

Optimal Attack Strategies Subject to Detection Constraints Against Cyber-Physical Systems

Yuan Chen, Soumya Kar, and José M. F. Moura

Abstract—This paper studies an attacker against a cyber-physical system (CPS) whose goal is to move the state of a CPS to a target state while ensuring that his or her probability of being detected does not exceed a given bound. The attacker's probability of being detected is related to the nonnegative bias induced by his or her attack on the CPS's detection statistic. We formulate a linear quadratic cost function that captures the attacker's control goal and establish constraints on the induced bias that reflect the attacker's detection-avoidance objectives. When the attacker is constrained to be detected at the false-alarm rate of the detector, we show that the optimal attack strategy reduces to a linear feedback of the attacker's state estimate. In the case that the attacker's bias is upper bounded by a positive constant, we provide two algorithms – an optimal algorithm and a sub-optimal, less computationally intensive algorithm – to find suitable attack sequences. Finally, we illustrate our attack strategies in numerical examples based on a remotely-controlled helicopter under attack.

I. INTRODUCTION

Security vulnerabilities in cyber-physical systems (CPS), systems that interface sensing, communication, and control with an underlying physical process, allow for sophisticated cyber attacks that cause catastrophic physical harm. In the past, events such as StuxNet [1] and the Maroochy Sewage Control Incident [2] have demonstrated the vulnerability of industrial processes. More recently, cyber-physical attacks have targeted automobiles [3], military vehicles [4], and commercial drones [5]. These examples show that CPS remain susceptible to cyber-attacks, and, in response, there have been significant efforts to improve the security of CPS.

Part of the effort in improving cyber-physical security has been devoted to categorizing different types of attacks and developing security countermeasures for each type [1], [6]. One particular type of attack is the integrity attack, in which an attacker manipulates the CPS's sensor readings and alters its actuator control signals [6], [7]. Prior work has analyzed CPS security sensor attacks, determining the fundamental limits of attack detection [8] and developing methods to reconstruct

sensor attacks [9]–[11]. Existing work has also studied the capabilities of an integrity attacker, relating the ability of the attacker to perform undetectable attacks to certain geometric control-theoretic properties of the CPS [12]–[14]. For systems affected by process and sensor noise, references [7] and [15] characterize the state estimation error caused by an attacker who tries to avoid detection.

In addition to analyzing the *ability* of an integrity attacker to cause damage and evade detection, prior work has also studied *how* an attacker should behave in order to achieve his or her objectives. Reference [16] considers a noisy CPS and designs an attack to optimally disrupt the system's feedback controller, while avoiding detection. Instead of attackers who seek to cause general disruption and damage to a CPS, our previous work studies attackers with specific control objectives [17], [18]. In [18], we considered an attacker whose goal is move the CPS to a target state while evading detection, formulated a cost function that penalized the deviation from the target state and the magnitude of the detection statistic, and we determined the optimal attack for such a cost function reduced to a linear feedback of the attacker's state estimate.

This paper studies an attacker who wishes to move the system to a target state, but, unlike [18], we impose an explicit bound on the probability of an attack being detected. We model the CPS as a linear dynamical system subject to sensor and process noise equipped with a Kalman filter for state estimation and an LQG controller. The CPS uses a χ^2 detector as an attack detector, which reports an attack if the energy of the Kalman filter innovation exceeds a certain threshold [19]. This model has been used in the literature to model CPS under attack (see, e.g., [20], [21]). The attacker's goal is to design a sequence of attacks that counters the system's LQG controller and minimizes the deviation of the CPS's state from the target state subject to a bound on the (non-negative) bias induced on the χ^2 detection statistic.

We define a linear quadratic cost function that captures the attacker's control objective by penalizing the distance between the system's state and the target state. Then, we formulate the attack design problem as an optimization problem of finding a sequence of attacks that minimizes the cost function subject to an upper bound constraint on the bias induced in the detection statistic. This differs from our previous work [18] that studies *unconstrained* attack design where the attacker's detection avoidance goals is an additional term of the overall cost function. This paper, unlike [18], requires attacks at each time step to satisfy explicit constraints. Since we compute the optimal attack sequence in a causal manner, we must ensure that attacks at each time step are *recursively feasible* [22]

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to guarantee that it is possible to satisfy the constraints of future time steps. We use geometric control properties (similar to those studied in [23]) of the CPS model to express, the constraint placed on the detection statistic bias as a linear constraint on the attack at each time step. From a practical perspective, this paper provides guarantees on the optimal attacks' probability of being detected (reference [18] does not provide such guarantees).

We consider separately two cases: 1) when the induced bias is constrained to be zero and 2) in which the induced bias is upper bounded by a positive constant. When the bias is zero, which restricts the attacker to be detected at the false alarm rate of the detector, we apply constrained dynamic programming to show that the optimal attack reduces to a linear feedback strategy. For bounded bias, we provide two algorithms to determine a suitable sequence of attacks. The first algorithm is more computationally intensive but finds an optimal sequence of attacks. The second, less computationally intensive, algorithm finds a (sub-optimal) sequence of attacks that satisfies the detection constraint. A preliminary version of part of this work appears in [24], designing attacks constrained to be detected at the false alarm rate, but when the CPS are *not* equipped with LQG controllers. When the CPS is equipped with its own controller, which we consider here, the attacker must account for system input in designing his or her attack.

The rest of the paper is organized as follows. Section II provides the model and assumptions for the CPS and attacker, reviews the χ^2 detector and the concept of recursive feasibility, and formally states the problems we address. In Section III, we determine the set of all recursively feasible attacks at each time step. In Section IV, we use dynamic programming to find an optimal strategy when the attacker's probability of being detected is constrained to be the detector's false alarm rate. Section V studies the case when the bias induced in the detection statistic is upper bounded by a positive constant; we provide two algorithms for computing attack sequences that achieve the attacker's objectives. We provide numerical examples of a remotely-controlled helicopter under attack (from each of our proposed strategies) in Section VI, and we conclude in Section VII.

II. BACKGROUND

A. Notation

Let \mathbb{R} denote the reals, \mathbb{R}^n denote the space of n -dimensional real (column) vectors, and $\mathbb{R}^{m \times n}$ denote the space of real m by n matrices. The multivariate Gaussian distribution with mean μ and covariance Σ is denoted as $\mathcal{N}(\mu, \Sigma)$. The n by n identity matrix is denoted as I_n . For a matrix M , $\mathcal{R}(M)$ denotes the range space of M , $\mathcal{N}(M)$ denotes the null space of M , and M^\dagger denotes the Moore-Penrose pseudoinverse. For a symmetric matrix $S = S^T$, $S \succeq 0$ denotes that S is positive semidefinite, and $S \succ 0$ denotes that S is positive definite. For $S \succ 0$, let $\|\cdot\|_S$ denote the S -weighted 2-norm. That is, for $S \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^n$, $\|x\|_S = \left\| S^{\frac{1}{2}} x \right\|_2 = \sqrt{x^T S x}$.

B. System Model

We use the same, linearized¹ CPS model as [18]:

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + \Gamma e_t + w_t, \\ y_t &= Cx_t + \Psi e_t + v_t, \end{aligned} \quad (1)$$

where $x_t \in \mathbb{R}^n$ describes the system's state, $u_t \in \mathbb{R}^m$ is the system input, $e_t \in \mathbb{R}^s$ is the attacker's input², and w_t and v_t are the process and sensor noise, respectively. The sensor and process noise are independently, identically, distributed (i.i.d.) in time and mutually independent; w_t has distribution $\mathcal{N}(0, \Sigma_w)$, and v_t has distribution $\mathcal{N}(0, \Sigma_v)$, with $\Sigma_w, \Sigma_v \succ 0$. The system starts running at time $t = -\infty$, and the initial state of the system $x_{-\infty}$ has distribution $\mathcal{N}(0, \Sigma_x)$ with $\Sigma_x \succ 0$ and is independent of the noise processes. The pair (A, C) is observable, and the pair (A, B) is controllable. The matrices Γ and Ψ describe the attacker. The model (1) is commonly adopted in studies of CPS under attack [18], [20], [21].

The system knows the matrices A, B , and C and the statistics of the noise processes and initial state, but does not know the matrices Γ and Ψ (since they describe the attacker). The system causally knows the system input u_t and the sensor output y_t , but not the attack e_t . We assume the system's goal is to regulate the system state to the origin. Because the system cannot directly observe the state x_t , it uses its sensor measurements y_t to construct an estimate of the state using a Kalman filter. Then, the system performs feedback control on the state estimate to regulate the state to the origin. The system constructs its Kalman filter and controller assuming nominal operating conditions (i.e., $e_t = 0$ for all t).

Under nominal operating conditions, the system's Kalman filter calculates \hat{x}_t , the minimum mean square error (MMSE) estimate of x_t given all sensor measurements up to time t and input up to time $t-1$. Since the system starts at $t = -\infty$, the Kalman filter has fixed gain:

$$K = PC^T(CPC^T + \Sigma_v)^{-1}, \quad (2)$$

$$\begin{aligned} P &= APA^T + \Sigma_w \\ &\quad - APC^T(CPC^T + \Sigma_v)^{-1}CPA^T, \end{aligned} \quad (3)$$

$$\hat{x}_t = \hat{x}_{t|t-1} + K(y_t - C\hat{x}_{t|t-1}), \quad (4)$$

$$\hat{x}_{t+1|t} = A\hat{x}_t + Bu_t. \quad (5)$$

To regulate the state x_t , the system has a feedback controller of the form

$$u_t = L\hat{x}_t, \quad (6)$$

where the feedback matrix L is chosen such that $A + BL$ is stable. One controller that takes the form of equation (6) is the infinite horizon LQG controller that minimizes the cost function $J_{\text{CPS}} = \lim_{T \rightarrow \infty} \frac{1}{2T+1} \mathbb{E} \left[\sum_{t=-T}^T x_t^T Q' x_t + u_t^T R' u_t \right]$, where $Q' \succeq 0$, $R' \succ 0$, and the pair (A, Q') is observable.

¹If the CPS is nonlinear, the model (1) represents its dynamics after linearization about an operating point.

²The attack e_t also models the case in which the attacker may separately attack the CPS's actuators and sensors. Define $e_t = \begin{bmatrix} e_t^d & e_t^s \end{bmatrix}^T$, where e_t^d and e_t^s are the actuator and sensor attacks, respectively, and define $\Gamma = \begin{bmatrix} \Gamma & 0 \end{bmatrix}$ and $\Psi = \begin{bmatrix} 0 & \Psi \end{bmatrix}$.

The CPS is equipped with a χ^2 detector to determine if, for some t , $e_t \neq 0$. The χ^2 attack detector [19] uses the innovations sequence of the Kalman filter, ν_t , defined as $\nu_t = y_t - C\hat{x}_{t|t-1}$, to determine whether or not an attack has occurred. The term $\hat{x}_{t|t-1}$ is the MMSE estimate of x_t given all sensor measurements and system input up to time $t-1$, assuming nominal operating conditions. When there is no attack (i.e., $e_t = 0$ for all t), the innovations sequence is i.i.d. $\mathcal{N}(0, \Sigma_\nu)$, where $\Sigma_\nu = CPC^T + \Sigma_v$, and ν_t is orthogonal to $\hat{x}_{t|t-1}$ [25]. The χ^2 reports an attack if the statistic $g_t = \sum_{k=t-t'+1}^t \nu_k^T \Sigma_\nu^{-1} \nu_k$, where S is the window size of the detector, exceeds a threshold τ , which is chosen a priori to balance the false alarm and missed detection probabilities [19]. In this paper, we consider a χ^2 detector with window size $t' = 1$, so $g_t = \nu_t^T \Sigma_\nu^{-1} \nu_t$.

There are attack detectors other than the χ^2 attack detector (see, e.g., [9], [12], [26].) These detectors require noiselessness [9], [12], bounded energy noise [26], or batch measurements [9], [26]. This paper studies attacks against CPS under broader conditions on the noise and provides a recursive solution. We consider an on-line χ^2 detector for systems with process and sensor noise. The linear, state space model with a Kalman filter, feedback controller, and χ^2 attack detector with window size 1 is a standard model for a CPS subject to attack [18], [20], [21].

C. Attacker Model

The attacker knows the system model and statistical properties, the controller feedback matrix L , and which sensors and actuators he or she can attack (i.e. the matrices Γ and Ψ)³. Following [18], we assume, without loss of generality, that the matrix $\begin{bmatrix} \Gamma \\ \Psi \end{bmatrix}$ is injective. The attacker causally knows the sensor output y_t and the attack e_t . Additionally, the attacker causally knows $\tilde{y}_t = Cx_t + v_t$, the value of the sensor output at time t before it is altered by the attack at time t .

The attacker performs Kalman filtering, separately from the system, to estimate the state. The attacker also uses his or her knowledge to compute the estimate produced by the system's Kalman filter. The attacker knows L and the system's state estimate, so he or she knows u_t . We design attack strategies that depend on the attacker knowing the system's input. In general, so long as the attacker knows u_t for all t , the CPS's control input need not be restricted to the form of equation (6). For this paper, we only consider the case of feedback control, but our methodology may be tailored toward other control laws. The attack begins at time $t = 0$, i.e., for $t = -\infty, \dots, -1$, $e_t = 0$. During the time interval $t = -\infty, \dots, 0$, the attacker observes the system output and keeps track of the state estimate \hat{x}_t .

The attacker's objective is to design an attack sequence over the finite time interval $t = 0$ to $t = N$ that moves the system

state to a target state x^* while satisfying a detection-avoidance constraint. The attacker chooses the sequence

$$\gamma(0, N) = \{e_0, \dots, e_N\},$$

to accomplish his or her goals such that at time t , the attack e_t only depends on the attacker's available information at time t , \mathcal{I}_t . Following [18], \mathcal{I}_t is the classical information pattern [25]: $\mathcal{I}_0 = \{\tilde{y}_0\}$, $\mathcal{I}_{t+1} = \{\mathcal{I}_t, \tilde{y}_{t+1}, e_t\}$. If a nonzero attack occurs, the attacker's Kalman filter then produces a different estimate than the system's Kalman filter and becomes:

$$\tilde{x}_{t+1|t} = A\hat{x}_t + Bu_t + \Gamma e_t, \quad (7)$$

$$\tilde{x}_t = \hat{x}_{t|t-1} + K(\tilde{y}_t - C\hat{x}_{t|t-1}), \quad (8)$$

The attacker's Kalman filter produces the MMSE state estimate given \mathcal{I}_t , i.e., $\tilde{x}_t = \mathbb{E}[x_t | \mathcal{I}_t]$.

The attack $\gamma(0, t)$ induces a bias ϵ_t in the system's innovation ν_t . Under an attack $\gamma(0, t)$, we have $\nu_t = \nu_t^0 + \epsilon_t$, where ν_t^0 is the value of the system's innovation in the case that there had been no attack (i.e., $e_0 = e_1 = \dots = e_t = 0$). The following state space dynamical system describes the relationship between $\gamma(0, t)$ and ϵ_t [18]:

$$\begin{aligned} \theta_{t+1} &= \hat{A}\theta_t + \hat{B}e_t, \\ \epsilon_t &= \hat{C}\theta_t + \hat{D}e_t, \end{aligned} \quad (9)$$

where $\hat{A} = \begin{bmatrix} (I_n - KC)A + BL & KC & KC \\ ABL & A & 0 \\ 0 & 0 & A \end{bmatrix}$, $\hat{B}^T = \begin{bmatrix} (K\Psi)^T & 0^T & \Gamma^T \end{bmatrix}$, $\hat{C} = \begin{bmatrix} -CA & C & C \end{bmatrix}$, $\hat{D} = \Psi$, and $\theta_0 = 0$.

The Σ_ν^{-1} -weighted 2-norm of ϵ_t relates to the probability of the attack being detected at time t [27]. Let $P_{D,t} = \mathbb{P}(g_t > \tau)$ be the detection probability at time t . If $\|\epsilon_t\|_{\Sigma_\nu^{-1}}^2 = 0$, then, for any positive detector window size, the probability of detection at time t is equal to the false alarm probability of the χ^2 detector, since there is no induced bias in ν_t . For nonzero bias (and detector window size 1), the following lemma relates the bound on $\|\epsilon_t\|_{\Sigma_\nu^{-1}}^2$ to the probability of being detected.

Lemma 1 (Detection Probability Bound [27]). *For any $\delta \in (0, \tau)$, if $\|\epsilon_t\|_{\Sigma_\nu^{-1}}^2 \leq \delta$, then*

$$P_{D,t} \leq \mathbb{P}\left(g_t^0 > \left(\sqrt{\tau} - \sqrt{\delta}\right)^2\right),$$

where g_t^0 is the value of the statistic g_t when there is no attack⁴.

To model the attacker's control objectives, define the cost function:

$$J = \mathbb{E}\left[\sum_{t=0}^N \|(x_t - x^*)\|_{Q_t}^2\right], \quad (10)$$

with $Q_t \succ 0$. The cost function J penalizes deviation of the state from the target state. The attacker's goal is to design an attack $\gamma(0, N)$ that achieves cost

$$J^* = \min_{\gamma(0, N)} \mathbb{E}\left[\sum_{t=0}^N \|(x_t - x^*)\|_{Q_t}^2\right], \quad \text{s.t.} \quad \|\epsilon_t\|_{\Sigma_\nu^{-1}}^2 \leq \delta, \forall t = 0, \dots, N \quad (11)$$

³In future work, we will consider defense strategies against such attackers and study the interaction between the attacker and CPS in a game theoretic framework. Thus, in this paper, we assume the worst-case, most powerful attacker who knows the system model perfectly. In addition, future work will study attack strategies that are robust to imperfections in the attacker's knowledge of the system model.

⁴The statistic g_t^0 is i.i.d. (in time) χ^2 with p degrees of freedom.

the minimum cost of J subject to constraints on the Σ_ν^{-1} -weighted 2-norm of ϵ_t . The constraints in the optimization problem (11) model the attacker's goals of evading detection.

D. Recursive Feasibility

The attacker designs the attack in real time: at time t , the attacker chooses the attack e_t based on his or her information \mathcal{I}_t . Note that the constraint $\|\epsilon_t\|_{\Sigma_\nu^{-1}}^2 \leq \delta$ in (11) is for all times $t = 0, \dots, N$. It is necessary that the attack be recursively feasible [22]: the attack e_t must be chosen such that $\|\epsilon_t\|_{\Sigma_\nu^{-1}}^2 \leq \delta$ and, for all future times $t+1, \dots, N$, there exist attacks e_{t+1}, \dots, e_N such that $\|\epsilon_{t+1}\|_{\Sigma_\nu^{-1}}^2 \leq \delta, \dots, \|\epsilon_N\|_{\Sigma_\nu^{-1}}^2 \leq \delta$. The recursive feasibility of (11) is related to the output minimization problem presented in [23]:

Lemma 2 ([23]). *Consider the system in (9) with arbitrary initial state θ_0 . Then, for any $k = 1, 2, \dots$,*

$$\min_{e_0, \dots, e_{k-1}} \sum_{t=0}^{k-1} \|\epsilon_t\|_{\Sigma_\nu^{-1}}^2 = \theta_0^T \hat{\mathbf{P}}_k \theta_0, \quad (12)$$

where $\hat{\mathbf{P}}_k$ follows the solution to the Riccati equation

$$\begin{aligned} \hat{\mathbf{P}}_{k+1} = & \hat{\mathbf{A}}^T \hat{\mathbf{P}}_k \hat{\mathbf{A}} + \hat{\mathbf{C}}^T \Sigma_\nu^{-1} \hat{\mathbf{C}} - \left(\hat{\mathbf{D}}^T \Sigma_\nu^{-1} \hat{\mathbf{C}} + \hat{\mathbf{B}}^T \hat{\mathbf{P}}_k \hat{\mathbf{A}} \right)^T \\ & \times \left(\hat{\mathbf{D}}^T \Sigma_\nu^{-1} \hat{\mathbf{D}} + \hat{\mathbf{B}}^T \hat{\mathbf{P}}_k \hat{\mathbf{B}} \right)^\dagger \left(\hat{\mathbf{D}}^T \Sigma_\nu^{-1} \hat{\mathbf{C}} + \hat{\mathbf{B}}^T \hat{\mathbf{P}}_k \hat{\mathbf{A}} \right), \end{aligned} \quad (13)$$

with $\hat{\mathbf{P}}_0 = 0$.

Furthermore, the matrix $\hat{\mathbf{P}}_k$ is positive semidefinite, and $\min_{e_0, \dots, e_{k-1}} \sum_{t=0}^{k-1} \|\epsilon_t\|_{\Sigma_\nu^{-1}}^2 = 0$ if and only if $\hat{\mathbf{P}}_k \theta_0 = 0$ [23].

E. Augmented State Space Notation

For the remainder of this paper, we use the augmented state space description of the cyber-physical system and attacker provided in [18]. Define the augmented state

$$\xi_t^T = \begin{bmatrix} \tilde{x}_t^T & \theta_t^T & (\hat{x}_t^0)^T & x^{*T} \end{bmatrix}, \quad (14)$$

where \hat{x}_t^0 denotes the system's state estimate in the case that $e_0 = \dots = e_t = 0$. The state ξ_t follows the dynamics

$$\xi_{t+1} = \mathcal{A}\xi_t + \mathcal{B}e_t + \mathcal{K}\tilde{\nu}_{t+1}, \quad (15)$$

where $\tilde{\nu}_{t+1}$ denotes the attacker's innovation at time

$$\begin{aligned} t+1, \quad \mathcal{A} = & \begin{bmatrix} A & BL\Omega & BL & 0 \\ 0 & \hat{A} & 0 & 0 \\ 0 & 0 & A+BL & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}, \quad \mathcal{B}^T = \\ & \begin{bmatrix} \tilde{B}^T & \hat{B}^T & 0^T & 0^T \end{bmatrix}, \quad \mathcal{K}^T = \begin{bmatrix} K^T & 0 & K^T & 0 \end{bmatrix}^T, \\ \tilde{\Omega} = & \begin{bmatrix} (I_n - KC)A + BL & KC & KC \end{bmatrix}, \quad \text{and} \\ \tilde{B} = & \Gamma + BLK\Psi. \end{aligned}$$

Further define $\tilde{\mathcal{C}} = \begin{bmatrix} 0 & \hat{\mathcal{C}} & 0 & 0 \end{bmatrix}$, $\tilde{\mathcal{D}} = \Psi$, and $\mathcal{H} = \begin{bmatrix} I_n & 0 & 0 & -I_n \end{bmatrix}$. Then, we have

$$\epsilon_t = \tilde{\mathcal{C}}\xi_t + \tilde{\mathcal{D}}e_t, \quad (16)$$

$$\tilde{x}_t - x^* = \mathcal{H}\xi_t. \quad (17)$$

One important property of ξ_t is that, given \mathcal{I}_t , the attacker can exactly determine the value of ξ_t [18]. Accordingly, the attacker can use ξ_t to determine his or her attack at time t .

Following [18] and [25], we manipulate the cost function J by substituting $x_t = \tilde{x}_t + n_t$, where n_t is the estimation error. It is well known [25] that, given \mathcal{I}_t , n_t is conditionally distributed as $\mathcal{N}(0, \hat{P})$, where $\hat{P} = P - PC^T \Sigma_\nu^{-1} CP$, and n_t is conditionally orthogonal to \hat{x}_t . Performing this substitution, the optimal attack design problem becomes.

$$\begin{aligned} \min_{\gamma(0, N)} \quad & \sum_{t=0}^N \text{trace}(\hat{P}Q_t) + \mathbb{E} \left[\sum_{t=0}^N \|\mathcal{H}\xi_t\|_{Q_t}^2 \right], \\ \text{s.t.} \quad & \|\epsilon_t\|_{\Sigma_\nu^{-1}}^2 \leq \delta, \forall t = 0, \dots, N \end{aligned} \quad (18)$$

where $\sum_{t=0}^N \text{trace}(\hat{P}Q_t)$ does not depend on $\gamma(0, N)$.

F. Problem Statement

This paper addresses three main problems. Consider the optimal attack design problem (18). First, determine, for any $\delta \geq 0$ and any time $t = 0, \dots, N$, the set of recursively feasible attacks. Second, find an optimal attack sequence $\gamma(0, N)$ when $\delta = 0$. This corresponds to finding the optimal attack under the constraint that the probability of being detected at any time t is equal to the false alarm probability of the detector. Third, find an optimal attack sequence $\gamma(0, N)$ when $\delta > 0$.

III. FEASIBILITY SETS

In this section, we determine which attacks e_t are recursively feasible at time t . Recursively feasible attacks are attacks e_t such that $\|\epsilon_t\|_{\Sigma_\nu^{-1}}^2 \leq \delta$ and there exists e_{t+1}, \dots, e_N such that $\|\epsilon_{t+1}\|_{\Sigma_\nu^{-1}}^2 \leq \delta, \dots, \|\epsilon_N\|_{\Sigma_\nu^{-1}}^2 \leq \delta$. From equations (15) and (16), we see that the recursive feasibility of an attack e_t depends on the state ξ_t . Define the sets Ξ_t , $t = 0, \dots, N$ as follows:

$$\begin{aligned} \Xi_N = & \left\{ \xi_N \in \mathbb{R}^{6n} \mid \exists e_N, \|\tilde{\mathcal{C}}\xi_N + \tilde{\mathcal{D}}e_N\|_{\Sigma_\nu^{-1}}^2 \leq \delta \right\}, \\ \Xi_t = & \left\{ \xi_t \in \mathbb{R}^{6n} \mid \exists e_t, \|\tilde{\mathcal{C}}\xi_t + \tilde{\mathcal{D}}e_t\|_{\Sigma_\nu^{-1}}^2 \leq \delta, \right. \\ & \left. \mathcal{A}\xi_t + \mathcal{B}e_t \in \Xi_{t+1} \right\}, \quad t = 0, \dots, N-1. \end{aligned} \quad (19)$$

In the definition of Ξ_t , $t = 0, \dots, N-1$, we have the condition $\mathcal{A}\xi_t + \mathcal{B}e_t \in \Xi_{t+1}$, which ignores the term $\mathcal{K}\nu_{t+1}$. From the structure of \mathcal{A} , \mathcal{C} , and \mathcal{K} , we see that membership in Ξ_t depends only on the θ_t component of ξ_t , which is unaffected by $\mathcal{K}\nu_{t+1}$. That is, we have $\mathcal{A}\xi_t + \mathcal{B}e_t \in \Xi_{t+1}$ if and only if $\mathcal{A}\xi_t + \mathcal{B}e_t + \mathcal{K}\nu_{t+1} \in \Xi_{t+1}$ for any $\nu_{t+1} \in \mathbb{R}^p$.

We use the sets Ξ_t to determine the existence of recursively feasible attacks at time t .

Lemma 3. *There exists a recursively feasible attack e_t if and only if $\xi_t \in \Xi_t$. That is, there exists a sequence of attacks $\gamma(t, N) = \{e_t, \dots, e_N\}$ such that $\|\epsilon_t\|_{\Sigma_\nu^{-1}}^2 \leq \delta, \dots, \|\epsilon_N\|_{\Sigma_\nu^{-1}}^2 \leq \delta$ if and only if $\xi_t \in \Xi_t$.*

The proof of Lemma 3 is found in the appendix. The set Ξ_t is nonempty for all $t = 0, \dots, N$ - if the θ_t component of

ξ_t is equal to 0, then $\xi_t \in \Xi_t$. This is because, if $\theta_t = 0$, then, following system (9), the attack sequence $\gamma(t, N) = \{0, \dots, 0\}$ is one such that $\|\epsilon_t\|_{\Sigma_t^{-1}}^2 = \dots = \|\epsilon_N\|_{\Sigma_N^{-1}}^2 = 0$. Recall that system (9) has initial state $\theta_0 = 0$, so we have $\xi_0 \in \Xi_0$. This means that the optimization problem (18) is feasible for any nonnegative value of δ , i.e., the attacker can always satisfy the detection constraint by choosing not to attack the system.

IV. ATTACKS UNDER FALSE ALARM CONSTRAINTS

In this section, we find an attack sequence $\gamma(0, N)$ that minimizes the cost function J under the constraint that $\|\epsilon_t\|_{\Sigma_t^{-1}}^2 = 0$, corresponding to finding the optimal attack under the restriction that the probability of being detected is equal to the false alarm probability of the detector. For the case of $\delta = 0$, we can relate the sets Ξ_t to the output minimization problem presented in Lemma 2 and [23]. Define

$$\mathcal{G} = \begin{bmatrix} 0 & I_{3n} & 0 & 0 \end{bmatrix}. \quad (20)$$

The matrix \mathcal{G} selects the variable θ_t from ξ_t (i.e., $\mathcal{G}\xi_t = \theta_t$).

Lemma 4. *For $\delta = 0$ and for $t = 0, \dots, N$, the set Ξ_t is the null space of $\hat{\mathbf{P}}_{N-t+1}\mathcal{G}$. That is, $\Xi_t = \mathcal{N}(\hat{\mathbf{P}}_{N-t+1}\mathcal{G})$.*

The proof of Lemma 4 is found in the appendix.

The following theorem gives the optimal sequence of attacks when $\delta = 0$.

Theorem 1 (Optimal Attack Strategy with $\delta = 0$ Detection Constraint). *An attack sequence $\gamma(0, N)$ that solves (18) with $\delta = 0$ is*

$$e_t = -\mathcal{F}_t (\mathcal{F}_t^T \mathcal{B}^T \mathbf{Q}_{t+1} \mathcal{B} \mathcal{F}_t)^\dagger \mathcal{F}_t^T \mathcal{B}^T \times \mathbf{Q}_{t+1} (\mathcal{A} - \mathcal{B} \mathcal{D}_t^\dagger \mathcal{C}_t) \xi_t - \mathcal{D}_t^\dagger \mathcal{C}_t \xi_t, \quad (21)$$

where

$$\mathcal{C}_N = \tilde{\mathcal{C}}, \quad \mathcal{D}_N = \tilde{\mathcal{D}}, \quad \mathcal{F}_N = I_s - \mathcal{D}_N^\dagger \mathcal{D}_N, \quad (22)$$

and, for $t = 0, \dots, N-1$,

$$\mathcal{C}_t = \begin{bmatrix} \hat{\mathbf{P}}_{N-t} \mathcal{G} \mathcal{A} \\ \tilde{\mathcal{C}} \end{bmatrix}, \quad \mathcal{D}_t = \begin{bmatrix} \hat{\mathbf{P}}_{N-t} \hat{\mathcal{B}} \\ \tilde{\mathcal{D}} \end{bmatrix}, \quad (23)$$

$$\mathcal{F}_t = I_s - \mathcal{D}_t^\dagger \mathcal{D}_t. \quad (24)$$

The matrix \mathbf{Q}_t is given recursively backward in time by

$$\begin{aligned} \mathbf{Q}_t &= \mathcal{H}^T \mathcal{Q}_t \mathcal{H} + (\mathcal{A} - \mathcal{B} \mathcal{D}_t^\dagger \mathcal{C}_t)^T \mathbf{Q}_{t+1} (\mathcal{A} - \mathcal{B} \mathcal{D}_t^\dagger \mathcal{C}_t) \\ &\quad - (\mathcal{A} - \mathcal{B} \mathcal{D}_t^\dagger \mathcal{C}_t)^T \mathbf{Q}_{t+1} \mathcal{B} \mathcal{F}_t (\mathcal{F}_t^T \mathcal{B}^T \mathbf{Q}_{t+1} \mathcal{B} \mathcal{F}_t)^\dagger \\ &\quad \times \mathcal{F}_t^T \mathcal{B}^T \mathbf{Q}_{t+1} (\mathcal{A} - \mathcal{B} \mathcal{D}_t^\dagger \mathcal{C}_t), \end{aligned} \quad (25)$$

with terminal condition $\mathbf{Q}_{N+1} = 0$.

Theorem 1 states that the optimal attack under the $\delta = 0$ detection constraint is a linear feedback of the state ξ_t , which is exactly determined by the attacker information \mathcal{I}_t . Equation (21) shows that the optimal attack e_t depends on the matrix \mathcal{F}_t , which in turn depends on the matrix \mathcal{D}_t . If the matrix \mathcal{D}_t has full column rank, then, $\mathcal{F}_t = 0$, since, by

definition, \mathcal{F}_t is the orthogonal projector onto $\mathcal{N}(\mathcal{D}_t)$. If the matrix \mathcal{D}_t has full column rank for all $t = 0, \dots, N-1$, then the optimal attack becomes $\gamma(0, N) = \{0, \dots, 0\}$. This corresponds to the case in which the attacker is not powerful enough, and his or her only option to satisfy the $\delta = 0$ detection constraint is to not attack the system.

Before we prove Theorem 1, we provide intermediate results that show that the optimal attack exists and that the optimal attack sequence e_0, \dots, e_{N-1} is unique (the attack e_N may not be unique). The proofs are found in the appendix.

Lemma 5. *For all $t = 0, \dots, N$, there exists $\mathcal{U}_t \succeq 0$ such that $\mathbf{Q}_t = \mathcal{H}^T \mathcal{Q}_t \mathcal{H} + \mathcal{U}_t$.*

Lemma 6. *For all $t = 0, \dots, N-1$, $\mathcal{R}(\mathcal{F}_t^T \mathcal{B}^T) = \mathcal{R}(\mathcal{F}_t^T \mathcal{B}^T \mathbf{Q}_{t+1} \mathcal{B} \mathcal{F}_t)$.*

Define the set

$$Z_t(\psi) = \{z \in \mathbb{R}^s \mid -\mathcal{F}_t^T \mathcal{B}^T \mathbf{Q}_{t+1} \mathcal{B} \mathcal{F}_t z = \mathcal{F}_t^T \mathcal{B}^T \psi\}. \quad (26)$$

One consequence of Lemma 6 is that $Z_t(\psi)$ is nonempty for all $t = 0, \dots, N-1$ and for all $\psi \in \mathbb{R}^{6n}$.

Lemma 7. *For any $t = 0, \dots, N-1$ and for any $\psi \in \mathbb{R}^{6n}$, if $z_1, z_2 \in Z_t(\psi)$, then $\mathcal{F}_t z_1 = \mathcal{F}_t z_2$.*

Proof (Theorem 1): We resort to dynamic programming to solve (18) with $\delta = 0$. The term $\sum_{t=0}^N \text{trace}(\hat{\mathbf{P}} \mathcal{Q}_t)$ in (18) does not depend on $\gamma(0, N)$. Define the optimal cost-to-go function for information \mathcal{I}_t as follows:

$$J_N^*(\mathcal{I}_N) = \min_{\substack{e_N \\ \text{s.t. } \epsilon_N = 0}} \mathbb{E} \left[\|\mathcal{H} \xi_N\|_{\mathcal{Q}_N}^2 \mid \mathcal{I}_N \right], \quad (27)$$

$$J_t^*(\mathcal{I}_t) = \min_{e_t} \mathbb{E} \left[\|\mathcal{H} \xi_t\|_{\mathcal{Q}_t}^2 + J_{t+1}^*(\mathcal{I}_{t+1}) \mid \mathcal{I}_t \right] \text{ s.t. } \epsilon_t = 0, \mathcal{A} \xi_t + \mathcal{B} e_t \in \Xi_{t+1} \quad (28)$$

Equations (27) and (28) restrict the attack at each time t to be recursively feasible.

We begin with $t = N$. At time N , the attack e_N does not affect the value of $\mathbb{E} \left[\|\mathcal{H} \xi_N\|_{\mathcal{Q}_N}^2 \right]$, so we choose e_N only to satisfy the constraint $\epsilon_N = \mathcal{C}_N \xi_N + \mathcal{D}_N e_N = 0$. Thus, we have $e_N = -\mathcal{D}_N^\dagger \mathcal{C}_N \xi_N$, and $J_N^*(\mathcal{I}_N) = \xi_N^T \mathbf{Q}_N \xi_N + \Pi_N$, where $\mathbf{Q}_N = \mathcal{H}^T \mathcal{Q}_N \mathcal{H}$ and $\Pi_N = 0$. Proceeding to $N-1$, we first reformulate the constraints. Applying Lemma 4, the constraint $\mathcal{A} \xi_{N-1} + \mathcal{B} e_{N-1} \in \Xi_N$ becomes

$$\hat{\mathbf{P}}_1 \mathcal{G} (\mathcal{A} \xi_t + \mathcal{B} e_t) = 0. \quad (29)$$

Combining (29) with the constraint $\epsilon_{N-1} = 0$ and using the fact that $\mathcal{G} \mathcal{B} = \hat{\mathcal{B}}$, we have

$$\mathcal{C}_{N-1} \xi_{N-1} + \mathcal{D}_{N-1} e_{N-1} = 0, \quad (30)$$

where \mathcal{C}_{N-1} and \mathcal{D}_{N-1} are given by (23). To solve (28), we eliminate the constraint in (30) (following [28]) and consider attacks e_{N-1} of the form

$$e_{N-1} = \mathcal{F}_{N-1} z_{N-1} - \mathcal{D}_{N-1}^\dagger \mathcal{C}_{N-1} \xi_{N-1}, \quad (31)$$

where $\mathcal{F}_{N-1} = I_s - \mathcal{D}_{N-1}^\dagger \mathcal{D}_{N-1}$. Equation (31) describes all recursively feasible e_{N-1} since $\mathcal{R}(\mathcal{F}_{N-1}) = \mathcal{N}(\mathcal{D}_{N-1})$.

After eliminating constraints and performing algebraic manipulations, (28) becomes

$$J_{N-1}^*(\mathcal{I}_{N-1}) = \xi_{N-1}^T \mathcal{H}^T \mathbf{Q}_{N-1} \mathcal{H} \xi_{N-1} + \Pi_N + \text{trace}(\Sigma_\nu \mathcal{K}^T \mathbf{Q}_N \mathcal{K}) + \min_{z_{N-1}} \bar{\xi}_{N-1}^T \mathbf{Q}_N \bar{\xi}_{N-1}, \quad (32)$$

where $\bar{\xi}_{N-1} = (\mathcal{A} - \mathcal{B}\mathcal{D}_{N-1}^\dagger \mathcal{C}_{N-1}) \xi_{N-1} + \mathcal{B}\mathcal{F}_{N-1} z_{N-1}$. The optimal z_{N-1} satisfies

$$0 = \mathcal{F}_{N-1}^T \mathcal{B} \mathbf{Q}_N \bar{\xi}_{N-1}. \quad (33)$$

As a consequence of Lemma 6, such a z_{N-1} exists. One particular z_{N-1} that satisfies (33) is

$$z_{N-1} = -(\mathcal{F}_{N-1}^T \mathcal{B}^T \mathbf{Q}_N \mathcal{B} \mathcal{F}_{N-1})^\dagger \mathcal{F}_{N-1}^T \mathcal{B}^T \mathbf{Q}_N \times (\mathcal{A} - \mathcal{B}\mathcal{D}_{N-1}^\dagger \mathcal{C}) \xi_{N-1}. \quad (34)$$

There may be more than one z_{N-1} that satisfies (34). Manipulating (34), we have that z_{N-1} satisfies

$$-(\mathcal{F}_{N-1}^T \mathcal{B}^T \mathbf{Q}_N \mathcal{B} \mathcal{F}_{N-1}) z_{N-1} = \mathcal{F}_{N-1}^T \mathcal{B}^T \psi, \quad (35)$$

with $\psi = \mathbf{Q}_N (\mathcal{A} - \mathcal{B}\mathcal{D}_{N-1}^\dagger \mathcal{C}) \xi_{N-1}$. By definition, all z_{N-1} that satisfy (35) belong to $Z_{N-1}(\psi)$. Then, since $e_{N-1} = \mathcal{F}_{N-1} z_{N-1} - \mathcal{D}_{N-1}^\dagger \mathcal{C}_{N-1} \xi_{N-1}$, we have, from Lemma 7, that the optimal attack e_{N-1} is unique.

Substituting (34) into (32) and performing algebraic manipulations, we have

$$J_{N-1}^*(\mathcal{I}_{N-1}) = \xi_{N-1}^T \mathcal{H}^T \mathbf{Q}_{N-1} \mathcal{H} \xi_{N-1} + \Pi_{N-1}, \quad (36)$$

where

$$\begin{aligned} \mathbf{Q}_{N-1} &= \mathcal{H}^T \mathbf{Q}_{N-1} \mathcal{H} + (\mathcal{A} - \mathcal{B}\mathcal{D}_{N-1}^\dagger \mathcal{C}_{N-1})^T \mathbf{Q}_N \times \\ &(\mathcal{A} - \mathcal{B}\mathcal{D}_{N-1}^\dagger \mathcal{C}_{N-1}) - (\mathcal{A} - \mathcal{B}\mathcal{D}_{N-1}^\dagger \mathcal{C}_{N-1})^T \times \\ &\mathbf{Q}_N \mathcal{B} \mathcal{F}_{N-1} (\mathcal{F}_{N-1}^T \mathcal{B}^T \mathbf{Q}_N \mathcal{B} \mathcal{F}_{N-1})^\dagger \mathcal{F}_{N-1}^T \mathcal{B}^T \mathbf{Q}_N \times \\ &(\mathcal{A} - \mathcal{B}\mathcal{D}_{N-1}^\dagger \mathcal{C}_{N-1}), \end{aligned} \quad (37)$$

and $\Pi_{N-1} = \Pi_N + \text{trace}(\Sigma_\nu \mathcal{K}^T \mathbf{Q}_N \mathcal{K})$. Repeating the dynamic programming procedure for $t = N-2, \dots, 0$, we find that the optimal attack has the same form as (34), where we replace $N-1$ with t . ■

V. ATTACKS UNDER GENERAL DETECTION CONSTRAINTS

In this section, we solve (18) with positive δ . We design a procedure to find the sequence $\gamma(0, N)$ that minimizes J under the constraint $\|\epsilon_t\|_{\Sigma_\nu^{-1}}^2 \leq \delta$ for $t = 0, \dots, N$ (the optimal attack does not have a closed form). This procedure becomes computationally intensive for large N . Thus, we also design a less computationally-intensive procedure that finds a sub-optimal and feasible attack sequence.

A. Optimal Attack with $\delta > 0$

For this section only, we introduce the following notation: let $\mathbb{E}_{\xi_t}[\cdot]$ denote the expectation taken over ξ_t , and let $\mathbb{E}_{\{\xi_k\}_{t+1}^N}[\cdot]$ denote the expectation taken over ξ_t, \dots, ξ_N . Further, define the operator π_t as $\pi_t(\gamma(t, N)) = e_t$. That is, π_t is an operator that takes an attack sequence over $N-t+1$ time steps and returns the first attack. To solve (18) with $\delta > 0$, we consider, for $t = 0, \dots, N-1$, the problem

$$\begin{aligned} \gamma_t^*(t, N) = & \underset{\gamma_t(t, N)}{\text{argmin}} \mathbb{E}_{\{\xi_k\}_{t+1}^N} \left[\sum_{k=t}^N \|\mathcal{H}\xi_k\|_{Q_k}^2 \right] \\ \text{s.t.} \quad & \|\epsilon_k\|_{\Sigma_\nu^{-1}}^2 \leq \delta, k = t, \dots, N \end{aligned} \quad (38)$$

where $\gamma_t(t, N) = \{e_t, \dots, e_N\}$ is an attack sequence in which each attack e_t, \dots, e_N only depends on ξ_t ⁵. This differs from the definition of $\gamma(t, N)$, in which each attack e_t, \dots, e_N depends on ξ_t, \dots, ξ_N , respectively. Problem (38) has a convex objective and convex constraints, so it can be efficiently solved.

Theorem 2 (Optimal Attack Strategy with $\delta > 0$ Detection Constraint). *Algorithm 1 gives an attack sequence $\gamma(0, N)$ that solves (18) with $\delta > 0$.*

Algorithm 1 Optimal Attack with $\delta > 0$

- 1: **Initialize:** $\mathcal{I}_0 \leftarrow \{\tilde{y}_0\}$
 - 2: **for** $t = 0, 1, \dots, N-1$ **do**
 - 3: Solve (38), $e_t \leftarrow \pi_t(\gamma_t^*(t, N))$, $\mathcal{I}_{t+1} \leftarrow \{\mathcal{I}_t, \tilde{y}_{t+1}, e_t\}$
 - 4: **end for**
 - 5: $e_N \leftarrow \gamma_{N-1}^*(N, N)$
-

Algorithm 1 works as follows. At time step t , for $t = 0, 1, \dots, N-1$, we find $\gamma_t^*(t, N)$, the sequence of attacks depending only on ξ_t that solves problem (38). The attack e_t is then set as the first attack in the sequence $\gamma_t^*(t, N)$. In the last $((N+1)^{\text{th}})$ time step, the attack e_N is set as the last attack component of the sequence $\gamma_{N-1}^*(N-1, N)$. By construction, every attack e_t produced by Algorithm 1 is recursively feasible: after attacking the system with e_t , the subsequence $\gamma_t^*(t+1, N)$ is a feasible attack sequence at time $t+1$. In order to prove Theorem 2, we require the following Lemma from [25]:

Lemma 8 ([25]). *Let $g(\xi, u)$ be a function such that, for any ξ , $\min_{u \in \mathcal{U}} g(\xi, u)$ exists and \mathcal{U} is a class of functions for which $\mathbb{E}_\xi[g(\xi, u)]$ exists. Then, $\min_{u(\xi) \in \mathcal{U}} \mathbb{E}_\xi[g(\xi, u(\xi))] = \mathbb{E}_\xi[\min_{u \in \mathcal{U}} g(\xi, u)]$.*

Proof (Theorem 2): From problem (18), we have that the optimal cost-to-go function at time t , $J_t^*(\xi_t)$, is defined as

$$J_N^*(\xi_N) = \min_{e_N} \|\mathcal{H}\xi_N\|_{Q_N}^2, \quad \text{s.t.} \quad \|\epsilon_N\|_{\Sigma_\nu^{-1}}^2 \leq \delta, \quad (39)$$

$$J_t^*(\xi_t) = \min_{e_t} \mathbb{E}_{\xi_{t+1}} [\|\mathcal{H}\xi_{t+1}\|_{Q_{t+1}}^2 + J_{t+1}^*(\xi_{t+1}) | \xi_t], \quad \text{s.t.} \quad \|\epsilon_t\|_{\Sigma_\nu^{-1}}^2 \leq \delta, \mathcal{A}\xi_t + \mathcal{B}e_t \in \Xi_{t+1} \quad (40)$$

⁵The state ξ_t is a sufficient statistic for the information set \mathcal{I}_t .

Let $\tilde{\gamma}(0, N) = \{\tilde{e}_0, \dots, \tilde{e}_N\}$ denote the attack sequence produced by Algorithm 1. The attack sequence has the form

$$\tilde{\gamma}(0, N) = \{\pi_0(\gamma_0^*(0, N)), \dots, \pi_N(\gamma_N^*(N, N))\}. \quad (41)$$

To show that $\tilde{\gamma}(0, N)$ is an optimal attack sequence, we show that each attack $\tilde{e}_t = \pi_t(\gamma_t^*(t, N))$ is the optimal attack at time t , for $t = 0, \dots, N-1$ ⁶.

In order to show that \tilde{e}_t is the optimal attack, we prove the intermediate result that, for $t = 0, \dots, N-1$,

$$J_t^*(\xi_t) = \min_{\gamma_{t+1}(t+1, N)} \mathbb{E}_{\{\xi_k\}_{t+1}^N} \left[\sum_{k=t}^N \|\mathcal{H}\xi_k\|_{Q_k}^2 \right] \quad \text{s.t.} \quad \|\epsilon_k\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta, k = t, \dots, N \quad (42)$$

We resort to induction. In the base case, we show that (42) is true for $t = N-1$. ($J_N^*(\xi_N) = \|\mathcal{H}\xi_N\|_{Q_N}^2$.) Consider the right hand side of (42) for $t = N-1$. Expressed in terms of ξ_{N-1} , (42) becomes

$$\min_{\gamma_{N-1}(N-1, N)} \mathbb{E}_{\xi_N} \left[\sum_{k=N-1}^N \|\mathcal{H}\xi_k\|_{Q_k}^2 \right] \quad \text{s.t.} \quad \|\epsilon_k\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta, k = N-1, N \quad (43)$$

$$= \min_{\gamma_{N-1}(N-1, N-1)} \left[\|\mathcal{H}\xi_{N-1}\|_{Q_{N-1}}^2 + \mathbb{E}_{\xi_N} [\|\mathcal{H}\xi_N\|_{Q_N}^2] \right] \quad \text{s.t.} \quad \|\epsilon_{N-1}\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta, \xi_N \in \Xi_N$$

$$= \min_{e_{N-1}} \left[\|\mathcal{H}\xi_{N-1}\|_{Q_{N-1}}^2 + \mathbb{E}_{\xi_N} [J_N(\xi_N) | \xi_{N-1}] \right] \quad \text{s.t.} \quad \|\epsilon_{N-1}\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta, \quad \mathcal{A}\xi_{N-1} + \mathcal{B}e_{N-1} \in \Xi_N \quad (44)$$

$$= J_{N-1}^*(\xi_{N-1}). \quad (45)$$

Equation (43) follows from the right hand side of (42) because the term $\|\mathcal{H}\xi_{N-1}\|_{Q_{N-1}}^2$ is not affected by $\gamma_{N-1}(N-1, N)$ and because the term $\gamma_{N-1}(N, N)$ does not affect the objective but only needs to satisfy the constraint $\|\epsilon_N\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta$. Equation (44) follows from (43) because the minimization in (43) is over $\gamma_{N-1}(N-1, N-1)$, which, by definition, is a function of ξ_{N-1} . Thus, the expectation over ξ_N in (43) refers to the conditional expectation over ξ_N given ξ_{N-1} .

In the induction step, we assume that (42) is true for $t+1$ and show that it is true for t . The right hand side of (42) for t becomes,

$$\min_{e_t} \left[\|\mathcal{H}\xi_t\|_{Q_t}^2 + \tilde{J}_t(\xi_t) \right] \quad \text{s.t.} \quad \|\epsilon_t\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta, \mathcal{A}\xi_t + \mathcal{B}e_t \in \Xi_{t+1} \quad (46)$$

where $\tilde{J}_t(\xi_t)$ is defined as

$$\min_{\gamma_{t+1}(t+1, N)} \mathbb{E}_{\{\xi_k\}_{t+1}^N} \left[\sum_{k=t+1}^N \|\mathcal{H}\xi_k\|_{Q_k}^2 \right] \quad \text{s.t.} \quad \|\epsilon_k\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta, k = t+1, \dots, N \quad (47)$$

The expression in (46) follows from the right hand side of (42) because $\gamma_t(t, N)$ can be partitioned as $\{e_t, \gamma_t(t+1, N)\}$, ξ_t does not depend on $\gamma_t(t+1, N)$, and a feasible $\gamma_t(t+1, N)$ in (47) exists if and only if e_t is recursively feasible.

⁶We ignore the attack \tilde{e}_N because it does not affect the cost associated with $\tilde{\gamma}(0, N)$.

Further manipulating (47), we have

$$\tilde{J}_t(\xi_t) = \mathbb{E}_{\xi_{t+1}} \left[\hat{J}_{t+1}(\xi_{t+1}) | \xi_t \right], \quad (48)$$

where $\hat{J}_{t+1}(\xi_{t+1})$ is defined as

$$\min_{\gamma_{t+1}(t+1, N)} \mathbb{E}_{\{\xi_k\}_{t+2}^N} \left[\sum_{k=t+1}^N \|\mathcal{H}\xi_k\|_{Q_k}^2 \right] \quad \text{s.t.} \quad \|\epsilon_k\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta, k = t+1, \dots, N \quad (49)$$

Equation (48) follows from (47) because the minimization in (47) is over $\gamma_t(t+1, N)$, which, by definition, is a function of ξ_t , so the expectation over ξ_{t+1}, \dots, ξ_N refers to the conditional expectation given ξ_t . We also use Lemma 8 to exchange the minimization operation and expectation over ξ_{t+1} in (47) to derive (48). By the induction hypothesis, we have $\hat{J}_{t+1}(\xi_{t+1}) = J_{t+1}^*(\xi_{t+1})$. Substituting back into (48) and (46) shows that (42) is true for t .

To conclude the proof of Theorem 2, we note that, as a result of (42), for $t = 0, \dots, N-1$,

$$\tilde{e}_t = \underset{e_t}{\operatorname{argmin}} \mathbb{E}_{\xi_{t+1}} [J_{t+1}^*(\xi_{t+1}) | \xi_t] \quad \text{s.t.} \quad \|\epsilon_t\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta, \mathcal{A}\xi_t + \mathcal{B}e_t \in \Xi_{t+1} \quad (50)$$

Because \tilde{e}_t is optimal for $t = 0, \dots, t = N-1$ and e_N does not affect the cost, $\tilde{\gamma}(0, N)$ is an optimal attack sequence. ■

B. Windowed Attack Algorithm with $\delta > 0$

Although Algorithm 1 gives the optimal attack sequence with $\delta > 0$, it is also computationally expensive. In the first time step ($t = 0$), in order to find the optimal attack, Algorithm 1 aims to find an optimal N -length attack sequence subject to N constraints. In each subsequent time step, the length of the attack sequence over which the optimization occurs and the number of constraints in the optimization only decreases by 1. Even though (38) is a convex optimization, if N is large, then, in order to find the optimal attack, Algorithm 1 must repeatedly solve large (convex) optimization problems, each with a large number of constraints.

To find an attack sequence in a less computationally-intensive manner, we consider the windowed attacker optimization problem:

$$\begin{aligned} \hat{\gamma}_t(t, t+W-1) = \\ \underset{\gamma_t(t, t+W-1)}{\operatorname{argmin}} \quad \mathbb{E}_{\{\xi_k\}_{t+1}^{t+W-1}} \left[\sum_{k=t}^{t+W-1} \|\mathcal{H}\xi_k\|_{Q_k}^2 \right] \\ \text{s.t.} \quad \|\epsilon_k\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta, k = t, \dots, t+W-1, \\ \left\| \hat{\mathbf{P}}_{N-(t+W-1)} \mathcal{G}\xi_{t+W} \right\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta, \end{aligned} \quad (51)$$

where $W \in \{2, \dots, N+1\}$ is a predetermined window size chosen by the attacker⁷. The goal of problem (51) is to find a W -length attack sequence that minimizes the attacker's cost over W time steps.

We use problem (51) to find a suboptimal attack sequence in less computationally intensive manner than Algorithm 1.

⁷We do not consider the case of $W = 1$ since, for any time t , the current attack e_t does not affect the value of ξ_t .

Algorithm 2 Windowed Attack with $\delta > 0$

```

1: Initialize:  $\mathcal{I}_0 \leftarrow \{\tilde{y}_0\}$ 
2: for  $t = 0, 1, \dots, N - W$  do
3:   Solve (51),  $e_t \leftarrow \pi_{N-W+1}(\hat{\gamma}(t, t + W - 1))$ ,
    $\mathcal{I}_{t+1} \leftarrow \{\mathcal{I}_t, \tilde{y}_{t+1}, e_t\}$ 
4: end for
5: for  $t = N - W + 1, \dots, N$  do
6:   Solve (38),  $e_t \leftarrow \pi_t(\gamma_t^*(t, N))$ ,  $\mathcal{I}_{t+1} \leftarrow \{\mathcal{I}_t, \tilde{y}_{t+1}, e_t\}$ 
7: end for
8:  $e_N \leftarrow \gamma_{N-1}^*(N, N)$ 

```

Algorithm 2 works as follows. At each time step t , for $t = 0, \dots, N - W$, we find a W -length attack sequence (that depends only on ξ_t) that minimizes the attacker's cost over W time steps. We set the attack e_t to be the first component of the attack sequence $\hat{\gamma}_t(t, t + W - 1)$. At each time step t , for $t = N - W + 1, \dots, N$, we determine the attack e_t in the same way as in Algorithm 1.

The additional constraint $\|\hat{\mathbf{P}}_{N-(t+W-1)} \mathcal{G} \xi_{t+W}\|_{\Sigma_v^{-1}}^2 \leq \delta$ ensures that the attack e_t , as determined by Algorithm 2, is recursively feasible. The attack e_t is the first component of a sequence $\hat{\gamma}_t(t, t + W - 1)$ that satisfies all constraints in (51). The additional constraint requires that, for the state ξ_{t+W} , which depends on $\hat{\gamma}_t(t, t + W - 1)$, there exists an attack sequence $\hat{\gamma}_t(t + W, N)$ such that $\sum_{k=t+W}^N \|\epsilon_k\|_{\Sigma_v^{-1}}^2 \leq \delta$. Since the sum of $\|\epsilon_k\|_{\Sigma_v^{-1}}^2$ from $k = t + W$ to $k = N$ is no greater than δ , we have $\|\epsilon_k\|_{\Sigma_v^{-1}}^2 \leq \delta$ for all $k = t + W, \dots, N$. Thus, the attack sequence $\{\hat{\gamma}_t(t, t + W - 1), \hat{\gamma}_t(t + W, N)\}$ satisfies $\|\epsilon_k\|_{\Sigma_v^{-1}}^2 \leq \delta$ for all $k = t, \dots, N$, so the attack e_t must be recursively feasible.

In each time step of Algorithm 2, we minimize cost over an attack sequence of length no greater than W , and subject to no more than $W + 1$ constraints. Thus, even if N is large, an attacker can use Algorithm 2 to find a feasible attack sequence without the computational expense of Algorithm 1 by choosing W to be small. One drawback of a small window size W is that the resulting attack sequence incurs a larger cost J . We verify the trade off between window size and optimality gap via numerical simulation in Section VI.

VI. NUMERICAL EXAMPLES

We demonstrate the proposed attack strategies under detection constraints with separate examples for the $\delta = 0$ and $\delta > 0$ cases. We consider the linearized state space model of a helicopter provided by [29]. The model is comprised of 10 states and 4 actuators. Due to space constraints, we refer the reader to [29] for a detailed explanation of the model states and the numerical values of the A and B matrices. The helicopter has sensors for each of the state variables ($C = I_{10}$). In our examples, we consider the following statistical properties: $\bar{x}_{t_0} = 0$, $\Sigma_x = 5I_{10}$, $\Sigma_v = 10^{-3}I_{10}$, $\Sigma_w = 10^{-4}\text{diag}(6, .1, 2, 2, .1, 6, 2, 2, 2, .1)$.

For all numerical examples, the attacker has target state

$$x^* = [0 \ 4 \ 0 \ 0 \ 0 \ 8.2 \ 0 \ 0 \ 0 \ 0]^T.$$

The attacker has the following Q_t matrix for all t :

$$Q_t = \text{diag}(.1, \ 3, \ .1, \ .1, \ .1, \ 4, \ .1, \ .1, \ .1, \ .1),$$

which means that the attacker only cares about manipulating the $x_{(2)}$ and $x_{(6)}$ components of the helicopter's state, corresponding to vertical and lateral velocity, respectively. The system starts running at $t = -75$, and the attacker attacks the system from $t = 0$ to $t = 75$ (i.e., $N = 75$).

First, we consider an attacker, denoted as "A1", that can attack all of the actuators and eight of the sensors – the attacker cannot alter the sensors measuring $x_{(8)}$ (yaw rate) and $x_{(9)}$ (roll angle). Figure 1 shows the effect of the optimal attack (A1) with the $\delta = 0$ constraint, and Figure 2 provides a component-wise description of the optimal attack over time.

From time $t = 0$ to $t = 75$, the attack computed using

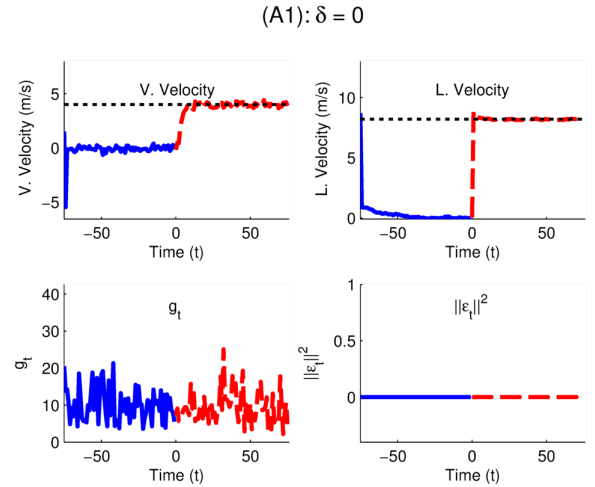


Fig. 1: Effect of the optimal attack (A1) under $\delta = 0$ constraint. Top: system states versus time. The black dotted line is the target state. Bottom: detection statistic versus time

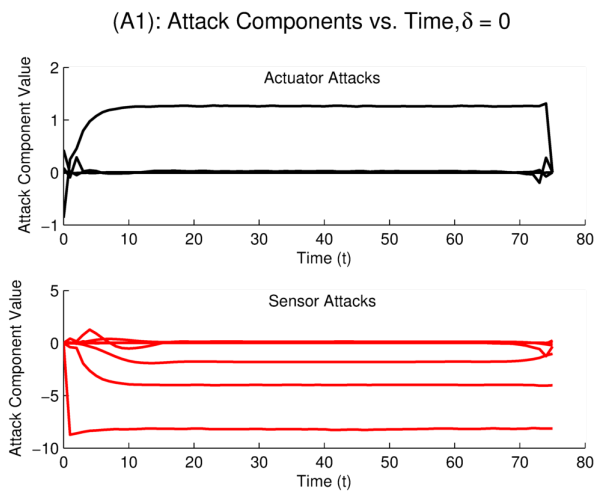


Fig. 2: Component-wise description of the optimal attack (A1) under the $\delta = 0$ constraint.

Theorem 1 moves the system to the target state while satisfying the $\|\epsilon_t\|_{\Sigma_v^{-1}}^2 = 0$ constraint.

Second, we consider an attacker, denoted as "A2", that can attack inputs $u_{(3)}$ and $u_{(4)}$ and manipulate the sensor values

measuring $x_{(2)}, x_{(6)}, x_{(7)}$, and $x_{(10)}$. Figure 3 shows that, for

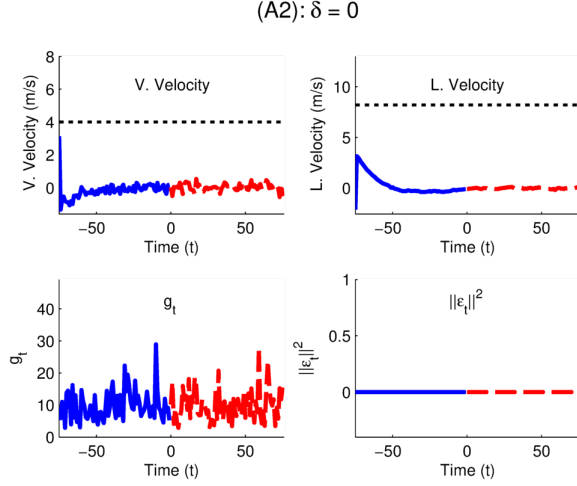


Fig. 3: Effect of the optimal attack (A2) under $\delta = 0$ constraint. Top: system states versus time. The black dotted line is the target state. Bottom: detection statistic versus time

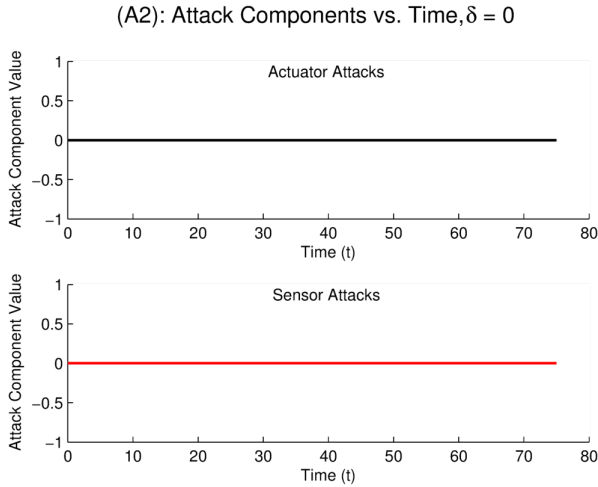


Fig. 4: Component-wise description of the optimal attack (A2) under the $\delta = 0$ constraint. The only feasible attack for (A2) is to not attack the system.

the attacker (A2), the optimal attack with $\delta = 0$ constraint does not successfully move the system to the target state. This is because the attacker is not powerful enough and cannot attack enough sensors and actuators. Indeed, as Figure 4 shows, the optimal strategy for (A2) under the $\delta = 0$ constraint is to not attack the system.

For demonstrating the attack strategies with $\delta > 0$, we consider the attacker (A2). In the implementation of Algorithms 1 and 2, we solve the optimization problems (38) and (51) using MOSEK [30]. First, we consider the optimal attack from Algorithm 1 under the constraint $\delta = 1$. Figure 5 shows that, by following Algorithm 1, the attacker (A2) is able to move the system to the target state while satisfying the $\|\epsilon_t\|_{\Sigma_\nu}^2 \leq 1$ constraint. Figure 6 provides a component-wise description of the optimal attack under the inequality constraint.

We then consider the attacker (A2) using the windowed attack algorithm (Algorithm 2) with window size $W = 5$.

(A2, Opt): $\delta > 0$

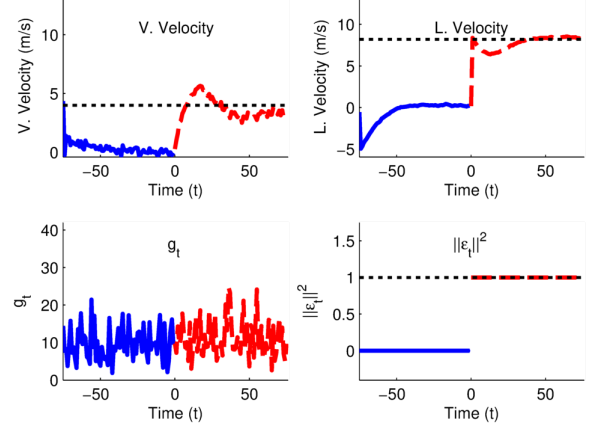


Fig. 5: Effect of the optimal attack (A2) under $\delta > 0$ constraint. Top: system states versus time. The black dotted line is the target state. Bottom: detection statistic versus time. The black dotted line is δ .

(A2, Opt): Attack Components vs. Time, $\delta > 0$

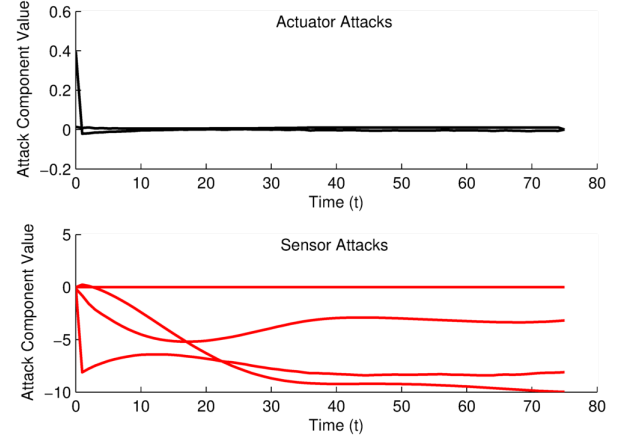


Fig. 6: Component-wise description of the optimal attack (A2) under the $\delta > 0$ constraint.

Figure 7 shows that, like the optimal attack, the suboptimal attack computed using Algorithm 2 also successfully brings the system state to the target state while satisfying the $\|\epsilon_t\|_{\Sigma_\nu}^2 \leq 1$ constraint. Moreover, Figure 8 shows that the windowed attack resembles the optimal inequality constrained attack. Both the optimal and windowed attack satisfy the $\|\epsilon_t\|_{\Sigma_\nu}^2 \leq 1$ constraint with inequality, which shows that the optimal cost is achieved at the boundary of the constraint set. The optimal attack strategy induces as large a bias ϵ_t as allowed by the explicit detection avoidance constraints.

The performance of Algorithm 2 depends on the window size W . We evaluate the cost attained by Algorithm 2 (by attacker (A2)) as a function of the window size W and the value of the constraint bound δ . For each W and each value of δ , we compute the optimality gap of Algorithm 2 as the average cost of 10000 simulations. The optimal cost is the one obtained by Algorithm 1, which we compute as the average of 10000 simulations. In general, for a fixed target state x^* , the optimality gap between the optimal attack and windowed

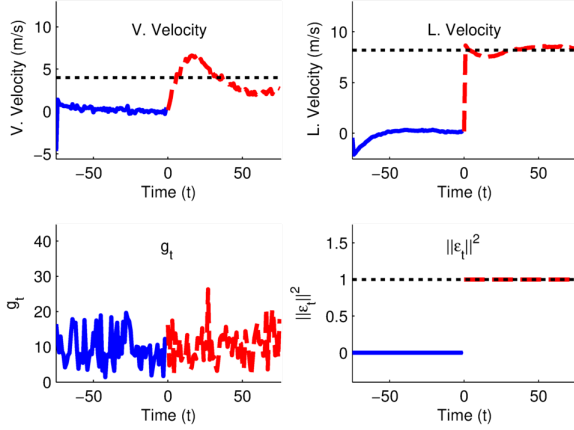
(A2, Win): $\delta > 0$ 

Fig. 7: Effect of the windowed attack (A2) under $\delta > 0$ constraint. Top: system states versus time. The black dotted line is the target state. Bottom: detection statistic versus time. The black dotted line is δ .

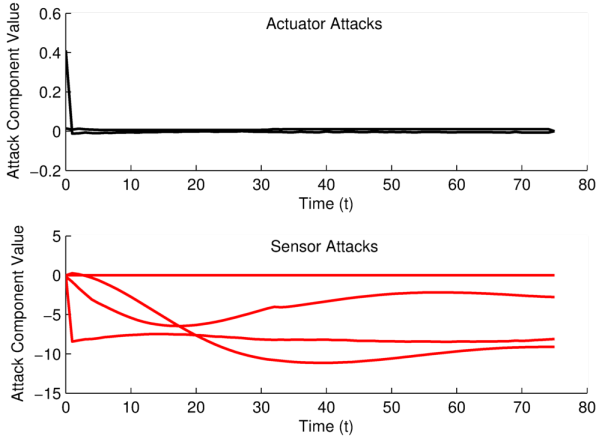
(A2, Win): Attack Components vs. Time, $\delta > 0$ 

Fig. 8: Component-wise description of the windowed attack (A2) under the $\delta > 0$ constraint.

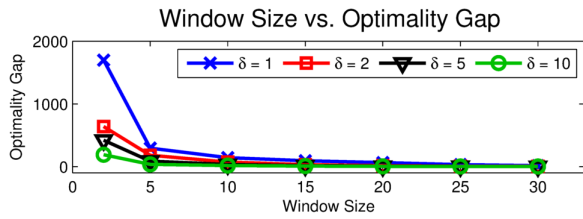


Fig. 9: Performance of Algorithm 2 as a function of window size.

attack depends on the window size W and the bound δ . Figure 9 shows that, as window size W increases, the cost achieved by Algorithm 2 approaches the optimal cost. For the smallest window size $W = 2$, which corresponds to the greedy attack strategy of minimizing the one-step-ahead cost under recursive feasibility constraints, tighter detection avoidance constraints (i.e., lower values of δ) incur a noticeably larger optimality gap. One possible explanation is that, for tighter constraints, there is a limited set of feasible attacks at any time step, and the greedy strategy further limits the set of

feasible attacks at future time-steps, resulting in a larger overall optimality gap. That is, the impact of the greedy strategy on the optimality gap is greater for tighter constraints.

VII. CONCLUSION

In this paper we studied attackers with control objectives and detection constraints against CPS. We formulated a cost function that captures the attacker's control objectives and defined constraints that relate to the probability of the attack being detected. In the case that the attacker's probability of being detected is constrained to be the false alarm rate of the detector, we showed that the optimal attack strategy is a linear feedback of an augmented system state calculated from the attacker's information set. Under more general constraints to the attacker's effect on the CPS's detection statistic, we provided an algorithm to find the optimal attack sequence and a second, less computationally intensive, algorithm to find a feasible, sub-optimal attack. Finally, we illustrated our attack strategies through numerical examples involving a remotely-controlled helicopter under attack.

APPENDIX

A. Proof of Lemma 3

Proof: We resort to induction. The base case of $t = N$ is true by the definition of Ξ_N . In the induction step, we assume there exists a sequence of attacks $\gamma(t+1, N)$ such that $\|\epsilon_{t+1}\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta, \dots, \|\epsilon_N\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta$ if and only if $\xi_{t+1} \in \Xi_{t+1}$, and we show that there exists a recursively feasible e_t if and only if $\xi_t \in \Xi_t$.

(If) Let $\xi_t \in \Xi_t$. Then there exists e_t such that $\|\tilde{C}\xi_t + \tilde{D}e_t\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta$ and $\xi_{t+1} = \mathcal{A}\xi_t + \mathcal{B}e_t + \mathcal{K}\nu_{t+1} \in \Xi_{t+1}$. Since $\xi_{t+1} \in \Xi_{t+1}$, by the induction hypothesis, there exists $\gamma(t+1, N)$ such that $\|\epsilon_{t+1}\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta, \dots, \|\epsilon_N\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta$. Concatenating e_t and $\gamma(t+1, N)$, we have that $\gamma(t, N) = \{e_t, \gamma(t+1, N)\}$ is an attack such that $\|\epsilon_t\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta, \dots, \|\epsilon_N\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta$, which means that e_t is recursively feasible.

(Only If) Let e_t be a recursively feasible attack. Then, there exists $\gamma(t, N) = \{e_t, \gamma(t+1, N)\}$ such that $\|\epsilon_t\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta, \dots, \|\epsilon_N\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta$. Since the subsequence $\gamma(t+1, N)$ satisfies $\|\epsilon_{t+1}\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta, \dots, \|\epsilon_N\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta$, we have, by the induction hypothesis, that $\xi_{t+1} = \mathcal{A}\xi_t + \mathcal{B}e_t + \mathcal{K}\nu_{t+1} \in \Xi_{t+1}$. Since $\|\epsilon_t\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta$, we have $\|\tilde{C}\xi_t + \tilde{D}e_t\|_{\Sigma_{\nu}^{-1}}^2 \leq \delta$. This means that $\xi_t \in \Xi_t$. ■

B. Proof of Lemma 4

Proof: We resort to induction. In the base case, we show that $\Xi_N = \mathcal{N}(\hat{\mathbf{P}}_1 \mathcal{G})$. By definition of Ξ_N , we have $\xi_N \in \Xi_N$ if and only if there exists e_N such that $\|\tilde{C}\xi_N + \tilde{D}e_N\|_{\Sigma_{\nu}^{-1}}^2 = 0$. Applying the result of Lemma 2 and noting that $\mathcal{G}\xi_N = \theta_N$, we have that $\|\tilde{C}\mathcal{G}\xi_N + \tilde{D}e_N\|_{\Sigma_{\nu}^{-1}}^2 = 0$ if and only if $\hat{\mathbf{P}}_1 \mathcal{G}\xi_N = 0$. Thus, we have $\xi_N \in \Xi_N$ if and only if $\hat{\mathbf{P}}_1 \mathcal{G}\xi_N = 0$, which shows that $\Xi_N = \mathcal{N}(\hat{\mathbf{P}}_1 \mathcal{G})$.

In the induction step, we assume that $\Xi_{t+1} = \mathcal{N}(\hat{\mathbf{P}}_{N-t}\mathcal{G})$, and we show that $\Xi_t = \mathcal{N}(\hat{\mathbf{P}}_{N-t+1}\mathcal{G})$. By definition of Ξ_t , we have $\xi_t \in \Xi_t$ if and only if there exists e_t such that $\|\tilde{\mathcal{C}}\xi_t + \tilde{\mathcal{D}}e_t\|_{\Sigma_t^{-1}}^2 = 0$ and $\xi_{t+1} = \mathcal{A}\xi_t + \mathcal{B}e_t + \mathcal{K}\nu_{t+1} \in \Xi_{t+1}$. By the induction hypothesis $\Xi_{t+1} = \mathcal{N}(\hat{\mathbf{P}}_{N-t}\mathcal{G})$. Thus, we have $\xi_{t+1} \in \Xi_{t+1}$ if and only if $\hat{\mathbf{P}}_{N-t}\mathcal{G}\xi_{t+1} = 0$. Applying the results of Lemma 3, we then have that $\xi_{t+1} \in \Xi_{t+1}$ if and only if there exists an attack sequence $\gamma(t+1, N)$ such that, starting from state ξ_{t+1} , we have $\|\epsilon_{t+1}\|_{\Sigma_{t+1}^{-1}}^2 = \dots = \|\epsilon_N\|_{\Sigma_N^{-1}}^2 = 0$. Concatenating e_t and $\gamma(t+1, N)$, we have that $\gamma(t, N) = \{e_t, \gamma(t+1, N)\}$ is an attack sequence such that, starting from state ξ_t , we have $\|\epsilon_t\|_{\Sigma_t^{-1}}^2 = \dots = \|\epsilon_N\|_{\Sigma_N^{-1}}^2 = 0$. Since there exists such an attack sequence $\gamma(t, N)$, we have $\min_{\gamma(t, N)} \sum_{k=t}^N \|\epsilon_k\|_{\Sigma_k^{-1}}^2 = 0$. Applying the results of Lemma 2, we have $\xi_t \in \Xi_t$ if and only if $\hat{\mathbf{P}}_{N-t+1}\mathcal{G}\xi_t = 0$, which shows that $\Xi_t = \mathcal{N}(\hat{\mathbf{P}}_{N-t+1}\mathcal{G})$. ■

C. Proof of Lemma 5

Proof: We resort to induction. In the base case $t = N$, we have $\mathbf{Q}_N = \mathcal{H}^T Q_N \mathcal{H}$ by definition. In the induction step, we assume that $\mathbf{Q}_{t+1} = \mathcal{H}^T Q_{t+1} \mathcal{H} + \mathcal{U}_{t+1}$ for some $\mathcal{U}_{t+1} \succeq 0$, and we show that there exists $\mathcal{U}_t \succeq 0$ such that $\mathbf{Q}_t = \mathcal{H}^T Q_t \mathcal{H} + \mathcal{U}_t$. From the induction hypothesis, we have that $\mathbf{Q}_{t+1} \succeq 0$ since $Q_{t+1} \succ 0$ and $\mathcal{U}_{t+1} \succeq 0$. Then, by algebraic manipulation of equation (25), we have

$$\mathbf{Q}_t = \mathcal{H}^T Q_t \mathcal{H} + \mathcal{X}_t^T \mathbf{Q}_{t+1} \mathcal{X}_t, \quad (52)$$

where $\mathcal{X}_t = \left(\mathcal{A} - \mathcal{B}\mathcal{D}_t^\dagger \mathcal{C}_t \right) - \mathcal{B}\mathcal{F}_t \left(\mathcal{F}_t^T \mathcal{B}^T \mathbf{Q}_{t+1} \mathcal{B}\mathcal{F}_t \right)^\dagger \times \mathcal{F}_t^T \mathcal{B}^T \mathbf{Q}_{t+1} \left(\mathcal{A} - \mathcal{B}\mathcal{D}_t^\dagger \mathcal{C}_t \right)$. Thus, $\mathcal{U}_t = \mathcal{X}_t^T \mathbf{Q}_{t+1} \mathcal{X}_t \succeq 0$, since $\mathbf{Q}_{t+1} \succeq 0$. ■

D. Proof of Lemma 6

Proof: Trivially, we have $\mathcal{R}(\mathcal{F}_t^T \mathcal{B}^T \mathbf{Q}_{t+1} \mathcal{B}\mathcal{F}_t) \subseteq \mathcal{R}(\mathcal{F}_t^T \mathcal{B}^T)$. We now show that $\mathcal{R}(\mathcal{F}_t^T \mathcal{B}^T) \subseteq \mathcal{R}(\mathcal{F}_t^T \mathcal{B}^T \mathbf{Q}_{t+1} \mathcal{B}\mathcal{F}_t)$ by showing that $\mathcal{N}(\mathcal{F}_t^T \mathcal{B}^T \mathbf{Q}_{t+1} \mathcal{B}\mathcal{F}_t) \subseteq \mathcal{N}(\mathcal{B}\mathcal{F}_t)$.

Let $\mu \in \mathcal{N}(\mathcal{F}_t^T \mathcal{B}^T \mathbf{Q}_{t+1} \mathcal{B}\mathcal{F}_t)$, and, by contradiction, suppose that $\mu \notin \mathcal{N}(\mathcal{B}\mathcal{F}_t)$. Then, we have $\mathcal{F}_t \mu \neq 0$. By definition of \mathcal{F}_t , we have $\mathcal{D}_t \mathcal{F}_t \mu = 0$, which means that $\Psi \mathcal{F}_t \mu = 0$. Since the matrix $\begin{bmatrix} \Gamma \\ \Psi \end{bmatrix}$ is injective, $\Psi \mathcal{F}_t \mu = 0$ means that $\tilde{\mu} = \Gamma \mathcal{F}_t \mu \neq 0$. Using the results of Lemma 5 and the structure of \mathcal{H} and \mathcal{B} , we have

$$\mu^T \mathcal{F}_t^T \mathcal{B}^T \mathbf{Q}_{t+1} \mathcal{B}\mathcal{F}_t \mu = \mu^T \mathcal{F}_t^T \mathcal{B}^T \mathcal{U}_{t+1} \mathcal{B}\mathcal{F}_t \mu + \tilde{\mu}^T Q_{t+1} \tilde{\mu}. \quad (53)$$

The first term on the right hand side of (53) is nonnegative since $\mathcal{U}_{t+1} \succeq 0$, and the second term on the right hand side of (53) is positive since $\tilde{\mu} \neq 0$ and $Q_{t+1} \succ 0$. Thus, we have $\mu^T \mathcal{F}_t^T \mathcal{B}^T \mathbf{Q}_{t+1} \mathcal{B}\mathcal{F}_t \mu > 0$, which contradicts the fact that $\mu \in \mathcal{N}(\mathcal{F}_t^T \mathcal{B}^T \mathbf{Q}_{t+1} \mathcal{B}\mathcal{F}_t)$. Thus, we have $\mu \in \mathcal{N}(\mathcal{B}\mathcal{F}_t)$ and $\mathcal{N}(\mathcal{F}_t^T \mathcal{B}^T \mathbf{Q}_{t+1} \mathcal{B}\mathcal{F}_t) \subseteq \mathcal{N}(\mathcal{B}\mathcal{F}_t)$. This means that $\mathcal{R}(\mathcal{F}_t^T \mathcal{B}^T) \subseteq \mathcal{R}(\mathcal{F}_t^T \mathcal{B}^T \mathbf{Q}_{t+1} \mathcal{B}\mathcal{F}_t)$. ■

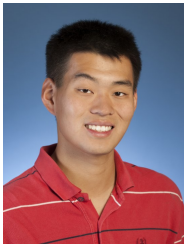
E. Proof of Lemma 7

Proof: Let $z_1, z_2 \in Z_t(\psi)$, and suppose, by contradiction, that $\mathcal{F}_t z_1 \neq \mathcal{F}_t z_2$. Let $\mu = z_1 - z_2$, which means $\mathcal{F}_t \mu \neq 0$. Since $z_1, z_2 \in Z_t(\psi)$, we have $\mathcal{F}_t^T \mathcal{B}^T \mathbf{Q}_{t+1} \mathcal{B}\mathcal{F}_t \mu = 0$. Following the proof of Lemma 6, we have that if $\mathcal{F}_t \mu \neq 0$, then $\mu^T \mathcal{F}_t^T \mathcal{B}^T \mathbf{Q}_{t+1} \mathcal{B}\mathcal{F}_t \mu > 0$, which contradicts the fact that $\mathcal{F}_t^T \mathcal{B}^T \mathbf{Q}_{t+1} \mathcal{B}\mathcal{F}_t \mu = 0$. Thus, we have $\mathcal{F}_t z_1 = \mathcal{F}_t z_2$. ■

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