Competitive Diffusion in Social Networks: Quality or Seeding?

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Abstract

In this paper, we study a strategic model of marketing and product consumption in social networks. We consider two firms in a market competing to maximize the consumption of their products. Firms have a limited budget which can be either invested on the quality of the product or spent on initial seeding in the network in order to better facilitate spread of the product. After the decision of firms, agents choose their consumptions following a myopic best response dynamics which results in a local, linear update for their consumption decision. We characterize the unique Nash equilibrium of the game between firms and study the effect of the budgets as well as the network structure on the optimal allocation. We show that at the equilibrium, firms invest more budget on quality when their budgets are close to each other. However, as the gap between budgets widens, competition in qualities becomes less effective and firms spend more of their budget on seeding. We also show that given equal budget of firms, if seeding budget is nonzero for a balanced graph, it will also be nonzero for any other graph, and if seeding budget is zero for a star graph it will be zero for any other graph as well. As a practical extension, we then consider a case where products have some preset qualities that can be only improved marginally. At some point in time, firms learn about the network structure and decide to utilize a limited budget to mount their market share by either improving the quality or new seeding some agents to incline consumers towards their products. We show that the optimal budget allocation in this case simplifies to a threshold strategy. Interestingly, we derive similar results to that of the original problem, in which preset qualities simulate the role that budgets had in the original setup.

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I. Introduction

Many recent studies have documented the role of social networks in individual purchasing decisions [2]–[4]. More data from online social networks and advances in information technologies have drawn the attention of firms to exploit this information for their marketing goals. As a result, firms have become more interested in models of influence spread in social networks in order to improve their marketing strategies. In particular, considering the relationship between people in social networks and their rational choices, many retailers are interested to know how to use the information about the dynamics of the spread in order to maximize their product consumption and achieve the most profit in a competitive market.

A main feature of product consumption in these settings is what is often called the "network effect" or positive externality. For such products, consumption of each agent incentivizes the neighboring agents to consume more as well, as the consumption decisions between agents and their neighbors are strategic complements of each other. There are diverse sets of examples for such products or services. New technologies and innovations, mobile applications (e.g., Viber, WhatsApp), online games (e.g., Warcraft), social network web sites (e.g., Facebook, Twitter) and online dating services (e.g., Zoosk, Match.com, OkCupid) are among many examples in which consuming from a common product or service is more preferable for people.

Also, a main property of many products is substitution. A substitute product is a product or service that satisfies the need of a consumer that another product or service fulfills (e.g. Viber and WhatsApp or Gmail and Yahoo email accounts). In all these examples, firms might be interested to utilize the social network among consumers and the positive externality of their products and services to incentivize a larger consumption of their products compared to rival substitute products. Therefore, it is important for firms to know how to shape their strategies in designing their products and offering them to a set of people in order to promote their products intelligently, and eventually achieving a larger share in the market.

In this paper, we study strategic competition between two firms trying to maximize their product consumption. Firms simultaneously allocate their fixed budgets between seeding a set of costumers embedded in a social network and designing the quality of their products. The consumption of each agent is the result of its myopic best response to the previous actions of its peers in the network. Therefore, a firm should provide enough incentives for spread of its product through the payoff that agents receive by consuming it. For this purpose and considering their budgets, firms should strategically design their products and know how to initially seed the network.

We model the above problem as a fixed-sum game between firms, where each firm tries to maximize discounted sum of its product consumption over time, considering its fixed budget. We describe the unique Nash equilibrium of the game between firms which depends on the budgets of the firms and the network structure. We show that at the Nash equilibrium, firms spend more budget on quality when their budgets are close. However, as the difference between budgets increases, firms spend more budget on seeding. We also show that given equal budget of firms, if seeding budget is nonzero for a balanced graph it will also be nonzero for any other graph, and if seeding budget is zero for a star graph it will be zero for any other graph too. Next, we study a different scenario in which firms produce products with some preset qualities. At some point in time, firms learn about the network structure and dedicate some budget to increase their product consumption. The budget can be spent on new seeding of agents in the social network and marginally improving the quality of the products. We derive a simple rule for optimal allocation of the budget between improving the quality and new seeding which in particular depends on the network structure and preset qualities of the products. We show that the optimal allocation of the budget depends on the entire centrality distribution of the graph. Specially, we show that maximum seeding occurs in a graph with maximum number of agents with centralities above a certain threshold. Also, the difference in qualities of firms plays an important role in the optimal allocation of the budget. In particular, we show that as the gap between the qualities of the products widens, the firms allocate more budget to seeding. We see that the budgets in the first scenario and preset qualities of the second scenario play similar roles in the optimal allocation.

It is worthwhile to note that the problem of influence and spread in networks has been extensively studied in the past few years [5]–[11]. Also, diffusion of new behaviors and strategies through local coordination games has been an active field of research [12]–[19]. Goyal and Kearns proposed a game theoretic model of product adoption in [20]. They computed upper bounds of the price of anarchy and showed how network structure may amplify the initial budget differences. Similarly, in [21] Bimpikis, Ozdaglar and Yildiz proposed a game theoretic model of competition between firms which can target their marketing budgets to individuals embedded in a social network. They provided conditions under which it is optimal for the firms to asymmetrically target a subset of the individuals. Also, Chasparis and Shamma assumed a dynamical model of preferences in [10] and computed optimal policies for finite and infinite horizon where endogenous network influences, competition between two firms and uncertainties in the network model were studied. The main contribution of our work is to explicitly study the tradeoff between investing on quality of a product and initial seeding in a social network. Our model is similar to the model proposed in [22], however, instead of pricing strategy in [22], the notion of quality is introduced and the tradeoff between quality and seeding is studied. Also, our model is tractable and allows us to characterize the exact product consumption at each time, instead of lower and upper bounds provided in [23], [24].

II. The Spread Dynamics

There are n agents $V = \{1, \ldots, n\}$ in a social network. The relationship among agents is represented by a directed graph $\mathcal{G} = (V, E)$ in which agents $i, j \in V$ are neighbors if $(i, j) \in E$. The weighted adjacency matrix of the graph $\mathcal{G}$ is denoted by a row stochastic matrix G where the ij-th entry of G, denoted by g_{ij}, represents the strength of the influence of agent j on i. For diagonal elements of matrix G, we have $g_{ii} = 0$ for all agents $i \in V$. We assume that there are two competing firms a and b producing product a and b. Each agent has one unit demand which can be supplied by either of the firms. We define the variable $0 \leq x_i(t) \leq 1$ and $0 \leq 1 - x_i(t) \leq 1$ as the consumption of the product a and b by agent i at time t.

Denote by $q_a, q_b \geq \epsilon > 0$ the quality of product a and b respectively, where ϵ has an infinitesimal value. The values of q_a and q_b can be interpreted as the payoff that any two agents i and j would achieve if they both consume the same product. In other words, we can assume

q_a and q_b are payoffs of the following game

	x_j	$1-x_j$
x_i	$q_a x_i x_j$	0
$1-x_i$	0	$q_b(1-x_i)(1-x_j)$

Since agents benefit from the same action of their neighbors, this game could be thought of as a local coordination game. From the above table it follows easily that the payoff of agents i and j from their interaction is

$$u_{ij}(x_i, x_j) = q_a x_i x_j + q_b(1-x_i)(1-x_j).$$

We also assume that each agent benefits from taking action x_i irrespective of actions taken by its neighbors. We assume the isolation payoff of consuming x_i and $1-x_i$ from product a and b is represented by the following quadratic form functions

$$u_{ii}^a = q_a(\alpha x_i - \beta x_i^2), \qquad u_{ii}^b = q_b[\alpha(1-x_i) - \beta(1-x_i)^2],$$

where α and β are parameters of the isolation payoff. This forms of payoff indicates that a product with higher quality has a higher isolation payoff. The total isolation payoff of agent i can be written as

$$u_{ii}(x_i) = \{q_a(\alpha x_i - \beta x_i^2)\} + \{q_b\left(\alpha(1-x_i) - \beta(1-x_i)^2\right)\}.$$

In order to have nonnegative isolation payoff for $x_i = 0$ and $x_i = 1$, we assume $\beta \leq \alpha$. Assuming quadratic form function for the isolation payoff not only makes the analysis more tractable, but also is a good second order approximation for the general class of concave payoff functions. By changing the variables $x_i = \frac{1}{2} + y_i$ after simplification we get

$$u_{ij}(y_i, y_j) = q_a(\frac{1}{2} + y_i)(\frac{1}{2} + y_j) + q_b(\frac{1}{2} - y_i)(\frac{1}{2} - y_j),$$
$$u_{ii}(y_i) = (q_a + q_b)(\frac{\alpha}{2} - \frac{\beta}{4} - \beta y_i^2) + (q_a - q_b)(\alpha - \beta)y_i.$$

Therefore, the total utility of agent i from taking action y_i is given by

$$\begin{aligned} U_i(y_i, \vec{y}_{-i}) &= (q_a + q_b)(\frac{\alpha}{2} - \frac{\beta}{4} - \beta y_i^2) + (q_a - q_b)(\alpha - \beta) y_i \\ &+ q_a \sum_{j=1}^{n} g_{ij}(\frac{1}{2} + y_i)(\frac{1}{2} + y_j) + q_b \sum_{j=1}^{n} g_{ij}(\frac{1}{2} - y_i)(\frac{1}{2} - y_j). \end{aligned} \tag{1}$$

In the above equation $\vec{y}_{-i}$ denotes an action vector of all agents other than agent i. From equation (1) we can see that product a and b have a positive externality effect in the network, meaning that the usage level of an agent has a positive impact on the usage level of its neighbors. Therefore, it follows that q_a and q_b in addition to the payoff of a local coordination game, can be interpreted as coefficients of network externality of product a and b respectively.

We assume agents repeatedly apply myopic best response to the actions of their neighbors. This means that each agent, considering its neighbors consumptions at the current period, chooses the amount of the product that maximizes its current payoff, as its consumption for the next period. In other words, consumption of agent i at time $t+1$ is updated as follows

$$y_i(t+1) = \arg\max_{y_i} \quad U_i(y_i(t), \vec{y}_{-i}(t)).$$

The above equation results in the following update dynamics

$$y_i(t+1) = (\frac{1}{2\beta}) \sum_{j=1}^{n} g_{ij} y_j(t) + (\frac{q_a - q_b}{4\beta(q_a + q_b)}) \sum_{j=1}^{n} g_{ij} + (\frac{(\alpha - \beta)(q_a - q_b)}{2\beta(q_a + q_b)}).$$

Therefore, the consumption of the product a can be written as the following linear update dynamics form

$$\vec{y}(t+1) = (\frac{1}{2\beta}) G \vec{y}(t) + (\frac{(1 + 2(\alpha - \beta))(q_a - q_b)}{4\beta(q_a + q_b)}) \vec{1}. \tag{2}$$

Similarly, for the consumption of the product b we have $1 - x_i(t) = \frac{1}{2} - y_i(t)$.

Assumption 1: We assume $1 + \alpha \leq 2\beta$. This assumption guaranties that $0 \leq x_i(t) \leq 1$ for all i and all t under the update rule (2).

Using the above assumption and defining

$$W \triangleq (\frac{1}{2\beta})G, \qquad \vec{u}_a \triangleq \left((\frac{1+2(\alpha-\beta)}{4\beta})(\frac{q_a-q_b}{q_a+q_b}) \right) \vec{1}, \tag{3}$$

equations (2) can be written as

$$\vec{y}(t+1) = W\vec{y}(t) + \vec{u}_a.$$

The above equation can be expanded as

$$\vec{y}(t) = W^t\vec{y}(0) + \sum_{k=0}^{t-1} W^k \vec{u}_a. \tag{4}$$

Therefore, the consumption of agents depends on the initial preferences, i.e. $\vec{y}(0)$, the quality of product a and b, i.e. q_a and q_b, and the structure of the network, i.e. the matrix G. In the next section we discuss how firms can exploit this information in order to maximize their product consumption and also characterize the unique Nash equilibrium of the game played between two firms.

III. Optimal Budget Allocation

In this section we describe the game between firms where each firm aims to maximize the consumption of its product over an infinite time horizon given a fixed budget. Each firm has an initial budget that it can either invest on "quality" or spend it on promoting its product by seeding some of the agents, or both. This initial seeding can be viewed as free offers to promote the products in the network. We define the utility of each firm as the discounted sum of its product consumption over time

$$U_a = \sum_{t=0}^{\infty} \delta^t \vec{1}^T((0.5)\vec{1} + \vec{y}(t)),$$
$$U_b = \sum_{t=0}^{\infty} \delta^t \vec{1}^T((0.5)\vec{1} - \vec{y}(t)).$$

Each firm has a limited budget K_a, K_b that can spend on either initial seeding, i.e. $\vec{S}_a$ and $\vec{S}_b$, or designing the quality of its product, i.e. q_a and q_b, or both. Seeding $\vec{S}_a$ and $\vec{S}_b$ will change the initial consumption of products a and b by $\vec{S}_a - \vec{S}_b$ and $\vec{S}_b - \vec{S}_a$ respectively. Therefore, the amount that agents initially consume from product a and b will be $\vec{x}(0) = (0.5)\vec{1} + \vec{S}_a - \vec{S}_b$ and $\vec{1} - \vec{x}(0) = (0.5)\vec{1} + \vec{S}_b - \vec{S}_a$. This means that if both firms seed an agent equally then the agent has no preference for one product over the other, i.e. $\vec{y}(0) = \vec{0}$. This assumption can be justified since agents should be initially indifferent between products before their consumption and realizing the quality of products if initial seedings by firms are equal. In order to have $0 \leq x_i(0) \leq 1$ and $0 \leq 1 - x_i(0) \leq 1$ for all agents i, we impose the constraints $\|\vec{S}_a\|_\infty \leq 0.5$ and $\|\vec{S}_b\|_\infty \leq 0.5$. This means that firms can initially seed agents up to their demand capacity which is 0.5 for all agents. Using equations (3) and (4) and defining the centrality vector v by $v = (I - \delta W^T)^{-1}\vec{1}$ where agents are ordered from the highest to the lowest centrality, i.e. $v_1 \geq v_2 \geq \cdots \geq v_n$, and noting that $\sum v_i = \frac{2\beta n}{2\beta - \delta}$, the utilities of firms can be written as

$$\begin{aligned} U_a &= (\frac{n}{2(1-\delta)}) + v^T\vec{S}_a - v^T\vec{S}_b + \lambda(\frac{q_a - q_b}{q_a + q_b}), \\ U_b &= (\frac{n}{2(1-\delta)}) + v^T\vec{S}_b - v^T\vec{S}_a + \lambda(\frac{q_b - q_a}{q_a + q_b}), \end{aligned} \tag{5}$$

where

$$\lambda = \frac{\delta(1 + 2(\alpha - \beta))n}{2(1-\delta)(2\beta - \delta)}. \tag{6}$$

We assume the cost of each unit of quality is given by c_q and the cost of each unit of initial seeding is given by c_s. Therefore, the game between the firms can be written as

$$\begin{aligned} \max_{\vec{S}_a, q_a} \quad & (\frac{n}{2(1-\delta)}) + v^T\vec{S}_a - v^T\vec{S}_b + \lambda(\frac{q_a - q_b}{q_a + q_b}), \\ \text{s.t.} \quad & c_s\|\vec{S}_a\|_1 + c_q q_a = K_a, \end{aligned}$$

for firm a, and

$$\begin{aligned} \max_{\vec{S}_b, q_b} \quad & (\frac{n}{2(1-\delta)}) + v^T\vec{S}_b - v^T\vec{S}_a + \lambda(\frac{q_b - q_a}{q_a + q_b}), \\ \text{s.t.} \quad & c_s\|\vec{S}_b\|_1 + c_q q_b = K_b, \end{aligned}$$

for firm b. Since the effect of the action of $\vec{S}_b$ is decoupled from $\vec{S}_a$ in U_a, therefore, the optimization problem of firm a is equivalent to

$$\begin{aligned} \max_{\vec{S}_a, q_a} \quad & v^T \vec{S}_a + \lambda(\frac{q_a - q_b}{q_a + q_b}), \\ \text{s.t.} \quad & c_s \|\vec{S}_a\|_1 + c_q q_a = K_a. \end{aligned}$$

Similarly, for firm b we have

$$\begin{aligned} \max_{\vec{S}_b, q_b} \quad & v^T \vec{S}_b + \lambda(\frac{q_b - q_a}{q_a + q_b}), \\ \text{s.t.} \quad & c_s \|\vec{S}_b\|_1 + c_q q_b = K_b. \end{aligned}$$

Remark 1: It can be easily shown that for a seeding budget $\|S_a\|_1$, the optimal seeding strategy is to seed the agents in the order of their centralities (from highest to lowest) until we either ran out of budget or all the agents are seeded. Therefore, an optimal action $(\vec{S}_a, q_a)$ is fully determined from $(\|S_a\|_1, q_a)$, thus reducing the action space of firm a to only q_a, given its budget constraint. Similar argument holds for firm b. Therefore, we may look at the utilities U_a and U_b as functions of (q_a, q_b) under the optimal seeding and fixed budgets.

In order to study the existence and uniqueness of the Nash equilibrium for the above game, we use a variation of the well-known Sion's minimax theorem (see [25] for the original Sion's theorem) as below.

Lemma 1: Consider a two person zero-sum game, on closed, bounded, and convex finite-dimensional action sets $\Omega_1 \times \Omega_2$, defined by the continuous function $L(x_1, x_2)$. Let $L(x_1, x_2)$ be strictly convex in x_1 for each $x_2 \in \Omega_2$ and strictly concave in x_2 for each $x_1 \in \Omega_1$. Then the game admits a unique pure strategy Nash equilibrium.

Proof: See Theorem $A.4$ on page 286 in [26]. ■

In the following theorem we characterize the Nash equilibrium of the game played between firms.

Theorem 1: Consider firms a and b with utility functions U_a and U_b as described in (5). The game between firms admits a unique Nash equilibrium of form

$$q_a^* = (2\lambda)(\frac{c_s}{c_q})(\frac{\tilde{v}_l}{(\tilde{v}_k+\tilde{v}_l)^2}), \qquad q_b^* = (2\lambda)(\frac{c_s}{c_q})(\frac{\tilde{v}_k}{(\tilde{v}_k+\tilde{v}_l)^2}),$$

$$S_{a_i}^* = \begin{cases} \frac{1}{2} & 1 \le i < k, \\ \frac{K_a}{c_s} - \frac{k-1}{2} - \frac{2\lambda\tilde{v}_l}{(\tilde{v}_k+\tilde{v}_l)^2} & i = k, \\ 0 & i > k, \end{cases} \qquad S_{b_i}^* = \begin{cases} \frac{1}{2} & 1 \le i < l, \\ \frac{K_b}{c_s} - \frac{l-1}{2} - \frac{2\lambda\tilde{v}_k}{(\tilde{v}_k+\tilde{v}_l)^2} & i = l, \\ 0 & i > l, \end{cases} \tag{7}$$

for some $v_k \le \tilde{v}_k \le v_{k-1}$ and $v_l \le \tilde{v}_l \le v_{l-1}$ that satisfy

$$\begin{aligned} 0 \le S_{a_k}^* &= \frac{K_a}{c_s} - \frac{k-1}{2} - \frac{2\lambda\tilde{v}_l}{(\tilde{v}_k+\tilde{v}_l)^2} < \frac{1}{2}, \\ 0 \le S_{b_l}^* &= \frac{K_b}{c_s} - \frac{l-1}{2} - \frac{2\lambda\tilde{v}_k}{(\tilde{v}_k+\tilde{v}_l)^2} < \frac{1}{2}, \end{aligned} \tag{8}$$

where $\tilde{v}_k = v_k$ if $S_{a_k}^* > 0$ and $\tilde{v}_l = v_l$ if $S_{a_l}^* > 0$. [1]

Proof: Given the optimal seeding of each firm, i.e. seeding agents from the highest to the lowest centrality, as discussed in Remark 1, the tradeoff between seeding amount and quality can be solved by optimizing U_a and U_b with respect to q_a and q_b respectively. The action space of firms, i.e. $\epsilon \le q_a \le \frac{K_a}{c_q}$ and $\epsilon \le q_b \le \frac{K_b}{c_q}$, is a closed, bounded, and convex finite-dimensional set. Also, $U_a + U_b = \frac{n}{(1-\delta)}$, hence, the game is a fixed-sum game and can be transformed to a zero sum game by subtracting the constant value of $\frac{n}{2(1-\delta)}$ from U_a and U_b. The term $v^T\vec{S}_a$ in U_a is piecewise linear in $\|S_a\|_1$ and thus in q_a, under optimal seeding. Using this, it is easy to see that $U_a(q_a, q_b)$ is strictly concave in q_a for each q_b, and strictly convex in q_b for each q_a via a similar argument. Therefore, based on Lemma 1, the game admits a unique Nash equilibrium. Assume that the first $(k-1)$ and $(l-1)$ agents are fully seeded by firms a and b respectively at equilibrium. Then, from the budget constraints we have $S_{a_k}^* = \frac{K_a}{c_s} - \frac{k-1}{2} - (\frac{c_q}{c_s})q_a$, and $S_{b_l}^* = \frac{K_b}{c_s} - \frac{l-1}{2} - (\frac{c_q}{c_s})q_b$, therefore, by plugging in the vector of optimal seeding S_a^* and S_b^*

[1] We define $v_0 \triangleq \infty$. If $S_{a_n}^* = \frac{1}{2}$ or $S_{b_n}^* = \frac{1}{2}$, then $\tilde{v}_n \le v_n$.

as described earlier, the optimization problem of firms is given by

$$\max_{\epsilon \le q_a \le \frac{K_a}{c_q}} \quad (\frac{1}{2})\sum_{i=1}^{k-1} v_i + (\frac{K_a}{c_s} - \frac{k-1}{2} - (\frac{c_q}{c_s})q_a)v_k + \lambda(\frac{q_a - q_b}{q_a + q_b}),$$
$$\max_{\epsilon \le q_b \le \frac{K_b}{c_q}} \quad (\frac{1}{2})\sum_{i=1}^{l-1} v_i + (\frac{K_b}{c_s} - \frac{l-1}{2} - (\frac{c_q}{c_s})q_b)v_l + \lambda(\frac{q_b - q_a}{q_a + q_b}).$$

If $0 < S^*_{a_k} < \frac{1}{2}$ and $0 < S^*_{b_l} < \frac{1}{2}$, the first order optimality condition requires taking the derivative of the two equations above with respect to q_a and q_b and setting them to zero

$$-(\frac{c_q}{c_s})v_k + (\frac{2\lambda q_b}{(q_a + q_b)^2}) = 0,$$
$$-(\frac{c_q}{c_s})v_l + (\frac{2\lambda q_a}{(q_a + q_b)^2}) = 0.$$

Solving equations above we get

$$q^*_a = (2\lambda)(\frac{c_s}{c_q})(\frac{v_l}{(v_k + v_l)^2}),$$
$$q^*_b = (2\lambda)(\frac{c_s}{c_q})(\frac{v_k}{(v_k + v_l)^2}),$$

where $1 \le k, l \le n$ are integers that must satisfy conditions in (8) for $\tilde{v}_k = v_k$ and $\tilde{v}_l = v_l$. If $S^*_{a_k} = 0$ and $S^*_{b_l} = 0$, the first order optimality condition is as follows

$$v_k \le \tilde{v}_k \le v_{k-1}, \qquad\qquad v_l \le \tilde{v}_l \le v_{l-1}, \tag{9}$$

where

$$\tilde{v}_k = (2\lambda)(\frac{c_s}{c_q})(\frac{q_b}{(q_a + q_b)^2}), \qquad\qquad \tilde{v}_l = (2\lambda)(\frac{c_s}{c_q})(\frac{q_a}{(q_a + q_b)^2}), \tag{10}$$

and if $S^*_{a_n} = \frac{1}{2}$ or $S^*_{b_n} = \frac{1}{2}$ then $\tilde{v}_n \le v_n$. We can solve q^*_a and q^*_b in terms of $\tilde{v}_k$ and $\tilde{v}_l$ as described in (7). ■

Corollary 1: If firms have equal budgets $K_a = K_b = K$, then in the unique symmetric Nash equilibrium of the game between firms we have

$$q_a^* = q_b^* = (\frac{\lambda}{2})(\frac{c_s}{c_q})(\frac{1}{\tilde{v}_l}), \qquad S^*_{a_i} = S^*_{b_i} = \begin{cases} \frac{1}{2} & 1 \le i < l, \\ \frac{K}{c_s} - \frac{l-1}{2} - \frac{\lambda}{2\tilde{v}_l} & i = l, \\ 0 & i > l, \end{cases} \tag{11}$$

for some $v_l \le \tilde{v}_l \le v_{l-1}$ that satisfy

$$0 \le S^*_{a_l} = S^*_{b_l} = \frac{K}{c_s} - \frac{l-1}{2} - \frac{\lambda}{2\tilde{v}_l} < \frac{1}{2}, \tag{12}$$

where $\tilde{v}_l = v_l$ if $S^*_{a_l} = S^*_{b_l} > 0$. [2]

Equation (7) indicates that the Nash equilibrium depends on both the budgets of the firms, i.e. K_a and K_b, centrality distribution of agents in the network, i.e. v. We will discuss the effect of each of these factors on the Nash equilibrium in the following subsections. All of our analysis here is for firm a and similar results can be shown for firm b as well. For simplicity, we only discuss seeding budget; quality budget can be found easily using the budget constraint.

A. *Effect of Budget of Firms on Firms' Decisions:*

In this subsection we study how the budget of each firm, i.e. K_a and K_b, can influence the Nash equilibrium. As it can be seen from (7), the Nash equilibrium depends on both $\tilde{v}_k$ and $\tilde{v}_l$, which in turn depend on firm's and its rival's budgets, i.e. both K_a and K_b. In the first proposition, we compare the seeding budget and quality of two firms at the Nash equilibrium with respect to their budgets. We first prove the following lemma.

Lemma 2: At the Nash equilibrium, if $q_a^* < q_b^*$, then $\|\vec{S}_a^*\|_1 \le \|\vec{S}_b^*\|_1$.

Proof: If $q_a^* < q_b^*$, then from (7), we have $\tilde{v}_l < \tilde{v}_k$. If $\tilde{v}_l < \tilde{v}_k$ then either $k < l$ or $l = k$. If $k < l$ it is obvious to see that $\vec{S}_a^* \le \vec{S}_b^*$. If $l = k$ then we have two cases: If $0 < S^*_{a_k} < \frac{1}{2}$, then based on (9) we have $\tilde{v}_k = v_k = v_l \le \tilde{v}_l$ which is a contradiction with $\tilde{v}_l < \tilde{v}_k$. If $S^*_{a_k} = 0$, then obviously $S^*_{a_k} \le S^*_{b_l}$ and therefore, $\vec{S}_a^* \le \vec{S}_b^*$. If $S^*_{a_n} = \frac{1}{2}$, then $\tilde{v}_l < \tilde{v}_k = \tilde{v}_n \le v_n$, hence, $S^*_{b_n} = \frac{1}{2}$. This finishes the proof. ■

[2] We define $v_0 \triangleq \infty$. If $S^*_{a_n} = S^*_{b_n} = \frac{1}{2}$, then $\tilde{v}_n \le v_n$.

The next proposition states that the firm with higher budget surpasses the rival in both quality and seeding.

Proposition 1: The firm with higher budget has higher seeding budget and quality, i.e. if $K_b \leq K_a$, then $\|\vec{S}_b^*\|_1 \leq \|\vec{S}_a^*\|_1$ and $q_b^* \leq q_a^*$.

Proof: Suppose that $q_a^* < q_b^*$. From Lemma 2 we have $\|\vec{S}_a^*\|_1 \leq \|\vec{S}_b^*\|_1$, which contradicts with $K_b \leq K_a$. Also, suppose $\|\vec{S}_a^*\|_1 < \|\vec{S}_b^*\|_1$, then from Lemma 2 we have $q_a^* \leq q_b^*$, which contradicts with $K_b \leq K_a$. This completes the proof. ■

In the next proposition we explain how the seeding budget and quality at the Nash equilibrium vary with K_a and K_b.

Proposition 2: Given a fixed graph, the optimal seeding $\|S_a^*\|_1$ and quality q_a^* at the Nash equilibrium are increasing functions of K_a. Furthermore, $\|S_a^*\|_1$ is a decreasing function of K_b if $K_b \leq K_a$ and an increasing function of K_b if $K_a \leq K_b$.

Proof: First note that $\|S_a^*\|_1$, $\|S_b^*\|_1$, q_a^*, q_b^* (and as a result $\tilde{v}_k$ and $\tilde{v}_l$) are continuous functions of K_a and K_b. To see this, let $B(q_a, q_b, K_a, K_b)$ denote the best response of the firms to qualities (q_a, q_b) when the budgets are (K_a, K_b). It follows from the continuity of the best response and compactness of action spaces that the set $\{(q_a^*, q_b^*)|B(q_a^*, q_b^*, K_a, K_b) = (q_a^*, q_b^*)\}$, that is the equilibrium space, is closed. This implies that the graphs of the functions $q_a^*(K_a, K_b)$ and $q_b^*(K_a, K_b)$ are closed and thus are continuous.

Now, if $0 < S_{a_k}^* < \frac{1}{2}$, then $\tilde{v}_k = v_k$. If K_a marginally increases, then, using the continuity of the equilibrium, the level k and as a result $\tilde{v}_k$ does not change. Thus, given fixed K_b, the constraint for firm b in (8) and hence $\tilde{v}_l$ does not change. Therefore, if K_a marginally increases, from the Nash equilibrium in (7), q_a^* does not change and $S_{a_k}^*$ marginally increases. If $S_{a_k}^* = 0$ and $v_k < \tilde{v}_k < v_{k-1}$, and K_a marginally increases, from the continuity of Nash we still have $v_k < \tilde{v}_k' < v_{k-1}$ and as a result $S_{a_k}^* = 0$ does not change and hence, q_a^* marginally increases. If $S_{a_k}^* = 0$ and $\tilde{v}_k = v_k$ or $\tilde{v}_k = v_{k-1}$, and K_a marginally increases, either we have $v_k < \tilde{v}_k' < v_{k-1}$ which means $S_{a_k}^* = 0$ does not change and q_a^* marginally increases, or $\tilde{v}_k$ does not change. In this latter case, given the fixed budget K_b, the constraint for firm b in (8) will remain unchanged and hence $\tilde{v}_l$ will not change. Therefore, from the Nash equilibrium in (7), q_a^* does not change and as a result $S^*_{a_k}$ marginally increases. It is to be noted here that the cases where $\tilde{v}_k$ moves above v_{k-1} or below v_k are not feasible as they will cause a jump in the seeding budget, contradicting the continuity of equilibrium. The analysis for the case when $S^*_{a_n} = \frac{1}{2}$ and $\tilde{v}_n \leq v_n$ is quite similar. Therefore, $\frac{\partial \|\vec{S}^*_a\|_1}{\partial K_a} \geq 0$ and $\frac{\partial q^*_a}{\partial K_a} \geq 0$.

For the second part of the proposition, if $S^*_{a_k} = 0$ and $v_k < \tilde{v}_k < v_{k-1}$ and K_b marginally increases, from continuity of $\tilde{v}_k$ we still have $v_k < \tilde{v}'_k < v_{k-1}$, and therefore, $S^*_{a_k} = 0$ and given the fixed K_a, q^*_a does not change. Hence, we only need to consider the case where $0 < S^*_{a_k} < \frac{1}{2}$ and $\tilde{v}_k = v_k$, or $S^*_{a_k} = 0$ and $\tilde{v}_k = v_k$ or $\tilde{v}_k = v_{k-1}$. In these cases, it is easy to see that either $\tilde{v}_k$ or $S^*_{a_k}$ remains unchanged. In the latter case, (given the fixed K_a) q^*_a does not change. Similar argument holds for when $S^*_{a_n} = \frac{1}{2}$ and $\tilde{v}_n \leq v_n$. Therefore, we only need to consider the case where $\tilde{v}_k$ does not change. From the first part of the proposition, q^*_b is an increasing function of K_b. Also, from Proposition 1, if $K_b \leq K_a$ ($K_a \leq K_b$), then $q^*_b \leq q^*_a$ ($q^*_a \leq q^*_b$). Therefore, if $K_b \leq K_a$ ($K_a \leq K_b$) and K_b marginally increases, equations (10) implies that q^*_a must marginally increase (decrease) or does not change so that $\tilde{v}_k$ remains fixed. Hence, given constant K_a, $S^*_{a_k}$ marginally decreases (increases) or does not change. Therefore, $\frac{\partial \|\vec{S}^*_a\|_1}{\partial K_b} \leq 0$, for $K_b \leq K_a$ and $\frac{\partial \|\vec{S}^*_a\|_1}{\partial K_b} \geq 0$, for $K_a \leq K_b$.

■

Proposition 2 implies that when $K_b \leq K_a$, the higher the budget of the rival firm, the lower the seeding budget of firm a, i.e., if $K_b \leq K'_b \leq K_a$ then, $\|\vec{S}^*_a(K'_b)\|_1 \leq \|\vec{S}^*_a(K_b)\|_1$. On the other hand, when competing with a firm which has a higher budget, i.e. $K_a \leq K_b$, the higher the budget of the rival firm, the higher firm a should spend on seeding. In other words, if $K_a \leq K_b \leq K'_b$ then, $\|\vec{S}^*_a(K_b)\|_1 \leq \|\vec{S}^*_a(K'_b)\|_1$.

Combining these two results, we can see that given a fixed value of K_a, the seeding budget of firm a is increasing with the difference $|K_a - K_b|$. The seeding budget attains its minimum when $K_b = K_a$, implying that the firm should allocate more budget to quality to distance itself from the rival firm. However, as the gap between budget widens, competition in qualities becomes less effective and firms spend more budget on seeding.

B. *Effect of Network Structure on Firms' Decisions*

In this subsection we study the effect of network structure on the Nash equilibrium. Since we already studied the effect of the budget on the Nash equilibrium, for the rest of this subsection we assume $K_a = K_b = K$ so that we can observe only the effect of the network structure. We first focus on two well studied graphs, i.e. star and balanced graphs, and highlight how they can reflect important properties of the seeding budget. Before continuing further, we first formally define these two graphs and find their network centralities in the next lemma.

Definition 1: A star graph is a directed graph in which there is an edge from any noncentral agent $i \in V - \{j\}$ to the central agent j with the weight $g_{ij} = 1$ and there are edges from the central agent j to all noncentral agents $i \in V - \{j\}$ such that $\sum_i g_{ji} = 1$.

Definition 2: A balanced graph is a directed graph in which the in-degree of each agent is equal to its out-degree, i.e. $\sum_j g_{ji} = \sum_j g_{ij} = 1$.

Lemma 3: The centrality of the agents in a balanced graph is given by $\bar{v} = \frac{2\beta}{2\beta - \delta}$. In a star graph, the centrality of the central agent is

$$v_h^s = \frac{1 + \frac{\delta(n-1)}{2\beta}}{1 - (\frac{\delta}{2\beta})^2},$$

and the centrality of non central agents is

$$v_l^s = \frac{1 + \frac{\delta}{2\beta(n-1)}}{1 - (\frac{\delta}{2\beta})^2}.$$

Moreover, for any arbitrary graph G, $\bar{v} \leq v_1 \leq v_h^s$.

Proof: First part simply follows from the fact that $v = (I - \delta W^T)^{-1}\vec{1}$, where W is given by (3), and that for any agent i in a balanced graph $\sum g_{ji} = \sum g_{ij} = 1$. For the star graph, noting that $v = \vec{1} + \delta W^T v$, we can obtain

$$v_h^s = 1 + \frac{\delta(n-1)v_l^s}{2\beta},$$
$$v_l^s = 1 + \frac{\delta v_h^s}{2\beta(n-1)},$$

solving which we can find v_h^s and v_l^s as given in the lemma.

Also, for any arbitrary graph G, $v_1 \geq \frac{\sum v_i}{n} = \bar{v}$. To show $v_1 \leq v_h^s$, using $v = \vec{1} + \delta W^T v$ for all $j \neq 1$ we can obtain

$$v_j \geq 1 + (\frac{\delta}{2\beta}) g_{1j} v_1.$$

This yields

$$\sum_{j=1}^{n} v_j \geq (n-1) + (1 + \frac{\delta}{2\beta}) v_1.$$

Applying simple algebra along with the fact that $\sum v_j = \frac{2\beta n}{2\beta - \delta}$ leads to $v_1 \leq v_h^s$. ■

The next proposition provides a condition for seeding profitability of any general graph. Also, the seeding budget of star and balanced graphs are compared and it is shown that the graph with higher seeding budget can be any of the two, depending on the budget.

Proposition 3: If seeding budget is nonzero for a balanced graph, it will be nonzero for any other graph too. On the other hand, if seeding budget is zero for a star graph, it will also be zero for any other graph. Moreover, if $\frac{1}{2} + \frac{\lambda}{2\bar{v}} < \frac{K}{c_s} < \frac{n}{2} + \frac{\lambda}{2v_l^s}$, a balanced graph has a larger seeding budget than a star graph, and if $\frac{\lambda}{2v_h^s} < \frac{K}{c_s} < \frac{1}{2} + \frac{\lambda}{2\bar{v}}$, a star graph has a larger seeding budget than a balanced graph. For $\frac{n}{2} + \frac{\lambda}{2v_l^s} < \frac{K}{c_s} < \frac{n}{2} + \frac{\lambda}{2}$ they have the same seeding budget.

Proof: If seeding budget is nonzero for a balanced graph, then according to (12) we have $\frac{\lambda}{2\bar{v}} < \frac{K}{c_s}$. As a result, for any other graph we will have $\frac{\lambda}{2v_1} < \frac{K}{c_s}$, since according to Lemma 3 $\bar{v} \leq v_1$. This means that there exists at least one agent that must be seeded. On the other hand, if seeding budget is zero for a star graph, then we must have $\frac{K}{c_s} \leq \frac{\lambda}{2v_h^s}$. Since we know $v_h^s \geq v_i$ for any agent i of any arbitrary graph, therefore, $\frac{K}{c_s} \leq \frac{\lambda}{2v_i}$ and no agent can be seeded in any other graph.

For the second part of the proposition, denote quality and seeding budget of balanced and star graphs by q_r, $\|\vec{S}_r\|_1$ and q_s, $\|\vec{S}_s\|_1$ respectively. If $\frac{1}{2} + \frac{\lambda}{2\bar{v}} < \frac{K}{c_s} < \frac{n}{2} + \frac{\lambda}{2v_l^s}$, then seeding budget is nonzero for balanced graph and hence, $q_r = (\frac{c_s}{c_q})\frac{\lambda}{2\bar{v}}$. This implies $\|\vec{S}_r\|_1 = \frac{K}{c_s} - \frac{\lambda}{2\bar{v}} > \frac{1}{2}$. For star graph $\frac{1}{2} + \frac{\lambda}{2\bar{v}} < \frac{K}{c_s} < \frac{n}{2} + \frac{\lambda}{2v_l^s}$ implies $\frac{1}{2} + \frac{\lambda}{2v_h^s} < \frac{K}{c_s}$. Therefore, the central agent in star graph must be seeded, i.e. $S_{a_1} = \frac{1}{2}$. If $S_{a_2} = 0$, then $\|\vec{S}_s\|_1 = \frac{1}{2}$ and clearly $\|\vec{S}_s\|_1 < \|\vec{S}_r\|_1$. If $S_{a_2} > 0$, then a noncentral agent must be seeded and we must have $q_s = (\frac{c_s}{c_q})\frac{\lambda}{2v_l^s}$. This implies $q_r < q_s$

and as a result $\|\vec{S}_s\|_1 < \|\vec{S}_r\|_1$.

If $\frac{\lambda}{2v_h^s} < \frac{K}{c_s} < \frac{1}{2} + \frac{\lambda}{2\bar{v}}$, then we have two cases. If $\frac{K}{c_s} \leq \frac{\lambda}{2\bar{v}}$ then $\|\vec{S}_r\|_1 = 0$. On the other hand $\|\vec{S}_s\|_1 > 0$ since $\frac{\lambda}{2v_h^s} < \frac{K}{c_s}$. Therefore, clearly $\|\vec{S}_r\|_1 < \|\vec{S}_s\|_1$. So let's assume $\frac{\lambda}{2\bar{v}} < \frac{K}{c_s} < \frac{1}{2} + \frac{\lambda}{2\bar{v}}$. This implies seeding budget is nonzero for balanced graph and hence, $q_r = (\frac{c_s}{c_q})\frac{\lambda}{2\bar{v}}$. As a result, $\|\vec{S}_r\|_1 = \frac{K}{c_s} - \frac{\lambda}{2\bar{v}} < \frac{1}{2}$. Now again consider two cases. If $\frac{K}{c_s} < \frac{1}{2} + \frac{\lambda}{2v_h^s}$, then $q_s = (\frac{c_s}{c_q})\frac{\lambda}{2v_h^s}$, and hence $q_s < q_r$. This implies $\|\vec{S}_r\|_1 < \|\vec{S}_s\|_1$. Otherwise, if $\frac{1}{2} + \frac{\lambda}{2v_h^s} \leq \frac{K}{c_s}$ then $\|\vec{S}_s\|_1 \geq \frac{1}{2}$. As a result, again we have $\|\vec{S}_r\|_1 < \frac{1}{2} \leq \|\vec{S}_s\|_1$.

If $\frac{n}{2} + \frac{\lambda}{2v_l^s} < \frac{K}{c_s} < \frac{n}{2} + \frac{\lambda}{2}$, then all agents in star graph are seeded up to agents maximum demand capacities which is 0.5 for each agent. Also, since $v_l^s < \bar{v}$, we have $\frac{n}{2} + \frac{\lambda}{2\bar{v}} < \frac{K}{c_s}$. Hence, all agents in balanced graph are also seeded up to agents maximum demand capacities. Therefore, both graphs have the same seeding budget. This completes the proof. ■

The next proposition provides us with a lower and an upper bound for minimum and maximum seeding budget. In order to characterize the graphs with maximum and minimum seedings for a given budget K, we need to introduce a few notations first.

Definition 3: Define $v_l^{max} = \max v_l$, i.e. the maximum of the l-th centrality v_l among all possible graphs subject to $\sum v_i = \frac{2\beta n}{2\beta - \delta}$. We can see that $v_1^{max} = v_h^s$ and

$$v_l^{max} = \frac{n\delta}{l(2\beta - \delta)} + 1, \tag{13}$$

for $l \geq 2$. Similarly, define $v_l^{min} = \min v_l$, i.e. the minimum of the l-th centrality v_l among all possible graphs subject to $\sum v_i = \frac{2\beta n}{2\beta - \delta}$. It is easy to see that $v_1^{min} = \bar{v}$, $v_2^{min} = v_l^s$, and $v_l^{min} = 1$ for $l \geq 3$.

Proposition 4: Let $(l, \tilde{v}_l^{max})$ be the unique pair satisfying condition (12) where $v_l^{max} \leq \tilde{v}_l^{max} \leq v_{l-1}^{max}$ and if $0 < S_{a_l}^*$ in (12), then $\tilde{v}_l^{max} = v_l^{max}$. The maximum seeding budget occurs in any graph for which $\tilde{v}_l = \tilde{v}_l^{max}$. An example for such a graph is an l-star graph if $\tilde{v}_l^{max} = v_l^{max}$ and an $(l-1)$-star graph if $\tilde{v}_l^{max} > v_l^{max}$. [3] Similarly, let $(l, \tilde{v}_l^{min})$ be the unique pair satisfying condition (12) where $v_l^{min} \leq \tilde{v}_l^{min} \leq v_{l-1}^{min}$ and if $0 < S_{a_l}^*$ in (12), then $\tilde{v}_l^{min} = v_l^{min}$. The

[3] $\tilde{v}_n^{max} \leq v_n^{max}$ if $S_{a_n}^* = \frac{1}{2}$.

minimum seeding budget occurs in any graph for which $\tilde{v}_l = \tilde{v}_l^{min}$. An example for such graphs is the balanced graph for $l = 1$, the star graph for $l = 2$, and any graph with $n - 2$ agents with centrality of one for $l \geq 3$. [4]

Proof: Let G be a graph attaining the maximum seeding (thus the minimum quality) and denote its corresponding equilibrium with $(l', \tilde{v}_{l'})$. Note that $l' \geq l$, since in a graph with $\tilde{v}_l = \tilde{v}_l^{max}$ the first $(l-1)$ agents are fully seeded. Now, if $l' > l$, then from $\tilde{v}_{l'} \leq v_{l'-1}^{max} \leq v_l^{max}$ and $v_l^{max} \leq \tilde{v}_l^{max}$ it follows that $\tilde{v}_{l'} \leq \tilde{v}_l^{max}$. But, then both pairs $(l', \tilde{v}_{l'})$ and $(l, \tilde{v}_l^{max})$ cannot satisfy (12). Therefore, in a graph with maximum seeding we should have $l' = l$. Now, if $\tilde{v}_l^{max} < \tilde{v}_{l'}$, then $v_l^{max} < \tilde{v}_{l'} \leq v_{l-1}^{max}$, which contradicts the uniqueness of the pair $(l, \tilde{v}_l^{max})$. To complete the proof, we also need to show that $\tilde{v}_l = \tilde{v}_l^{max}$ is achievable. It is quite straightforward to show that for $\tilde{v}_l^{max} = v_l^{max}$ an l-star graph with $v_1 = \ldots = v_l = v_l^{max}$, and for $\tilde{v}_l^{max} > v_l^{max}$ an $(l-1)$-star graph with $v_1 = \ldots = v_{l-1} = v_{l-1}^{max}$ admit $(l, \tilde{v}_l^{max})$ as the equilibrium. The proof for the minimum seeding budget is similar. ■

Example 1: As a numerical example for the minimum and maximum seeding budgets, we consider a network with $n = 15$ agents with budget $K = 2$, quality and seeding costs of $c_s = c_q = 1$ and parameters of $\alpha = \beta = 1$ and $\delta = 0.5$. For this example from equations (6) we have $\lambda = 5$. As a result, we can see that for $l = 3$ and $v_3^{max} = \frac{8}{3}$ from (13), condition $0 < S_{a_3}^* = \frac{1}{16} < \frac{1}{2}$ in (12) is satisfied. Therefore, a graph with the maximum seeding budget is a 3-star with the seeding budget of $\frac{17}{16}$ as illustrated in Fig. 1. Also, we can see that for $l = 1$ and $\bar{v} = \frac{4}{3}$, condition $0 < S_{a_1}^* = \frac{1}{8} < \frac{1}{2}$ in (12) is satisfied. Thus, balanced graph has the minimum seeding budget of $\frac{1}{8}$. Given $v_h^s = 4.8$ and $v_l^s = 1.08$, it can be seen that in star graph $\tilde{v}_2 = \frac{5}{3}$ and star graph has a seeding budget of 0.5 which is neither a minimum nor a maximum.

As we saw, the structure of the graphs with minimum and maximum seeding budget depends on the budget. However, for certain values of budget K the seeding budget will be independent of the structure of the graph, as described in the next proposition.

Proposition 5: If $\frac{K}{c_s} < \frac{\lambda}{2v_h^s}$ no graph can be seeded. On the other hand, if $\frac{K}{c_s} > \frac{n}{2} + \frac{\lambda}{2}$ all

[4] $\tilde{v}_n^{min} \leq v_n^{min}$ if $S_{a_n}^* = \frac{1}{2}$.

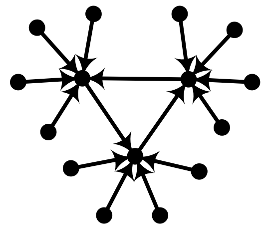

Fig. 1. A graph with maximum seeding budget

graphs can be seeded up to agents maximum demand capacities.

Proof: The maximum possible centrality happens for the central agent of the star graph as shown in Lemma 3. As a result, if $\frac{K}{c_s} < \frac{\lambda}{2v_h^s}$, then we have $\frac{K}{c_s} < \frac{\lambda}{2v_i}$ for all i in any graph and from condition (12) no agent can be seeded. Also, since from definition $1 \leq v_i$ for all i, if $\frac{K}{c_s} > \frac{1}{2} + \frac{\lambda}{2}$ then we have $\frac{K}{c_s} > \frac{n}{2} + \frac{\lambda}{2v_i}$ for all agents in any graph, and any graph can be seeded up to agents' maximum demand capacities which is 0.5 for each agent. ■

IV. Seeding Versus Quality Improvement

In this section we describe a scenario in which firms have already produced their products with some preset quality. We assume at some point in time, say $t = 0$, firms learn about the network structure and utilize a fixed budget to maximize their marginal utility by either marginally "improving the quality" of their products or new seeding some agents to change their consumption towards their products or both. Since the products have been in the market for a while, we assume agents have already decided on their consumption from products a and b which are denoted by $\vec{x}(0)$ and $\vec{1} - \vec{x}(0)$ respectively. Each firm has a limited budget, i.e. K_a and K_b, that can spend on either new seeding, i.e. $\vec{S}_a$ and $\vec{S}_b$, or enhancing the quality of its product, i.e. Δq_a and Δq_b, or both. New seeding $\vec{S}_a$ and $\vec{S}_b$ will change the initial consumption of products a and b by $\vec{S}_a - \vec{S}_b$ and $\vec{S}_b - \vec{S}_a$ respectively. In order to have $0 \leq x_i(0) \leq 1$ and $0 \leq 1 - x_i(0) \leq 1$ for all agents i, we impose the constraints $\|\vec{S}_a + \vec{y}(0)\|_\infty \leq 0.5$ and $\|\vec{S}_b - \vec{y}(0)\|_\infty \leq 0.5$. This means that firms can initially seed agents up to their demand capacity. From equation (5) the marginal change in the utility of firm a and b resulted from the new budget K_a and K_b are given by

$$\Delta U_a = v^T \vec{S}_a - v^T \vec{S}_b + \frac{2\lambda q_b \Delta q_a}{(q_a+q_b)^2} - \frac{2\lambda q_a \Delta q_b}{(q_a+q_b)^2},$$
$$\Delta U_b = v^T \vec{S}_b - v^T \vec{S}_a + \frac{2\lambda q_a \Delta q_b}{(q_a+q_b)^2} - \frac{2\lambda q_b \Delta q_a}{(q_a+q_b)^2}.$$

We assume the cost of improving the quality by Δq is given by $c_q \Delta q$ and c_q is a large number, and also the cost of each unit of new seeding is given by c_s. Each firm maximizes its marginal utility given its fixed budget. Since the effect of the action of firm b, i.e. $\vec{S}_b$ and Δq_b, is decoupled from that of the action of firm a in ΔU_a, thus firm a should solve the following optimization problem

$$\begin{aligned} \max_{\vec{S}_a, \Delta q_a} \quad & v^T \vec{S}_a + \frac{2\lambda q_b \Delta q_a}{(q_a+q_b)^2}, \\ \text{s.t.} \quad & c_s \|\vec{S}_a\|_1 + c_q \Delta q_a = K_a. \end{aligned} \tag{14}$$

Similarly, for the firm b we have

$$\begin{aligned} \max_{\vec{S}_b, \Delta q_b} \quad & v^T \vec{S}_b + \frac{2\lambda q_a \Delta q_b}{(q_a+q_b)^2}, \\ \text{s.t.} \quad & c_s \|\vec{S}_b\|_1 + c_q \Delta q_b = K_b. \end{aligned} \tag{15}$$

From equations (14) and (15) it can be seen that the optimal strategy of each firm is independent of the action of the other firm. It is to be noted that despite the independence of the actions, the optimal strategy of each firm depends on the state (i.e., quality) of the rival firm. This results in a Nash equilibrium to be simply the pair of the optimal actions of the firms. In the next Theorem we describe a simple rule for the optimal allocation of the budget for each firm and discuss the resulting Nash equilibrium.

Theorem 2: For firm a, it is more profitable to seed agent j rather than enhancing the quality of its product if $v_j > v_c^a$ where

$$v_c^a \triangleq (2\lambda)(\frac{c_s}{c_q})(\frac{q_b}{(q_a+q_b)^2}). \tag{16}$$

Similarly, for firm b, it is more profitable to seed agent j rather than enhancing the quality of its product if $v_j > v_c^b$ where

$$v_c^b \triangleq (2\lambda)(\frac{c_s}{c_q})(\frac{q_a}{(q_a+q_b)^2}). \tag{17}$$

Moreover, any pair of the optimal strategies of the firms described by the above threshold rules describes a Nash equilibrium.

Proof: From equation (14) and (15) the relative marginal utility to cost for spending budget to seed agent j is $\frac{v_j}{c_s}$. Therefore, it is always more profitable to seed an agent with higher centrality. Also, the relative marginal utility to cost for spending budget on enhancing quality of product a is $\frac{2\lambda q_b}{c_q(q_a+q_b)^2}$ according to (14). Therefore, for firm a it is more profitable to seed agent j rather than enhancing the quality of its product iff

$$\frac{v_j}{c_s} > \frac{2\lambda q_b}{c_q(q_a+q_b)^2}.$$

This completes the proof. Similar story holds for firm b. Moreover, since the best response of each firm resulting from equations (14) and (15) is independent of the action of the other firm, any Nash equilibrium of the game between firms is simply a pair of firms best responses. ■

Corollary 2: If firms have equal qualities $q_a = q_b = q$, for both firms a and b, it is more profitable to seed agent j rather than enhancing the quality of their products if $v_j > v_c$ where

$$v_c \triangleq (\frac{\lambda}{2})(\frac{c_s}{c_q})(\frac{1}{q}). \tag{18}$$

Moreover, any pair of the optimal strategies of the firms described by the above threshold rules describes a Nash equilibrium.

Remark 2: If we compare the thresholds

$$v_c^a \triangleq (2\lambda)(\frac{c_s}{c_q})(\frac{q_b}{(q_a+q_b)^2}), \qquad v_c^b \triangleq (2\lambda)(\frac{c_s}{c_q})(\frac{q_a}{(q_a+q_b)^2}), \tag{19}$$

with qualities in Section III

$$q_a^* = (2\lambda)(\frac{c_s}{c_q})(\frac{\tilde{v}_l}{(\tilde{v}_k+\tilde{v}_l)^2}), \qquad q_b^* = (2\lambda)(\frac{c_s}{c_q})(\frac{\tilde{v}_k}{(\tilde{v}_k+\tilde{v}_l)^2}), \tag{20}$$

we can see a similarity as follows: In equation (19), q_a and q_b determine v_c^a and v_c^b which in turn determine the trade off between $\vec{S}$ and Δq according to Theorem 2. In Section III, K_a and K_b determine $\tilde{v}_k$ and $\tilde{v}_l$ based on the inequalities in (8) and $\tilde{v}_k$ and $\tilde{v}_l$ determine q_a^* and q_a^* according to (20), which in turn determine the trade of between $\vec{S}$ and q based on the budget constraint. Therefore, as it will be discussed later, we can achieve similar results for the effect of q_a and q_b on the optimal budget allocation, as we did for the effect of K_a and K_b on the Nash equilibrium.

Following the above theorem, the optimal allocation of the budget for each firm is to follow a so called water-filling strategy, that is, to start seeding in the order of agents' centralities until the centrality falls below the threshold given by (16) for firm a or (17) for firm b (in which case the firm spends the rest of the budget on improving the quality), or the firm runs out of budget. Also, the amount that agents can be seeded is up to their demand capacity, i.e. $\vec{S}_a^{max} = (0.5)\vec{1} - \vec{y}(0) > 0$ and $\vec{S}_b^{max} = (0.5)\vec{1} + \vec{y}(0) > 0$. Also, note that if the centrality of any agent is equal to the threshold defined in (16) or (17), then firms are indifferent between seeding that agent and quality improvement. Equations (16) and (17) indicate that the optimal allocation depends on quality of products, i.e. q_a and q_b and centrality distribution of agents in the network, i.e. v. In what follows, we will study the effect of each of these factors on the optimal allocation of the firms in more details. All of our analysis here is for firm a and similar results can be shown for firm b as well. For simplicity, we only discuss optimal seeding budget; optimal quality improvement budget can be found easily using the budget constraint.

A. *Effect of Quality of Products on Firms' Decisions:*

In this subsection we study how the quality of each product, i.e. q_a and q_b, can influence the optimal allocation of seeding and quality improvement budgets.

As it can be seen from equation (16), the threshold v_c^a depends on both firm's and its rival's qualities, i.e. both q_a and q_b. In the next proposition, we compare the seeding budget of two firms in the optimal allocation with respect to their qualities.

Proposition 6: Given an equal budget, the firm with higher quality also has higher seeding budget, i.e. if $q_a \leq q_b$, then $\|\vec{S}_a^*\|_1 \leq \|\vec{S}_b^*\|_1$.

Proof: From equations (16) and (17) it can be easily seen that if $q_a \leq q_b$, then $v_c^b \leq v_c^a$.

As a result, more agents satisfy the condition (17) for firm b compared to firm a and therefore, $\|\vec{S}_a^*\|_1 \leq \|\vec{S}_b^*\|_1$. ■

This result is due to diminishing return of quality which means if a firm already has a high quality it would profit less by spending on quality improvement and it would be better for the firm to invest on seeding. Also, note that the result of Proposition 6 is similar to the result of Proposition 1. The only difference is that instead of budgets K_a and K_b in Proposition 1, qualities q_a and q_b in Proposition 6 play the role of the budgets while comparing the seedings of the firms.

In the next proposition we explain how the optimal seeding budget vary with q_a and q_b.

Proposition 7: Given a fixed graph, the optimal seeding budget is an increasing function of q_a. Furthermore, it is a decreasing function of q_b if $q_b \leq q_a$ and an increasing function of q_b if $q_a \leq q_b$.

Proof: The optimal seeding budget is a decreasing function of the threshold value v_c^a. Also, it is easy to see that the threshold value of v_c^a is a decreasing function of q_a. This implies the first part of proposition. For the second part, it is easy to see that the threshold value of v_c^a is a decreasing function of q_a. This implies the first part of proposition. For the second part, it is easy to see that the threshold value of v_c^a is an increasing function of the quality of product b, when $q_b \leq q_a$ and a decreasing function of the quality of product b, when $q_b \geq q_a$. This completes the proof. ■

Proposition 7 implies that a higher quality in a firm's product results in a higher seeding budget in the optimal allocation. This can be due to the diminishing return of quality: when quality is higher there is less need for quality improvement and it would be more profitable to spend on seeding. Furthermore, when $q_b \leq q_a$, the higher the quality of the rival firm's product, the lower the seeding budget of firm a, i.e., if $q_a \geq q_b' \geq q_b$ then, $\|\vec{S}_a^*(q_b')\|_1 \leq \|\vec{S}_a^*(q_b)\|_1$. On the other hand, when competing with a firm whose product has a higher quality, i.e. $q_b \geq q_a$, the higher the quality of the rival firm's product, the higher firm a should spend on seeding. In other words, if $q_a \leq q_b \leq q_b'$ then, $\|\vec{S}_a^*(q_b)\|_1 \leq \|\vec{S}_a^*(q_b')\|_1$.

Combining these two results, we can see that given a fixed value of q_a, the seeding budget of firm a is increasing with the difference $|q_a - q_b|$. The seeding budget attains its minimum when $q_b = q_a$, implying that the firm should allocate more budget to quality improvement to distance itself from the rival firm. However, as the gap between qualities widens, competition in qualities becomes less effective and firms spend more budget on seeding. Also, note that the result of Proposition 7 is similar to the result of Proposition 2. The only difference is that seeding budgets vary with q_a and q_b in Proposition 7, whereas they vary with K_a and K_b in Proposition 2.

B. *Effect of Network Structure on Firms' Decisions:*

In this subsection we study the effect of network structure on the optimal allocation of the budget for seeding and quality improvement. First we define seeding capacity of a graph.

Definition 4: The seeding capacity of a graph is the amount that it can be seeded in the optimal allocation when there is no budget constraint.

We first focus on two well studied graphs, i.e. star and balanced graphs, and highlight how they can reflect important properties of the seeding budget. The next proposition provides a condition for seeding profitability of any general graph. Also, the seeding capacity of star and balanced graphs are compared and it is shown that the graph with higher seeding capacity can be any of the two, depending on the threshold value of v_c^a in (16).

Proposition 8: If seeding capacity is nonzero for a balanced graph, it will be nonzero for any other graph too. On the other hand, if seeding capacity is zero for a star graph, it will also be zero for any other graph. Moreover, if $v_l^s < v_c^a < \bar{v}$, a balanced graph has a larger seeding capacity than a star graph, and if $\bar{v} < v_c^a < v_h^s$, a star graph has a larger seeding capacity than a balanced graph. For $1 < v_c^a < v_l^s$ they have the same seeding capacity.

Proof: If seeding capacity is nonzero for a balanced graph, then we have $v_c^a < \bar{v}$. As a result, for any other graph we will have $v_c^a < v_{max}$, where $v_{max} = \max v_i$, since according to Lemma 3 $\bar{v} \leq v_{max}$. This means that there exists at least one agent that must be seeded. On the other hand, if seeding capacity is zero for a star graph, then we must have $v_c^a > v_h^s$. Since we know $v_h^s \geq v_i$ for any agent i of any arbitrary graph, therefore, $v_c^a > v_i$ and no agent can be seeded in any other graph. For the second part of the proposition, if $v_l^s < v_c^a < \bar{v}$, then seeding capacity for the star graph will be $S_{a_i}^{max}$, where $S_{a_1}^{max} \geq S_{a_2}^{max} \geq \cdots \geq S_{a_n}^{max}$ are elements of the demand capacity vector $\vec{S}_a^{max}$ and agent i is the central agent. However, for the balanced graph all agents can be seeded up to their maximum demand capacities and the seeding capacity will be $\|\vec{S}_a^{max}\|_1$. On the other hand, if $\bar{v} < v_c^a < v_h^s$, still seeding for the star graph will be $S_{a_i}^{max}$, however, no agent can be seeded in the balanced graph. For $1 < v_c^a < v_l^s$, agents in both graphs can be seeded up to $\|\vec{S}_a^{max}\|_1$. ■

It is easy to see that Proposition 8 presents very similar results as Proposition 3. The next proposition provides us with a lower and an upper bound for minimum and maximum seeding capacities.

Proposition 9: If $1 < v_c^a < v_h^s$, the maximum seeding capacity is given by

$$\|\vec{S}_a^*\|_1^{max} = \sum_{i=1}^{k} S_{a_i}^{max},$$

where

$$k = \min\{\lfloor \frac{n\delta}{(v_c^a - 1)(2\beta - \delta)} \rfloor, n\}. \tag{21}$$

On the other hand, the minimum seeding capacity is $S_{a_n}^{max} + S_{a_{n-1}}^{max}$ if $1 < v_c^a < v_l^s$, is $S_{a_n}^{max}$ if $v_l^s < v_c^a < \bar{v}$, and is zero if $\bar{v} < v_c^a < v_h^s$.

Proof: From condition (16) the more agents with centralities above the threshold v_c^a, the more seeding budget can be allocated. Therefore, the maximum number of k agents with centralities above the threshold v_c^a must be found. Since $v_i \geq 1$ for all agents, first a centrality of one is given to each agent and then the remainder of the centrality sum is distributed among maximum number of agents so that each agent receives at least $v_c^a - 1$, making its overall centrality greater than v_c^a. It is easy to see that the number of such agents is upper bounded by $\lfloor \frac{\frac{2\beta n}{2\beta - \delta} - n}{v_c^a - 1} \rfloor$. This along with the fact that $1 \leq k \leq n$ results in (21). Note that, in order to complete the proof, we should also provide an example achieving this maximum capacity. For $k = 1$, the maximum seeding capacity is clearly achieved by the star graph with the seeding capacity of $S_{a_1}^{max}$. For $k \geq 2$, a graph with largest seeding capacity is the one with k central agents having the largest demand capacities and with equal centralities of

$$\tilde{v}_h^s = \frac{n\delta}{k(2\beta - \delta)} + 1,$$

where k is given in (21), and the remainder $n-k$ agents with the minimum centrality of $\tilde{v}_l^s = 1$. For the graph with minimum seeding capacity, similar to the proof of Proposition 8, we have minimum seeding capacity of $S_{a_n}^{max}$ in star graph if $v_l^s < v_c^a < \bar{v}$, and zero in balanced graph if $\bar{v} < v_c^a < v_h^s$. For the case where $1 < v_c^a < v_l^s$, let i be the agent with the highest centrality. Clearly, $v_l^s < v_i \leq v_h^s$. Now, considering the fact that sum of the centralities is fixed, there is an agent $j \in V - \{i\}$ for which $v_l^s \leq v_j$. This implies that there exist at least two agents whose centralities are above v_c^a. An example of a graph with exactly two centralities above v_c^a is a directed star graph where all edges are directed towards the center except one edge which goes both ways. ■

It can be seen from both Proposition 9 and Proposition 4 that graphs with similar structures attain maximum and minimum seeding in both scenarios.

Example 2: As a numerical example for the minimum and maximum seeding capacities, we consider a network with $n = 15$ agents with demand capacity vectors of $\vec{S}_a^{max} = \vec{S}_b^{max} = (0.5)\vec{1}$, qualities of $q_a = q_b = 1$, quality and seeding costs of $c_s = c_q = 1$ and parameters of $\alpha = \beta = 1$ and $\delta = 0.5$. For this example from equations (16) and (17) we have $v_c^a = v_c^b = 2.5$ and as a result, from equation (21) we get $k = 3$. Therefore, a graph with the maximum seeding capacity is a 3-star with seeding capacity of 1.5 as illustrated in Fig. 1. Also, since $\bar{v} = \frac{4}{3} < v_c^a, v_c^b < v_h^s = 4.8$, a balanced graph has the minimum seeding capacity of zero. A star graph has a seeding capacity of 0.5 which is neither a minimum nor a maximum.

Note that in both Example 1 and Example 2 a graph with the maximum seeding budget and capacity is a 3-star graph and a graph with the minimum seeding budget and capacity is a balanced graph. A star graph has neither a minimum nor a maximum seeding budget and capacity in both examples.

As we saw, the structure of the graphs with minimum and maximum seeding capacity depends on the threshold value of v_c^a. However, for certain values of v_c^a the seeding capacity will be independent of the structure of the graph, as described in the next proposition.

Proposition 10: If $v_c^a > v_h^s$ no graph can be seeded. On the other hand, if $v_c^a < 1$ all graphs can be seeded equally up to agents' maximum demand capacities.

Proof: The maximum possible centrality happens for the central agent of the star graph as shown in Lemma 3. As a result, if $v_c^a > v_h^s$, then we have $v_c^a > v_i$ for all i in any graph and from condition (16) no agent can be seeded. Also, since from definition $1 \leq v_i$ for all i, if $v_c^a < 1$ then we have $v_c^a < v_i$ for all agents in any graph, and any graph can be seeded up to agents' maximum demand capacities, given the availability of budget. ■

It is easy to see that Proposition 10 presents very similar results as Proposition 5.

V. Conclusion

We proposed and studied a strategic model of marketing and product consumption in social networks. Two firms compete for maximizing the consumption of their products in a social network. Initially firms utilize a limited budget to either design the quality of their products or initially seed a set of agents in the social network. Agents are myopic yet utility maximizing, given the qualities of the products and actions of their neighbors. This myopic best response results in a local, linear update dynamics for the consumptions of the agents. We characterized the unique Nash equilibrium of the game between firms. We showed that at the Nash equilibrium, firms invest more budget on quality when their budgets are close. However, as the difference between budgets increases, firms spend more budget on seeding. We also showed that given equal budget of firms, if seeding budget is nonzero for a balanced graph it will also be nonzero for any other graph, and if seeding budget is zero for a star graph it will be zero for any other graph too. Afterwards, we considered a different scenario in which firms produce two products with some preset qualities that can only be improved marginally. At some point in time, firms spend a limited budget to marginally improve the quality of their products and to give free offers to a set of agents in the network in order to promote their products. We derived a simple threshold rule for the optimal allocation of the budget between new seedings and quality improvement. We showed that the optimal allocation of the budget in particular depends on the entire centrality distribution of the graph and the qualities of the products. Furthermore, we derived similar results to the original setup for this scenario, in which preset qualities resemble the role of budgets.

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