

Totally Asynchronous Primal-Dual Convex Optimization in Blocks

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Abstract

We present a parallelized primal-dual algorithm for solving constrained convex optimization problems. The algorithm is “block-based,” in that vectors of primal and dual variables are partitioned into blocks, each of which is updated only by a single processor. We consider four possible forms of asynchrony: in updates to primal variables, updates to dual variables, communications of primal variables, and communications of dual variables. We show that any amount of asynchrony in the communications of dual variables can preclude convergence, though the other forms of asynchrony are permitted. A first-order primal-dual update law is then developed and shown to be robust to these other forms of asynchrony. We next derive convergence rates to a Lagrangian saddle point in terms of the operations agents execute, without specifying any timing or pattern with which they must be executed. These convergence rates include an “asynchrony penalty” that we quantify and present ways to mitigate. Numerical results illustrate these developments.

I. INTRODUCTION

A wide variety of machine learning problems can be formalized as convex programs [5], [8], [33], [34]. Large-scale machine learning then requires solutions to large-scale convex programs, which can be accelerated through parallelized solvers running on networks of processors. In large networks, it can be difficult to synchronize their computations, which generate new information, and communications, which share this new information with other processors. Accordingly, we are interested in asynchrony-tolerant large-scale optimization.

The challenge of asynchrony is that it causes disagreements among processors that result from generating and receiving different information at different times. One way to reduce disagreements is through repeated averaging of processors’ iterates. This approach dates back several decades [37], and approaches of this class include [13], [23], [26]–[28], [35], [36], [40]. However, these averaging-based methods require bounded delays in some form, often through requiring connectedness of agents’ communication graphs over intervals of a prescribed length [6, Chapter 7]. In some applications, delays are outside agents’ control, e.g., in a contested environment where communications are jammed, and delay bounds cannot be easily enforced. Moreover, graph connectivity cannot be easily checked individual agents, meaning even satisfaction or violation of connectivity bounds is not readily ascertained. In

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addition, these methods require multiple processors to update each decision variable, which can be prohibitive, e.g., in learning problems with billions of data points.

Therefore, in this paper we develop a totally asynchronous parallelized primal-dual method for solving large constrained convex optimization problems. The term “totally asynchronous” dates back to [6] and describes scenarios in which both computations and communications are executed without any assumptions on delay bounds. By “parallelized,” we mean that each decision variable is updated only by a single processor. As problems grow, this has the advantage of keeping each processor’s computational burden approximately constant. The decision variables assigned to each processor are referred to as a “block,” and asynchronous block-based algorithms date back several decades as well [4], [6], [37]. Those early works solve unconstrained or set-constrained problems, in addition to select problems with functional constraints. Recent asynchronous block-based algorithms have also been developed for some specific classes of problems with set or functional constraints [10], [11], [25], [31], [32]. To bring parallelization to arbitrary constrained problems, we develop a primal-dual approach that does not require constraints to have a specific form.

Block-based methods have previously been shown to tolerate arbitrarily long delays in both communications and computations in some unconstrained problems [4], [22], [38], eliminating the need to enforce and verify delay boundedness assumptions. For constrained problems of a general form, block-based methods have been paired with primal-dual algorithms with centralized dual updates [17], [19] and/or synchronous primal updates [24]. To the best of our knowledge, arbitrarily asynchronous block-based updates have not been developed for convex programs of a general form. A counterexample in [19] showed that arbitrarily asynchronous communications of dual variables can preclude convergence, though that example leaves open the extent to which more limited dual asynchrony is compatible with convergence.

In this paper, we present a primal-dual optimization algorithm that permits arbitrary asynchrony in primal variables, while accommodating dual asynchrony to the extent possible. Four types of asynchrony are possible: (i) asynchrony in primal computations, (ii) asynchrony in communicating primal variables, (iii) asynchrony in dual computations, and (iv) asynchrony in communicating dual variables. The first contribution of this paper is to show that item (iv) is fundamentally problematic. Specifically, we show that arbitrarily small disagreements among dual variables can cause primal computations to disagree by arbitrarily large amounts. For this reason, we rule out asynchrony in communicating dual variables. However, we permit all other forms of asynchrony, and, relative to existing work, this is the first to permit arbitrarily asynchronous computations of dual variables in blocks.

The second contribution of this paper is to establish convergence rates. These rates are shown to depend upon problem parameters, which lets us calibrate their values to improve convergence. Moreover, we show that convergence can be inexact due to dual asynchrony, and thus the scalability of parallelization comes at the expense of a potentially inexact solution. We term this inexactness the “asynchrony penalty,” and we give an explicit bound on it, as well as methods to mitigate it. Simulation results show convergence of this algorithm and illustrate that the asynchrony penalty is mild.

This paper is an extension of the conference paper [20]. This paper extends all previous results on scalar blocks to non-scalar blocks, provides bounds on regularization error, provides techniques to mitigate the asynchrony penalty,

and gives a simplified convergence analysis.

The rest of the paper is organized as follows. Section II provides background and a formal problem statement. Section III presents our asynchronous algorithm. Convergence rates are developed in Section IV. Section V presents simulation results, and Section VI concludes.

II. BACKGROUND AND PROBLEM STATEMENT

Real-world applications of multi-agent optimization may face challenges that prevent agents from computing or communicating at specified times or intervals. For example, very large networks of processors may face difficulty in synchronizing all of their clocks, and networks of autonomous agents in a contested environment may face jammed communications that make information sharing sporadic. This asynchrony in computations and communications motivates the development of algorithms that tolerate as much asynchrony as possible. Thus, we study the following form of optimization problem.

Problem 1: Given $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$, and $X \subset \mathbb{R}^n$, asynchronously solve

$$\begin{aligned} & \text{minimize } f(x) \\ & \text{subject to } g(x) \leq 0 \\ & x \in X. \end{aligned} \quad \diamond$$

We assume the following about f .

Assumption 1: The objective function f is twice continuously differentiable and convex. \triangle

We make a similar assumption about the constraints g .

Assumption 2: For all $j \in \{1, \dots, m\}$, the function g_j is twice continuously differentiable and convex. And g satisfies Slater's condition, i.e., there exists $\bar{x} \in X$ such that $g(\bar{x}) < 0$. \triangle

Assumptions 1 and 2 permit a wide range of functions to be used, such as all convex polynomials of all orders. We impose the following assumption on the constraint set.

Assumption 3: The set X is non-empty, compact, and convex. It can be decomposed via $X = X_1 \times \dots \times X_{N_p}$ where N_p is the number of agents optimizing over x . \triangle

Assumption 3 permits many sets to be used, such as box constraints, which often arise in multi-agent optimization [29]. This assumption allows agents to project their blocks of the decision variable onto the corresponding part of the constraint set, i.e., for all i , agent i is able to project its updates onto X_i , which ensures that $x \in X$ overall. This property enables a distributed projected update law in which each agent ensures set constraint satisfaction of its block of the decision variable. This form of decomposition has been used in [10], [11], [24], [25], [32], [40] (and other works) for the same purpose.

We will solve Problem 1 using a primal-dual approach. This allows the problem to be parallelized across many processors by re-encoding constraints through Karush-Kuhn-Tucker (KKT) multipliers. In particular, because the constraints g couple the agents' computations, they can be difficult to enforce in a distributed way. By introducing KKT multipliers to encode constraints, we can solve an equivalent, higher-dimensional unconstrained problem.

An ordinary primal-dual approach would find a saddle point of the Lagrangian associated with Problem 1, defined as $L(x, \mu) = f(x) + \mu^T g(x)$, where $\mu \geq 0$. That is, one would solve $\min_{x \in X} \max_{\mu \geq 0} L(x, \mu)$, and, under Assumptions 1-3, this would furnish a solution to Problem 1. However, L is affine in μ , which implies that $L(x, \cdot)$ is concave but not strongly concave. Strong convexity has been shown to provide robustness to asynchrony in minimization problems [6], and thus we wish to endow the maximization over μ with strong concavity. We use a Tikhonov regularization [14] in μ to form

$$L_\delta(x, \mu) = f(x) + \mu^T g(x) - \frac{\delta}{2} \|\mu\|^2, \quad (1)$$

where $\delta > 0$ and $\|\cdot\|$ denotes the Euclidean norm. This ensures δ -strong concavity in μ . Thus, we will find a saddle point $(\hat{x}_\delta, \hat{\mu}_\delta)$ of L_δ , which is approximately equal to a saddle point of L and thus approximately solves Problem 1. We bound the error introduced by regularizing in Theorem 1 below.

One challenge in designing and analyzing an algorithm is that $\hat{\mu}_\delta$ is contained in the unbounded domain \mathbb{R}_+^m , which is the non-negative orthant of \mathbb{R}^m . Because this domain is unbounded, gradients with respect to the dual variable are unbounded. Specifically, dual iterates may not be within a bounded distance of the optimum and hence they may produce gradients that are arbitrarily large. To remedy this problem, we will confine dual variables to a set $\mathcal{M} \subset \mathbb{R}_+^m$, defined as

$$\mathcal{M} := \{\mu \in \mathbb{R}_+^m : \|\mu\|_1 \leq B\}, \quad B := \frac{f(\bar{x}) - f^*}{\min_{1 \leq j \leq m} -g_j(\bar{x})}, \quad (2)$$

where \bar{x} is any Slater point. Here, f^* denotes the optimal objective function value over X (but without g), though any lower-bound for this value will suffice. For example, if f is non-negative, then one can substitute 0 in place of f^* . We will show below in Lemma 1 that using \mathcal{M} does not affect the final answer that is computed.

Instead of regularizing with respect to the primal variable x , we impose the following assumption in terms of the Hessian $H(x, \mu) := \nabla_x^2 L_\delta(x, \mu)$. When convenient, we suppress the arguments x and μ and simply write H .

Assumption 4 (Diagonal Dominance): The Hessian matrix $H = \nabla_x^2 L_\delta(x, \mu)$ is β -diagonally dominant for all $\mu \in \mathcal{M}$. That is, $|H_{ii}| - \beta \geq \sum_{j=1, j \neq i}^n |H_{ij}|, \forall i = 1, \dots, n$. \triangle

It has been observed in the literature that Assumption 4 or a similar variant of diagonal dominance is necessary to ensure the convergence of totally asynchronous algorithms [6, Section 6.3.2]. If this assumption does not hold, the Lagrangian can be regularized with respect to x to help provide H 's diagonal dominance. If this is done, there may be cases in which the regularization parameter required to satisfy Assumption 4 is large, and this can introduce large regularization errors, which can be undesirable; see [24] for bounds on regularization error when both primal and dual regularizations are used. Fortunately, numerous problems satisfy this assumption without regularizing in x [16], and, for such problems, we proceed without regularizing in x to avoid unnecessarily introducing regularization error. Diagonal dominance has been shown to arise in sum of squares problems [1], linear systems with sparse graphs [9], matrix scaling and balancing [12], and quadratic programs [38]. We show in Section V that diagonal dominance improves convergence of our algorithm, and this is in line with existing algorithms [1], [2], [9], [12], [15], [39].

Using the definition of \mathcal{M} in (2) and Assumption 4, we now observe that \mathcal{M} contains the optimum $\hat{\mu}_\delta$.

Lemma 1: Let Assumptions 1-4 hold. Then $\hat{\mu}_\delta \in \mathcal{M}$.

Proof: Follows Section II-C in [18]. ■

We now present the following saddle point problem that will be the focus of the rest of the paper.

Problem 2: Let Assumptions 1-4 hold and fix $\delta > 0$. For L_δ defined in (1), asynchronously compute

$$(\hat{x}_\delta, \hat{\mu}_\delta) := \arg \min_{x \in X} \arg \max_{\mu \in \mathcal{M}} L_\delta(x, \mu). \quad \diamond$$

The strong convexity of $L_\delta(\cdot, \mu)$ and strong concavity of $L_\delta(x, \cdot)$ imply that $(\hat{x}_\delta, \hat{\mu}_\delta)$ is unique. Due to regularizing, the solution to Problem 2 may not equal that of Problem 1, and regularization could also introduce constraint violations. We next bound both regularization error in solutions and constraint violations in terms of the regularization parameter δ .

Theorem 1: Let Assumptions 1-4 hold. Let $(\hat{x}, \hat{\mu})$ denote a saddle point of L (without regularization applied). Then the regularization error introduced by the Tikhonov regularization in (1) is bounded by $\|\hat{x}_\delta - \hat{x}\|^2 \leq \frac{\delta}{\beta} B^2$, where δ is the regularization parameter and B is defined in Lemma 1. Furthermore, possible constraint violations are bounded via $g_j(\hat{x}_\delta) \leq M_j B \sqrt{\frac{\delta}{\beta}}$, where $M_j := \max_{x \in X} \|\nabla g_j(x)\|$.

Proof: See Appendix A. ■

The error in solutions is $O(\delta)$ and the potential constraint violation is $O(\sqrt{\delta})$, and thus both can be made arbitrarily small. Moreover, if it is essential that a feasible point be computed, then, for all j , one can replace the constraint $g_j(x) \leq 0$ with $\tilde{g}_j(x) = g_j(x) - M_j B \sqrt{\frac{\delta}{\beta}} \leq 0$, which will ensure the generation of a feasible point. And Slater's condition in Assumption 2 implies that, for sufficiently small δ , there exist points that satisfy \tilde{g}_j .

III. ASYNCHRONOUS PRIMAL-DUAL ALGORITHM

Solving Problem 2 asynchronously requires an update law that we expect to be robust to asynchrony and simple to implement in a distributed way. In this context, first-order gradient-based methods offer some degree of inherent robustness, as well as computations that are simpler than other methods, such as Newton-type methods. We apply a projected gradient method to both the primal and dual variables, based on the seminal Uzawa algorithm [3]. Recall that, given some $x(0)$ and $\mu(0)$, at iteration $k + 1$ the Uzawa algorithm computes the primal update, $x(k + 1)$, and dual update, $\mu(k + 1)$, using

$$x(k + 1) = \Pi_X[x(k) - \gamma \nabla_x L_\delta(x(k), \mu(k))] \quad (3)$$

$$\mu(k + 1) = \Pi_{\mathcal{M}}[\mu(k) + \rho \nabla_\mu L_\delta(x(k), \mu(k))], \quad (4)$$

where $\gamma, \rho > 0$ are stepsizes, Π_X is the Euclidean projection onto X , and $\Pi_{\mathcal{M}}$ is the Euclidean projection onto \mathcal{M} .

A. Overview of Approach

The Uzawa algorithm is centralized, and we will decentralize (3) and (4) among a number of agents while allowing them to generate and share information as asynchronously as possible. We consider N agents indexed over $i \in \mathcal{I} := \{1, \dots, N\}$. We also define the sets $\mathcal{I}_p := \{1, \dots, N_p\}$ and $\mathcal{I}_d := \{1, \dots, N_d\}$, where $N_p + N_d = N$. The set \mathcal{I}_p contains indices of ‘‘primal agents’’ that update primal blocks (contained in x), while \mathcal{I}_d contains indices

of “dual agents” that update dual blocks (contained in μ)¹. Thus, $x \in \mathbb{R}^n$ is divided into N_p blocks and $\mu \in \mathbb{R}^m$ into N_d blocks.

Primal agent i updates the i^{th} primal block, $x_{[i]}$, and dual agent c updates the c^{th} dual block, $\mu_{[c]}$. Let n_i denote the length of primal agent i 's block and m_c the length of dual agent c 's block. Then $n = \sum_{i=1}^{N_p} n_i$ and $m = \sum_{c=1}^{N_d} m_c$. The block of constraints g that correspond to $\mu_{[c]}$ is denoted by $g_{[c]} : \mathbb{R}^n \rightarrow \mathbb{R}^{m_c}$ and each dual agent projects its computations onto a set \mathcal{M}_c derived from Lemma 1, namely $\mathcal{M}_c = \{\nu \in \mathbb{R}_+^{m_c} : \|\nu\|_1 \leq B\}$.

Using a primal-dual approach, there are four behaviors that could be asynchronous: (i) computations of primal variables, (ii) communications of the values of primal variables, (iii) computations of dual variables, and (iv) communications of the values of dual variables. In all cases, we assume that communications arrive in finite time and are received in the order they were sent. We examine these four behaviors here:

(i) *Computations of Updates to Primal Variables:* When parallelizing (3) across the N_p primal agents, we index all primal agents' computations using the same iteration counter, $k \in \mathbb{N}$. However, they may compute and communicate at different times and they do not necessarily do either at all k . The subset of times at which primal agent $i \in \mathcal{I}_p$ computes an update is denoted by $K^i \subset \mathbb{N}$. For distinct $i, j \in \mathcal{I}_p$, we allow $K^i \neq K^j$. These sets are used only for analysis and need not be known to agents.

(ii) *Communications of Primal Variables:* Primal variable communications are also totally asynchronous. A primal block's current value may or may not be sent to other primal and dual agents that need it at each time k . Thus, one agent may have onboard an old value of a primal block computed by another agent. We use $\mathcal{N}_i \subset \mathcal{I}_p$ to denote the set of indices of primal agents whose decision variables are needed for agent i 's computations. Formally, $j \in \mathcal{N}_i$ if and only if $\nabla_{x_i} L_\delta(x, \mu)$ explicitly depends on $x_{[j]}$. The set \mathcal{N}_i is referred to as agent i 's *essential neighbors*, and only agent i 's essential neighbors need to communicate to agent i . In particular, primal communications are not all-to-all. We use $\tau_j^i(k)$ to denote the time at which primal agent j originally computed the value of $x_{[j]}$ stored onboard primal agent i at time k . We use $\sigma_j^c(k)$ to denote the time at which primal agent j originally computed the value of $x_{[j]}$ stored onboard dual agent c at time k . These functions are used only for analysis, i.e., agents do not need to know the values of $\tau_j^i(k)$ or $\sigma_j^c(k)$.

We impose the following assumption.

Assumption 5 (Primal Updates and Communications): For all $i \in \mathcal{I}_p$, the set K^i is infinite. If $\{k_n\}_{n \in \mathbb{N}}$ is an increasing set of times in K^i , then $\lim_{n \rightarrow \infty} \tau_i^j(k_n) = \infty$ for all $j \in \mathcal{I}_p$ such that $i \in \mathcal{N}_j$ and $\lim_{n \rightarrow \infty} \sigma_i^c(k_n) = \infty$ for all $c \in \mathcal{I}_d$ such that $x_{[i]}$ is constrained by $g_{[c]}$. \triangle

This simply ensures that, for all $i \in \mathcal{I}_p$, primal agent i never stops computing or communicating, though delays can be arbitrarily large.

(iii) *Computations of Updates to Dual Variables:* Dual agents wait for every primal agent's updated block before computing an update to a dual variable. Dual agents may perform computations at different times because

¹Although the same index may be contained in both \mathcal{I}_p and \mathcal{I}_d , we define the sets in this way to avoid non-consecutive numbering of primal agents and dual agents, which would be cumbersome in the forthcoming analysis. The meaning of an index will always be made unambiguous by specifying whether it is contained in \mathcal{I}_p or \mathcal{I}_d .

they may receive updates to primal blocks at different times. In some cases, a dual agent may receive multiple updates from a subset of primal agents prior to receiving all required primal updates. In this case, only the most recently received update from a primal agent will be used in the dual agent's computation. For all $c \in \mathcal{I}_d$, dual agent c keeps an iteration count t_c to track the number of updates it has completed.

(iv) *Communications of Updated Dual Variables:* Previous work [19, Section VI] has shown that allowing primal agents to disagree arbitrarily about dual variables can preclude convergence. In particular, that work provides an example problem in which such disagreements lead to oscillations in the primal variables that do not decay with time, and thus agents do not even converge. This is explained by the following: fix $\mu^1, \mu^2 \in \mathcal{M}$. Then a primal agent with μ^1 onboard is minimizing $L_\delta(\cdot, \mu^1)$, while a primal agent with μ^2 onboard is minimizing $L_\delta(\cdot, \mu^2)$. If μ^1 and μ^2 can be arbitrarily far apart, then it is not surprising that the minima of $L(\cdot, \mu^1)$ and $L(\cdot, \mu^2)$ are arbitrarily far apart, which is what is observed in [19, Section VI]. One may then conjecture that small disagreements in dual variables lead to small distances between these minima. We next show that this is false.

Theorem 2: Fix any $\epsilon > 0$ and any $L > \epsilon$. Then, under Assumptions 1-4, there exists a problem and points $\mu^1, \mu^2 \in \mathcal{M}$ such that $\|\mu^1 - \mu^2\| < \epsilon$ and $\|\hat{x}_1 - \hat{x}_2\| > L$, where $\hat{x}_1 = \arg \min_{x \in X} L_\delta(x, \mu^1)$ and $\hat{x}_2 = \arg \min_{x \in X} L_\delta(x, \mu^2)$.

Proof: See Theorem 1 in the preliminary version of this work [21]. ■

The counterexample in [19, Section VI] shows that primal agents need not even converge if they use different dual variables that take them to minima that are far apart. Theorem 2 shows that arbitrarily small disagreements in the values of dual variables can drive primal agents' computations to points that are arbitrarily far apart. Thus, by combining these two results, we see that *any* disagreement in the dual variables can preclude convergence. Therefore, primal agents are allowed to operate totally asynchronously, but their computations must use the same dual variable, formalized as follows.

Assumption 6 (Dual Communications): Any transmission sent from primal agent i to primal agent j while they both have $\mu(t)$ onboard is only used by primal agent j in its own computations if it is received before the next dual update. △

However, we emphasize that this *does not* mean that the values of dual variables must be communicated synchronously. Instead, primal agents can use any method to ensure that their computations use the same dual variables. For example, when dual agent c sends its updated dual block to primal agents, it can also send its iteration count t_c . Primal agents can use these t_c values to annotate which version of μ is used in their updates, e.g., by appending the value of t_c for each c to the end of the vector of primal variables they communicate. To ensure that further primal updates rely upon the same dual value, other primal agents will disregard any received primal updates that are annotated with an old iteration count for any block of the dual variable.

The algorithm we present is unchanged if any other method is used to ensure that the primal agents use the same dual variable in their computations, e.g., primal agents may send each other acknowledgments of a new dual variable prior to computing new iterations.

We also note that for many problems, dual agents may not be required to communicate with all primal agents. Problems with a constraint function g_c that depends only on a subset of primal variables will result in a dual

entry μ_c that needs to be sent only to the corresponding subset of primal agents, which we illustrate next.

Example 1 (Dual-to-Primal Communications): Consider a problem with any objective function f , $x \in \mathbb{R}^3$, and constraints $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by

$$g_1(x) = x_1 + x_2 - b_1, \quad g_2(x) = x_2 - b_2, \quad g_3(x) = x_3 - b_3,$$

where b_1 , b_2 , and b_3 are some constants. The regularized Lagrangian associated with this problem is

$$L_\delta(x, \mu) = f(x) + \mu_1(x_1 + x_2 - b_1) + \mu_2(x_2 - b_2) + \mu_3(x_3 - b_3) - \frac{\delta}{2} \|\mu\|^2.$$

We observe that, among μ_1 , μ_2 , and μ_3 , $\nabla_{x_1} L_\delta(x, \mu)$ depends only on μ_1 . Similarly, $\nabla_{x_2} L_\delta(x, \mu)$ depends only on μ_1 and μ_2 , and $\nabla_{x_3} L_\delta(x, \mu)$ depends only on μ_3 . Therefore, only primal agents computing x_1 and x_2 would need to receive μ_1 , a primal agent computing x_2 would need μ_2 , and only a primal agent computing x_3 would need μ_3 . For primal blocks that are scalar values, this leads to the required dual-to-primal communications shown in Figure 1.

Furthermore, the primal and dual variables may instead be divided into the blocks shown in Figure 2, leading to an even simpler communication requirement. In that case, only Primal Agent 1 needs updates from Dual Agent 1 and only Primal Agent 2 needs updates from Dual Agent 2.

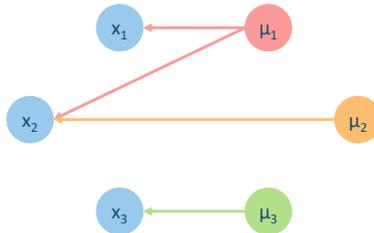


Fig. 1: Required dual-to-primal communications in Example 1 with scalar blocks. This illustrates that some constraint formulations will only require dual agents to communicate to a subset of primal agents.

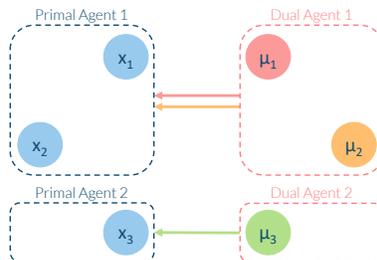


Fig. 2: Required dual-to-primal communications in Example 1 when separating the primal and dual variables into non-scalar blocks. By dividing blocks according to constraints, required dual-primal communications may be reduced even further.

The principle illustrated by this example is that, by using non-scalar blocks and exploiting the structure of a problem, the required dual communications may be significantly reduced. Specifically, a dual block $\mu_{[c]}$ only needs to be sent to the primal agents whose decision variables appear in the block of constraints $g_{[c]}$. This is reflected in Step 7 our statement of Algorithm 1 below.

B. Glossary of Notation in Algorithm 1

The following glossary contains the notation used in our algorithm statement:

- k The iteration count used by all primal agents.
- K^i The set of times at which primal agent i computes updates.
- $\nabla_{x_{[i]}}$ The derivative with respect to the i -th block of x . That is, $\nabla_{x_{[i]}} := \frac{\partial}{\partial x_{[i]}}$.
- \mathcal{N}_i Essential neighborhood of primal agent i .
- \mathcal{I}_d Set containing the indices of all dual agents.
- \mathcal{I}_p Set containing the indices of all primal agents.
- $\sigma_j^c(k)$ Time at which primal agent j originally computed the value of $x_{[j]}$ onboard dual agent c at time k .
- $\tau_j^i(k)$ Time at which primal agent j originally computed the value of $x_{[j]}$ onboard primal agent i at time k . Note that $\tau_i^i(k) = k$ for all $i \in \mathcal{I}_p$.
- t The vector of dual agent iteration counts. The c^{th} entry, t_c , is the iteration count for dual agent c 's updates.
- $x_{[j]}^i$ Primal or dual agent i 's value for the primal block j , which is updated/sent by primal agent j . If agent i is primal, it is indexed by both k and t ; if agent i is dual, it is indexed only by t .
- \hat{x}_δ The primal component of the saddle point of L_δ . Part of the optimal solution pair $(\hat{x}_\delta, \hat{\mu}_\delta)$.
- $\mu_{[d]}^c$ Primal or dual agent c 's copy of dual block d , which is updated/sent by dual agent d .
- $\hat{\mu}_\delta$ The dual component of the saddle point of L_δ , $(\hat{x}_\delta, \hat{\mu}_\delta)$.

C. Statement of Algorithm

We now state the asynchronous primal-dual algorithm.

Algorithm 1:

Step 0: Initialize all primal and dual agents with $x(0) \in X$ and $\mu(0) \in \mathcal{M}$. Set $t = 0 \in \mathbb{R}^{N_d}$ and $k = 0 \in \mathbb{N}$.

Step 1: For all $i \in \mathcal{I}_p$ and all $c \in \mathcal{I}_d$, if primal agent i receives a dual variable update from dual agent c , it sets

$$\mu_{[c]}^i(t_c) = \mu_{[c]}^c(t_c).$$

Step 2: For all $i \in \mathcal{I}_p$, if $k \in K^i$, primal agent i executes

$$x_{[i]}^i(k+1; t) = \Pi_{X_i} [x_{[i]}^i(k; t) - \gamma \nabla_{x_{[i]}} L_\delta(x^i(k; t), \mu^i(t))].$$

If $k \notin K^i$, then $x_{[i]}^i(k+1; t) = x_{[i]}^i(k; t)$.

Step 3: For all $i \in \mathcal{I}_p$ and all $j \in \mathcal{N}_i$,

$$x_{[j]}^i(k+1; t) = \begin{cases} x_{[j]}^j(\tau_j^i(k+1); t) & i \text{ receives } x_{[j]}^j \text{ at } k+1 \\ x_{[j]}^i(k; t) & \text{otherwise} \end{cases}$$

Step 4: For all $i \in \mathcal{I}_p$, primal agent i may send $x_{[i]}^i(k+1; t)$ to any primal or dual agent. Due to communication delays, it may not be received for some time. Set $k := k + 1$.

Step 5: For $c \in \mathcal{I}_d$ and $i \in \mathcal{I}_p$, if dual agent c receives an update from primal agent i computed with dual update t , it sets

$$x_{[i]}^c(t_c) = x_{[i]}^i(\sigma_i^c(k); t).$$

Otherwise, $x_{[i]}^c(t_c)$ remains constant.

Step 6: For $c \in \mathcal{I}_d$, if dual agent c has received an update from every primal agent constrained by $g_{[c]}$ that was computed with the latest dual iteration t , dual agent c executes

$$\mu_{[c]}^c(t_c + 1) = \Pi_{\mathcal{M}_c} [\mu_{[c]}^c(t_c) + \rho \frac{\partial L_\delta}{\partial \mu_{[c]}}(x^c(t_c), \mu^c(t_c))].$$

Step 7: If dual agent c updated in Step 6, it sends $\mu_{[c]}^c(t_c + 1)$ to all primal agents that are constrained by $g_{[c]}$. Due to asynchrony, it may not be received for some time. Set $t_c := t_c + 1$.

Step 8: Return to Step 1.

IV. OVERALL CONVERGENCE AND REDUCING THE ASYNCHRONY PENALTY

In this section, we present our main convergence result and strategies for reducing the asynchrony penalty, which is an error term in that result that is due to asynchronous operations. First, let $H(x, \mu) = \nabla_x^2 L_\delta(x, \mu)$ and choose the primal stepsize $\gamma > 0$ to satisfy

$$\gamma < \frac{1}{\max_i \max_{x \in X} \max_{\mu \in \mathcal{M}} \sum_{j=1}^n |H_{ij}(x, \mu)|}. \quad (5)$$

Recall that x and μ both take values in compact sets (cf. Assumption 3 and Lemma 1), and thus each entry H_{ij} is bounded. In particular, the upper bound on γ is positive.

The main convergence result for Algorithm 1 is in terms of the number of operations that agents have executed, counted in a specific order as follows. Upon the first primal agent's receipt of a dual variable with iteration vector t , we set $\text{ops}(k, t) = 0$. Then, after all primal agents have computed an update to their decision variable with $\mu(t)$ and sent it to and had it received by all other primal agents in their essential neighborhoods, say by time k' , we increment ops to $\text{ops}(k', t) = 1$. After $\text{ops}(k', t) = 1$, we then wait until all primal agents have subsequently computed a new update (still using the same dual variable indexed with t) and it has been sent to and received by all primal agents' essential neighbors. If this occurs at time k'' , then we set $\text{ops}(k'', t) = 2$, and then this process continues. If at some time k''' , primal agents receive an updated μ (whether just a single dual agent sent an update or multiple dual agents send updates) with an iteration vector of t' , then the count would begin again with $\text{ops}(k''', t') = 0$.

A. Main Result

We now present our main result on the convergence of $x^i(k; t)$ to \hat{x}_δ . Recall that δ is the dual regularization parameter, γ is the primal stepsize given in (5), and ρ is the dual stepsize.

Theorem 3: Let Assumptions 1-6 hold and fix $\delta > 0$. Choose $0 < \rho < \frac{2\delta}{\delta^2+2}$. Let $T(t) := \min_c t_c$ be the minimum number of updates any one dual agent has performed by time t and let $K(t)$ be the minimum value of ops that was reached for any primal block used to compute any dual block from $\mu(0)$ to $\mu(t)$. Then for agents executing Algorithm 1, for all i , all k , and all t , we have

$$\|x^i(k; t) - \hat{x}_\delta\|^2 \leq q_p^{2\text{ops}(k,t)} 2nD_x^2 + q_d^{T(t)} \frac{2M^2}{\beta^2} \|\mu(0) - \hat{\mu}_\delta\|^2 + q_p^{2K(t)} C_1 + q_p^{K(t)} C_2 + C_3,$$

where C_1 , C_2 , and C_3 are positive constants given by

$$C_1 := \frac{2nN_dM^4D_x^2(q_d - \rho^2)}{\beta^2(1 - q_d)}, \quad C_2 := \frac{4\rho^2\sqrt{n}N_dM^4D_x^2}{\beta^2(1 - q_d)}, \quad C_3 := \frac{2N_dM^4D_x^2(q_d - \rho^2)}{\beta^2(1 - q_d)},$$

and $q_d := (1 - \rho\delta)^2 + 2\rho^2 \in [0, 1)$, $q_p := (1 - \gamma\beta) \in [0, 1)$, $M := \max_{x \in X} \|\nabla g(x)\|$, $D_x := \max_{x, y \in X} \|x - y\|$, n is the length of the primal variable x , and N_d is the number of dual agents.

Proof: See Appendix B. ■

Remark 1: The term C_3 in Theorem 3 is termed the ‘‘asynchrony penalty’’ because it is an offset from reaching a solution, and it is not reduced by changing the value of ops . It is due to asynchronously computing dual variables and is absent when dual updates are centralized [19], [24]. In Corollary 1 below, we suggest methods to mitigate this penalty.

Note that the terms premultiplied by q_p are minimized when the exponent (determined by primal operations through the ops term) is allowed to grow. In particular, if more primal operations occur before communications are sent to dual agents, the terms are reduced. Similarly, if dual updates occur frequently and asynchronously, the exponent will be reduced and hence the terms containing q_p will become larger.

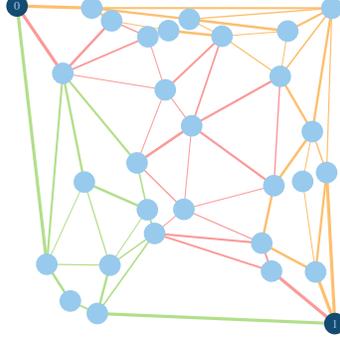


Fig. 3: Network graph where Node 0 is the source and Node 1 is the target and edge thicknesses correspond to their flow capacities. The paths may be divided into three groups (green, pink, and orange) such that there are no shared edges for paths in different groups.

B. Reducing the Asynchrony Penalty

The asynchrony penalty may be addressed in a few ways: by completing a certain number of primal operations prior to sending updates to dual agents, by completing more dual updates, and finally by choosing ρ and δ . These are discussed explicitly in Corollary 1 below.

Corollary 1: Let all conditions and definitions of Theorem 3 hold. Let positive error bounds ϵ_1 and ϵ_2 be fixed. Then there exist values of $T(t)$, $K(t)$, the dual stepsize ρ , and the regularization parameter δ such that

$$\|x^i(k; t) - \hat{x}_\delta\|^2 \leq \epsilon_1 + \epsilon_2.$$

In particular, set $K(t) \geq \frac{\log(\epsilon_1) - \log(4nD_x^2 + 2C_1 + 2C_2)}{\log(q_p)}$, $T(t) \geq \frac{\log(\epsilon_1\beta^2) - \log(4M^2\|\mu(0) - \hat{\mu}_\delta\|^2)}{\log(q_d)}$, $\rho = \frac{\delta}{1 + \delta^2}$, and $\delta^2 \geq \frac{2N_d M^4 D_x^2}{\epsilon_2 \beta^2 (1 - q_d)} - 1$.

Proof: See Appendix C. ■

The lower bound on δ is dependent on the chosen ϵ_2 . If $\epsilon_2 \geq \frac{2N_d M^4 D_x^2}{\beta^2 (1 - q_d)}$, then δ^2 may take any positive value. However, if $\epsilon_2 \leq \frac{2N_d M^4 D_x^2}{\beta^2 (1 - q_d)}$, δ^2 is lower bounded by a positive number. This illustrates the potential trade-off between the asynchrony penalty and regularization error by showing that one cannot have arbitrarily small values of both. However, given a desired value of one, it is possible to compute the other using Theorem 1, and, if needed, make further adjustments to the numbers of updates and choices of parameters to ensure satisfactory performance.

V. NUMERICAL EXAMPLE

We consider a network flow problem where agents are attempting to route data over given paths from the source (node 0) to the target (node 1)². We consider an example whose network graph is given in Figure 3, where the edge widths represent the edge capacities. We consider a problem in which we must route 15 different paths from the source to the target along some combination of the 66 edges present. The primal variable is composed of

²The network graph in Figure 3 was generated using [30]. All other code for this section can be found at www.github.com/kathendrickson/dacoa.

the traffic assigned to each path (with $n = 15$) and the limits for traffic sent along each edge are the constraints (thus, $m = 66$). By construction, the paths and edges can be divided into three groups, with the edges in each group being used only by that group’s paths. Each edge group (indicated by the green, pink, and orange colors in Figure 3) contains five paths, corresponding to five entries of the primal variable. The objective function is

$$f(x) = -W \sum_{i=1}^n \log(1 + x_i),$$

where W is a positive constant. Primal variables x_i are allowed to take any value between 0 and 10. The constraints are given by $Ax \leq b$, where the edge capacities, b , take random values between 5 and 40, with edges connected to the source and target having capacities of 50. The matrix A is given by

$$A_{k,i} = \begin{cases} 1 & \text{if flow path } i \text{ traverses edge } k \\ 0 & \text{otherwise} \end{cases}.$$

Thus the regularized Lagrangian is given by

$$L_\delta(x, \mu) = -W \sum_{i=1}^n \log(1 + x_i) + \mu^T (Ax - b) - \frac{\delta}{2} \|\mu\|^2,$$

where the Hessian matrix H is β -diagonally dominant and $\beta = \frac{W}{11^2}$. We choose algorithm parameters $\gamma = 0.01$, $\delta = 0.1$ and $\rho = \frac{\delta}{\delta^2 + 1} \approx .099$. Communications between agents occur with a random probability called the “communication rate,” which we vary across simulation runs below. We collect each primal agent’s block into the combined primal variable $x = (x_{[1]}^{1T}, \dots, x_{[N_p]}^{N_p T})^T$ to measure convergence.

We use this simulation example to explore the benefit of using non-scalar blocks over our previous work with scalar blocks [20]. Additionally, we examine the effect that the magnitude of diagonal dominance has on convergence. We are also able to vary the communication rate and study its effect on convergence. Finally, this example demonstrates the effectiveness of our algorithm with large-scale problems and the ease with which it is scaled up and distributed.

We begin by comparing scalar blocks to non-scalar blocks where primal agents have a 50% chance of computing an update at every time k . When using scalar blocks, we assign one primal agent to each flow path and one dual agent to each edge constraint. Thus, we have 15 primal agents and 66 dual agents in the scalar block case. For dividing among non-scalar blocks, we assign one primal agent to compute all five flow paths in each network group (indicated by the different colors in Figure 3). We then assign one dual agent to each group to handle all of the edge constraints for that group. Thus, we have 3 primal agents and 3 dual agents in the non-scalar block case. For a communication rate of 0.75 (primal agents have a 75% chance of communicating the latest update to another agent at each time step), non-scalar blocks provide an advantage when considering the number of time steps needed to converge, shown in Figure 4. In both cases, the algorithm converges to \tilde{x} such that $\|\tilde{x} - \hat{x}\| \leq 0.38$, where \hat{x} is the unregularized solution. This result is representative of other simulations by the authors in which the asynchrony penalty is small in practice and the bound provided in Theorem 3 is loose.

Next we use the non-scalar blocks with primal agents performing updates at every time k to isolate the effects of diagonal dominance and communication rate (communications do not necessarily happen at every k). We first vary β over $\beta \in \{0.10, 0.25, 0.75\}$. To measure convergence, we take the 2-norm of the difference between x

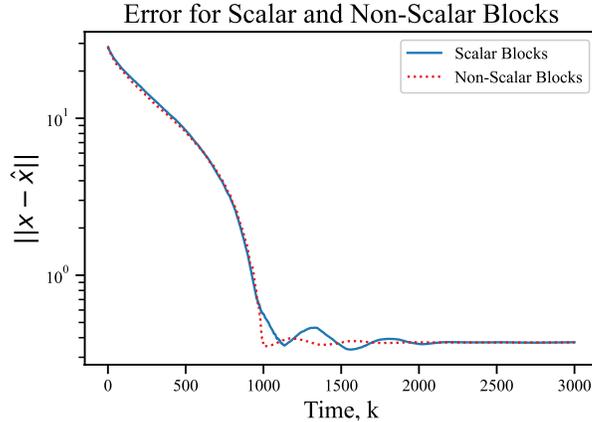


Fig. 4: Convergence for scalar and non-scalar blocks. Non-scalar blocks provide a significant advantage over scalar blocks when considering the number of time steps needed to reach a solution, as indicated by the red dotted line versus the solid blue line.

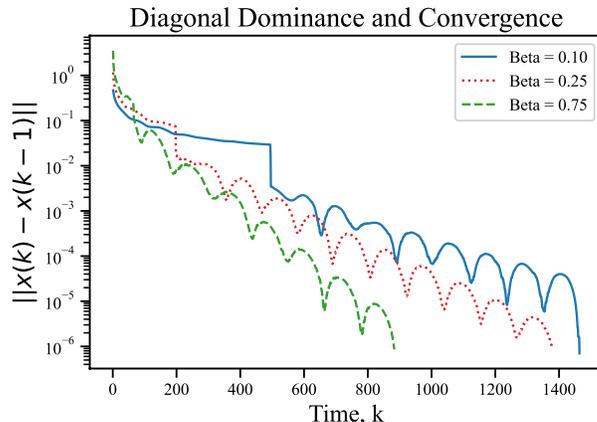


Fig. 5: Effect of diagonal dominance on convergence. Here, we see that larger values of β lead to faster convergence.

at consecutive time steps. Figure 5 plots the time step k versus this distance for the varying values of β and a communication rate of 0.75. As predicted by Theorem 3, a larger β correlates with faster convergence in general.

Varying the communication rate has a significant impact on the number of time steps required to converge as shown in Figure 6. However, a solution is still eventually reached. This reveals that faster convergence can be achieved both by increasing communication rates and increasing the diagonal dominance of the problem. In these two plots, the abrupt decreases in distances between successive iterates are due to primal agents' computations reaching the boundary of the feasible region defined by X and g , which causes the iterates to make only small progress afterwards (as the dual variables continue to slowly change).

VI. CONCLUSION

Algorithm 1 presents a primal-dual approach that is asynchronous in primal updates and communications and

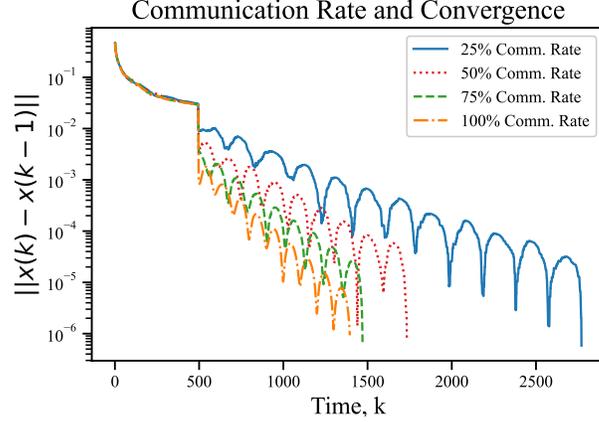


Fig. 6: Effect of communication rate on convergence. Less frequent communication leads to slower convergence.

asynchronous in distributed dual updates. The error due to regularization was bounded and convergence rates were established. Methods for mitigating the resulting asynchrony penalty were presented. A numerical example illustrates the benefit of non-scalar blocks and the effect diagonal dominance has with other parameters upon convergence. Future work will examine additional applications for the algorithm and implementation techniques to reduce error and improve convergence.

APPENDIX

A. Proof of Theorem 1

The following proof generally follows that of Proposition 3.1 in [24], with differences resulting from this work only regularizing with respect to the dual variable rather than regularizing with respect to both the primal and dual variables. Let $(\hat{x}, \hat{\mu})$ denote a saddle point of the unregularized Lagrangian L . Let $(\hat{x}_\delta, \hat{\mu}_\delta)$ denote a saddle point of the dual-regularized Lagrangian L_δ . Then because $(\hat{x}_\delta, \hat{\mu}_\delta)$ is a saddle point, for all $x \in X, \mu \in \mathbb{R}_+^m$ we have the two inequalities $L_\delta(\hat{x}_\delta, \mu) \leq L_\delta(\hat{x}_\delta, \hat{\mu}_\delta) \leq L_\delta(x, \hat{\mu}_\delta)$. Using $\hat{\mu} \in \mathbb{R}_+^m$ and we can write

$$0 \leq L_\delta(\hat{x}_\delta, \hat{\mu}_\delta) - L_\delta(\hat{x}_\delta, \hat{\mu}) = \sum_j (\hat{\mu}_{\delta,j} - \hat{\mu}_j) g_j(\hat{x}_\delta) - \frac{\delta}{2} \|\hat{\mu}_\delta\|^2 + \frac{\delta}{2} \|\hat{\mu}\|^2. \quad (6)$$

Because each g_j is convex, we have

$$g_j(\hat{x}_\delta) \leq g_j(\hat{x}) + \nabla g_j(\hat{x}_\delta)^T (\hat{x}_\delta - \hat{x}) \leq \nabla g_j(\hat{x}_\delta)^T (\hat{x}_\delta - \hat{x}), \quad (7)$$

where the last inequality follows from $g_j(\hat{x}) \leq 0$ (which holds since \hat{x} solves Problem 1). Additionally, because all dual variables are non-negative, we can multiply by $\hat{\mu}_{\delta,j}$ to get

$$\sum_j \hat{\mu}_{\delta,j} g_j(\hat{x}_\delta) \leq \sum_j \hat{\mu}_{\delta,j} \nabla g_j(\hat{x}_\delta)^T (\hat{x}_\delta - \hat{x}). \quad (8)$$

By definition of L_δ , the right-hand side can be expanded as

$$\sum_j \hat{\mu}_{\delta,j} \nabla g_j(\hat{x}_\delta)^T (\hat{x}_\delta - \hat{x}) = \nabla_x L_\delta(\hat{x}_\delta, \hat{\mu}_\delta)^T (\hat{x}_\delta - \hat{x}) - \nabla f(\hat{x}_\delta)^T (\hat{x}_\delta - \hat{x}). \quad (9)$$

Because \hat{x}_δ minimizes $L_\delta(\cdot, \hat{\mu}_\delta)$, for all $x \in X$ we have $\nabla_x L_\delta(\hat{x}_\delta, \hat{\mu}_\delta)^T(\hat{x}_\delta - x) \leq 0$. In particular, we can set $x = \hat{x} \in X$ to find $\nabla_x L_\delta(\hat{x}_\delta, \hat{\mu}_\delta)^T(\hat{x}_\delta - \hat{x}) \leq 0$. Combining this with (9) and (8) gives

$$\sum_j \hat{\mu}_{\delta,j} g_j(\hat{x}_\delta) \leq -\nabla f(\hat{x}_\delta)^T(\hat{x}_\delta - \hat{x}). \quad (10)$$

By the convexity of each g_j , we have $g_j(\hat{x}_\delta) \geq g_j(\hat{x}) + \nabla g_j(\hat{x})^T(\hat{x}_\delta - \hat{x})$. By multiplying this inequality with the non-positive $-\hat{\mu}_j$ and summing over j , we obtain

$$-\sum_j \hat{\mu}_j g_j(\hat{x}_\delta) \leq -\sum_j \hat{\mu}_j g_j(\hat{x}) - \sum_j \hat{\mu}_j \nabla g_j(\hat{x})^T(\hat{x}_\delta - \hat{x}).$$

By complementary slackness, we have $\hat{\mu}^T g(\hat{x}) = 0$ and thus

$$-\sum_j \hat{\mu}_j g_j(\hat{x}_\delta) \leq -\sum_j \hat{\mu}_j \nabla g_j(\hat{x})^T(\hat{x}_\delta - \hat{x}). \quad (11)$$

Expanding $\nabla_x L(\hat{x}, \hat{\mu}) = \nabla f(\hat{x}) + \sum_j \hat{\mu}_j \nabla g_j(\hat{x})$, we see that

$$\sum_j \hat{\mu}_j \nabla g_j(\hat{x})^T(\hat{x} - \hat{x}_\delta) = \nabla_x L(\hat{x}, \hat{\mu})^T(\hat{x} - \hat{x}_\delta) - \nabla f(\hat{x})^T(\hat{x} - \hat{x}_\delta) \leq -\nabla f(\hat{x})^T(\hat{x} - \hat{x}_\delta).$$

This follows from the fact that \hat{x} minimizes $L(\cdot, \hat{\mu})$ over all $x \in X$ and thus $\nabla_x L(\hat{x}, \hat{\mu})^T(\hat{x} - x) \leq 0$ for all $x \in X$. Then setting $x = \hat{x}_\delta$ gives the above bound. Combining this with (11) gives

$$-\sum_j \hat{\mu}_j g_j(\hat{x}_\delta) \leq \nabla f(\hat{x})^T(\hat{x}_\delta - \hat{x}). \quad (12)$$

Adding (10) and (12) gives

$$(\hat{\mu}_\delta - \hat{\mu})^T g(\hat{x}_\delta) \leq (\nabla f(\hat{x}) - \nabla f(\hat{x}_\delta))^T(\hat{x}_\delta - \hat{x}) \leq -\frac{\beta}{2} \|\hat{x}_\delta - \hat{x}\|^2,$$

where the last inequality is from the β -strong convexity of f (follows from Assumption 4 which holds for $\mu = 0$). Applying this to (6),

$$0 \leq -\frac{\beta}{2} \|\hat{x}_\delta - \hat{x}\|^2 - \frac{\delta}{2} \|\hat{\mu}_\delta\|^2 + \frac{\delta}{2} \|\hat{\mu}\|^2.$$

This implies the final result $\|\hat{x}_\delta - \hat{x}\|^2 \leq \frac{\delta}{\beta} (\|\hat{\mu}\|^2 - \|\hat{\mu}_\delta\|^2)$. Furthermore, we can use (7) to bound possible constraint violations, where

$$g_j(\hat{x}_\delta) \leq \nabla g_j(\hat{x}_\delta)^T(\hat{x}_\delta - \hat{x}) \leq \|\nabla g_j(\hat{x}_\delta)\| \|\hat{x}_\delta - \hat{x}\| \leq \max_{x \in X} \|\nabla g_j(x)\| \sqrt{\frac{\delta}{\beta} (\|\hat{\mu}\|^2 - \|\hat{\mu}_\delta\|^2)} \leq M_j B \sqrt{\frac{\delta}{\beta}}. \quad \blacksquare$$

B. Main Result Proofs

1) *Primal Convergence*: Towards defining an overall convergence rate to the regularized optimal solution $(\hat{x}_\delta, \hat{\mu}_\delta)$, we first find the primal convergence rate for a fixed dual variable. Given a fixed $\mu(t)$, centralized projected gradient descent for minimizing $L_\delta(\cdot, \mu(t))$ can be written as

$$h(x) = \Pi_X [x - \gamma \nabla_x L_\delta(x, \mu(t))], \quad (13)$$

where $\gamma > 0$. The fixed point of h is the minimizer of $L_\delta(\cdot, \mu(t))$ and is denoted by $\hat{x}_\delta(t) = \arg \min_{x \in X} L_\delta(x, \mu(t))$.

Leveraging some existing theoretical tools in the study of optimization algorithms [4], [7], we can study h in a way that elucidates its behavior under asynchrony in a distributed implementation. According to [7], the assumption of diagonal dominance guarantees that h has the contraction property

$$\|h(x) - \hat{x}_\delta(t)\|_\infty \leq \alpha \|x - \hat{x}_\delta(t)\|_\infty$$

for all $x \in X$, where $\|v\|_\infty = \max_i |v_i|$ for $v \in \mathbb{R}^n$, $\alpha \in [0, 1)$ and $\hat{x}_\delta(t)$ is a fixed point of h , which depends on the choice of fixed $\mu(t)$. However, the value of α is not specified in [7], and it is precisely that value that governs the rate of convergence to a solution. We therefore compute α explicitly.

Following the method in [4], two $n \times n$ matrices G and F must also be defined.

Definition 1: Define the $n \times n$ matrices G and F as

$$G = \begin{bmatrix} |H_{11}| & -|H_{12}| & \dots & -|H_{1n}| \\ \vdots & \vdots & \ddots & \vdots \\ -|H_{n1}| & -|H_{n2}| & \dots & |H_{nn}| \end{bmatrix} \text{ and } F = I - \gamma G,$$

where I is the $n \times n$ identity matrix.

We now have the following.

Lemma 2: Let h , G , and F be as above and let Assumptions 1-6 hold. Then $|h(x) - h(y)| \leq F|x - y|$ for all $x, y \in \mathbb{R}^n$, where $|v|$ denotes the element-wise absolute value of the vector $v \in \mathbb{R}^n$ and the inequality holds component-wise.

Proof: We proceed by showing the satisfaction of three conditions in [4]: (i) γ is sufficiently small, (ii) G is positive definite, and (iii) F is positive definite.

(i) *γ is sufficiently small:* Results in [7] require $\gamma \sum_{j=1}^n |H_{ij}| < 1$ for all $i \in \{1, \dots, n\}$, which here follows immediately from (5).

(ii) *G is positive definite:* By definition, G has only positive diagonal entries. By H 's diagonal dominance we have the following inequality for all $i \in \{1, \dots, n\}$:

$$|G_{ii}| = |H_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^n |H_{ij}| + \beta > \sum_{\substack{j=1 \\ j \neq i}}^n |H_{ij}| = \sum_{\substack{j=1 \\ j \neq i}}^n |G_{ij}|.$$

Because G has positive diagonal entries, is symmetric, and is strictly diagonally dominant, G is positive definite by Gershgorin's Circle Theorem.

(iii) *F is positive definite:* Eq. (5) ensures the diagonal entries of F are always positive. And F is diagonally dominant if, for all $i \in \{1, \dots, n\}$,

$$|F_{ii}| = 1 - \gamma |H_{ii}| > \gamma \sum_{\substack{j=1 \\ j \neq i}}^n |H_{ij}| = \sum_{\substack{j=1 \\ j \neq i}}^n |F_{ij}|.$$

This requirement can be rewritten as $\gamma \sum_{j=1}^n |H_{ij}| < 1$, which was satisfied under (i). Because F has positive diagonal entries, is symmetric, and is strictly diagonally dominant, F is positive definite by Gershgorin's Circle Theorem. ■

We next show that the gradient update law h in (13) converges with asynchronous, distributed computations. Furthermore, we quantify the rate of convergence.

Lemma 3: Let γ and h be as defined in (5) and (13). Let Assumptions 1-6 hold and fix $\mu(t) \in \mathcal{M}$. Then for the fixed point $\hat{x}_\delta(t)$ of h and for all $x \in X$,

$$\|h(x) - h(\hat{x}_\delta(t))\|_\infty \leq q_p \|x - \hat{x}_\delta(t)\|_\infty,$$

where $q_p := (1 - \gamma\beta) \in [0, 1)$.

Proof: For all i , Assumption 4 and the definition of F give

$$\sum_{j=1}^n F_{ij} = 1 - \gamma \left(|H_{ii}| - \sum_{\substack{j=1 \\ j \neq i}}^n |H_{ij}| \right) \leq 1 - \gamma\beta.$$

This result, the definition of $\|\cdot\|_\infty$, and Lemma 2 give

$$\begin{aligned} \|h(x) - h(\hat{x}_\delta(t))\|_\infty &= \max_i |h_i(x) - h_i(\hat{x}_\delta(t))| \\ &\leq \max_i \sum_{j=1}^n F_{ij} |x_j - \hat{x}_{\delta,j}(t)| \\ &\leq \max_l |x_l - \hat{x}_{\delta,l}(t)| \max_i \sum_{j=1}^n F_{ij} \\ &\leq \max_l |x_l - \hat{x}_{\delta,l}(t)| (1 - \gamma\beta) \\ &= (1 - \gamma\beta) \|x - \hat{x}_\delta(t)\|_\infty, \end{aligned}$$

where the last inequality follows from Lemma 2. All that remains is to show $(1 - \gamma\beta) \in [0, 1)$. From (5) and the inequality $|H_{ii}| \geq \beta$, for all $x \in X$ and $\mu(t) \in M$,

$$\gamma\beta < \frac{\beta}{\max_i \sum_{j=1}^n |H_{ij}(x, \mu(t))|} \leq \frac{\beta}{\max_i |H_{ii}(x, \mu(t))|} = 1. \quad \blacksquare$$

Lemma 4: Let Assumptions 1-6 hold. Let $\mu(t)$ be the dual vector onboard all primal agents at some time k and let k_0^t denote the latest time that any primal agent received the dual variable $\mu(t)$ that agents currently have onboard. Then, with primal agents asynchronously executing the gradient update law h , agent i has

$$\|x^i(k; t) - \hat{x}_\delta(t)\|_\infty \leq q_p^{\text{ops}(k,t)} \max_j \|x^j(k_0^t; t) - \hat{x}_\delta(t)\|_\infty,$$

where $\hat{x}_\delta(t)$ is the fixed point of h with $\mu(t)$ held constant.

Proof: From Lemma 3 we see that h is a q_p -contraction mapping with respect to the norm $\|\cdot\|_\infty$. From Section 6.3 in [7], this property implies that there exist sets of the form

$$\begin{aligned} X(k) &= \{x \in \mathbb{R}^n \mid \|x - \hat{x}_\delta(t)\|_\infty \\ &\leq q_p^k \max_j \|x^j(k_0^t; t) - \hat{x}_\delta(t)\|_\infty\} \end{aligned}$$

that satisfy the following criteria from [19]:

- i. $\dots \subset X(k+1) \subset X(k) \subset \dots \subset X$

ii. $\lim_{k \rightarrow \infty} X(k) = \{\hat{x}_\delta(t)\}$

iii. For all i , there are sets $X_i(k) \subset X_i$ satisfying

$$X(k) = X_1(k) \times \cdots \times X_N(k)$$

iv. For all $y \in X(k)$ and all $i \in \mathcal{I}_p$, $h_i(y) \in X_i(k+1)$, where $h_i(y) = \Pi_{X_i} [y_i - \gamma \nabla_{x_{[i]}} L_\delta(y, \mu(t))]$.

We will use these properties to compute the desired convergence rate. Suppose all agents have a fixed $\mu(t)$ onboard. Upon receipt of this $\mu(t)$, agent i has $x^i(k_0^i; t) \in X(0)$ by definition. Suppose at time ℓ_i that agent i computes an update. Then $x_{[i]}^i(\ell_i + 1; t) \in X_i(1)$. For $m = \max_{i \in \mathcal{I}_p} \ell_i + 1$, we find that $x_{[i]}^i(m; t) \in X_i(1)$ for all i . Next, suppose that, after all updates have been computed, these updated values are sent to and received by all agents that need them, say at time m' . Then, for any $i \in \mathcal{I}_p$, agent i has $x_{[j]}^i(m'; t) \in X_j(1)$ for all $j \in \mathcal{I}_p$. In particular, $x^i(m'; t) \in X(1)$, and this is satisfied precisely when ops has incremented by one. Iterating this argument completes the proof. \blacksquare

2) *Dual Convergence*: Towards defining the behavior during an update of a single dual block, we consider the number of operations primal agents compute before communications are sent to a dual agent. In particular, we are interested in defining the oldest primal block a dual agent uses in its own computations. Each of the received primal blocks was sent when some number of operations had been completed by primal agents using the prior dual update. Towards quantifying this, we are interested in defining the primal computation time of the oldest primal block used by dual agent c when it computes update $t_c + 1$.

Definition 2: For dual agent c computing update $t_c + 1$, let $\kappa(c, t_c)$ denote the earliest primal computation time for all blocks in $x^c(t_c)$. That is, for all primal blocks used by dual agent c during update $t_c + 1$, $\kappa(c, t_c)$ is the oldest primal time any were computed.

Thus, the minimum number of operations completed by any primal agent for the blocks used by dual agent c during update $t_c + 1$ is equal to $\text{ops}(\kappa(c, t_c), t)$. We next derive a block-wise convergence rate for the dual variable.

Lemma 5: Let Assumptions 1-6 hold. Let the dual stepsize ρ be defined such that $\rho < \frac{2\delta}{\delta^2 + 2}$. Let $t_c \geq 0$ and consider the case where dual agent c performs a single update denoted with the iteration counter $t_c + 1$. Then the distance from the optimal value for block c is bounded by

$$\|\mu_{[c]}^c(t_c + 1) - \hat{\mu}_{\delta, [c]}\|^2 \leq q_d \|\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta, [c]}\|^2 + q_p^{2\text{ops}(\kappa(c, t_c), t)} E_1(c) + q_p^{\text{ops}(\kappa(c, t_c), t)} E_2(c) + E_3(c),$$

where $E_1(c) := (q_d - \rho^2)nM_{[c]}^2 D_x^2$, $E_2(c) := 2\rho^2 \sqrt{n}M_{[c]}^2 D_x^2$ and $E_3(c) := (q_d - \rho^2)M_{[c]}^2 D_x^2$, $q_d := (1 - \rho\delta)^2 + 2\rho^2 \in [0, 1)$, $M_{[c]} := \max_{x \in X} \|\nabla g_{[c]}(x)\|$, $D_x := \max_{x, y \in X} \|x - y\|$, and n is the length of the primal variable x .

Proof: Define $\hat{x}_\delta(t) = \arg \min_{x \in X} L_\delta(x, \mu(t))$ and $\hat{x}_\delta = \arg \min_{x \in X} L_\delta(x, \hat{\mu}_\delta)$. Let $x_t^c := x^c(t_c)$ for brevity of

notation. Expanding the dual update law and using the non-expansiveness of $\Pi_{\mathcal{M}}$, we find

$$\begin{aligned}
\|\mu_{[c]}^c(t_c+1) - \hat{\mu}_{\delta,[c]}\|^2 &= \|\Pi_{\mathcal{M}_c}[\mu_{[c]}^c(t_c) + \rho(g_{[c]}(x_t^c) - \delta\mu_{[c]}^c(t_c))] - \Pi_{\mathcal{M}_c}[\hat{\mu}_{\delta,[c]} + \rho(g_{[c]}(\hat{x}_\delta) - \delta\hat{\mu}_{\delta,[c]})]\|^2 \\
&\leq \|\mu_{[c]}^c(t_c) + \rho(g_{[c]}(x_t^c) - \delta\mu_{[c]}^c(t_c)) - \hat{\mu}_{\delta,[c]} - \rho(g_{[c]}(\hat{x}_\delta) - \delta\hat{\mu}_{\delta,[c]})\|^2 \\
&= \|(1 - \rho\delta)(\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta,[c]}) - \rho(g_{[c]}(\hat{x}_\delta) - g_{[c]}(x_t^c))\|^2 \\
&\leq (1 - \rho\delta)^2\|\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta,[c]}\|^2 + \rho^2\|g_{[c]}(x_t^c) - g_{[c]}(\hat{x}_\delta)\|^2 \\
&\quad - 2\rho(1 - \rho\delta)(\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta,[c]})^T(g_{[c]}(\hat{x}_\delta) - g_{[c]}(x_t^c)).
\end{aligned}$$

Adding $g_{[c]}(\hat{x}_\delta(t)) - g_{[c]}(\hat{x}_\delta(t))$ inside the last set of parentheses gives

$$\begin{aligned}
\|\mu_{[c]}^c(t_c+1) - \hat{\mu}_{\delta,[c]}\|^2 &\leq (1 - \rho\delta)^2\|\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta,[c]}\|^2 + \rho^2\|g_{[c]}(x_t^c) - g_{[c]}(\hat{x}_\delta)\|^2 \\
&\quad - 2\rho(1 - \rho\delta)(\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta,[c]})^T(g_{[c]}(\hat{x}_\delta) - g_{[c]}(\hat{x}_\delta(t))) \\
&\quad - 2\rho(1 - \rho\delta)(\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta,[c]})^T(g_{[c]}(\hat{x}_\delta(t)) - g_{[c]}(x_t^c)). \tag{14}
\end{aligned}$$

We can write

$$0 \leq \|(1 - \rho\delta)(g_{[c]}(\hat{x}_\delta) - g_{[c]}(\hat{x}_\delta(t))) + \rho(\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta,[c]})\|^2,$$

which can be expanded and rearranged to give

$$\begin{aligned}
&-2\rho(1 - \rho\delta)(\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta,[c]})^T(g_{[c]}(\hat{x}_\delta) - g_{[c]}(\hat{x}_\delta(t))) \\
&\leq (1 - \rho\delta)^2\|g_{[c]}(\hat{x}_\delta) - g_{[c]}(\hat{x}_\delta(t))\|^2 + \rho^2\|\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta,[c]}\|^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&-2\rho(1 - \rho\delta)(\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta,[c]})^T(g_{[c]}(\hat{x}_\delta(t)) - g_{[c]}(x_t^c)) \\
&\leq (1 - \rho\delta)^2\|g_{[c]}(\hat{x}_\delta(t)) - g_{[c]}(x_t^c)\|^2 + \rho^2\|\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta,[c]}\|^2.
\end{aligned}$$

Applying these inequalities to (14) gives

$$\begin{aligned}
\|\mu_{[c]}^c(t_c+1) - \hat{\mu}_{\delta,[c]}\|^2 &\leq (1 - \rho\delta)^2\|\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta,[c]}\|^2 \\
&\quad + \rho^2\|g_{[c]}(x_t^c) - g_{[c]}(\hat{x}_\delta)\|^2 + (1 - \rho\delta)^2\|g_{[c]}(\hat{x}_\delta) - g_{[c]}(\hat{x}_\delta(t))\|^2 \\
&\quad + 2\rho^2\|\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta,[c]}\|^2 + (1 - \rho\delta)^2\|g_{[c]}(\hat{x}_\delta(t)) - g_{[c]}(x_t^c)\|^2 \\
&\leq ((1 - \rho\delta)^2 + 2\rho^2)\|\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta,[c]}\|^2 + \rho^2\|g_{[c]}(x_t^c) - g_{[c]}(\hat{x}_\delta)\|^2 \\
&\quad + (1 - \rho\delta)^2\|g_{[c]}(\hat{x}_\delta) - g_{[c]}(\hat{x}_\delta(t))\|^2 + (1 - \rho\delta)^2\|g_{[c]}(\hat{x}_\delta(t)) - g_{[c]}(x_t^c)\|^2. \tag{15}
\end{aligned}$$

In (15), we next use $\rho^2\|g_{[c]}(x_t^c) - g_{[c]}(\hat{x}_\delta)\|^2 = \rho^2\|g_{[c]}(x_t^c) - g_{[c]}(\hat{x}_\delta(t)) + g_{[c]}(\hat{x}_\delta(t)) - g_{[c]}(\hat{x}_\delta)\|^2$, then expand, and combine like terms to find

$$\begin{aligned}
\|\mu_{[c]}^c(t_c+1) - \hat{\mu}_{\delta,[c]}\|^2 &\leq ((1 - \rho\delta)^2 + 2\rho^2)\|\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta,[c]}\|^2 + ((1 - \rho\delta)^2 + \rho^2)\|g_{[c]}(x_t^c) - g_{[c]}(\hat{x}_\delta(t))\|^2 \\
&\quad + 2\rho^2\|g_{[c]}(x_t^c) - g_{[c]}(\hat{x}_\delta(t))\| \|g_{[c]}(\hat{x}_\delta(t)) - g_{[c]}(\hat{x}_\delta)\| \\
&\quad + ((1 - \rho\delta)^2 + \rho^2)\|g_{[c]}(\hat{x}_\delta(t)) - g_{[c]}(\hat{x}_\delta)\|^2.
\end{aligned}$$

Using the Lipschitz property of $g_{[c]}$, we can write

$$\begin{aligned} \|\mu_{[c]}^c(t_c + 1) - \hat{\mu}_{\delta,[c]}\|^2 &\leq ((1-\rho\delta)^2 + 2\rho^2)\|\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta,[c]}\|^2 + ((1-\rho\delta)^2 + \rho^2)M_{[c]}^2\|x_t^c - \hat{x}_\delta(t)\|^2 \\ &\quad + 2\rho^2M_{[c]}^2\|x_t^c - \hat{x}_\delta(t)\|\|\hat{x}_\delta(t) - \hat{x}_\delta\| + ((1-\rho\delta)^2 + \rho^2)M_{[c]}^2\|\hat{x}_\delta(t) - \hat{x}_\delta\|^2. \end{aligned}$$

Using $\|\hat{x}_\delta(t) - \hat{x}_\delta\| \leq D_x$, the inequality simplifies to

$$\begin{aligned} \|\mu_{[c]}^c(t_c + 1) - \hat{\mu}_{\delta,[c]}\|^2 &\leq ((1-\rho\delta)^2 + 2\rho^2)\|\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta,[c]}\|^2 + ((1-\rho\delta)^2 + \rho^2)M_{[c]}^2\|x_t^c - \hat{x}_\delta(t)\|^2 \\ &\quad + 2\rho^2M_{[c]}^2D_x\|x_t^c - \hat{x}_\delta(t)\| + ((1-\rho\delta)^2 + \rho^2)M_{[c]}^2D_x^2. \end{aligned} \quad (16)$$

Using Definition 2, define $\tilde{x}^c(t_c)$ as the primal variable whose distance is greatest from the optimal value at primal time $\kappa(c, t_c)$. That is, $\tilde{x}^c(t_c) := \max_{j \in \mathcal{I}_p} \|x^j(\kappa(c, t_c), t_c) - \hat{x}_\delta(t)\|$. Using this, the contraction property of primal updates from Lemma 4, and the definition of D_x , we find

$$\|x_t^c - \hat{x}_\delta(t)\| \leq \|\tilde{x}^c(t_c) - \hat{x}_\delta(t)\| \leq \sqrt{n}\|\tilde{x}^c(t_c) - \hat{x}_\delta(t)\|_\infty \leq q_p^{\text{ops}(\kappa(c, t_c), t)}\sqrt{n}D_x.$$

Applying this result to (16) above gives

$$\begin{aligned} \|\mu_{[c]}^c(t_c + 1) - \hat{\mu}_{\delta,[c]}\|^2 &\leq ((1-\rho\delta)^2 + 2\rho^2)\|\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta,[c]}\|^2 + ((1-\rho\delta)^2 + \rho^2)nM_{[c]}^2q_p^{2\text{ops}(\kappa(c, t_c), t)}D_x^2 \\ &\quad + 2\rho^2\sqrt{n}M_{[c]}^2D_xq_p^{\text{ops}(\kappa(c, t_c), t)} + ((1-\rho\delta)^2 + \rho^2)M_{[c]}^2D_x^2. \end{aligned}$$

Using $\rho < \frac{2\delta}{\delta^2+2}$, we have $q_d = (1-\rho\delta)^2 + 2\rho^2 \in (0, 1)$, completing the proof. \blacksquare

Lemma 6: Let all conditions and definitions of Lemma 5 hold. Let $T(t) = \min_c t_c$ be the minimum number of updates any one dual agent has performed by time t and let $K(t)$ be the minimum number of operations primal agents completed on any primal block used to compute any dual block from $\mu(0)$ to $\mu(t)$. Then, Algorithm 1's convergence for μ obeys

$$\begin{aligned} \|\mu(t) - \hat{\mu}_\delta\|^2 &\leq q_d^{T(t)}\|\mu(0) - \hat{\mu}_\delta\|^2 + \left(q_p^{2K(t)}(q_d - \rho^2)nN_dM^2D_x^2 \right. \\ &\quad \left. + q_p^{K(t)}2\rho^2\sqrt{n}N_dM^2D_x^2 + (q_d - \rho^2)N_dM^2D_x^2 \right) \frac{1}{1 - q_d}, \end{aligned}$$

where $M := \max_{x \in X} \|\nabla g(x)\|$ and N_d is the number of dual agents.

Proof: Let $K_c(t_c)$ be the minimum number of operations primal agents completed on any primal block used to compute $\mu_{[c]}$ from $\mu_{[c]}(0)$ to $\mu_{[c]}(t_c)$. Then recursively applying Lemma 5 and using the definition of $K_c(t_c)$ gives

$$\begin{aligned} \|\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta,[c]}\|^2 &\leq q_d\|\mu_{[c]}^c(t_c - 1) - \hat{\mu}_{\delta,[c]}\|^2 + q_p^{2K_c(t_c)}E_1(c) + q_p^{K_c(t_c)}E_2(c) + E_3(c) \\ &\leq q_d^{t_c}\|\mu_{[c]}^c(0) - \hat{\mu}_{\delta,[c]}\|^2 + \sum_{i=0}^{t_c-1} q_d^i \left(q_p^{2K_c(t_c)}E_1(c) + q_p^{K_c(t_c)}E_2(c) + E_3(c) \right) \\ &\leq q_d^{t_c}\|\mu_{[c]}^c(0) - \hat{\mu}_{\delta,[c]}\|^2 + \left(q_p^{2K_c(t_c)}E_1(c) + q_p^{K_c(t_c)}E_2(c) + E_3(c) \right) \frac{1 - q_d^{t_c}}{1 - q_d}, \end{aligned} \quad (17)$$

where the last inequality uses $q_d \in [0, 1)$ and sums the geometric series. We now derive a bound on the entire μ vector at time t . Expanding $\|\mu(t) - \hat{\mu}_\delta\|^2$ allows us to write

$$\begin{aligned} \|\mu(t) - \hat{\mu}_\delta\|^2 &= \sum_{c=1}^{N_d} \|\mu_{[c]}^c(t_c) - \hat{\mu}_{\delta, [c]}\|^2 \\ &\leq \sum_{c=1}^{N_d} q_d^{t_c} \|\mu_{[c]}^c(0) - \hat{\mu}_{\delta, [c]}\|^2 + \left(q_p^{2K_c(t_c)} E_1(c) + q_p^{K_c(t_c)} E_2(c) + E_3(c) \right) \frac{1 - q_d^{t_c}}{1 - q_d} \\ &\leq \sum_{c=1}^{N_d} q_d^{t_c} \|\mu_{[c]}^c(0) - \hat{\mu}_{\delta, [c]}\|^2 + \left(q_p^{2K(t)} (q_d - \rho^2) n N_d M^2 D_x^2 \right. \\ &\quad \left. + q_p^{K(t)} 2\rho^2 \sqrt{n} N_d M^2 D_x^2 + (q_d - \rho^2) N_d M^2 D_x^2 \right) \frac{1}{1 - q_d}, \end{aligned}$$

where the first inequality applies (17) and the second uses $K_c(t_c) \geq K(t)$, $M_{[c]}^2 \leq N_d M^2$, and simplifies. Applying the summation and definition of $T(t)$ completes the proof. \blacksquare

3) *Proof of Theorem 3:* We see that

$$\begin{aligned} \|x^i(k; t) - \hat{x}_\delta\|^2 &= \|x^i(k; t) - \hat{x}_\delta(t) + \hat{x}_\delta(t) - \hat{x}_\delta\|^2 \\ &\leq 2\|x^i(k; t) - \hat{x}_\delta(t)\|^2 + 2\|\hat{x}_\delta(t) - \hat{x}_\delta\|^2 \\ &\leq 2n\|x^i(k; t) - \hat{x}_\delta(t)\|_\infty^2 + \frac{2M^2}{\beta^2} \|\mu(t) - \hat{\mu}_\delta\|^2, \end{aligned}$$

where the last line applies Lemma 4.1 in [24]. Next, applying Lemmas 4 and 6 gives

$$\begin{aligned} \|x^i(k; t) - \hat{x}_\delta\|^2 &\leq 2nq_p^{2\text{ops}(k, t)} \max_j \|x^j(k_0^t; t) - \hat{x}_\delta(t)\|_\infty^2 \\ &\quad + q_d^{T(t)} \frac{2M^2}{\beta^2} \|\mu(0) - \hat{\mu}_\delta\|^2 + \left(q_p^{2K(t)} \frac{2nN_d M^4 D_x^2 (q_d - \rho^2)}{\beta^2} \right. \\ &\quad \left. + q_p^{K(t)} \frac{4\rho^2 \sqrt{n} N_d M^4 D_x^2}{\beta^2} + \frac{2N_d M^4 D_x^2 (q_d - \rho^2)}{\beta^2} \right) \frac{1}{1 - q_d}. \end{aligned}$$

Defining C_1 , C_2 , and C_3 completes the proof. \blacksquare

C. Proof of Corollary 1:

We first simplify by noting that $2\text{ops}(k, t) \geq K(t)$ and $2K(t) \geq K(t)$. This allows us to factor the bound in Theorem 3 with $q_p^{K(t)} \left(2n \max_j \|x^j(k_0^t; t) - \hat{x}_\delta(t)\|_\infty^2 + C_1 + C_2 \right)$. Setting this less than or equal to $\frac{\epsilon_1}{2}$ and solving gives the lower bound on $K(t)$. Similarly, setting $q_d^{T(t)} \frac{2M^2}{\beta^2} \|\mu(0) - \hat{\mu}_\delta\|^2 \leq \frac{\epsilon_1}{2}$ gives the lower bound on $T(t)$. Finally, we set $\rho = \frac{\delta}{1+\delta^2}$ which results in $(q_d - \rho^2) = \frac{1}{1+\delta^2}$. Applying this to C_3 and setting less than or equal to ϵ_2 gives the final bound on δ^2 . \blacksquare

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