Transmission-Constrained Consensus of Multiagent Networks

Xiaotian Wang, Housheng Su

***Abstract*— This paper studies the consensus problem for multiagent systems with transmission constraints. A novel model of multiagent systems is proposed where the information transmissions between agents are disturbed by irregular distortions or interferences (named transmission constraint functions), and this model is universal which can be applied in many cases, such as interval consensus and discarded consensus. In the transmission-constrained consensus problem, we obtain the necessary and sufficient condition that agents can converge to state consensus. Furthermore, a more general case is studied in which the system reaches an equilibrium. Based on some techniques of algebraic topology and stability theory, the existence, uniqueness and stability of the system equilibrium point can be proven, which means the system can reach an asymptotically stable equilibrium. Moreover, the state values of the equilibrium are only decided by the network structure and transmission constraint functions, but not the agents' initial states. Finally, numerical simulations are presented to illustrate the proposed theorems and corollaries.**

***Index Terms*— Multiagent system, Consensus, Transmission constraint, Directed graph, Asymptotically stable.**

I. Introduction

IN the past few years, distributed coordination of multiagent systems (MASs) has attracted much attention due to its broad application prospects in civil, military and other fields. One of its fundamental problems is consensus, which requires that agents achieve agreement about certain quantities of interest that depends on all agents' states. Many scientific problems of consensus have emerged, and lots of control protocols are proposed in this area, such as consensus tracking [1], average consensus [2] and robust consensus [3], [4].

In most of the above consensus problems, agents' states are not constrained. However, there are various state constraints on agents in many real-world scenarios, such as restricted actuators and limited communication distance. Imposing state constraints on agents has notable significance and great research value. The constrained consensus problems have been studied from different perspectives. For example, the constrained consensus and optimization problems have been studied, where agents' states are constrained in closed convex sets [5]. A novel state-constrained consensus (named interval consensus) problem has been proposed and studied in [6]. Moreover, alternative approaches for imposing state constraints have been proposed in [7], [8].

Information sharing is a necessary condition for MASs to admit a consensus solution. In the process of information transmitting, there are various unfavorable factors such as environmental interference, noise, and attenuation. Therefore, it is impractical and ideal to assume that agents can receive neighbors' information without distortions and noise disturbances. Previous researches on the imperfect information transmission often focused on switching topology, communication delay, communication link fault, packet loss, etc. For example, average consensus problem is studied in [9], where time-varying delay and packet loss are considered under the undirected communication network.

In the above constrained consensus problems, the constraints are imposed on agents' states or inputs, directly. In this work, we impose constraints on the information transmitting to make the agent's received information different from the original information of its neighbour, and research the effect of this difference on system stability. To distinguish our problem from other constrained consensus problems, we call it **transmission-constrained consensus**. In this problem, the deformed transmitted information is depicted by heterogeneous functions (named transmission constraint function). And a variety of functions can be chosen as transmission constraint function, hence this study is so universal that it can be applied in many cases. Some applications and motivating examples are given in subsection II-C.

The first part of this work can be regarded as a study on the consensus conditions for the MAS with distorted transmitted state information (attenuation or saturation). The necessary and sufficient conditions for the ranges of transmission constraint functions are obtained. A non-empty intersection of constraints is an important condition for MASs to achieve consensus. In many state-constrained consensus problems, it is assumed that the constraints have a non-empty intersection. However, in this paper, we also consider the non-empty intersection case, where the multiagent systems may not achieve consensus, but an equilibrium. In the second part of this study, it is proved that when the transmission constraint functions are distributed in a specific range (i.e., satisfy the conditions of Theorem 4 or 5), the system will reach an asymptotically stable equilibrium

This work was supported by the National Natural Science Foundation of China under Grant Nos. 61991412, 61873318, and U1913602, and the Program for HUST Academic Frontier Youth Team under Grant No. 2018QYTD07.

The authors are with the School of Artificial Intelligence and Automation, Huazhong University of Science and Technology, and also with the Key Laboratory of Image Processing and Intelligent Control of Education Ministry of China, Wuhan 430074, China. Email: XiaotianWangEmail@gmail.com, houshengsu@gmail.com.

even though that intersection of constraints is empty.

Compared with the existing works about the consensus of networked systems with constraints, the contributions of this work can be obtained as follows.

1) This work first studies the transmission-constrained consensus of multiagent networks. The transmission-constrained consensus model studied in this paper does not have a definite form, so that it can be regarded as a paradigm. The multiagent systems that can be translated to this model are able to make the system achieve consensus under the necessary conditions, such as interval consensus [6]. Unlike traditional constrained consensus problems, transmission-constrained consensus problem has the following features:
   a) each link in the interaction network is limited by an individual constraint function, which is more general in reality;
   b) constraint functions do not have a uniform type, and they can be various functions, such as trigonometric function, saturation function and Sigmoid function, etc.

   Those features make the transmission-constrained model have a wide range of application, but also bring heterogeneity into the dynamics, which increases the difficulty of analysis.
2) For the transmission-constrained consensus problem, we obtain some consensus conditions, in which a necessary and sufficient condition limits the distribution of constraint functions. As a more general case than the consensus case, equilibrium of MAS is seldom studied. We investigate this phenomenon and obtain conditions of equilibrium's existence, uniqueness and stability. Due to the novel model, where the unknown transmission constraints make the dynamics nonlinear, the analysis of MAS's stability is quite a challenge. We design some linear boundaries and propose the corresponding lemmas to analyze the boundedness of dynamics. Then, we analyze the limit points of multiple solutions to prove the convergence of MASs. By coordinate transformations, another Lyapunov function is constructed to study the stability and uniqueness of equilibrium.

The paper is organized as follows. Preliminaries and problem statement are given in Section II. Main results are provided in Section III. Supports of numerical examples are provided in Section IV, and Section V concludes this work. Finally, we put all proofs in appendixes.

II. PRELIMINARIES

A. Graph Theory

This paper studies the problem of transmission-constrained consensus of multiagent networks. Consider a MAS with n agents, and denote $\mathbf{N} = \{1, 2, \dots, n\}$. The finite vertex set is denoted by $\mathcal{V} = \{v_1, \dots, v_n\}$, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes edge set where $(v_j, v_i) \in \mathcal{E}$ means that there exists a communication link from agent j to agent i. The adjacency weight matrix $\mathcal{A} \in \mathbb{R}^{n \times n}$ is defined as $a_{ij} > 0$ if and only if $(v_j, v_i) \in \mathcal{E}$, and $a_{ij} = 0$, otherwise. Then, the underlying interaction network of MAS is described by a (weighted) graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ which is a triple. (v_j, v_i) is defined as the directed edge from agent j to agent i, and $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$ denotes the neighbor set of agent i. Denote $\alpha_i = \sum_{j=1}^{n} a_{ij}$ as the row sum of $\mathcal{A}$, and $\bar{a} = \max \alpha_i$.

B. Problem Statement

For any $i \in \mathbf{N}$, denote the state of v_i by $x_i(t) \in \mathbb{R}$. Then consider the continuous-time dynamics of single-integrator MAS with n agents: $\dot{x}_i(t) = u_i(t)$, $i \in \mathbf{N}$, where $u_i(t) \in \mathbb{R}$ is the control input.

The problem studied in this work is different from the general MAS dynamics. In this problem, the information transmissions between agents are disturbed by interference functions (or attenuation functions), i.e., the transmission of state $x_i(t)$ is replaced by a transmission constrain function $f_{ij}\big(x_i(t)\big)$. f_{ij} represents the transmission constraint imposed on the link from i to j. Then, the transmission-constrained consensus algorithm of $x_i(t)$ is

$$\dot{x}_i(t) = u_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij} \Big[f_{ji}\big(x_j(t)\big) - x_i(t) \Big]. \qquad (1)$$

Assumption 1: $\forall\, i, j \in \mathbf{N}$, the transmission constrain functions $f_{ij}(x)$ are piecewise continuous.

Remark 1: Since $\forall\, i, j \in \mathbf{N}$, $f_{ij}(x)$ are piecewise continuous, the autonomous system (1) may have multiple solutions. However, the following theorems and corollaries apply to both unique and multiple solutions.

Assumption 2: The directed graph $\mathcal{G}$ is strongly connected.

Assumption 3: There exists an interval $[\partial_m, \partial_M]$, a value $\partial \in [\partial_m, \partial_M]$ and two rays

$$\begin{aligned} L_1(x) &= k_1(x - \partial) + \partial, \quad x \in (-\infty, \partial]; \\ L_2(x) &= k_2(x - \partial) + \partial, \quad x \in [\partial, +\infty), \end{aligned}$$

where $k_1, k_2 < 0$, such that $\forall\, j \in \mathbf{N}$, $i \in \mathcal{N}_j$,

$$\begin{aligned} x \le f_{ij}(x) < L_1(x), \quad x \in (-\infty, \partial_m); \\ L_2(x) < f_{ij}(x) \le x, \quad x \in (\partial_M, +\infty). \end{aligned}$$

Assumption 4: For any $x' \in (-\infty, \partial_m) \cup (\partial_M, +\infty)$, there eixst $j \in \mathbf{N}$ and $i \in \mathcal{N}_j$, such that $f_{ij}(x)$ is continuous on x' and $f_{ij}(x') \neq x'$.

This paper aims to find which transmission constraints could make MAS stable and obtain the consensus conditions for MAS (1).

C. Applications and Motivating Examples

The distortion (attenuation or saturation) in information transmission or detection is an actual embodiment of transmission constraints. Those transmission constraints may be caused by objective physical constraints, or those constraints are added on purpose.

1) Objective Constraints: There are three kinds of objective constraints to show those transmission constraints are common in real world scenarios.

- *Information distortion caused by transmission.*
  Energy loss exists during the signal transmission, which may cause information distortion. The voltage drop on wires is an inevitable phenomenon during signal transmission. If state of agent (or device) are represented by voltage of a signal, and this signal is transmitted on wires, then we can get an information distortion
  $$f_{ij}(x_i) = \frac{R_r}{R_L + R_r} x_i,$$
  where R_L is the resistance of wires, and R_r is the equivalent resistance of the port.
- *Information distortion caused by detection.*
  In real world scenarios, agents use sensors to get themselves or neighbors' states. However, except for noise interference, state information cannot be obtained precisely due to the inherent characteristics of sensors. For example, temperature offset leads to signal fluctuation in ultrasonic distance measurements [10]. Likewise, the saturation characteristic of hall sensor may cause information distortion, i.e., $f_{ij}(x_i) = \text{sat}(x_i)$ [11].
- *Information distortion caused by privacy protection.*
  In social networks, individuals may express an opinion that is different from his/her private opinion, due to the pressure of conforming to a group standard or norm [12], [13]. Hence, x_i could represent the private opinion, and $f_{ij}(x_i)$ is the expressed opinion.

2) Subjective Constraints: As a class of constraints, transmission constraints could make agent's states converge into the expected set (see Theorem 2 and Remark 3). Especially, Since we do not specify the formula of transmission constraints, different transmission constraints can be designed to suit different scenarios, such as interval consensus [6] and discarded consensus [14]. Related discussions are in remarks 5 and 6.

Those above examples show that information distortion during transmission is a common phenomenon in the real world. Hence, study consensus under transmission constraints is necessary.

D. Notations and Some Definitions

Notations: The set of positive integers is denoted by $\mathbb{N}^+$. Consider a matrix $B = [b_{ij}] \in M_{m,n}$ and denote $|B| = [|b_{ij}|]$ (i.e., element-wise absolute value of matrix B). $d^+Z(t)$ denotes the upper right Dini derivative of $Z(t)$. The arrow '$\Longrightarrow$' means 'implies', and the arrow '$\Longleftrightarrow$' means 'if and only if'. Denote sign function

$$\text{sign}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

The distance between interval $[\partial_m, \partial_M]^n$ and vector $\mathbf{x}(t)$ is denoted by

$$\textit{distance}\big([\partial_m, \partial_M]^n, \mathbf{x}(t)\big) = \min_{c \in [\partial_m, \partial_M]^n} \|\mathbf{x}(t) - c\|.$$

Denote $\mathbf{e} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}^T$ to be an equilibrium of MAS (1), then it can be concluded that for all $i \in \mathbf{N}$,

$$\dot{x}_i(t)\big|_{\mathbf{e}_i} = \sum_{j \in \mathcal{N}_i} a_{ij}\big(f_{ji}(\mathbf{e}_j) - \mathbf{e}_i\big) = 0.$$

Denote the error between the state $\mathbf{x}(t)$ and equilibrium $\mathbf{e}$ by $\varepsilon_i(t) = x_i(t) - \mathbf{e}_i$, $\forall\, i \in \mathbf{N}$.

Introduce the definition of consensus zone, and this work can be divided into two parts: the part of non-empty consensus zone and the part of empty consensus zone.

Definition 1: For MAS (1), denote $\Theta_{ij} = \{x : f_{ij}(x) = x\}$, and **consensus zone** $\Phi = \bigcap_{(v_j, v_i) \in \mathcal{E}} \Theta_{ij}$. Consensus zone means the transmission constraints vanish when all the states of MAS are in this consensus zone.

Theorem 2 shows that under the given conditions, MAS (1) reaches consensus and its states fall into consensus zone.

Remark 2: For any time, if an agent's state is in consensus zone, then the information it transmits to its neighbors is without transmission constraints. If all agents' initial states are in consensus zone, then the MAS becomes a standard consensus dynamics. In that case, under a strongly connected digraph (or the digraph has a spanning tree), the MAS will reach consensus and the consensus value is in the consensus zone. That is why we name it consensus zone.

III. Main Results

In this section, we first analyze the convergence of MAS in subsection III-A. Secondly, for the non-empty consensus zone, we get the consensus conditions in subsection III-B. Thirdly, for the empty consensus zone, the system's states may achieve an equilibrium, and the existence, stability and uniqueness of equilibrium are studied in subsection III-C.

A. Convergence analysis

The following theorem states the conditions where the states of the multiagent system are bounded, and gives the boundary. Furthermore, the conclusion of Theorem 1 plays an important role in the proofs of Theorems 2 and 3.

Theorem 1: Along the system (1), suppose Assumptions 1, 2, 3 and 4 hold, and $k_1 k_2 = 1$. Then for any MAS (1) and initial state $\mathbf{x}(t_0) \in \mathbb{R}^n$,

$$\lim_{t \to \infty} \text{distance}\big([\partial_m, \partial_M]^n, \mathbf{x}(t)\big) = 0,$$

if and only if for any MAS (1), initial state $\mathbf{x}(t_0) \in \mathbb{R}^n$, and $j \in \mathbf{N}$, $i \in \mathcal{N}_j$, $\partial_m \le f_{ij}(x) \le \partial_M$, $x \in [\partial_m, \partial_M]$.

B. Nonempty Consensus Zone: Consensus

Then, we introduce conditions of transmission-constrained consensus. The following theorem states the consensus conditions for MAS, and the condition about range of constraint functions is necessary and sufficient.

Theorem 2: Along the system (1), suppose Assumptions 1, 2, 3 and 4 hold, and $f_{ij}(x) = x$, $x \in [\partial_m, \partial_M]$. Then, for any MAS (1), $i \in \mathbf{N}$, $\lim_{t \to \infty} x_i(t) = v^*$, $v^* \in [\partial_m, \partial_M]$ **if and only if** for any MAS (1), $k_1 k_2 \le 1$.

Remark 3: Theorem 2 shows that although the initial state is not in the consensus zone, the transmission constraints will limit the final state of the system. Hence, it shows that imposing constraints on links of the interaction networks can indirectly limit agents' states.

Remark 4: Under the conditions in Theorem 2, the constraint functions can be various functions. For example, constraint functions can be Sigmoid function or tanh function, which have many applications such as activation function in artificial neural networks, logistic function in biology, etc. More candidate constraint functions are given in the numeral example section.

Remark 5: In [6], the interval consensus problem is studied. We propose a smooth interval consensus model:

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}\big[T_j\big(x_j(t)\big) - x_i(t)\big],\ \ i \in \mathbf{N},$$

$$T(x) = \begin{cases} \rho x + (1-\rho)q, & \text{if } x > q, \\ x, & \text{if } p \le x \le q, \\ \rho x + (1-\rho)p, & \text{if } x < p, \end{cases}$$

where $\rho \in (0,1)$ is a constant. By Theorem 2, it can be concluded that the system will reach interval consensus.

Remark 6: The discarded consensus problem is studied in [14], but the initial states of system must be in the constraint set. Another discarded consensus model can be proposed:

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i, x_j(t)\in\Omega_{c_i}} a_{ij}x_j(t) - \sum_{j\in\mathcal{N}_i} a_{ij}x_i(t), \qquad (2)$$

where $\Omega_{c_i} = [-c_i, c_i]$ is the constraint interval of agent i [14]. By Theorem 2, we can get that the MAS (2) will reach discarded consensus with arbitrarily initial states.

Remark 7: Consider the following multiagent system:

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}\big[\sin\big(x_j(t)+\pi\big) - x_i(t)\big],\ \ i \in \mathbf{N}. \qquad (3)$$

Due to the consensus zone of MAS (3) being $\Phi = \{0\}$, Theorem 2 shows that the agents' states will converge to 0, when the underlying directed graph $\mathcal{G}$ is strongly connected.

Corollary 1: Along the system (1), suppose following conditions hold:

1) Assumption 2 holds;
2) the consensus zone $\Phi \neq \emptyset$;
3) for any $j \in \mathbf{N}$, $i \in \mathcal{N}_j$ and $\omega \neq 0$,

$$-1 < \frac{f_{ij}(x+\omega) - f_{ij}(x)}{\omega} \le 1, \quad x \in \mathbb{R}.$$

Then, $\forall\, i \in \mathbf{N}$, $\lim_{t\to\infty} x_i(t) = v^*$, $v^* \in \Phi$.

C. Empty Consensus Zone: Existence, Stability and Uniqueness of Equilibria

In this part, the existence, stability and uniqueness of equilibria are discussed and proved.

Theorem 3 gives the existence conditions of equilibrium, which is a prerequisite for the Theorem 5.

Theorem 3: Suppose $\forall\, j \in \mathbf{N}$, $i \in \mathcal{N}_j$, $f_{ij}(x)$ is a continuous function. If there exists an interval $[\partial_m, \partial_M]$ and $\forall\, j \in \mathbf{N}$, $i \in \mathcal{N}_j$,

$$\partial_m \le f_{ij}(x) \le \partial_M, \quad x \in [\partial_m, \partial_M],$$

then the system (1) exists at least one equilibrium. In fact, all equilibria of the system lie within $[\partial_m, \partial_M]^n$, if the following conditions hold:

1) Assumptions 2, 3 and 4 hold;
2) $k_1k_2 = 1$, and $\partial_m \le f_{ij}(x) \le \partial_M$, $x \in [\partial_m, \partial_M]$;
3) $f_{ij}(x)$ is a continuous function, $\forall\, j \in \mathbf{N}$, $i \in \mathcal{N}_j$.

Theorem 3 establishes the existence of equilibrium and gives the region where all equilibria exist. But it does not illustrate whether the MAS will reach equilibria, not to mention the stability of equilibria. Theorem 4 indicates that the system will converge to an asymptotically stable equilibrium, if some conditions hold.

Theorem 4: Along the system (1), suppose following conditions hold:

1) Assumption 2 holds;
2) there exists a equilibrium $\mathbf{e} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}^T$, i.e.,

$$\dot{x}_i(t)\big|_{\mathbf{e}_i} = \sum_{j\in\mathcal{N}_i} a_{ij}\big(f_{ji}(\mathbf{e}_j) - \mathbf{e}_i\big) = 0,\ \ \forall\, i \in \mathbf{N}.$$

3) there exist two rays

$$L_{e1}(\varepsilon) = k_{e1}\varepsilon, \quad \varepsilon \in (-\infty, 0];$$
$$L_{e2}(\varepsilon) = k_{e2}\varepsilon, \quad \varepsilon \in [0, +\infty),$$

where $k_{e1}, k_{e2} < 0$ and $k_{e1}k_{e2} = 1$, such that $\forall\, j \in \mathbf{N}$, $i \in \mathcal{N}_j$,

$$\varepsilon \le f_{ij}(\mathbf{e}_i + \varepsilon) - f_{ij}(\mathbf{e}_i) < L_{e1}(\varepsilon), \quad \varepsilon \in (-\infty, 0);$$
$$L_{e2}(\varepsilon) < f_{ij}(\mathbf{e}_i + \varepsilon) - f_{ij}(\mathbf{e}_i) \le \varepsilon, \quad \varepsilon \in (0, +\infty).$$

4) for any $\varepsilon' \neq 0$, there exist $j \in \mathbf{N}, i \in \mathcal{N}_j$, such that $f_{ij}(\mathbf{e}_i + \varepsilon')$ is continuous on ε' and $f_{ij}(\mathbf{e}_i + \varepsilon') - f_{ij}(\mathbf{e}_i) \neq \varepsilon'$.

Then, the equilibrium $\mathbf{e}$ is unique and asymptotically stable, i.e., $\lim_{t\to\infty} x_i(t) = \mathbf{e}_i$, $\forall\, i \in \mathbf{N}$.

Remark 8: Unlike Theorem 2, the boundary rays in Theorem 4 are two clusters of parallel lines with the same slopes, but the endpoints may be different. The auxiliary lines in Fig. 5 are the illustrations of two clusters of parallel lines.

Remark 9: The unique equilibrium's values are only decided by the network structure and transmission constraint functions, but not related to the initial states of MAS (1).

The following theorem can be regarded as a combination of theorems 3 and 4, which gives the conditions for the system to converge to an asymptotically stable equilibrium.

Theorem 5: Along the system (1), suppose following conditions hold:

1) Assumption 2 holds;
2) the consensus zone $\Phi = \emptyset$;
3) for any $j \in \mathbf{N}$, $i \in \mathcal{N}_j$ and $\omega \neq 0$,

$$-1 < \frac{f_{ij}(x+\omega) - f_{ij}(x)}{\omega} < 1, \quad x \notin \Theta_{ij}.$$

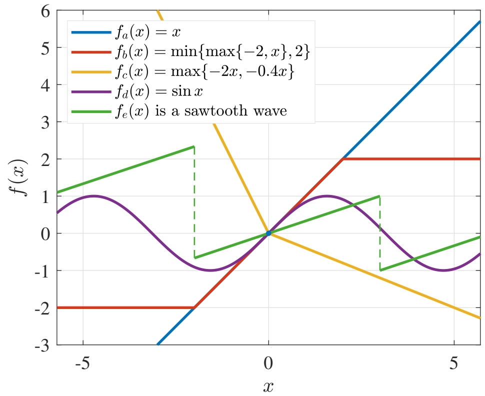

Fig. 1. The constraint functions in Example 1.

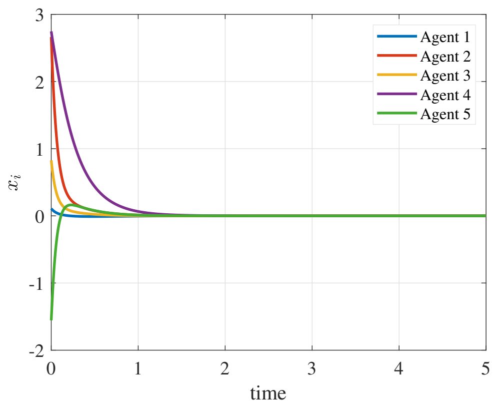

Fig. 2. The trajectories of $\mathbf{x}(t)$ in Example 1.

Then there exists a unique, asymptotically stable equilibrium of the MAS (1).

Remark 10: Theorem 4 requires a known equilibrium of MAS. In contrast, Theorem 5 relaxes the condition that the equilibrium is known, i.e., we just need to know the constraints functions and the connectivity of interaction networks, then we can predict the trajectory of MAS. In conclusion, Theorem 5 states the existence, stability and uniqueness of Equilibria.

Corollary 2: Suppose the directed graph $\mathcal{G}$ is strongly connected and the consensus zone $\Phi = \emptyset$. If for all $j \in \mathbf{N}$, $i \in \mathcal{N}_j$, $f_{ij}(x_i) = k_{ij}(x_i)x_i + m_{ij}(x_i)$ is a continuous and piecewise linear function with its slopes $k_{ij} \in (-1, 1]$ and $m_{ij} = 0$ when $k_{ij} = 1$, then the MAS (1) has a unique, asymptotically stable equilibrium point.

IV. Numeral Example

In this section, four numeral examples are presented to illustrate the theorems and corollaries proposed in this paper. Examples 1 and 2 illustrate the consensus theorem. Additionally, example 3 illustrates the theorem for stability, uniqueness of equilibrium, i.e., Theorem 5.

In all following examples, the number of agents $n = 5$ with underlying strongly connected graphs. For simplicity, in examples 2 and 3, let $f_{ij}(x) = f_i(x)$, $\forall\, j \in \mathbf{N}$.

TABLE I
Configuration of Transmission Constraints in Example 1

Transmission Constraints f_{ij}					
f_{ij} \ j ; i	1	2	3	4	5
1	—	—	$(f_b + f_d)/2$	—	—
2	—	—	f_a	f_b	$(f_e + f_a)/2$
3	f_d	—	—	—	f_c
4	f_a	f_b	$(f_c + f_d)/2$	—	—
5	f_c	f_e	—	—	—

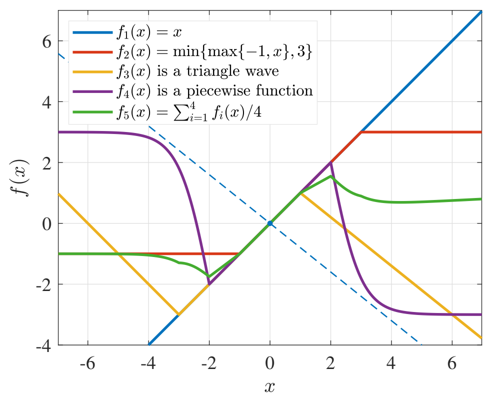

Fig. 3. The constraint functions in Example 2.

Example 1: The adjacency matrix in this example is

$$\mathcal{A}_1 = \begin{bmatrix} 0 & 0 & 3.6 & 0 & 0 \\ 0 & 0 & 4.6 & 1.3 & 6.5 \\ 3.6 & 0 & 0 & 0 & 7.6 \\ 0.5 & 1.4 & 2.1 & 0 & 0 \\ 2.9 & 6.5 & 0 & 0 & 0 \end{bmatrix},$$

Fig. 1 shows candidates for the constraint function imposed on the information transmissions, and the configuration of constraint functions is shown in Table I. It shows that the consensus zone $\Phi = \{0\}$. Fig. 2 shows that MAS achieves transmission-constrained consensus with $\lim_{t\to\infty} x(t) = 0$. In this example, the constraint function $f_e(x)$ is a piecewise continuous function similar to a sawtooth wave. And the constraint function $f_c(x)$ can be chosen approximately as the boundary rays since it satisfies the Condition (ii) of Theorem 2 and $k_1 k_2 = 0.8 < 1$.

Example 2: The adjacency matrix in this example is

$$\mathcal{A}_2 = \begin{bmatrix} 0 & 2.5 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4.5 \\ 0 & 5.6 & 0 & 3.3 & 0 \\ 0.5 & 0 & 0 & 0 & 0 \\ 1.9 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In Fig. 3, we can get that the consensus zone $\Phi = [-1, 1]^5$. The auxiliary line in Fig. 3 represents the boundary rays with $k_1 k_2 = 0.64 < 1$. Fig. 4 shows that $x(t) \to \Phi$ as $t \to \infty$.

Example 3: The adjacency matrix in this continuous-time example is $\mathcal{A}_2$. Fig. 6 shows that no matter which initial states of agents are, MAS will reach the same equilibrium,

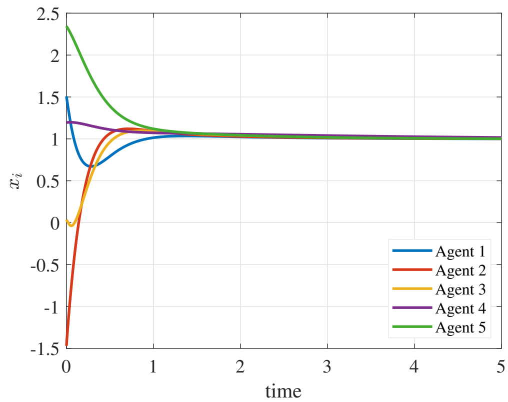

Fig. 4. The trajectories of $\mathbf{x}(t)$ in Example 2.

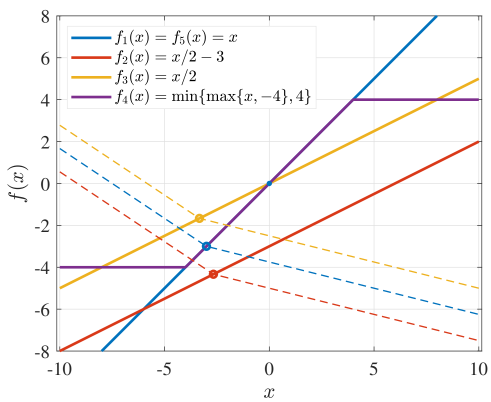

Fig. 5. The constraint functions in Example 3.

i.e., unique equilibrium (see the points $(\mathbf{e}_i, f_i(\mathbf{e}_i))$ shown by circles in Fig. 5). It is easy to know that for any equilibrium $\mathbf{e}$, there exist two clusters of rays (see auxiliary lines in Fig. 5) satisfying the Condition (iii) of Theorem 4 or the Condition (iii) of Theorem 5, which means that the system will converge to a unique, asymptotically stable equilibrium, even though we do not know the value of equilibrium.

V. Conclusion

This paper focuses on the transmission-constrained consensus problem of multiagent networks, where information transmissions between agents are affected by irregular constraint functions. We obtain the necessary and sufficient conditions about the range of transmission constraint functions where agents' states can converge to consensus. Due to the piecewise continuous constraint functions, the LaSalle invariance principle is not applicable in those proofs. We construct a sophisticated Lyapunov function and discuss the boundaries of multiple limit points of MAS states to facilitate the convergence analysis. Meanwhile, in some cases where the system cannot achieve consensus, there is an asymptotically stable and unique equilibrium independent of the initial values of agents' states. Finally, the numerical simulations are presented to verify the effectiveness of theoretical results.

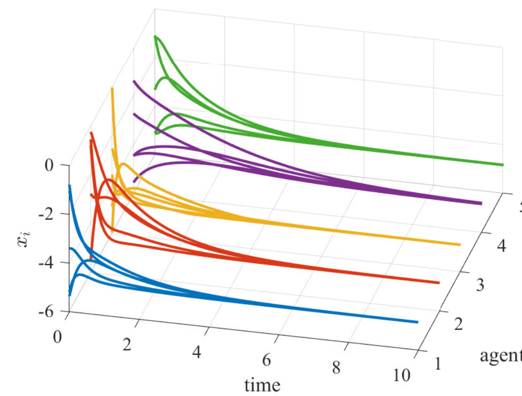

Fig. 6. The trajectories of $\mathbf{x}(t)$ in Example 3.

Appendix I
Proof of Theorem 1

A. Technical lemmas

First of all, we introduce some technical lemmas.

Lemma 1: **(Lemma 2.2 in [15])** If for all $i \in \mathbf{N}$, $Z_i(\mathbf{x}) : \mathbb{R}^d \to \mathbb{R}$ is of class C^1, and denote $Z(\mathbf{y}) = \max_{i\in\mathbf{N}} Z_i(\mathbf{y})$. Denote $\mathsf{N}_m(t) = \{i \in \mathbf{N} : Z(\mathbf{y}) = Z_i(\mathbf{y})\}$ the indices set in which the maximum is reached at time t. Then it turns out that $d^+Z(\mathbf{y}(t)) = \max_{i\in\mathsf{N}_m(t)} \dot{Z}_i(\mathbf{y}(t))$.

Lemma 2: If $\partial \in [\partial_m, \partial_M]$ and $k_1k_2 = 1$, then

$$\partial_M - \partial_m \geq \min\{(1-k_2)(\partial_M - \partial), (1-k_1)(\partial - \partial_m)\}.$$

Proof: When $\partial_M - \partial_m = 0$ or $(\partial_M - \partial)(\partial - \partial_m) = 0$, the conclusion is obvious.

When $\partial_M - \partial_m > 0$ and $(\partial_M - \partial)(\partial - \partial_m) > 0$, we use a contradiction argument to prove it. Let $\partial_M - \partial_m < \min\{(1-k_2)(\partial_M - \partial), (1-k_1)(\partial - \partial_m)\}$, i.e.,

$$\begin{cases} \partial_M - \partial_m < (1-k_2)(\partial_M - \partial); \\ \partial_M - \partial_m < (1-k_1)(\partial - \partial_m). \end{cases} \tag{4}$$

Since $\partial \in (\partial_m, \partial_M)$, we let $\partial = \rho\partial_M + (1-\rho)\partial_m$ with $\rho \in (0,1)$. Then, (4) can be rewritten as

$$\begin{cases} \partial_M - \partial_m < (1-k_2)(1-\rho)(\partial_M - \partial_m); \\ \partial_M - \partial_m < (1-k_1)\rho(\partial_M - \partial_m). \end{cases}$$

Then, we can get that $-k_2 > \frac{\rho}{1-\rho} > 0$ and $-k_1 > \frac{1-\rho}{\rho} > 0$, which implies that $k_1k_2 > 1$. We get a contradiction, and prove the Lemma 2. ■

The following lemma shows the implicit inequality from the given condition, and it helps us discuss the boundedness of transmission-constrained consensus dynamics.

Lemma 3: Denote $x_m(t) = \min_{i\in\mathbf{N}} x_i(t)$, $x_M(t) = \max_{i\in\mathbf{N}} x_i(t)$, $L_1(x_i(t)) = k_1x_i(t) + (1-k_1)\partial$, $L_2(x_i(t)) = k_2x_i(t) + (1-k_2)\partial$ and

$$Y(t) = \max\left\{L_1(x_m(t)) - x_m(t),\ x_M(t) - L_2(x_M(t))\right\},$$

where ∂ is a constant value. If $k_1, k_2 < 0$ and $k_1k_2 = 1$, then $\forall\, i \in \mathbf{N}$,

1) $Y(t) = x_M(t) - L_2(x_M(t)) \implies x_M(t) \geq L_1(x_i(t))$;

2) $Y(t) = L_1\big(x_m(t)\big) - x_m(t) \implies x_m(t) \le L_2\big(x_i(t)\big)$.

Proof: We first discuss Item 1, i.e., the case where $Y(t) = x_M(t) - L_2\big(x_M(t)\big)$, which means that $x_M(t) \ge \partial$.

Note that for any $i \in \mathbf{N}$, if $x_i(t) \ge \partial$, it is easy to know that,

$$x_M(t) \ge \partial \ge \partial + (1-k_1)\big(\partial - x_i(t)\big) = L_1(x_i(t)).$$

For any $i \in \mathbf{N}$, if $x_i(t) < \partial$,

$$\begin{aligned} &Y(t) = x_M(t) - L_2\big(x_M(t)\big) \\ \iff &(1-k_2)\big(x_M(t) - \partial\big) \ge (1-k_1)\big(\partial - x_i(t)\big) \\ \iff &x_M(t) - \partial \ge \frac{1-k_1}{1-k_2}\big(\partial - x_i(t)\big) \\ \iff &x_M(t) - \partial \ge k_1(x_i - \partial). \end{aligned}$$

Then we can get that $x_M(t) - k_1 x_i(t) - (1-k_1)\partial \ge 0$, which implies

$$x_M(t) \ge L_1\big(x_i(t)\big) = k_1 x_i(t) + (1-k_1)\partial, \ \ \forall\, i \in \mathbf{N}.$$

Therefore, the proof of Item 1 is completed. The proof method of Item 2 is similar to that of Item 1, and hence is omitted here. Hence, Lemma 3 is proved. ■

Lemma 4: For the MAS (1), if there exists an interval $[\partial_m, \partial_M]^n$ such that for all $j \in \mathbf{N}$, $i \in \mathcal{N}_j$,

$$\partial_m \le f_{ij}(x) \le \partial_M, \ \ x \in [\partial_m, \partial_M],$$

then $[\partial_m, \partial_M]^n$ is a positively invariant set.

Proof: The dynamics of MAS (1) can be rewritten as

$$\dot{x}(t) = h\big(x(t)\big) = \Big(h_1\big(x(t)\big), \ldots, h_n\big(x(t)\big)\Big)^T, \qquad (5)$$

where $h_i\big(x(t)\big) = \sum_{j \in \mathcal{N}_i} a_{ij}\Big[f_{ji}\big(x_j(t)\big) - x_i(t)\Big]$.

The initial states of (5) is $x_0 = x(t_0)$. Assume that $x(t_0) \in [\partial_m, \partial_M]^n$. Since the vector field h is pointing inwards $[\partial_m, \partial_M]^n$ that is an n-dimensional cube, it concludes that

$$x(t) \in [\partial_m, \partial_M]^n, \ \ \forall\, t \ge t_0.$$

It shows that $[\partial_m, \partial_M]^n$ is a positively invariant set and the proof is completed. ■

Lemma 5: Along the system (1), suppose there exist an interval $[\partial_m, \partial_M]$, a value $\partial \in [\partial_m, \partial_M]$ and two rays

$$\begin{aligned} L_1(x) &= k_1(x - \partial) + \partial, \quad x \in (-\infty, \partial]; \\ L_2(x) &= k_2(x - \partial) + \partial, \quad x \in [\partial, +\infty), \end{aligned}$$

where $k_1 k_2 = 1$, such that $\forall\, j \in \mathbf{N}$, $i \in \mathcal{N}_j$,

$$\begin{aligned} x \le f_{ij}(x) &\le L_1(x), \quad x \in (-\infty, \partial_m); \\ \partial_m \le f_{ij}(x) &\le \partial_M, \quad x \in [\partial_m, \partial_M]; \\ L_2(x) \le f_{ij}(x) &\le x, \quad x \in (\partial_M, +\infty). \end{aligned}$$

Denote $x_m(t) = \min\limits_{i \in \mathbf{N}} x_i(t)$, $x_M(t) = \max\limits_{i \in \mathbf{N}} x_i(t)$ and

$$\begin{aligned} Y(t) = \max\big\{&\partial_M - \partial_m, x_M(t) - \partial_m, \partial_M - x_m(t), \\ &x_M(t) - L_2\big(x_M(t)\big), L_1\big(x_m(t)\big) - x_m(t)\big\}. \end{aligned}$$

If $\partial_M - \partial_m \ge \max\{(1-k_2)(\partial_M - \partial), (1-k_1)(\partial - \partial_m)\}$, then $Y(t)$ is a non-increasing function for any initial state $x_* \in \mathbb{R}^n$.

Proof: By the structure of $Y(t)$, there exists five cases:

1) $Y(t) = Y_1(t) = x_M(t) - L_2\big(x_M(t)\big)$;
2) $Y(t) = Y_2(t) = x_M(t) - \partial_m$;
3) $Y(t) = Y_3(t) = L_1\big(x_m(t)\big) - x_m(t)$;
4) $Y(t) = Y_4(t) = \partial_M - x_m(t)$;
5) $Y(t) = Y_5(t) = \partial_M - \partial_m$.

The Case 1 is analyzed firstly. Since

$$x_M(t) - L_2\big(x_M(t)\big) \ge \partial_M - \partial_m \ge (1-k_2)(\partial_M - \partial),$$

it implies that $x_M(t) \ge \partial_M$.

Denote $\mathcal{I}_M(t) = \big\{k : x_k(t) = \max\limits_{l \in \mathbf{N}} x_l(t)\big\}$. By Lemma 1, we have

$$\begin{aligned} &d^+ Y_1(t) = d^+ \max_{i \in \mathbf{N}} \Big\{(1-k_2)\big(x_i(t) - \partial\big)\Big\} \\ &= \max_{i \in \mathcal{I}_M(t)} \Big\{(1-k_2) \sum_{j \in \mathcal{N}_i(t)} a_{ij}\Big(f_{ji}\big(x_j(t)\big) - x_i(t)\Big)\Big\}. \end{aligned}$$

For all $i \in \mathcal{I}_M(t)$, which means that $x_i(t) = x_M(t)$, and we conduct the following analysis:

$$\begin{cases} x_M \ge L_1(x_j) > f_{ji}(x_j) & \text{if } x_j \le \partial_m; \\ x_M \ge \partial_M \ge f_{ji}(x_j) & \text{if } x_j \in [\partial_m, \partial_M]; \\ x_M \ge x_j \ge f_{ji}(x_j) & \text{if } x_j > \partial_M, \end{cases}$$

where the first inequality follows from Lemma 3 and the fact that $x_M(t) - L_2\big(x_M(t)\big) \ge L_1\big(x_m(t)\big) - x_m(t)$.

Then, it can be concluded that $d^+ Y_1(t) \le 0$ when $Y(t) = Y_1(t) = x_M(t) - L_2\big(x_M(t)\big)$.

Secondly, the Case 2 is discussed. Since $x_M(t) - \partial_m \ge \partial_M - \partial_m$, it implies that $x_M(t) \ge \partial_M$.

Since $x_M(t) - \partial_m \ge L_1\big(x_m(t)\big) - x_m(t)$, we can get that

$$\begin{aligned} x_M(t) \ge& k_1 x_m(t) + (1-k_1)\partial + \partial_m - x_m(t) \\ =& L_1\big(x_m(t)\big) + \partial_m - x_m(t) \\ \ge& L_1\big(x_i(t)\big) + \partial_m - x_m(t), \ \ \forall\, i \in \mathbf{N}. \end{aligned}$$

It turns out that

$$\begin{cases} x_M \ge L_1(x_j) \ge f_{ji}(x_j) & \text{if } x_j \le \partial_m; \\ x_M \ge \partial_M \ge f_{ji}(x_j) & \text{if } x_j \in [\partial_m, \partial_M]; \\ x_M \ge x_j \ge f_{ji}(x_j) & \text{if } x_j > \partial_M, \end{cases}$$

where the first inequality follows from $x_M \ge L_1(x_i) + \partial_m - x_m \ge L_1(x_i)$ when $x_m \le \partial_m$.

Then, it shows that

$$d^+ Y_2(t) = \max_{i \in \mathcal{I}_M(t)} \Big\{ \sum_{j \in \mathcal{N}_i(t)} a_{ij}\Big(f_{ji}\big(x_j(t)\big) - x_i(t)\Big)\Big\} \le 0.$$

The analyses of cases 3 and 4 are symmetric to those of cases 1 and 2, hence they are omitted. As for Case 5, the conclusion is obvious.

Therefore, by the above five cases, it can be concluded that $Y(t)$ is a non-increasing function. ■

Lemma 6: Suppose the MAS (1) satisfies the conditions in Lemma 5. Denote the initial time t_0. Then for any $t \ge t_0$, we have

1) $Y(t_0) = \partial_M - \partial_m \implies \partial_m \le x_i(t) \le \partial_M, \forall\, i \in \mathbf{N}$;
2) $Y(t_0) = x_M(t_0) - L_2\big(x_M(t_0)\big) \implies x_m(t) \ge L_2\big(x_M(t_0)\big)$;

3) $Y(t_0) = L_1\big(x_m(t_0)\big) - x_m(t_0) \implies x_M(t) \leq L_1\big(x_m(t_0)\big)$;
4) $Y(t_0) = x_M(t_0) - \partial_m \implies x_m(t) \geq \min\{x_m(t_0), \partial_m\}$;
5) $Y(t_0) = \partial_M - x_m(t_0) \implies x_M(t) \leq \max\{x_M(t_0), \partial_M\}$.

Proof: $Y(t_0) = \partial_M - \partial_m$ means that $\mathbf{x}(t_0) \in [\partial_m, \partial_M]^n$. By Lemma 4, we can get that $[\partial_m, \partial_M]^n$ is a positively invariant set, and this case is proven trivially.

When $Y(t_0) = x_M(t_0) - L_2\big(x_M(t_0)\big)$, if $x_m(t) < L_2\big(x_M(t_0)\big)$, we have

$$\begin{aligned}
&Y(t) \geq L_1\big(x_m(t)\big) - x_m(t) > (1-k_1)\big(\partial - L_2(x_M(t_0))\big)\\
=&-k_2(1-k_1)\big(x_M(t_0) - \partial\big) = (1-k_2)\big(x_M(t_0) - \partial\big)\\
=&Y(t_0),
\end{aligned}$$

which contradicts the Lemma 5. By symmetry, the case where $Y(t_0) = L_1\big(x_m(t_0)\big) - x_m(t_0)$ is also proven.

When $Y(t_0) = x_M(t_0) - \partial_m$, it means that $L_2\big(x_M(t_0)\big) \geq \partial_m$. Since $x_M(t) \leq x_M(t_0)$, $\forall\, t \geq t_0$, we have $L_2\big(x_i(t)\big) \geq L_2\big(x_M(t_0)\big)$, $\forall\, i \in \mathbf{N}$, $t \geq t_0$. Hence, it can be concluded that $f_{ji}\big(x_j(t)\big) \geq \min\{x_j(t), \partial_m, L_2\big(x_j(t)\big)\} \geq \min\{x_m(t), \partial_m\}$, $\forall\, i, j \in \mathbf{N}$. Therefore, we can get $x_m(t) \geq \min\{x_m(t_0), \partial_m\}$. By symmetry, the case where $Y(t_0) = \partial_M - x_m(t_0)$ is also proven. This proof is completed.

When $Y(t_0) = \partial_M - x_m(t_0)$, it shows that $L_1\big(x_m(t_0)\big) \leq \partial_M$. Since $x_m(t) \geq x_m(t_0)$, $\forall\, t \geq t_0$. Hence, it can be concluded that $f_{ji}\big(x_j(t)\big) \leq \max\{x_j(t), \partial_M, L_1\big(x_m(t)\big)\} \leq \max\{x_M(t), \partial_M\}$, $\forall\, i, j \in \mathbf{N}$. Then, a contradiction argument is used to prove that $x_M(t) \leq \max\{x_M(t_0), \partial_M\}$, $\forall\, t \geq t_0$. Assume that $\exists\, t_* \geq t_0$, $x_M(t_*) > \max\{x_M(t_0), \partial_M\}$. Hence, there exists a $T \geq t_0$, such that for any $t \in [t_0, T]$,

$$\begin{cases}
x_M(t) \leq \max\{x_M(t_0), \partial_M\},\\
x_M(T) = \max\{x_M(t_0), \partial_M\},\\
d^+ x_M(T) > 0.
\end{cases}$$

Since

$$\begin{aligned}
&d^+ x_M(T) = \max_{i \in \mathcal{I}_M(T)} \dot{x}_i(T)\\
=&\max_{i \in \mathcal{I}_M(T)} \sum_{j \in \mathcal{N}_i(T)} a_{ij}\Big(f_{ji}\big(x_j(T)\big) - x_i(T)\Big)\\
\leq&\max_{i \in \mathcal{I}_M(T)} \sum_{j \in \mathcal{N}_i(T)} a_{ij}\big(\max\{x_i(T), \partial_M\} - x_i(T)\big)\\
=&\max_{i \in \mathcal{I}_M(T)} \sum_{j \in \mathcal{N}_i(T)} a_{ij}\big(\max\{x_M(t_0), \partial_M\}\\
&\quad - \max\{x_M(t_0), \partial_M\}\big) = 0,
\end{aligned}$$

which leads to a contradiction and it shows that $x_M(t) \leq \max\{x_M(t_0), \partial_M\}$, $\forall\, t \geq t_0$. By symmetry, the case where $Y(t_0) = \partial_M - x_m(t_0)$ is also proven. ∎

B. Proof of Theorem 1

Proof:

1) Necessity: A contradiction argument is applied to prove the necessity.

For simplicity, we assume that there are only two agents in MAS (1), i.e., $\mathbf{N} = \{1, 2\}$. Since $\mathcal{G}$ is strongly connected, it turns out that $\mathcal{N}_1 = \{2\}$ and $\mathcal{N}_2 = \{1\}$.

Denote the initial time $t_0 \geq 0$. Suppose there exist $j \in \mathbf{N}, i \in \mathcal{N}_j$ and $x_j(t_0) \in [\partial_m, \partial_M]$ such that

$$f_{ji}\big(x_j(t_0)\big) = \partial_M + \omega, \;\; \omega > 0.$$

Without loss of generality, assume that $j = 1$ and $i = 2$, i.e., $f_{12}\big(x_1(t_0)\big) = \partial_M + \omega$ in which $x_1(t_0) \in [\partial_m, \partial_M]$. Let $x_2(t_0) = f_{12}\big(x_1(t_0)\big)$ and $f_{21}\big(x_2(t_0)\big) = x_1(t_0)$, then it can be concluded that for all $t \geq t_0$,

$$\begin{aligned}
\dot{x}_1(t) &= a_{12}\Big(f_{21}\big(x_2(t)\big) - x_1(t)\Big) = 0,\\
\dot{x}_2(t) &= a_{21}\Big(f_{12}\big(x_1(t)\big) - x_2(t)\Big) = 0.
\end{aligned}$$

which implies that $x_1(t) = x_1(t_0)$, $x_2(t) = x_2(t_0)$, $\forall\, t \geq t_0$.

Moreover, because

$$\begin{cases}
x_1(t_0) \in [\partial_m, \partial_M],\\
f_{12}\big(x_1(t_0)\big) = \partial_M + \omega > \partial_M,\\
x_2(t_0) = f_{12}\big(x_1(t_0)\big) = \partial_M + \omega > \partial_M,\\
f_{21}\big(x_2(t_0)\big) = x_1(t_0) \leq \partial_M,
\end{cases}$$

it is easy to find two rays L_1 and L_2 satisfying the Assumptions 3, 4, and the condition $k_1 k_2 = 1$ is also satisfied.

Since $\forall\, t \geq t_0$, $x_2(t) = \partial_M + \omega > \partial_M$, it shows that

$$\lim_{t \to \infty} \text{distance}\big([\partial_m, \partial_M]^n, \mathbf{x}(t)\big) \neq 0.$$

Hence, we get a contradiction and the proof for the necessity statement of Theorem 1 is proved.

2) Sufficiency: We prove it in three steps.

Step 1: Since $\partial \in [\partial_m, \partial_M]$ and $k_1 k_2 < 1$, by Lemma 2, there exist two possibilities:

1) $\partial_M - \partial_m \geq \max\{(1-k_2)(\partial_M - \partial), (1-k_1)(\partial - \partial_m)\}$;
2) $\partial_M - \partial_m < (1-k_2)(\partial_M - \partial)$ or $\partial_M - \partial_m < (1-k_1)(\partial - \partial_m)$.

We discuss the Possibility 1) in the rest of Step 1, and the Possibility 2) is analyzed in Step 2.

Assume that $\partial_M - \partial_m \geq \max\{(1-k_2)(\partial_M - \partial), (1-k_1)(\partial - \partial_m)\}$. Form Lemma 5, we have that $Y(t)$ is a non-increasing function.

Let n be the number of agents. Since

$$\begin{aligned}
Y(t) =& \max\big\{\partial_M - \partial_m, x_M(t) - \partial_m, \partial_M - x_m(t),\\
&x_M(t) - L_2\big(x_M(t)\big), L_1\big(x_m(t)\big) - x_m(t)\big\},
\end{aligned}$$

we continue this proof case by case.

Case 1: $Y(t_0) = Y_1(t_0) = x_M(t_0) - L_2\big(x_M(t_0)\big)$.

By Lemma 6, we have for any $t \geq t_0$,

$$\begin{aligned}
&L_1\big(x_m(t)\big) = k_1\big(x_m(t) - \partial\big) + \partial\\
\leq&k_1 k_2 x_M(t_0) + k_1(1-k_2)\partial + (1-k_1)\partial = x_M(t_0).
\end{aligned}$$

Choose $i_0 \in \mathcal{I}_0 := \{i : x_i(t_0) = x_m(t_0)\}$. For any $j \in \mathcal{N}_{i_0}$, we can get that

$$f_{ji_0}\big(x(t)\big) \leq \max\big\{x_M(t), L_1\big(x_m(t)\big), \partial_M\big\} \leq x_M(t_0).$$

It implies that

$$\dot{x}_{i_0}(t) = \sum_{j\in\mathcal{N}_{i_0}} a_{i_0j}\big[f_{ji_0}\big(x_j(t)\big) - x_{i_0}(t)\big] \\ \le \alpha_{i_0}[x_M(t_0) - x_{i_0}(t)],$$

which implies that

$$x_{i_0}(t) \le e^{-\alpha_{i_0}(t-t_0)}x_m(t_0) + [1 - e^{-\alpha_{i_0}(t-t_0)}]x_M(t_0).$$

If $t \in [t_0, t_0+\tau]$, then we have for any $i_0 \in \mathcal{I}_0$,

$$x_{i_0}(t) \le \gamma_0 x_m(t_0) + (1-\gamma_0)x_M(t_0), \tag{6}$$

where $\gamma_0 = e^{-\tau\bar{a}}$.

Choose $i_1 \in \mathcal{I}_1 := \{i : \exists j \in \mathcal{I}_0,\ j \in \mathcal{N}_i\}$. By the conditions of Theorem 1 and the equation (6), it is trivial to get that $f_{i_0i_1}\big(x_{i_0}(t)\big) < x_M(t_0)$. Hence, for any $t \in [t_0, t_0+\tau/n]$, there exists a constant $\gamma_0' \in (0,1)$ such that

$$f_{i_0i_1}\big(x_{i_0}(t)\big) \le \gamma_0' x_m(t_0) + (1-\gamma_0')x_M(t_0).$$

Then, we can get that

$$\begin{aligned}
&x_{i_1}(t_0+\frac{\tau}{n}) \\
\le& e^{-\alpha_{i_1}\frac{\tau}{n}}x_{i_1}(t_0) + \big[a_{i_1i_0}\gamma_0'\big(x_m(t_0)-x_M(t_0)\big) \\
&+ \alpha_{i_1}x_M(t_0)\big]\int_{t_0}^{t_0+\frac{\tau}{n}} e^{-\alpha_{i_1}(t_0+\frac{\tau}{n}-s)}ds \\
\le& e^{-\alpha_{i_1}\frac{\tau}{n}}x_M(t_0) + (1-e^{-\alpha_{i_1}\frac{\tau}{n}})x_M(t_0) \\
&+ a_{i_1i_0}\gamma_0'\big(x_m(t_0)-x_M(t_0)\big)\int_{t_0}^{t_0+\frac{\tau}{n}} e^{-\alpha_{i_1}(t_0+\frac{\tau}{n}-s)}ds \\
=& x_M(t_0) + \frac{a_{i_1i_0}}{\alpha_{i_1}}(1-e^{-\alpha_{i_1}\frac{\tau}{n}})\gamma_0'\big(x_m(t_0)-x_M(t_0)\big).
\end{aligned}$$

Since there exists a constant $\rho_1 > 0$ such that for any $i_1 \in \mathcal{I}_1$, $i_0 \in \mathcal{N}_{i_1}$, $\rho_1 \le \frac{a_{i_1i_0}}{\alpha_{i_1}}(1-e^{-\alpha_{i_1}\frac{\tau}{n}})$. Therefore, it means $x_{i_1}(t_0+\frac{\tau}{n}) \le \rho_1\gamma_0' x_m(t_0) + (1-\rho_1\gamma_0')x_M(t_0)$. Similar to (6), we can get that for any $i_1 \in \mathcal{I}_1$, $t \in [t_0+\frac{\tau}{n}, t_0+\tau]$,

$$x_{i_1}(t) \le \gamma_1 x_m(t_0) + (1-\gamma_1)x_M(t_0), \tag{7}$$

where $\gamma_1 = \rho_1\gamma_0'\gamma_0$. Continuing the above analysis over $[t_0+\frac{\tau}{n}m, t_0+\tau]$, $\forall\, m = 1,2,\ldots,n-1$, it can be concluded that for all $i \in \mathbf{N}$,

$$x_i(t_0+\tau) \le \gamma_{n-1}x_m(t_0) + (1-\gamma_{n-1})x_M(t_0),$$

where $\gamma_{n-1} = \rho_{n-1}\gamma_{n-2}'\gamma_0$.

If $x_m(t_0) < x_M(t_0)$, then there exists a constant $\omega \in (0,1]$ such that $x_m(t_0) \le \omega\partial + (1-\omega)x_M(t_0)$. Here, we have

$$Y_1(t_0+\tau) = (1-k_2)\big(x_M(t_0+\tau)-\partial\big) \le (1-\omega\gamma_{n-1})Y_1(t_0).$$

If $x_m(t_0) = x_M(t_0) > \partial_M$, we use the Assumption 4 to get the convergence. Since there exists i_0, i_1 such that $f_{i_0i_1}\big(x_{i_0}(t_0)\big)$ is continuous on $x_{i_0}(t_0) = x_M(t_0)$ and $f_{i_0i_1}\big(x_{i_0}(t_0)\big) < x_M(t_0)$. Hence, there exists $T(\omega')$ such that $\forall\, t \in [t_0, t_0+T(\omega')]$, $f_{i_0i_1}\big(x_{i_0}(t)\big) \le \omega'\partial + (1-\omega')x_M(t_0)$, where the constant $\omega' \in (0,1]$. Similar to (7), we have $\forall\, t \in [t_0+T(\omega'), t_0+nT(\omega')]$,

$$x_{i_1}(t) \le \omega_1\partial + (1-\omega_1)x_M(t_0),$$

where $\omega_1 = \rho_1\omega'\gamma_0$. Furthermore, it shows that

$$Y_1\big(t_0+nT(\omega')\big) \le (1-\omega_{n-1})Y_1(t_0),$$

where $\omega_{n-1} = \rho_{n-1}\omega_{n-2}'\gamma_0$.

The analysis of Case 1 is completed. The case $Y(t_0) = Y_3(t_0) = L_1\big(x_m(t_0)\big) - x_m(t_0)$ is symmetric to Case 1, so we omit its analysis.

Case 2: $Y(t_0) = Y_2(t_0) = x_M(t_0) - \partial_m$.

By Lemma 6, we can get that $\forall\, i \in \mathbf{N}$, $t \ge t_0$, $x_M(t_0) \ge L_1\big(x_i(t)\big)$. Hence, it is trivial to get that

$$x_i(t_0+\tau) \le \gamma_{n-1}x_m(t_0) + (1-\gamma_{n-1})x_M(t_0),\ \ \forall\, i \in \mathbf{N}.$$

Furthermore, we can get that $Y_2(t_0+\tau) \le (1-\omega\gamma_{n-1})Y_2(t_0)$ or $Y_2\big(t_0+nT(\omega')\big) \le (1-\omega_{n-1})Y_2(t_0)$. The case $Y(t_0) = Y_4(t_0) = \partial_M - x_m(t_0)$ is symmetric to this case, so we omit its analysis.

Finally, we get $x(t) \to [\partial_m, \partial_M]^n$ as $t \to \infty$, and the proof of the Possibility 1 is completed.

Step 2: In this step, we will complete the proof of Possibility 2, i.e., $\partial_M - \partial_m < (1-k_2)(\partial_M - \partial)$ or $\partial_M - \partial_m < (1-k_1)(\partial - \partial_m)$. By symmetry, let $\partial_M - \partial_m < (1-k_1)(\partial-\partial_m)$, which implies that $\exists\, \partial_M' > \partial_M$, such that $\partial_M' - \partial_m = (1-k_1)(\partial - \partial_m)$. Denote

$$\begin{aligned}
Y'(t) =& \max\big\{\partial_M' - \partial_m, x_M(t) - \partial_m, \partial_M' - x_m(t), \\
&(1-k_2)\big(x_M(t)-\partial\big), (1-k_1)\big(\partial - x_m(t)\big)\big\}.
\end{aligned}$$

Similar to the proof in Step 1, it concludes that $\mathbf{x}(t) \to [\partial_m, \partial_M']^n$ as $t \to \infty$. Then, we use a contradiction argument to prove that for any solution $\mathbf{x}(t) \to [\partial_m, \partial_M]^n$ as $t \to \infty$.

Assume that there exist a solution $\hat{x}(t)$ and $i^* \in \mathbf{N}$, such that $\hat{x}_{i^*}(t) \to (\partial_M, \partial_M']$ as $t \to \infty$. Then, there is a T^* such that for any $t > T^*$ and $j \in \mathbf{N}$, $f_{i^*j}\big(\hat{x}_{i^*}(t)\big) > \partial_m$.

Since the directed graph $\mathcal{G}$ is strongly connected, we can get that there exists a $T' > T^*$ such that $\forall\, t > T'$, $i \in \mathbf{N}$, $\hat{x}_i(t) > \partial_m$.

Denote $Z(t) = \max\{\hat{x}_M(t), \partial_M\}$. Repeat the analysis of Step 1, it shows that $\lim\limits_{t\to\infty} Z(t) = \partial_M$, which implies that

$$\limsup_{t\to\infty} \hat{x}_{i^*}(t) \le \partial_M.$$

Therefore, the trajectory of $\hat{x}_{i^*}(t)$ cannot converge to $(\partial_M, \partial_M']$ as $t \to \infty$. Here we prove that for any solution $\mathbf{x}(t) \to [\partial_m, \partial_M]^n$ as $t \to \infty$.

The proof is completed. ■

Remark 11: In the sufficiency proof, the idea of constructing auxiliary variables to analyze the boundedness of MAS is inspired by [6]. However, since our dynamics is not Lipschitz continuous, the LaSalle invariance principle is not applicable. We design some linear boundaries and propose the corresponding lemmas to eliminate nonlinearity, and analyze the states' trending to obtain the boundedness of agents' states.

APPENDIX II
PROOF OF THEOREM 2

A. Technical lemma

Lemma 7: **(Proposition 4.10 in [3])** Let graph $\mathcal{G}$ has a directed spanning tree, and consider the dynamics of MAS defined over $\mathcal{G}$:

$$\dot{x}_i(t) = \sum_{j=1}^{N} a_{ij}\big(x_j(t) - x_i(t)\big) + \theta_i(t), \quad i = \mathbf{N},$$

in which $\theta_i(t)$ is piecewise continuous on $[t_0, \infty)$ and is finite. If $\lim_{t\to\infty} \theta_i(t) = 0$, $\forall\, i \in \mathbf{N}$, then $\lim_{t\to\infty} x_i(t) - x_j(t) = 0$, $\forall\, i, j \in \mathbf{N}$.

B. Proof of Theorem 2

Proof:

1) Necessity: Antagonistic interaction means that the underlying edges between agents have negative weights. A signed graph $\mathcal{G}_A$ is **structurally balanced** if there exists a bipartition $\{\mathcal{V}_1, \mathcal{V}_2\}$ of the nodes, where $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$, $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, such that $a_{ij} \geq 0\ \forall\, v_i, v_j \in \mathcal{V}_m$ $(m \in \{1,2\})$ and $a_{ij} \leq 0\ \forall\, v_i \in \mathcal{V}_m$, $v_j \in \mathcal{V}_l$, $m \neq l$ $(m, l \in \{1,2\})$ [16]. We use a contradiction argument. Consider the MAS with antagonistic interactions

$$\dot{x}_i(t) = \sum_{j=1}^{N} |a^s_{ij}|\big(\mathrm{sign}(a^s_{ij}) \cdot x_j(t) - x_i(t)\big), \tag{8}$$

in which the signed graph $\mathcal{G}_s = \{\mathcal{V}_s, \mathcal{E}_s, \mathcal{A}_s = [a^s_{ij}]\}$, and suppose the signed graph $\mathcal{G}_s$ is strongly connected and structurally balanced. By the bipartite consensus theorem (Theorem 2 in [16]), it shows that the system (8) reaches bipartite consensus but not consensus, in which agents' state values are the same except for the sign.

Let $\mathcal{A}' = |\mathcal{A}_s| = [|a^s_{ij}|]$ and $\mathcal{G}' = \{\mathcal{V}_s, \mathcal{E}_s, \mathcal{A}'\}$, i.e., $\mathcal{G}'$ is a strongly connected graph with only cooperative interactions. Assume the MAS (1) is under the graph $\mathcal{G}'$, and for all $i \in \mathbf{N}$, $j \in \mathcal{N}_i$, $f_{ji}\big(x_j(t)\big) = \mathrm{sgn}(a^s_{ij}) \cdot x_j(t)$, which implies that $k_1 k_2 > 1$. With the above assumption, the dynamics of MAS (1) is equivalent to the dynamics of MAS (8). Therefore, it is turns out that MAS (1) cannot achieves consensus.

On the other hand, it is obvious that under the above assumption, the system (1) satisfies all conditions in Theorem 2. By Theorem 2, the states of agents will converge to a consensus value. Hence, we get a contradiction and the proof for the necessity statement of Theorem 2 is proved.

2) Sufficiency: Applying Theorem 1, we have:

$$\mathbf{x}(t) \to [\partial_m, \partial_M]^n, \text{ as } t \to \infty. \tag{9}$$

Notice that if $\partial_m = \partial_M$, then it turns out that $\lim_{t\to+\infty} x_i(t) = \partial_m = \partial_M = v^*$, $\forall\, i \in \mathbf{N}$, and therefore the sufficiency statement of Theorem 2 is proved.

Hence, we continue our proof in the condition that $\partial_m < \partial_M$. Assume $\partial_m < \partial_M$ in the following.

Step 1: In this step, it shows that the states of agents tend to achieve consensus.

Denote $\theta_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}\Big(f_{ji}\big(x_j(t)\big) - x_j(t)\Big)$, and the dynamics of MAS (1) can be rewritten as

$$\frac{d}{dt}x_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}\big(x_j(t) - x_i(t)\big) + \theta_i(t). \tag{10}$$

Then, we use a contradiction argument to prove that $\lim_{t\to\infty} \theta_i(t) = 0$, $\forall\, i \in \mathbf{N}$. Without loss of generality, assume that there exists a solution $x(t)$, such that $\forall\, i \in \mathbf{N}$, $\liminf_{t\to\infty} \theta_i(t) \geq 0$ and $\exists\, i^* \in \mathbf{N}$, $\liminf_{t\to\infty} \theta_{i^*}(t) > 0$.

$\mathcal{P} = \{\chi_1, \chi_2, \dots\}$ denotes the set of all limit points of $x_{i^*}(t)$ as $t \to \infty$, i.e., there are time sequences $\{t'_n\}$ with $\lim_{n\to\infty} t'_n = \infty$ and $\lim_{n\to\infty} x_{i^*}(t'_n) = \chi_1$, and $\{t''_n\}$ with $\lim_{n\to\infty} t''_n = \infty$ and $\lim_{n\to\infty} x_{i^*}(t''_n) = \chi_2$, etc. Since $\liminf_{t\to\infty} \theta_{i^*}(t) > 0$, it turns out that $\exists\, i' : i^* \in \mathcal{N}_{i'}$, such that

$$\sup_{\chi_i \in \mathcal{P}} \Big\{ \limsup_{x \to \chi_i} \big(f_{i^*i'}(x) - x\big) \Big\} > 0.$$

Furthermore, since $x_{i^*}(t) \to [\partial_m, \partial_M]$ as $t \to \infty$, it can be concluded that $\limsup_{x\to\partial_m} f_{i^*i'}(x) > \partial_m$.

Therefore, there is a time sequence $\{t^*_n\} \to \infty$ with $\{x_{i^*}(t^*_n)\} \to \partial_m$ and $\lim_{n\to\infty} f_{i^*i'}\big(x_{i^*}(t^*_n)\big) > \partial_m$. It is clear that $\partial_m \in \mathcal{P}$.

Since $f_{i^*i'}(x)$ is continuous in $[\partial_m, \partial_M]$ and has finite breaks in $(-\infty, \partial_m)$, there exists a time sequence $\{t^\sim_n\} \subseteq \{t^*_n\}$ and $f_{i^*i'}(x)$ is continuous on $x_{i^*}(t)$, $\forall\, t \in \{t^\sim_n\}$, which implies that $f_{i^*i'}\big(x_{i^*}(t)\big)$ is continuous on $\{t^\sim_n\}$.

Further, there exist $\epsilon > 0$ and $T(\epsilon)$, such that $f_{i^*i'}\big(x_{i^*}(t)\big) > \partial_m$ and $f_{i^*i'}\big(x_{i^*}(t)\big)$ is continuous in $(t - \epsilon, t + \epsilon)$, for all $t > T(\epsilon)$ and $t \in \{t^\sim_n\}$. Denote closed and connected interval $\mathbf{I}_k = [t_k - \frac{\epsilon}{2}, t_k + \frac{\epsilon}{2}]$, where $t_k > T(\epsilon)$ and $t_k \in \{t^\sim_n\}$, $k \in \mathbb{N}^+$.

Since $\liminf_{t\to\infty} \theta_i(t) \geq 0$, $\forall\, i \in \mathbf{N}$, it can be concluded that $\liminf_{t\to\infty} f_{ij}\big(x_i(t)\big) \geq \partial_m$, $\forall\, i, j \in \mathbf{N}$.

Notice that $\int_{\mathbf{I}_k} f_{i^*i'}\big(x_{i^*}(t)\big) > \partial_m$ and repeat the analysis of Theorem 1, we can get that $\liminf_{t\to\infty} x_{i^*}(t) > \partial_m$. Then, it turns out that $\partial_m \notin \mathcal{P}$. Here, we find a contradiction.

Hence, we have proven that $\lim_{t\to\infty} \theta_i(t) = 0$, $\forall\, i \in \mathbf{N}$.

Applying Lemma 7, we can get that

$$\lim_{t\to+\infty} x_i(t) - x_j(t) = 0, \quad \forall\, i, j \in \mathbf{N}. \tag{11}$$

Step 2: In Step 1, it shows that the states of agents will converge to consensus. In this step, by the fact that $x(t) \to [\partial_m, \partial_M]^n$, we prove that for any $i \in \mathbf{N}$, $\lim_{t\to\infty} x_i(t) = v^*$ and $v^* \in [\partial_m, \partial_M]$.

By (9), it turns out that for any $\omega_1 > 0$, there exists a finite $T_1 > 0$, which holds the following inequation:

$$\partial_m - \omega_1 \leq x_i(t) \leq \partial_M + \omega_1, \quad \forall\, t \geq T_1,\, i \in \mathbf{N}. \tag{12}$$

Without loss of generality, assume that $\frac{\partial_m + \partial_M}{2} \leq x_k(T_1) \leq \partial_M + \omega_1$, where k is a fixed node.

Similarly, by (11), $\exists\, T_2 > 0$ which is finite, there holds

$$|x_i(t) - x_k(t)| \leq \omega_2, \quad \forall\, t \geq T_2,\, i \in \mathbf{N}. \tag{13}$$

According to (12) and (13), let ω_1 and ω_2 be sufficiently small, and we get that $\partial_m < x_i(T_*) < \partial_M + \omega_1 + \omega_2$, $\forall\, i \in \mathbf{N}$, where $T_* > \max\{T_1, T_2\}$.

Depending on whether $\exists\, l \in \mathbf{N}$, $x_l(T_*) \geq \partial_M$ or not, there are two cases in the following proof.

1) $\exists\, l \in \mathbf{N}$, $x_l(T_*) \geq \partial_M$. Repeating the analysis in Step 1 in the proof of Theorem 1, we can get that

$$\bar{Y}(t) = \max_{i\in\mathbf{N}} \Big\{ (1-k_2)\big(x_i(t) - \partial\big) \Big\}$$

is non-increasing for $t \geq T_*$. Since $1 - k_2 > 0$, it turns out that $\max_{i\in\mathbf{N}} x_i(t)$ is non-increasing for $t \geq T_*$.

2) $\forall\, i \in \mathbf{N}$, $\partial_m < x_i(T_*) < \partial_M$. It is easy to get $f_{ij}\big(x_i(T_*)\big) = x_i(T_*)$, $\forall\, i,j \in \mathbf{N}$, and the system degenerates into a standard multiagent system at time T_*. Therefore, it is easy to know that $\max_{i\in\mathbf{N}} x_i(t)$ is non-increasing for $t \geq T_*$.

Combining the above analyses, we can conclude that $\max_{i\in\mathbf{N}} x_i(t)$ is non-increasing for $t \geq T_*$. Furthermore, $\max_{i\in\mathbf{N}} x_i(t)$ converges to a finite limit value (denote the value by $\overline{v}$). According to (11), $\min_{i\in\mathbf{N}} x_i(t)$ must converge to the same limit value $\overline{v}$. Since $\min_{i\in\mathbf{N}} x_i(t) \leq x_j(t) \leq \max_{i\in\mathbf{N}} x_i(t)$, $\forall\, j \in \mathbf{N}$, it is trivial to get that $\lim_{t\to\infty} x_i(t) = \overline{v} = v^*$, $\forall\, i \in \mathbf{N}$.

Using (9), we can conclude that $v^* \in [\partial_m, \partial_M]$. Based on the above analysis, it is shown that all $x_i(t)$ will converge to a finite limit v^* and $v^* \in [\partial_m, \partial_M]$. ■

Remark 12: The robust consensus idea is inspired by [6]. However, since our dynamics is not Lipschitz continuous, the system may have multiple solutions. We analyze the limit points of multiple solutions and integrate the relevant variables over a short period to analyze the system's convergence.

Appendix III
Proof of Theorem 3

Proof: By Lemma 4, it shows $[\partial_m, \partial_M]^n$ is a positively invariant set. By the Brouwer fixed point Theorem extended to dynamical systems [17], we can conclude that there exists an equilibrium in $[\partial_m, \partial_M]^n$. Hence, along the system (1), the existence of equilibria is proven. By Theorem 1, it shows that

$$\lim_{t\to\infty} \text{distance}\big([\partial_m, \partial_M]^n, \mathbf{x}(t)\big) = 0,$$

which implies that every equilibrium $\mathbf{e} \in [\partial_m, \partial_M]^n$. ■

Appendix IV
Proof of Theorem 4

Proof: Denote

$$\begin{aligned} V(t) &= \max_{i\in\mathbf{N}} \big\{ (1-k_{e1})\big(\mathbf{e}_i - x_i(t)\big), (1-k_{e2})\big(x_i(t) - \mathbf{e}_i\big) \big\} \\ &= \max_{i\in\mathbf{N}} \big\{ (1-k_{e1})\big(-\varepsilon_i(t)\big), (1-k_{e2})\varepsilon_i(t) \big\}, \end{aligned}$$

and clearly V is Lipschitz continuous. At first, we will prove that $V(t)$ is a non-increasing function.

Denote $\varepsilon_m(t) = \min_{i\in\mathbf{N}} \varepsilon_i(t)$, $\varepsilon_M(t) = \max_{i\in\mathbf{N}} \varepsilon_i(t)$.

By the structure of $V(t)$, there exists two cases:

1) $V(t) = (1-k_{e2})\varepsilon_M(t)$;
2) $V(t) = (1-k_{e1})\big(-\varepsilon_m(t)\big)$.

We first consider the Case 1, which implies that $\exists\, t^*$, $V(t^*) = \max_{i\in\mathbf{N}} \big\{ (1-k_{e2})\varepsilon_i(t^*) \big\}$ and $\varepsilon_M(t^*) > 0$. Denote $\mathcal{I}_e(t) = \big\{ k : \varepsilon_k(t) = \max_{i\in\mathbf{N}} \varepsilon_i(t) \big\}$. By Lemma 1, we have

$$\begin{aligned} & d^+V(t^*) = d^+ \max_{i\in\mathbf{N}} \Big\{ (1-k_{e2})\varepsilon_i(t^*) \Big\} \\ &= \max_{i\in\mathcal{I}_e(t^*)} \Big\{ (1-k_{e2}) \sum_{j\in\mathcal{N}_i} a_{ij} \Big(f_{ji}\big(x_j(t^*)\big) - x_i(t^*) \Big) \Big\} \\ &= \max_{i\in\mathcal{I}_e(t^*)} \Big\{ (1-k_{e2}) \sum_{j\in\mathcal{N}_i} a_{ij} \Big(f_{ji}\big(\mathbf{e}_j + \varepsilon_j(t^*)\big) - \mathbf{e}_i - \varepsilon_i(t^*) \Big) \Big\}. \end{aligned}$$

Furthermore, noticing that $\sum_{j\in\mathcal{N}_i} a_{ij}\big(f_{ji}(\mathbf{e}_j) - \mathbf{e}_i\big) = 0$, we can conclude that

$$\dot{\varepsilon}_i(t^*) = \sum_{j\in\mathcal{N}_i} a_{ij} \big(f_{ji}\big(\mathbf{e}_j + \varepsilon_j(t^*)\big) - f_{ji}(\mathbf{e}_j) - \varepsilon_i(t^*) \big). \tag{14}$$

Let $\varepsilon_{i'} = \varepsilon_M$, which implies $i' \in \mathcal{I}_e(t^*)$. Applying Lemma 3 on (14) and under condition 2, we can get that

$$\begin{cases} \varepsilon_{i'} \geq \varepsilon_j \geq f_{ji'}\big(\mathbf{e}_j + \varepsilon_j\big) - f_{ji'}\big(\mathbf{e}_j\big) & \text{if } \varepsilon_j \geq 0; \\ \varepsilon_{i'} \geq L_{e1}(\varepsilon_j) > f_{ji'}\big(\mathbf{e}_j + \varepsilon_j\big) - f_{ji'}\big(\mathbf{e}_j\big) & \text{if } \varepsilon_j < 0. \end{cases}$$

By the above two analysis, it concludes that $d^+V(t^*) \leq 0$ when $(1-k_{e2})\varepsilon_M(t^*) > (1-k_{e1})\big(-\varepsilon_m(t^*)\big)$.

The proof of Case 2 is similar to the above proof, and hence are omitted here. Combining the two cases, we have proved that $d^+V(t) \leq 0$ for all $t \geq t_0$.

Repeating the analysis of Theorem 1, we have $\mathbf{x}(t) \to \mathbf{e}$, as $t \to \infty$. Since the equilibrium $\mathbf{e}$ is asymptotically stable and the initial states $\mathbf{x}(t_0)$ can be arbitrary, $\mathbf{e}$ is a unique equilibrium. The proof is completed. ■

Appendix V
Proofs of Theorem 5 and Corollary 2

A. Proof of Theorem 5

Proof: We first prove the existence of equilibria.

By conditions 2 and 3, we can get that for any $j \in \mathbf{N}$, $i \in \mathcal{N}_j$ and $\omega \neq 0$,

$$-1 < k^* \leq \frac{f_{ij}(x+\omega) - f_{ij}(x)}{\omega} \leq 1, \quad x \in \mathbb{R}, \tag{15}$$

which implies that f_{ij} is a continuous function. For simplicity, let $k^* \in (-1, 0)$.

By (15), it can be concluded that there exists at least one intersection between functions f_{ij} and $f(x) = x$, i.e., $\Theta_{ij} \neq \emptyset$. Denote $X_M = \max\big\{x : x \in \bigcup \Theta_{ij}\big\}$ and $X_m = \min\big\{x : x \in \bigcup \Theta_{ij}\big\}$. Since $\Theta_{ij} \neq \emptyset$, let $x^*_{ij} \in \Theta_{ij}$, i.e., $f_{ij}(x^*_{ij}) = x^*_{ij}$. By (15), it shows that

$$\begin{cases} f_{ij}(x^*_{ij} + \omega) \leq f_{ij}(x^*_{ij}) + \omega = x^*_{ij} + \omega, & \omega > 0; \\ f_{ij}(x^*_{ij} + \omega) \geq f_{ij}(x^*_{ij}) + \omega = x^*_{ij} + \omega, & \omega < 0, \end{cases}$$

which implies that

$$\begin{cases} f_{ij}(x) \leq x, & x \geq X_M; \\ f_{ij}(x) \geq x, & x \leq X_m. \end{cases} \tag{16}$$

There are two parallel lines with slope $k^* \in (-1,0)$:

$$\begin{cases} L_m(x) = k^*x + (1-k^*)X_m, \\ L_M(x) = k^*x + (1-k^*)X_M, \end{cases}$$

and it can be concluded that

$$\begin{cases} f_{ij}(x) \le L_M(x), & x \le X_M; \\ f_{ij}(x) \ge L_m(x), & x \ge X_m. \end{cases}$$

Let $L^*(x) = -x + X_M + X_m$ with slope $k = -1$.

Since $-1 < k^* < 0$, it is easy to know that between L^* and the parallel lines L_M, L_m exist two intersections (y_m, y_M) and (y_M, y_m) where $y_M \ge y_m$, which implies that $L_M(y_m) = y_M$ and $L_m(y_M) = y_m$. And it is easy to know that $y_m < X_m \le X_M < y_M$. Then, it can be concluded that

$$\begin{cases} f_{ij}(x) \le L_M(x) \le y_M, & y_m \le x \le X_M; \\ f_{ij}(x) \ge L_m(x) \ge y_m, & X_m \le x \le y_M. \end{cases} \tag{17}$$

Combine (16) and (17), we can get that $y_m \le f_{ij}(x) \le y_M$, $y_m \le x \le y_M$. By Theorem 3, it shows that the system (1) has at least one equilibrium. Then, we will show that the MAS (1) has only one asymptotically stable equilibrium.

Assume one of equilibria is $\mathbf{e}^* = \{\mathbf{e}_1^*, \ldots, \mathbf{e}_n^*\}^T$, and denote the error between $x(t)$ and $\mathbf{e}^*$ by $\varepsilon_i(t)^* = x_i(t) - \mathbf{e}_i^*$. By (15), we have for all $j \in \mathbf{N}$, $i \in \mathcal{N}_j$,

$$\begin{aligned} \varepsilon_i^* \le f_{ij}(\mathbf{e}_i + \varepsilon_i^*) - f_{ij}(\mathbf{e}_i) \le k^*\varepsilon_i^*, \quad \varepsilon_i^* < 0; \\ k^*\varepsilon_i^* \le f_{ij}(\mathbf{e}_i + \varepsilon_i^*) - f_{ij}(\mathbf{e}_i) \le \varepsilon_i^*, \quad \varepsilon_i^* > 0. \end{aligned}$$

Since $k^* \in (-1,0)$, it shows that $k^*k^* < 1$. Hence, the Condition 3 of Theorem 4 holds. Because $\bigcap_{(v_j,v_i)\in\mathcal{E}} \Theta_{ij} = \emptyset$ and for any $j \in \mathbf{N}$, $i \in \mathcal{N}_j$ and $\omega \ne 0$,

$$-1 < \frac{f_{ij}(x+\omega) - f_{ij}(x)}{\omega} < 1, \quad x \notin \Theta_{ij},$$

we can conclude that for any $\varepsilon^* \ne 0$, there exist $j \in \mathbf{N}, i \in \mathcal{N}_j$ and $\delta > 0$, such that $f_{ij}(\mathbf{e}_i + \varepsilon') - f_{ij}(\mathbf{e}_i) \ne \varepsilon'$, $\forall \varepsilon' \in (\varepsilon^* - \delta, \varepsilon^* + \delta)$. Hence, the Condition 4 of Theorem 4 holds.

Apply Theorem 4, it shows that the equilibrium $\mathbf{e}$ is a unique, asymptotically stable equilibrium. ■

B. Proof of Corollary 2

Proof: Because $f_{ij}(x_i)$ is a continuous and piecewise linear function with its slopes $k_{ij} \in (-1,1]$ and $m_{ij} = 0$ when $k_{ij} = 1$, it turns out that for any $\omega \ne 0$, $-1 < \frac{f_{ij}(x+\omega)-f_{ij}(x)}{\omega} < 1, x \notin \Theta_{ij}$. Apply Theorem 5, this corollary is proved. ■

References

[1] Y. Hong, J. Hu, and L. Gao, “Tracking control for multi-agent consensus with an active leader and variable topology,” *Automatica*, vol. 42, no. 7, pp. 1177–1182, 2006.

[2] D. Silvestre, J. P. Hespanha, and C. Silvestre, “Broadcast and gossip stochastic average consensus algorithms in directed topologies,” *IEEE Transactions on Control of Network Systems*, vol. 6, no. 2, pp. 474–486, 2019.

[3] G. Shi and K. H. Johansson, “Robust consensus for continuous-time multiagent dynamics,” *SIAM Journal on Control and Optimization*, vol. 51, no. 5, pp. 3673–3691, 2013.

[4] H. Moradian and S. S. Kia, “On robustness analysis of a dynamic average consensus algorithm to communication delay,” *IEEE Transactions on Control of Network Systems*, vol. 6, no. 2, pp. 633–641, 2019.

[5] A. Nedic, A. Ozdaglar, and P. A. Parrilo, “Constrained consensus and optimization in multi-agent networks,” *IEEE Transactions on Automatic Control*, vol. 55, no. 4, pp. 922–938, 2010.

[6] A. Fontan, G. Shi, X. Hu, and C. Altafini, “Interval consensus for multiagent networks,” *IEEE Transactions on Automatic Control*, vol. 65, no. 5, pp. 1855–1869, 2019.

[7] W. Meng, Q. Yang, J. Si, and Y. Sun, “Consensus control of nonlinear multiagent systems with time-varying state constraints,” *IEEE Transactions on Cybernetics*, vol. 47, no. 8, pp. 2110–2120, 2016.

[8] C. Sun, C. J. Ong, and J. K. White, “Consensus control of multi-agent system with constraint-the scalar case,” in *52nd IEEE Conference on Decision and Control*. IEEE, 2013, pp. 7345–7350.

[9] J. Wu and Y. Shi, “Average consensus in multi-agent systems with time-varying delays and packet losses,” in *2012 American Control Conference (ACC)*. IEEE, 2012, pp. 1579–1584.

[10] A. Carullo and M. Parvis, “An ultrasonic sensor for distance measurement in automotive applications,” *IEEE Sensors journal*, vol. 1, no. 2, p. 143, 2001.

[11] E. Ramsden, *Hall-effect sensors: theory and application*. Elsevier, 2011.

[12] M. Ye, Y. Qin, A. Govaert, B. D. Anderson, and M. Cao, “An influence network model to study discrepancies in expressed and private opinions,” *Automatica*, vol. 107, pp. 371–381, 2019.

[13] J. Hou, W. Li, and M. Jiang, “Opinion dynamics in modified expressed and private model with bounded confidence,” *Physica A: Statistical Mechanics and its Applications*, vol. 574, p. 125968, 2021.

[14] Z.-X. Liu and Z.-Q. Chen, “Discarded consensus of network of agents with state constraint,” *IEEE Transactions on Automatic Control*, vol. 57, no. 11, pp. 2869–2874, 2012.

[15] Z. Lin, B. Francis, and M. Maggiore, “State agreement for continuous-time coupled nonlinear systems,” *SIAM Journal on Control and Optimization*, vol. 46, no. 1, pp. 288–307, 2007.

[16] C. Altafini, “Consensus problems on networks with antagonistic interactions,” *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 935–946, 2012.

[17] W. Basener, B. P. Brooks, and D. Ross, “The brouwer fixed point theorem applied to rumour transmission,” *Applied Mathematics Letters*, vol. 19, no. 8, pp. 841–842, 2006.