pmatrix} -3 & 2 & 0 \\ 0 & 0 & 1 \\ 2 & 2 & 1 \end{pmatrix}, \ B_{i_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$C'_{i_3} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ C'_{m,i_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ G_{i_3} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

$$A_{i_4} = \begin{pmatrix} -2 & 3 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix}, \ B_{i_4} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$C'_{i_4} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ C'_{m,i_4} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ G_{i_4} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

and $i_1 \in \{1, 2, 3\}$, $i_2 \in \{4, 5, 6\}$, $i_3 \in \{7, 8\}$, $i_4 \in \{9, 10\}$. The network topology is given by the directed graph presented in Fig. 1(a). The external disturbances are chosen as $\bar{w}_1(t) = \sin(9t)$, $\bar{w}_2(t) = 1.5$, $\bar{w}_3(t) = \cos(6t)$, $\bar{w}_4(t) = 1$, $\bar{w}_5(t) = \cos(3t)$, $\bar{w}_6(t) = \sin(6t)$, $\bar{w}_7(t) = \sin(5t)$, $\bar{w}_8(t) = \cos(t)$, $\bar{w}_9(t) = \sin(10t)$, and $\bar{w}_{10}(t) = \cos(2t)$. The degree of infinite zeros for each agent is equal to 2.

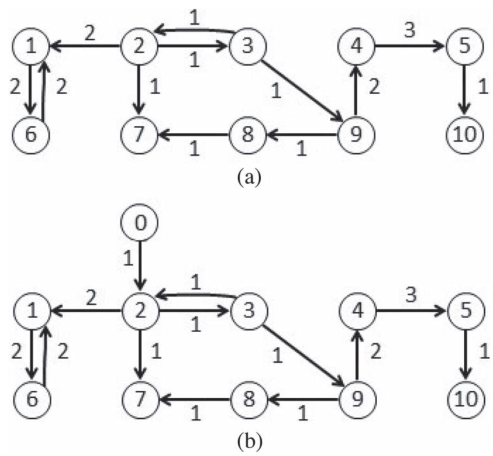

Fig. 1. Topologies of the primary and augmented network. (a) Primary network. (b) Augmented network.

By choosing $R = [-2, -3]$, and $M = 1$, the precompensators for the aforementioned systems in the form of (6) are designed as follows:

$$\begin{cases} \dot{x}_{p,i_1} = \begin{pmatrix} -6 & -3 & 0 \\ -7 & -7 & 1 \\ -7 & -8 & 1 \end{pmatrix} x_{p,i_1} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_{i_1} + \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} z_{m,i_1}, \\ \bar{u}_{i_1} = \begin{pmatrix} -2 & -3 & -4 \end{pmatrix} x_{p,i_1} + u_{i_1} \end{cases}$$

$$\begin{cases} \dot{x}_{p,i_2} = \begin{pmatrix} -7 & 1 & -5 \\ -7 & 0 & -6 \\ -8 & 1 & -7 \end{pmatrix} x_{p,i_2} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_{i_2} + \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} z_{m,i_2}, \\ \bar{u}_{i_2} = \begin{pmatrix} -1 & -3 & -5 \end{pmatrix} x_{p,i_2} + u_{i_2} \end{cases}$$

$$\begin{cases} \dot{x}_{p,i_3} = \begin{pmatrix} -3 & 2 & -5 \\ 0 & 0 & -6 \\ 2 & 2 & -14 \end{pmatrix} x_{p,i_3} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_{i_3} + \begin{pmatrix} 5 \\ 7 \\ 15 \end{pmatrix} z_{m,i_3}, \\ \bar{u}_{i_3} = \begin{pmatrix} -2 & -4 & -4 \end{pmatrix} x_{p,i_3} + u_{i_3} \end{cases}$$

$$\begin{cases} \dot{x}_{p,i_4} = \begin{pmatrix} -7 & -2 & 0 \\ -7 & -7 & 1 \\ -27 & -28 & 3 \end{pmatrix} x_{p,i_4} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_{i_4} + \begin{pmatrix} 5 \\ 7 \\ 30 \end{pmatrix} z_{m,i_4}, \\ \bar{u}_{i_4} = \begin{pmatrix} -3 & -4 & -6 \end{pmatrix} x_{p,i_4} + u_{i_4} \end{cases}$$

where i_1, i_2, i_3, i_4 are defined in the same way as before. Moreover, the agents have dynamics of the form (11) with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

$\mathcal{H}_\infty$ almost output synchronization: Our protocol for this example is given by (13) with $K_\varepsilon = \varepsilon^{-1} S_\varepsilon^{-1} K$ and $F_{\delta\varepsilon} = \varepsilon^{-2} F_\delta S_\varepsilon$. By choosing $K = (3\ 2)$ and $\delta = 10^{-5}$, then, if $\varepsilon = 0.05$, we obtain $K_\varepsilon = [60; 800]$, $F_{\delta\varepsilon} = [-1.7889\ \ -2.6765]$ while, if $\varepsilon = 0.008$, we obtain $K_\varepsilon = [375; 31250]$, $F_{\delta\varepsilon} = [-69.8771\ \ -16.7279]$. Fig. 2 shows the results for $\varepsilon = 0.05$ and $\varepsilon = 0.008$. When ε is smaller, the effect of disturbance on the network disagreement is squeezed much more.

$\mathcal{H}_\infty$ almost regulated output synchronization: In this case, the reference system $y_0 = \cos(0.5t)$ is connected to the root agent 2, shown in Fig. 1(b). So in (29), $R_0 = [-0.5^2\ 0]$, $M_0 = 1$, A, B, and C are the same as before. The protocol for this example is given by (30). Here, K stays unchanged and we choose $\delta = 10^{-4}$. When $\varepsilon = 0.04$, $K_\varepsilon = [75; 1250]$, $F_{\delta\varepsilon} = [-8.8388\ \ -5.9565]$. And when $\varepsilon = 0.01$, $K_\varepsilon = [300; 20000]$, $F_{\delta\varepsilon} = [-141.4214\ \ -23.8262]$. Fig. 3 shows that when ε is smaller, the network behaves much more like the reference.

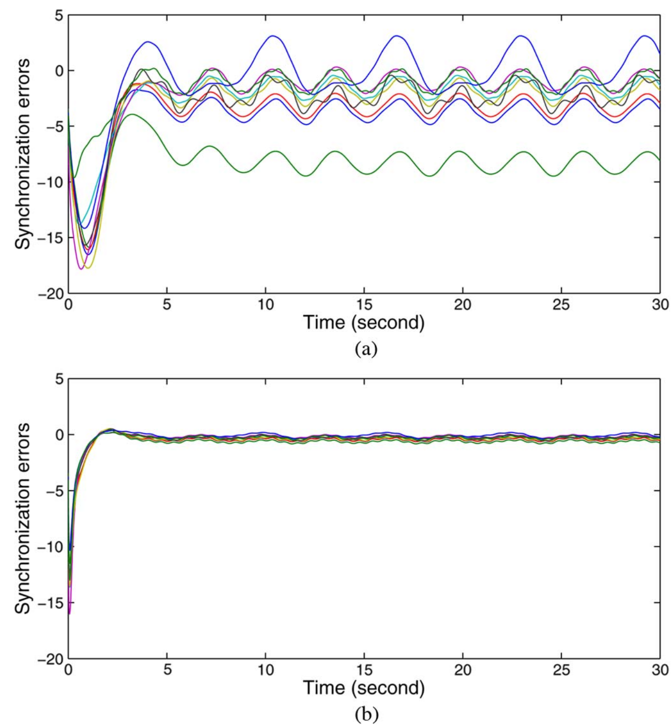

Fig. 2. $\mathcal{H}_\infty$ almost output synchronization. (a) $\mathcal{H}_\infty$ almost output synchroniztion with $\varepsilon = 0.05$, $\delta = 1e-5$; (b) $\mathcal{H}_\infty$ almost output synchronization with $\varepsilon = 0.008$, $\delta = 1e-5$.

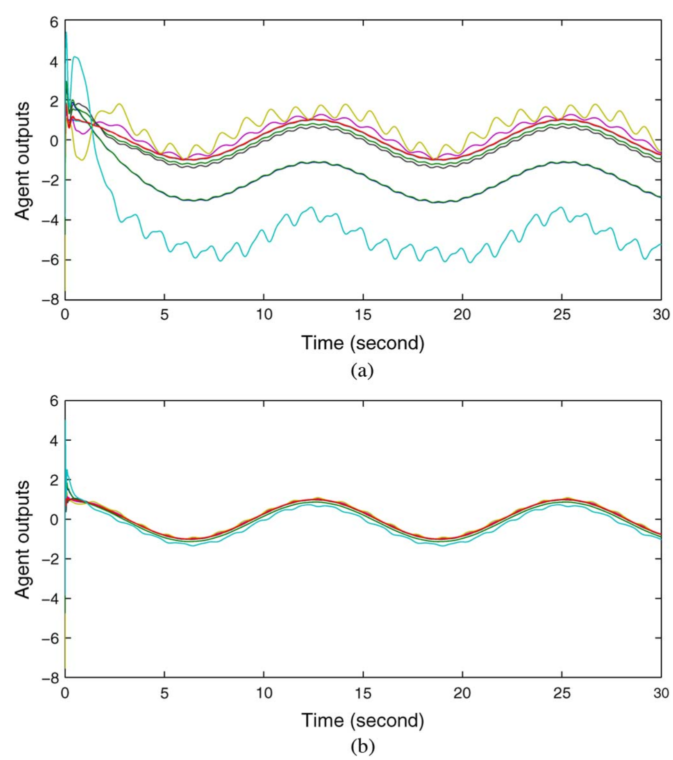

Fig. 3. $\mathcal{H}_\infty$ almost regulated output synchronization. (a) $\mathcal{H}_\infty$ almost regulated output synchroniztion with $\varepsilon = 0.04$, $\delta = 1e-4$; (b) $\mathcal{H}_\infty$ almost regulated output synchronization with $\varepsilon = 0.01$, $\delta = 1e-4$.

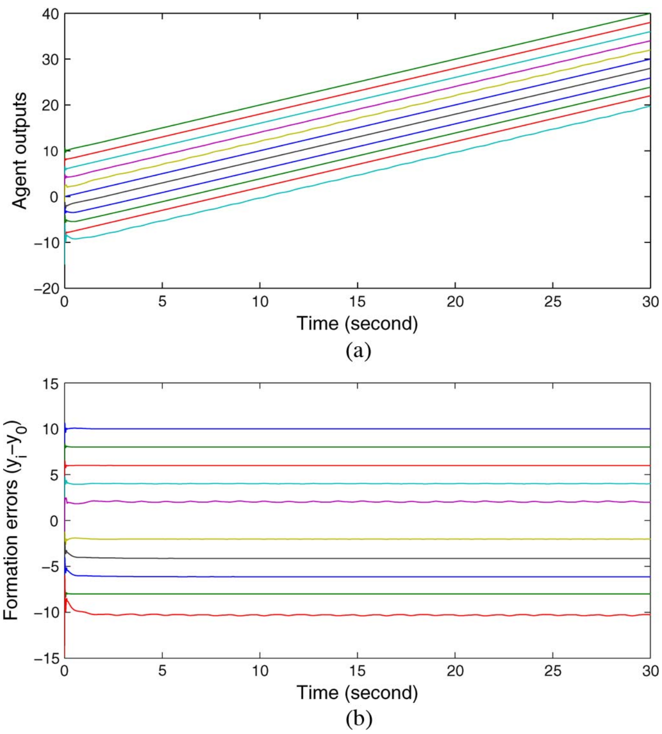

Fig. 4. $\mathcal{H}_\infty$ almost formation with regulated output. (a) $\mathcal{H}_\infty$ almost formation with regulated output with $\varepsilon = 0.01$, $\delta = 1e-4$. (b) Formation errors $y_i - y_0$.

$\mathcal{H}_\infty$ almost formation with output regulation: We use the same network as that in regulated output synchronization, but set a different reference $y_0 = t$. Agents in the network have to form a desired configuration and follow the reference trajectory as well. R_0 in (29) is [0 0], and the others are the same. And choose the formation set $\{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}\} = \{10, 8, 6, 4, 2, -2, -4, -6, -8, -10\}$. The protocol for this formation problem is given by (33). Here, K is unchanged and $\delta = 10^{-4}$, $\varepsilon = 0.01$. $K_\varepsilon = [300; 20000]$, $F_{\delta\varepsilon} = [-141.4214 \ \ -23.8262]$. Fig. 4 shows that all ten agents almost behave like the reference y_0 while they are in formation set.

Appendix

In this section, we briefly introduce how to build a precompensator to shape the general dynamic system (1) to the dynamics system (7). In Assumption 1, each agent is assumed to be right-invertible. Furthermore, each agent has the order of infinite zero $n_{q,i}$, $i = 1, \ldots, N$. Let $n_q \geq \max\{n_{q,i}\}$, for $i = 1, \ldots, N$. Then, we will first design a pre-ompensator to make the agent dynamics invertible and of equal rank n_q

$$\begin{cases} \dot{x}^1_{p,i} = A^1_{p,i} x^1_{p,i} + B^1_{p,i} u^1_i, \\ \bar{u}_i = C^1_{p,i} x^1_{p,i} x^1_{p,i} + D^1_{p,i} u^1_i \end{cases} \tag{35}$$

where $u^1_i \in \mathbb{R}^p$. The design procedures for invertibility and rank equalization were developed originally in [18] and [19], respectively, and they can be also found in detail in [27, App. A.2 and A.3]. Next, we will concentrate on transforming invertible, equal-ranking different system dynamics to the almost identical ones in the form of (7).

Let $\tilde{\bar{x}}_i = \mathrm{col}\{\breve{x}_i; x^1_{p,i}\}$. There always exists nonsingular state transformation $\Gamma_{i,x}$ and input transformation $\Gamma_{i,u}$ (please refer to [19]), such that

$$\tilde{\bar{x}}_i = \Gamma_{i,x} \bar{\bar{x}}_i, \quad u^1_i = \Gamma_{i,u} u^2_i \tag{36}$$

where $\bar{\bar{x}}_i := \mathrm{col}\{\bar{\bar{x}}_{i,a}; \bar{\bar{x}}_{i,d}\}$. Then, combining the system of (1) and (35) can be written in this form

$$\begin{cases} \dot{\bar{\bar{x}}}_{i,a} = \bar{A}_{i,a} \bar{\bar{x}}_{i,a} + \bar{L}_{i,a} y_i + E_{a,i} \bar{w}_i, \\ \dot{\bar{\bar{x}}}_{i,d} = A_d \bar{\bar{x}}_{i,d} + B_d \left(u^2_i + D_{i,a} \bar{\bar{x}}_{i,a} + D_{i,d} \bar{\bar{x}}_{i,d}\right) + E_{d,i} \bar{w}_i, \\ y_i = C_d \bar{\bar{x}}_{i,d} \end{cases} \tag{37}$$

where A_d, B_d, C_d have the same definitions of (8). It is worth noting that the zero dynamics is decoupled. So we could have internal stability if the system is nonminimum phase.

Note that the information $\bar{z}_{m,i} := \mathrm{col}\{z_{m,i}; x^1_{p,i}\}$ is available for agent i, and $\bar{z}_{m,i}$ can be represented in terms of $\bar{\bar{x}}_{i,a}$, $\bar{\bar{x}}_{i,d}$ as

$$\bar{z}_{m,i} = \bar{C}_{m,i} \begin{pmatrix} \bar{\bar{x}}_{i,a} \\ \bar{\bar{x}}_{i,d} \end{pmatrix}, \quad \text{where } \bar{C}_{m,i} = \begin{pmatrix} C_{m,i} & 0 \\ 0 & I \end{pmatrix} \Gamma_{i,x}.$$

From (37), we define that for $i = 1, \ldots, N$

$$\bar{A}_i = \begin{pmatrix} \bar{A}_{i,a} & \bar{L}_{i,a} C_d \\ B_d D_{i,a} & A_d + B_d D_{i,d} \end{pmatrix}, \ \bar{B}_i = \begin{pmatrix} 0 \\ B_d \end{pmatrix}, \ E_i = \begin{pmatrix} E_{a,i} \\ E_{d,i} \end{pmatrix}.$$

From Assumption 1, $(C_{m,i}, A_i)$ is observable, which implies $(\bar{C}_{m,i}, \bar{A}_i)$ is observable. We then design an observer-based precompensator for the system (37) as

$$\begin{cases} \dot{\hat{\bar{\bar{x}}}}_i = \bar{A}_i \hat{\bar{\bar{x}}}_i + \bar{B}_i u^2_i - \bar{K}_i (\bar{z}_{m,i} - \bar{C}_{m,i} \hat{\bar{\bar{x}}}_i), \\ u^2_i = (-D_{i,a} \ \bar{F}_d - D_{i,d}) \hat{\bar{\bar{x}}}_i + u_i \end{cases} \tag{38}$$

where $u_i \in \mathbb{R}^p$, $\bar{K}_i$ is chosen such that $\bar{A}_i + \bar{K}_i \bar{C}_{m,i}$ is Hurwitz stable, and $\bar{F}_d$ is chosen such that $A_d + B_d \bar{F}_d$ has desired eigenvalues. Define the observer error $\tilde{x}_i = \bar{\bar{x}}_i - \hat{\bar{\bar{x}}}_i$. It is easy to see that the observer error dynamics is asymptotically stable. Moreover, the effect of $\bar{\bar{x}}_{i,a}$ on the dynamics $\bar{\bar{x}}_{i,d}$ is asymptotically canceled. Thus, the mapping from the new input u_i to the output y_i is given by

$$\begin{cases} \dot{\bar{\bar{x}}}_{i,d} = A_d \bar{\bar{x}}_{i,d} + B_d (\bar{F}_d \bar{\bar{x}}_{i,d} + u_i) + E_{d,i} \bar{w}_i + \rho_i, \\ y_i = C_d \bar{\bar{x}}_{i,d} \end{cases} \tag{39}$$

where

$$\begin{cases} \dot{\tilde{x}}_i = (\bar{A}_i + \bar{K}_i \bar{C}_{m,i}) \tilde{x}_i + E_i \bar{w}_i, \\ \rho_i = B_d (D_{i,a} \ D_{i,d} - \bar{F}_d) \tilde{x}_i. \end{cases} \tag{40}$$

Clearly, (39) and (40) correspond to (7) and (9), respectively. Furthermore, by combining (35), (36), and (38), we obtain the precompensator of the form (6) for each agent i, $i = 1, \ldots, N$.

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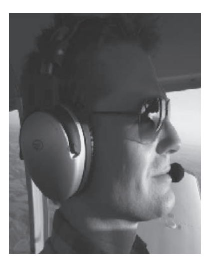

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