A Partially Distributed Fixed-Time Economic Dispatch Algorithm with Kron's Modeled Power Transmission Losses

Shivanshu Tripathi, Anoop Jain, and Abhisek K. Behera

***Abstract*—A partially distributed economic dispatch algorithm, which renders optimal value in fixed time with the objective of supplying the load requirement as well as the power transmission losses, is proposed in this paper. The transmission losses are modeled using Kron's $\mathcal{B}-$loss formula, under a standard assumption on the values of $\mathcal{B}-$coefficients. The total power supplied by the generators is subjected to time-varying equality constraints due to time-varying nature of the transmission losses. Using Lyapunov and optimization theory, we rigorously prove the convergence of the proposed algorithm and show that the optimal value of power is reached within a fixed-time, whose upper bound dependents on the values of $\mathcal{B}-$coefficients, parameters characterizing the convexity of the cost functions associated with each generator and the interaction topology among them. Finally, an example is simulated to illustrate the theoretical results.**

***Index Terms*—Fixed-time convergence, distributed control, economic dispatch, $\mathcal{B}$-loss coefficients, transmission loss.**

I. Introduction

A. Motivation and Literature Survey

The economic dispatch problem (EDP) has been a celebrated problem in the optimal operation and management of power systems. With the rapid integration of renewable energy sources in the microgrid, solving EDP becomes a challenging task due to the scalability of the power network. To encounter such systems with increased robustness, reliability, and efficiency, the centralized power generation infrastructure is slowly moving towards a distributed one [1]. The primary goal of EDP in a distributed infrastructure is to seek the minimum value of a collective cost function defined over a network of generators. This led to the requirement of an algorithm that works with renewable energy resources in a distributed manner. In this direction, there exist several approaches in the existing literature; for instance, [2, 3] discussed consensus-based algorithms; [4] described a distributed gradient-based algorithm; [5] studied initialization-free privacy-guaranteed distributed algorithm; [6] presented a gossip-based distributed algorithm; an adaptive event-triggered distributed algorithm is considered in [6] etc.

One of the main concerns in designing a distributed algorithm is that it must ensure a faster convergence rate, as the power output changes frequently due to the continuous use of distributed generation systems and dynamic pricing [7]. Addressing these facts, the efforts in existing literature have been towards developing algorithms with finite or fixed-time convergence based on [8, 9]. For example, [10] proposes a distributed finite time algorithm which can address the EDP in the smart grid with/without power generation constraints; [11] presents a distributed continuous-time algorithm to solve a convex optimization problem with equality constraint, which reaches the optimal value in fixed time; [12] extends these results by proposing a new lemma which guarantees finite-time convergence with a tighter upper bound on the convergence time; [13] discusses user-specified fixed-time consensus-based algorithm to solve EDP with time-varying topology.

In addition to supplying the load demand, it is equally important that the distributed algorithm must satisfy the constraints posed by the time-varying power transmission losses [14]. Existing works in this direction primarily consider a simplified model for the transmission losses and discuss asymptotic or exponential convergence to the optimal solution [15–20]. Further, [10–13] do not address the aspect of transmission losses. Unlike these works, in this paper, we propose an algorithm that accounts for Kron's modeled power transmission losses and reaches the optimal solution of the EDP in a fixed time.

B. Contributions

Aggregation of the Kron's modeled power transmission losses, by nature, poses an additional requirement of globally sharing the generated power information among the generators. Addressing this fact, the proposed algorithm in the paper considers that the generators have a two-layered communication topology−the generated power is shared globally in order to obtain the total power transmission losses, while the cost function and other auxiliary variables are shared locally. Such multi-layered topological considerations are motivated from many works [21–23] in this direction in the context of multi-agent systems, deployed for various collaborative missions. Further, our analysis is based on certain assumptions relying on an interplay between the eigenvalues of the matrix $\mathcal{B}$, and parameters characterizing the convexity of the cost functions associated with each generator. The main contributions of this work can be summarized as follows:

i) We propose a novel consensus-based partially distributed algorithm, which solves the EDP in the presence of power transmission losses characterized by the Kron's $\mathcal{B}$-loss formula [24].

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ii) Using tools from Lyapunov stability and optimization theory, we rigorously show that the optimal solution of the EDP is rendered in a fixed time. An analytical expression of the upper bound on the convergence time is obtained, which is independent of initial values of power and dependent on the eigenvalues of the Kron's $\mathcal{B}-$loss matrix, the convexity of the cost function associated with each generator and network topology among them.

C. Paper Structure

The paper unfolds as follows: Section II describes Kron's transmission loss formula, formulates the problem, and presents some preliminary results on finite-time stability. Section III derives a few introductory results, describes the proposed algorithm, and obtain an upper bound on the convergence time. Theoretical results are illustrated through a simulation example in Section IV. Finally, Section V concludes the paper and presents future directions of the work.

Notations: Throughout the paper, $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of real and non-negative real numbers, respectively. For any $x \in \mathbb{R}$, we define function $\text{sig}^{\mu} : \mathbb{R} \to \mathbb{R}$ as $\text{sig}^{\mu}(x) = |x|^{\mu}\text{sign}(x), \mu > 0$, where $\text{sign}(x)$ is the signum function of x. The Hadamard product (or element-wise product) of two matrices X and Y of the same dimension $m \times n$ is defined as $[(X \odot Y)_{ij}] := [X_{ij}][Y_{ij}]$. Let $\psi = [\psi_1, \ldots, \psi_N]^T \in \mathbb{R}^N$, then $\text{diag}\{\psi\}$ denotes the diagonal matrix with the entries of ψ along its principal diagonal. $\nabla f(\bullet)$ and $\nabla^2 f(\bullet)$ represent the gradient and Hessian of the function $f : \mathbb{R}^n \to \mathbb{R}$ with respect to its argument $\bullet$, respectively. The Jacobian of a function $g : \mathbb{R}^n \to \mathbb{R}^m$ is defined to be an $m \times n$ matrix whose $(i, j)^{\text{th}}$ entry is $J_{ij} = \partial g_i / \partial x_j$. We represent by $\mathbf{1}_N = [1, \ldots, 1]^T \in \mathbb{R}^N$ and $\mathbf{0}_N = [0, \ldots, 0]^T \in \mathbb{R}^N$, respectively. I_N denotes the identity matrix of order $N \times N$. We use symbols $\succeq, \preceq$ to represent element-wise comparison between two matrices of the same size.

An undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is a collection of node set $\mathcal{V} = \{1, \ldots, N\}$, the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, along with edge weights captured by the adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ with $a_{ij} = a_{ji} > 0$ if $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. The Laplacian of $\mathcal{G}$ is defined as $\mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{N \times N}$ with $\ell_{ii} = \sum_{j \in \mathcal{N}_i} a_{ij}$ and $\ell_{ij} = -a_{ij}, \ \forall i \neq j$, where $\mathcal{N}_i$ is the set of neighboring vertices of vertex i. For an undirected and connected graph, 0 is a simple eigenvalue of $\mathcal{L}$ with the corresponding eigenvector $\mathbf{1}_N$, and all the other eigenvalues are positive.

II. Kron's Formula, Problem Description, and Preliminary Results

This section reviews the Kron's $\mathcal{B}-$loss formula for power transmission losses, formulates the problem in this paper, and discuss some preliminary results.

A. Transmission Losses

Transmission losses in a power system network are often evaluated using Kron's approximated loss Formula. An expression for transmission losses in terms of source loading and a set of loss coefficients (usually referred to as $\mathcal{B}-$coefficients) is of the quadratic form:

$$P_L = \sum_{i=1}^{N} \sum_{j=1}^{N} P_i \mathcal{B}_{ij} P_j + \sum_{i=1}^{N} P_i \mathcal{B}_{i0} + \mathcal{B}_{00}, \tag{1}$$

where $\mathcal{B}_{ij}$, $\mathcal{B}_{i0}$ and $\mathcal{B}_{00}$ are constant $\mathcal{B}-$loss coefficients and can be evaluated using methods as discussed in [24, 25]. Further, P_i and P_j are the power outputs of generators i and j in megawatts, respectively. The expression (1) can be compactly re-written as $P_L = \sum_{i=1}^{N} P_{Li}$, where,

$$P_{Li} = \sum_{j=1}^{N} P_i \mathcal{B}_{ij} P_j + P_i \mathcal{B}_{i0} + \mathcal{B}_{00_i}, \tag{2}$$

is the power transmission loss associated with the i^{th} generator and $\mathcal{B}_{00} = \sum_{i=1}^{N} \mathcal{B}_{00_i}$.

B. Problem Formulation

Consider a network comprising N generators in a grid and the cost function of individual generators is given as $C_i(P_i)$. The main objective here is to cooperatively minimize the total cost, that is, the sum of all individual *local* objective functions $C_i(P_i)$, while maintaining an equality constraint, defined in terms of the load demand and power transmission losses P_L. Let D and P_T be the total load demand, and total power supplied by the system of generators, respectively. With this description, the economic dispatch problem can be formulated as:

$$\text{Min } C(P) = \sum_{i=1}^{N} C_i(P_i) \tag{3a}$$

$$\begin{aligned} \text{subject to } \sum_{i=1}^{N} P_i = D + P_L &= \sum_{i=1}^{N} D_i + \sum_{i=1}^{N} P_{L_i} \\ &= \sum_{i=1}^{N} D_{i0} + \sum_{i=1}^{N} P_{L_i} = P_T, \end{aligned} \tag{3b}$$

where D_{i0} is the initial value of the time-varying load demand D_i correspond to the i^{th} generator-load pair. It is assumed that the load demand is constant at all time, that is, $\sum_{i=1}^{N} D_i = \sum_{i=1}^{N} D_{i0}$ for all $t \geq 0$, which is often a standard assumption in power system networks [11]. It is worth noting that inclusion of power transmission losses P_L does not result in trivially regularizing the overall cost function (3a), instead, it affects the equality constraints (3b) of the optimization problem (3) and makes it challenging. Unless otherwise stated, $P_i \geq 0, \forall i$ in our analysis, as the generated power can not be negative.

C. Some Preliminary Results

Below we describe some useful results that will be helpful in the sequel.

Lemma 1 ([9]). *Consider the dynamical system $\dot{x} = f(x(t))$, where $x \in \mathbb{R}^n$, $f : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function with $f(\mathbf{0}_N) = \mathbf{0}_N$. Assume that the origin is the equilibrium point* *of the system. If there exist a continuous radially unbounded Lyapunov function $V : \mathbb{R}^N \to \mathbb{R}_+ \cup \{0\}$ such that $V(x) = 0 \Leftrightarrow x = 0$ and any solution of $x(t)$ of the system satisfies the inequality $\dot{V}(x(t)) \leq -(\alpha V^p(x(t)) + \beta V^q(x(t)))^k$ for some $\alpha, \beta, p, q, k > 0; pk < 1, qk > 1$, then the origin of the system is globally fixed-time stable, and the following estimates of the settling time holds:*

$$T_s \leq \frac{1}{\alpha^k(1-pk)} + \frac{1}{\beta^k(qk-1)}. \tag{4}$$

Lemma 2 ([26]). *Let $\zeta_i \geq 0$ for $i = \{1, \ldots, N\}$. Then*

$$\sum_{i=1}^{N} \zeta_i^m \geq \left(\sum_{i=1}^{N} \zeta_i\right)^m, \text{ if } 0 < m \leq 1, \tag{5a}$$

$$\sum_{i=1}^{N} \zeta_i^m \geq N^{(1-m)} \left(\sum_{i=1}^{N} \zeta_i\right)^m, \text{ if } 1 < m < \infty. \tag{5b}$$

III. Main Results

This section presents our main results by proposing an algorithm to solve the optimization problem (3a) with equality constraints (3b) in the presence of transmission losses (1). The proposed algorithm is as follows:

$$\begin{cases} \lambda_i = \dfrac{\partial C_i(P_i)}{\partial P_i} \\ H_i = \left(1 + \dfrac{\partial P_{L_i}}{\partial P_i}\right) \\ \dot{z}_i = -k_1 \operatorname{sig}\left[\displaystyle\sum_{j\in\mathcal{N}_i} a_{ij}\left(H_j\lambda_j - H_i\lambda_i\right)\right]^{\mu} \\ \qquad - k_2 \operatorname{sig}\left[\displaystyle\sum_{j\in\mathcal{N}_i} a_{ij}\left(H_j\lambda_j - H_i\lambda_i\right)\right]^{\nu} \\ P_i = \displaystyle\sum_{j\in\mathcal{N}_i} a_{ij}(z_j - z_i) + D_{i0} + P_{Li}, \end{cases} \tag{6}$$

where k_1, k_2, μ and ν are constants such that they satisfy conditions $0 < \mu < 1$ and $\nu > 1$; H_i, z_i and λ_i are intermediate variables. Unlike [10, 11], the algorithm (6) also accounts for transmission losses P_L by assimilation of an additional term H_i, which further influences the dynamics $\dot{z}_i$ of the auxiliary variable z_i. Later, we also discuss through simulations that proposed consensus dynamics of auxiliary variables z_i can handle a special class of bounded disturbances.

Remark 1. *As will be shown in below Lemma 3, the computation of term $\partial P_{L_i}/\partial P_i$ in algorithm* (6) *requires the information of power generated by all the generators. Thereby, the implementation of* (6) *requires global topology for obtaining H_i, and local topology for sharing the information about the cost function λ_i and the auxiliary variables z_i for each i. This is the reason we call it a partially distributed consensus algorithm. Although the problem can be solved in a completed distributed way by considering only a single local network for the simplified Korn's modeled transmission losses $P_L = \sum_{i=1}^{N} \mathcal{B}_i P_i^2$ as discussed in [15, 18], however, this would be a special case of the problem addressed in our paper.*

Before proceeding further, we incorporate the following assumptions in our analysis:

Assumption 1 (Network topology). *The generators have a two-layered network topology − the information (only) about generated powers is shared globally among them, and the information about their cost function and other auxiliary variables is shared locally, according to an undirected and connected topology.*

Assumption 2 (Cost function). *For $i = \{1, \ldots, N\}$, the cost function $C_i(P_i)$ is a strongly convex function such that $\nabla^2 C_i(P_i) \geq \sigma > 0$ for constant $\sigma \in \mathbb{R}_+$. Further, there exists a $\delta \in \mathbb{R} \setminus \{0\}$ such that $\nabla C_i(P_i) \geq \delta$, $\forall i$.*

Assumption 3 ($\mathcal{B}-$loss coefficients). *The kron's $\mathcal{B}-$loss coefficient matrix $\mathcal{B} = [\mathcal{B}_{ij}], \forall i, j = \{1, \ldots, N\}$ in* (1) *is symmetrical with all its elements $\mathcal{B}_{ij} \geq 0$ such that*

(A1) $0 \leq \partial P_{L_i}/\partial P_i < 1, \forall i = \{1, \ldots, N\}$.

(A2) *Let $b_1 \leq b_2 \leq \cdots \leq b_N$ be the eigenvalues of $\mathcal{B}$. Denote by $\rho = \min_i\{\mathcal{B}_{i0}\}$. Then, the parameters σ, δ and ρ are such that they satisfy: $(1+\rho)\sigma + b_1\delta > 0$, if $\delta > 0$, and $(1+\rho)\sigma + b_N\delta > 0$, if $\delta < 0$.*

Remark 2. *It is to be noted that Assumption* (A1) *is common for practical power system networks (for instance, please refer to [25, 27–29]). This is due to the fact that the values of $\mathcal{B}$-loss coefficients are usually very small such that total transmission losses P_L are negligible compared to the value of total load demand D. Following this, one can write from* (3b) *that $\sum_{i=1}^{N} P_i \approx \sum_{i=1}^{N} D_{i0} = \bar{D}$ (say) for small values of $\mathcal{B}$-coefficient. This implies that $P_i \leq \bar{D}$ for each i. According to Assumption* (A1)*, it follows from* (2) *that the inequality $\partial P_{L_i}/\partial P_i = \sum_{j=1,j\neq i}^{N} \mathcal{B}_{ij}P_j + 2\mathcal{B}_{ii}P_i + \mathcal{B}_{i0} < 1$ must hold true for all $t \geq 0$. For the given $\bar{D}$, this can be assured only if the $\mathcal{B}-$coefficients are such that $\sum_{j=1,j\neq i}^{N} \mathcal{B}_{ij} + 2\mathcal{B}_{ii} + \mathcal{B}_{i0}\bar{D}^{-1} < \bar{D}^{-1}$ for each i. In fact, the term $\partial P_{L_i}/\partial P_i$ can be obtained from the well known notion of penalty factor, defined by $1/(1-(\partial P_{L_i}/\partial P_i))$ in the literature [27–29], and justifies our assumption. Obviously, $\partial P_L/\partial P_i \geq 0$, as $P_i \geq 0$ for all i and $t \geq 0$.*

We now discuss the following lemmas before stating the main result.

Lemma 3. *Under Assumption 3, the following relation holds:*

$$\frac{\partial P_{L_i}}{\partial P_i} = \frac{1}{2}\frac{\partial P_L}{\partial P_i} + \mathcal{B}_{ii}P_i + \frac{\mathcal{B}_{i0}}{2}, \ \forall i. \tag{7}$$

Proof. Differentiating (1) and (2) with respect to P_i, we have

$$\frac{\partial P_L}{\partial P_i} = 2\sum_{j=1,j\neq i}^{N} \mathcal{B}_{ij}P_j + 2\mathcal{B}_{ii}P_i + \mathcal{B}_{i0}, \tag{8}$$

$$\frac{\partial P_{L_i}}{\partial P_i} = \sum_{j=1,j\neq i}^{N} \mathcal{B}_{ij}P_j + 2\mathcal{B}_{ii}P_i + \mathcal{B}_{i0}, \tag{9}$$

which leads to required result, as $[\mathcal{B}_{ij}] = [\mathcal{B}_{ji}], \forall i, j$. ☐

Lemma 4. *Let $P = [P_1, \ldots, P_N]^T$ be the vector of all generator bus net outputs. Define $\mathcal{F} = \nabla P_L(P) \odot \nabla \mathcal{C}(P)$, $\mathcal{M} = \nabla \mathcal{R}(P) \odot \nabla \mathcal{C}(P)$, and $\mathcal{Q} = \nabla^2 \mathcal{C}(P) + 0.5\nabla \mathcal{F} + \nabla \mathcal{M}$, where,*

$$\nabla \mathcal{R}(P) = \left[\left(\mathcal{B}_{11}P_1 + \frac{\mathcal{B}_{10}}{2}\right), \ldots, \left(\mathcal{B}_{NN}P_N + \frac{\mathcal{B}_{N0}}{2}\right)\right]^T.$$

Further, let $\mathcal{S}$ be an $N \times N$ matrix with diagonal entries $[\mathcal{S}_{ii}] = (1 + \mathcal{B}_{i0})\sigma + 2\mathcal{B}_{ii}\delta$ and off-diagonal entries $[S_{ij}] = \mathcal{B}_{ij}\delta$. Under Assumptions 2 and 3, the following properties hold:

- (R1) $[\nabla \mathcal{F}_{ii}] \geq 2\mathcal{B}_{ii}\delta + \mathcal{B}_{i0}\sigma$ *and* $[\nabla \mathcal{F}_{ij}] \geq 2\mathcal{B}_{ij}\delta$.
- (R2) $\nabla \mathcal{M}$ *is a diagonal matrix with* $[\nabla \mathcal{M}_{ii}] \geq \left(\mathcal{B}_{ii}\delta + \frac{B_{i0}}{2}\sigma\right)$.
- (R3) $[\mathcal{Q}_{ii}] \geq \sigma + 2\mathcal{B}_{ii}\delta + \mathcal{B}_{i0}\sigma$ *and* $[\mathcal{Q}_{ij}] \geq \mathcal{B}_{ij}\delta$.
- (R4) $\mathcal{S}$ *is a symmetric matrix satisfying* $\mathcal{S} \preccurlyeq \mathcal{Q}$.
- (R5) *Let $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_N$ be the eigenvalues of $\mathcal{S}$. Then, $b_1\delta \leq \tau_i - \sigma(1 + \mathcal{B}_{i0}) \leq b_N\delta$, if $\delta > 0$; and $b_N\delta \leq \tau_i - \sigma(1 + \mathcal{B}_{i0}) \leq b_1\delta$, if $\delta < 0$, for each i, where b_1 and b_N are the smallest and largest eigenvalues of $\mathcal{B}$, as defined in Assumption* (A2).

Please refer to Appendix for the proof. We are now ready to state the main result:

Theorem 1. *The algorithm* (6)*, under the Assumptions 1, 2 and 3, solves the economic load dispatch problem* (3) *in a fixed time.*

Proof. The sum of power supplied by each generator at any time instant satisfies

$$\begin{aligned}\sum_{i=1}^{N} P_i &= \sum_{i=1}^{N}\left(\sum_{j\in\mathcal{N}_i} a_{ij}(z_j - z_i) + D_{i0} + P_{L_i}\right)\\ &= \sum_{i=1}^{N} D_{i0} + \sum_{i=1}^{N} P_{L_i} = P_T, \end{aligned} \tag{10}$$

as $\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_j} a_{ij}(z_j - z_i) = 0$ for an undirected and connected graph with $a_{ij} = a_{ji}$. Clearly, (10) satisfies the desired equality constraint (3a). Substituting for P_i from (6), the optimization problem (3a) can be represented as the following unconstrained optimization problem:

$$\text{Min } C(z) = \sum_{i=1}^{N} C_i\left(\sum_{j\in N_i} a_{ij}(z_j - z_i) + D_{i0} + P_{Li}\right). \tag{11}$$

From (6), the derivative of P_i with respect to z_j is obtained as:

$$\frac{\partial P_i}{\partial z_j} = \begin{cases} -\sum_{j=1}^{N} a_{ij} + \frac{\partial P_{L_i}}{\partial z_j} & \text{if } j = i\\ a_{ij} + \frac{\partial P_{L_i}}{\partial z_j} & \text{if } j \neq i \end{cases} \tag{12}$$

$$= \begin{cases} -\sum_{j=1}^{N} a_{ij} + \frac{\partial P_{L_i}}{\partial P_i}\frac{\partial P_i}{\partial z_j} & \text{if } j = i\\ a_{ij} + \frac{\partial P_{L_i}}{\partial P_i}\frac{\partial P_i}{\partial z_j} & \text{if } j \neq i. \end{cases} \tag{13}$$

Once again using (12) in (13) for $\frac{\partial P_i}{\partial z_j}$, we have

$$\frac{\partial P_i}{\partial z_j} = \begin{cases} -\sum_{j=1}^{N} a_{ij} + \frac{\partial P_{L_i}}{\partial P_i} \\ \times\left[-\sum_{j=1}^{N} a_{ij} + \frac{\partial P_{Li}}{\partial P_i}\frac{\partial P_i}{\partial z_j}\right] & \text{if } j = i\\ a_{ij} + \frac{\partial P_{Li}}{\partial P_i}\left[a_{ij} + \frac{\partial P_{L_i}}{\partial P_i}\frac{\partial P_i}{\partial z_j}\right] & \text{if } j \neq i. \end{cases} \tag{14}$$

Continuing the substitution in each step, an infinite series is formed for $\partial P_i/\partial z_j$, as below:

$$\frac{\partial P_i}{\partial z_j} = \begin{cases} -\sum_{j=1}^{N} a_{ij} - \sum_{j=1}^{N} a_{ij}\frac{\partial P_{L_i}}{\partial P_i} - \\ \sum_{j=1}^{N} a_{ij}\left(\frac{\partial P_{L_i}}{\partial P_i}\right)^2 - \cdots & j = i\\ a_{ij} + a_{ij}\frac{\partial P_{Li}}{\partial P_i} + a_{ij}\left(\frac{\partial P_{L_i}}{\partial P_i}\right)^2 + \cdots & j \neq i. \end{cases} \tag{15}$$

Following Assumption (A1), the higher order terms are neglected to get:

$$\frac{\partial P_i}{\partial z_j} = \begin{cases} -\sum_{j=1}^{N} a_{ij}\left(1 + \frac{\partial P_{Li}}{\partial P_i}\right) & \text{if } j = i\\ a_{ij}\left(1 + \frac{\partial P_{Li}}{\partial P_i}\right) & \text{if } j \neq i, \end{cases} \tag{16}$$

which on substitution for $\partial P_{Li}/\partial P_i$ from Lemma 3 results in

$$\frac{\partial P_i}{\partial z_j} = \begin{cases} -\sum\limits_{j=1}^{N} a_{ij}\left(1 + \frac{1}{2}\frac{\partial P_L}{\partial P_i} + \mathcal{B}_{ii}P_i + \frac{\mathcal{B}_{i0}}{2}\right) & \text{if } j = i\\ a_{ij}\left(1 + \frac{1}{2}\frac{\partial P_L}{\partial P_i} + \mathcal{B}_{ii}P_i + \frac{\mathcal{B}_{i0}}{2}\right) & \text{if } j \neq i. \end{cases} \tag{17}$$

Similarly, it can be written from (6) about the cost function that:

$$\begin{aligned}\frac{\partial C_i}{\partial z_j} &= \frac{\partial C_i}{\partial P_i}\frac{\partial P_i}{\partial z_j}\\ &= \begin{cases} -\sum\limits_{j=1}^{N} a_{ij}\left(1 + \frac{1}{2}\frac{\partial P_L}{\partial P_i} + \mathcal{B}_{ii}P_i + \frac{\mathcal{B}_{i0}}{2}\right)\lambda_i & \text{if } j = i\\ a_{ij}\left(1 + \frac{1}{2}\frac{\partial P_L}{\partial P_i} + \mathcal{B}_{ii}P_i + \frac{\mathcal{B}_{i0}}{2}\right)\lambda_i & \text{if } j \neq i. \end{cases}\end{aligned} \tag{18}$$

Note that the gradient of P_L is $\nabla P_L(P) = \left[\frac{\partial P_L}{\partial P_1}, \ldots, \frac{\partial P_L}{\partial P_N}\right]^T$, using which, (17) can be expressed in the form of Jacobian as

$$\begin{aligned}\mathcal{J}_P &= \frac{\partial(P_1, \ldots, P_N)}{\partial(z_1, \ldots, z_N)}\\ &= -(I_N + 0.5\ \text{diag}\{\nabla P_L(P)\} + \text{diag}\{\nabla \mathcal{R}(P)\})\mathcal{L},\end{aligned} \tag{19}$$

where $\nabla \mathcal{R}(P)$ is defined in Lemma 4 and $\mathcal{L}$ is the Laplacian of the underlying topology. Further, the gradient of $C(z)$, using (18), is given by

$$\begin{aligned}\nabla C(z) = -\mathcal{L}\Bigg[\Bigg(&I_N + 0.5\ \text{diag}\{\nabla P_L(P)\}\\ &+ \text{diag}\{\nabla \mathcal{R}(P)\}\Bigg)\nabla C(P)\Bigg].\end{aligned} \tag{20}$$

We emphasize here that $\mathcal{J}_p$ is $N \times N$ matrix, while $\nabla C(z)$ is an $N \times 1$ vector, as $\nabla C(P) = \left[\frac{\partial C_1}{\partial P_1}, \ldots, \frac{\partial C_N}{\partial P_N}\right]^T$. The Hessian of (20) satisfies,

$$\begin{aligned}\nabla^2 C(z) = -\mathcal{L}\nabla\Bigg[\Bigg(&I_N + 0.5\ \text{diag}(\nabla P_L(P))\\ &+ \text{diag}(\nabla \mathcal{R}(P))\Bigg)\nabla C(P)\Bigg]\mathcal{J}_P,\end{aligned} \tag{21}$$

which, further simplifying the term inside the square bracket and using (19), yields

$$\nabla^2 C(z) = \mathcal{L}\bigg[\nabla^2 C(P) + 0.5\ \nabla(\nabla P_L(P) \odot \nabla C(P)) + \nabla(\nabla\mathcal{R}(P) \odot \nabla C(P))\bigg] \times (I_N + 0.5\ \mathrm{diag}\{\nabla P_L(P)\} + \mathrm{diag}\{\nabla\mathcal{R}(P)\})\mathcal{L}. \tag{22}$$

From Lemma 4, please note that $\nabla P_L(P) \odot \nabla C(P) = \mathcal{F}$ and $\nabla\mathcal{R}(P) \odot \nabla C(P) = \mathcal{M}$, which implies that

$$\nabla^2 C(z) = \mathcal{L}(\nabla^2 C(P) + 0.5\ \nabla\mathcal{F} + \nabla\mathcal{M}) \times (I_N + 0.5\ \mathrm{diag}\{\nabla P_L(P)\} + \mathrm{diag}\{\nabla\mathcal{R}(P)\})\mathcal{L}. \tag{23}$$

Let the optimal solution of convex optimization problem (11) be given as $z^* = [z_1^\star, \ldots, z_N^\star]$. The trivial solution is given by $z^* \in \beta\mathbf{1}_N$, where constant $\beta \in \mathbb{R}$. The focus of our analysis is on non-trivial case where solutions belong to the convex and compact set $\mathcal{Z} \subset \mathbb{R}^N \setminus \beta\mathbf{1}_N$. For any $z, \xi \in \mathcal{Z}$, it follows for the strongly convex functions from [30] that,

$$\mathcal{C}(z) = C(\xi) + \nabla^T C(\xi)(z - \xi) + \frac{1}{2}(z - \xi)^T \nabla^2 C(\hat{z})(z - \xi), \tag{24}$$

where $\hat{z} = \xi + \eta(z - \xi)$ with $\eta \in [0, 1]$. Replacing z, ξ by $z^\star, z$, respectively, (24) becomes

$$C(z^\star) = C(z) + \nabla^T C(z)(z^\star - z) + \frac{1}{2}(z^\star - z)^T \nabla^2 C(\tilde{z})(z^* - z), \tag{25}$$

where $\tilde{z} = z + \eta(z^* - z)$ with $\eta \in [0, 1]$. Rearranging (25) as

$$C(z) - C(z^\star) = \nabla^T C(z)(z - z^\star) - \frac{1}{2}(z^\star - z)^T \nabla^2 C(\tilde{z})(z^\star - z),$$

and using Assumption 2, it holds that

$$C(z) - \mathcal{C}(z^\star) \leq \nabla^T C(z)(z - z^\star). \tag{26}$$

Let $\phi_1, \phi_2, \cdots, \phi_N$ be the eigenvalues of Laplacian $\mathcal{L}$ such that $0 = \phi_1 \leq \phi_2 \leq \ldots \leq \phi_N$ with corresponding orthogonal eigenvectors $\mathbf{1}_N, v_2, \ldots, v_N$, where $||v_i|| = 1,\ i = 2, \ldots, N$. The vector $z - z^\star$ can be expressed as

$$z - z^\star = \kappa_1\mathbf{1}_N + \kappa_2 v_2 + \ldots + \kappa_N v_N, \tag{27}$$

where $\kappa_i, i = \{1, \ldots, N\}$ are constants. Using (27), (26) becomes

$$C(z) - C(z^\star) \leq \nabla^T C(z)(\kappa_1\mathbf{1}_N + v), \tag{28}$$

where $v = \kappa_2 v_2 + \ldots + \kappa_N v_n$. Note that $||v||^2 = \kappa_2^2 + \ldots + \kappa_N^2$. From (20) and (28), it follows that

$$\begin{aligned} C(z) - \mathcal{C}(z^\star) &\leq -\kappa_1\left[(I_N + \mathrm{diag}\{\nabla P_L(P)\})\nabla C(P)\right]^T \mathcal{L}\mathbf{1}_N + \nabla^T C(z)v \\ &= \nabla^T C(z)v \leq ||\nabla^T C(z)||||v||, \end{aligned} \tag{29}$$

as $\mathcal{L} = \mathcal{L}^T$ and $\mathcal{L}\mathbf{1}_N = \mathbf{0}_N$ for an undirected and connected graph. Now, substituting $\xi = z^\star$ in (24) and noting that $\nabla C(z^\star) = \mathbf{0}_N$, we have

$$C(z) - C(z^\star) = 0.5(z - z^\star)^T \nabla^2 C(\hat{z})(z - z^\star), \tag{30}$$

which, upon substitution from (23), gives

$$C(z) - C(z^*) = 0.5(\mathcal{L}(z - z^\star))^T(\nabla^2 C(P) + 0.5\ \nabla\mathcal{F} + \nabla\mathcal{M}) \times (I_N + 0.5\ \mathrm{diag}\{\nabla P_L(P)\} + \mathrm{diag}\{\nabla\mathcal{R}(P)\})(\mathcal{L}(z - z^\star)). \tag{31}$$

According to Assumption 3, and Lemmas 3 and 4, it is clear that $1 + 0.5\ \nabla P_L(P_i) + \nabla\mathcal{R}(P_i) = 1 + \sum_{j=1, j\neq i}^N \mathcal{B}_{ij}P_j + 2\mathcal{B}_{ii}P_i + \mathcal{B}_{i0} \geq (1 + \mathcal{B}_{i0}) \geq (1 + \rho)$ for each i, where $\rho = \min_i\{\mathcal{B}_{i0}\}$. This implies that the diagonal matrix $I_N + 0.5\ \mathrm{diag}\{\nabla P_L(P)\} + \mathrm{diag}\{\nabla\mathcal{R}(P)\} \succeq (1 + \rho)I_N$. Consequently, it holds from (31) that,

$$\begin{aligned} C(z) - C(z^*) &\geq 0.5(1 + \rho)(\mathcal{L}(z - z^\star))^T \\ &\quad \times (\nabla^2 C(P) + 0.5\ \nabla\mathcal{F} + \nabla\mathcal{M})(\mathcal{L}(z - z^\star)) \\ &= 0.5(1 + \rho)(\mathcal{L}(z - z^\star))^T \mathcal{Q}(\mathcal{L}(z - z^\star)), \end{aligned} \tag{32}$$

where $\mathcal{Q} = \nabla^2 C(P) + 0.5\ \nabla\mathcal{F} + \nabla\mathcal{M}$, according to Lemma 4. Further, one can write using result (R4) from Lemma 4 that:

$$C(z) - C(z^*) \geq 0.5(1 + \rho)(\mathcal{L}(z - z^\star))^T \mathcal{S}(\mathcal{L}(z - z^\star)), \tag{33}$$

where $\mathcal{S}$ is a symmetric matrix with eigenvalues $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_N$, as per Lemma 4. Using Courant-Fischer theorem [[31], Chapter 4, pg. 236] for the symmetric matrix $\mathcal{S}$, it holds for $z \in \mathcal{Z}$ that

$$\begin{aligned} &C(z) - C(z^*) \geq 0.5(1 + \rho)\tau_1(\mathcal{L}(z - z^\star))^T(\mathcal{L}(z - z^\star)) \\ &\geq 0.5(1 + \rho)\tau_1(\kappa_2\phi_2 v_2 + \kappa_3\phi_3 v_3 + \cdots + \kappa_N\phi_N v_N)^T \\ &\times (\kappa_2\phi_2 v_2 + \kappa_3\phi_3 v_3 + \cdots + \kappa_N\phi_N v_N) \\ &= 0.5(1 + \rho)\tau_1({\kappa_2}^2{\phi_2}^2 + {\kappa_3}^2{\phi_3}^2 + \cdots + {\kappa_N}^2{\phi_N}^2) \\ &\geq 0.5(1 + \rho)\tau_1\phi_2^2||v||^2, \end{aligned} \tag{34}$$

using (27). Now, it follows from (29) and (34) that

$$\begin{aligned} &||\nabla^T C(z)||^2||v||^2 \geq 0.5(1 + \rho)\tau_1\phi_2^2||v||^2(C(z) - C(z^\star)) \\ &\implies ||\nabla^T C(z)||^2 \geq 0.5(1 + \rho)\tau_1\phi_2^2(C(z) - C(z^*)), \end{aligned} \tag{35}$$

as $||v|| \neq 0$ for non-trivial optimal solution. It is worth noticing that $\tau_i > 0, \forall i$, under Assumption (A2). This follows from the fact (R5) in Lemma 4 that $\tau_i \geq (1 + \mathcal{B}_{i0})\sigma + b_1\delta \geq (1 + \rho)\sigma + b_1\delta > 0$ if $\delta > 0$; and $\tau_i \geq (1 + \mathcal{B}_{i0})\sigma + b_N\delta \geq (1 + \rho)\sigma + b_N\delta > 0$ if $\delta < 0$, as per Assumption (A2).

Next, we consider the candidate Lyapunov function

$$V = 0.5(C(z) - C(z^\star))^2, \tag{36}$$

whose time derivative is

$$\dot{V} = (C(z) - C(z^\star))\nabla^T C(z)\dot{z}. \tag{37}$$

Using (6) and (18), it yields that

$$\begin{aligned} \dot{z}_i &= -k_1\,\mathrm{sig}\left(\frac{\partial C_i}{\partial z_i} + \sum_{j=1, j\neq i}^N \frac{\partial C_i}{\partial z_i}\right)^\mu - k_2\,\mathrm{sig}\left(\frac{\partial C_i}{\partial z_i} + \sum_{j=1, j\neq i}^N \frac{\partial C_i}{\partial z_i}\right)^\nu \\ &= -k_1\,\mathrm{sig}\left(\frac{\partial C}{\partial z_i}\right)^\mu - k_2\,\mathrm{sig}\left(\frac{\partial C}{\partial z_i}\right)^\nu,\ \forall i. \end{aligned} \tag{38}$$

It can be rewritten in vector notations that

$$\dot{z} = -k_1 \operatorname{sig}(\nabla C(z))^{\mu} - k_2 \operatorname{sig}(\nabla C(z))^{\nu},$$

which on substitution in (37), yields

$$\dot{V} = (C(z) - C(z^\star))\nabla^T C(z)(-k_1 \operatorname{sig}(\nabla C(z))^{\mu} - k_2 \operatorname{sig}(\nabla C(z))^{\nu}).$$

Note that [11]

$$\nabla^T C(z) \operatorname{sig}(\nabla C(z))^{\mu} = \sum_{i=1}^{N} \left|\frac{\partial C}{\partial z_i}\right|^{(\mu+1)} = \sum_{i=1}^{N} \left(\frac{\partial C}{\partial z_i}\right)^{2\frac{(\mu+1)}{2}}$$

$$\nabla^T C(z) \operatorname{sig}(\nabla C(z))^{\nu} = \sum_{i=1}^{N} \left|\frac{\partial C}{\partial z_i}\right|^{(\nu+1)} = \sum_{i=1}^{N} \left(\frac{\partial C}{\partial z_i}\right)^{2\frac{(\nu+1)}{2}}.$$

Using these relations, we have

$$\dot{V} = (C(z) - C(z^\star))\times \left[-k_1 \sum_{i=1}^{N} \left(\frac{\partial C}{\partial z_i}\right)^{2\frac{(\mu+1)}{2}} - k_2 \sum_{i=1}^{N} \left(\frac{\partial C}{\partial z_i}\right)^{2\frac{(\nu+1)}{2}} \right]. \quad (39)$$

From Lemma 2,

$$\sum_{i=1}^{N} \left(\frac{\partial C}{\partial z_i}\right)^{2\left(\frac{1+\mu}{2}\right)} \geq \left(\sum_{i=1}^{N} \left(\frac{\partial C}{\partial z_i}\right)^{2}\right)^{\frac{1+\mu}{2}}$$

$$\sum_{i=1}^{N} \left(\frac{\partial C}{\partial z_i}\right)^{2\left(\frac{1+\nu}{2}\right)} \geq N^{\frac{1-\nu}{2}} \left(\sum_{i=1}^{N} \left(\frac{\partial C}{\partial z_i}\right)^{2}\right)^{\frac{1+\nu}{2}},$$

implying that

$$\begin{aligned}
\dot{V} \leq & -k_1 (C(z) - C(z^\star)) \left(\sum_{i=1}^{N} \left(\frac{\partial C}{\partial z_i}\right)^{2}\right)^{\frac{1+\mu}{2}} \\
& - k_2 (C(z) - C(z^*)) N^{\frac{1-\nu}{2}} \left(\sum_{i=1}^{N} \left(\frac{\partial C}{\partial z_i}\right)^{2}\right)^{\frac{1+\nu}{2}} \\
\leq & -k_1 (C(z) - C(z^*))(||\nabla C(z)||^2)^{\frac{1+\mu}{2}} \\
& - k_2 (C(z) - C(z^*)) N^{\frac{1-\nu}{2}} (||\nabla C(z)||^2)^{\frac{1+\nu}{2}}.
\end{aligned}$$

Using (35) and (36), we have

$$\begin{aligned}
\dot{V} \leq & -k_1 (C(z) - C(z^\star))[0.5(1+\rho)\tau_1 \phi_2^2 (C(z) - C(z^\star))]^{\frac{1+\mu}{2}} \\
& - k_2 N^{\frac{1-\nu}{2}} (C(z) - C(z^*))[0.5(1+\rho)\tau_1 \phi_2^2 (C(z) - C(z^\star))]^{\frac{1+\nu}{2}} \\
\leq & -k_1 (0.5(1+\rho)\tau_1 \phi_2^2)^{\frac{1+\mu}{2}} 2^{\frac{3+\mu}{4}} (V)^{\frac{3+\mu}{4}} \\
& - k_2 N^{\frac{1-\nu}{2}} (0.5(1+\rho)\tau_1 \phi_2^2)^{\frac{1+\nu}{2}} 2^{\frac{3+\nu}{4}} (V)^{\frac{3+\nu}{4}} \\
\leq & -k_1 ((1+\rho)\tau_1 \phi_2^2)^{\frac{1+\mu}{2}} 2^{\frac{1-\mu}{4}} (V)^{\frac{3+\mu}{4}} \\
& - k_2 N^{\frac{1-\nu}{2}} ((1+\rho)\tau_1 \phi_2^2)^{\frac{1+\nu}{2}} 2^{\frac{1-\nu}{4}} (V)^{\frac{3+\nu}{4}}.
\end{aligned}$$

Following Lemma 1, it can be concluded that $\dot{V} \leq -(\alpha V^p + \beta V^q)$ with $k = 1, \alpha = k_1((1+\rho)\tau_1\phi_2^2)^{\frac{1+\mu}{2}} 2^{\frac{1-\mu}{4}}, \beta = k_2 N^{\frac{1-\nu}{2}}((1+\rho)\tau_1\phi_2^2)^{\frac{1+\nu}{2}} 2^{\frac{1-\nu}{4}}$, and $0 < p = \frac{3+\mu}{4} < 1, q = \frac{3+\nu}{4} > 1$, and hence, the settling time is bounded by

$$T_s \leq \frac{4}{k_1((1+\rho)\tau_1\phi_2^2)^{\frac{1+\mu}{2}} 2^{\frac{1-\mu}{4}} (1-\mu)} + \frac{4}{k_2 N^{\frac{1-\nu}{2}}((1+\rho)\tau_1\phi_2^2)^{\frac{1+\nu}{2}} 2^{\frac{1-\nu}{4}} (\nu-1)}. \quad (40)$$

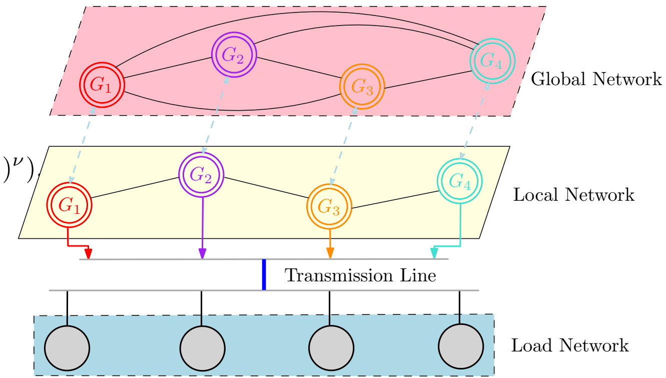


Fig. 1. Generator-load configuration with a two-layered interaction topology. Global network is used for sharing the generated power, while the local network for cost function and auxiliary variable in algorithm (6).

TABLE I
GENERATOR COST PARAMETERS.

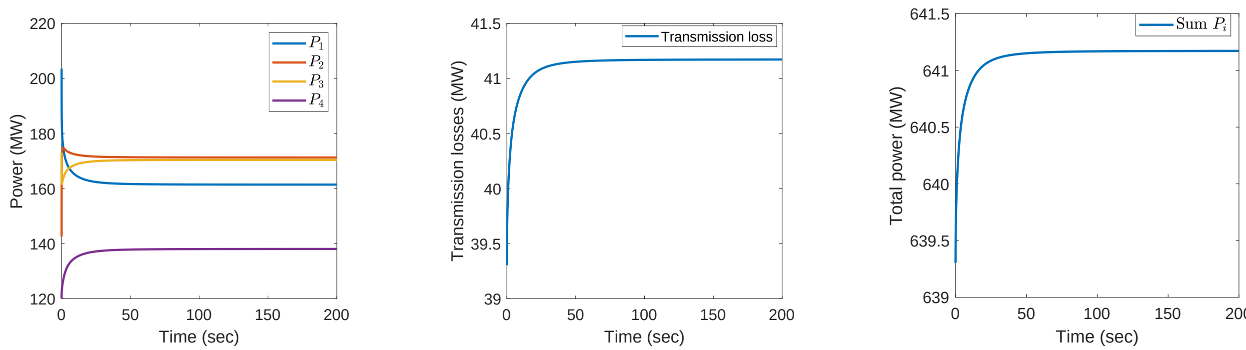
Bus	a_i(\$/h)	b_i(\$/MWh)	c_i(\$/MW^{2}h)
G_1	53	1.21	0.094
G_2	34	3.47	0.082
G_3	45	2.24	0.086
G_4	78	2.55	0.105

This implies that $V \to 0$ as $t \to T_s$, and hence it follows from (30) and (36) that $z = z^\star$ as $t \geq T_s$. This concludes the proof. □

Remark 3. *Note that the right-side of the inequality* (40) *is well-defined as $\tau_1 > 0$, under Assumption* (A2). *According to Lemma 4, since τ_1 depends upon the eigenvalues of matrix $\mathcal{B}$, coefficients $\mathcal{B}_{i0}$, and the constants σ, δ associated with the cost functions C_i, the settling time T_s shows a dependence on these parameters, and network topology because of the occurrence of ϕ_2 (the second smallest eigenvalue of the Laplacian $\mathcal{L}$ associated with the local network). In fact, the inequality* (40) *provides an estimate of the upper bound for the convergence time. Its value is robust to the changes in the initial conditions and power transmission losses. The actual convergence time may be much less than this estimated value.*

IV. Simulation Example

Consider a power system network of four generators comprising a two-layered interaction topology, as shown in Fig. 1. The generators share power outputs globally and the other auxiliary variables in algorithm (6) are shared locally. The power generation cost associated with each generator is characterized by the quadratic function $C_i(P_i) = c_i P_i^2 + b_i P_i + a_i$, where a_i, b_i, c_i are the cost coefficients. The economic dispatch problem can be described as: Min $C(P) = \sum_{i=1}^{4} c_i P_i^2 + b_i P_i + a_i$, subject to $\sum_{i=1}^{4} P_i = D + P_L$. The values of a_i, b_i, c_i are given in Table (I). Clearly, $\sigma = 2\min_i\{c_i\} = 0.164$ and $\delta = \min_i\{b_i\} = 1.21$. Let the total load demand be $D = 600$ MW. The initial power supplied by the generators are given by $P_1(0) = 170$ MW, $P_2(0) = 110$ MW, $P_3(0) = 140$ MW,



(a) P_i Vs time (b) P_L Vs time (c) $\sum_{i=1}^{N} P_i$ Vs time

Fig. 2. Evolution of generated power, transmission losses and total power during $0-200$ sec. Clearly, the power supplied by generator network is equal to the sum of load demand and power transmission losses at each instant of time.

$P_4(0) = 180$ MW. The power transmission losses (1) are obtained by setting the $\mathcal{B}-$loss coefficients as:

$$\mathcal{B} = \begin{bmatrix} 0.1200 & 0.0286 & 0.0481 & 0.0321 \\ 0.0286 & 0.1341 & 0.0511 & 0.1251 \\ 0.0481 & 0.0511 & 0.1539 & 0.1463 \\ 0.0321 & 0.1251 & 0.1463 & 0.1612 \end{bmatrix} \times 10^{-3},$$

which is symmetric with all positive entries and $\mathcal{B}_0 = [2.0, 1.0, 2.5, 1.5]^T \times 10^{-3}; B_{00} = 4$. It can be easily verified that the condition in Remark 2 holds for the given load demand $D = 600$ MW, and the chosen $\mathcal{B}-$coefficients.

- The algorithm (6) is simulated with control parameters as $k_1 = k_2 = 5$, and $\mu = 0.5, \nu = 2$. The optimal power supplied by the generators are obtained as $P_1^* = 161.4$ MW, $P_2^* = 171.3$ MW, $P_3^* = 170.4$ MW and $P_4^* = 138.1$ MW, as shown in Fig. 2 (a). The total power supplied is $P_T = 641.2$ MW, meeting the load demand $D = 600$ MW and the power transmission losses 41.2 MW at the optimal solution (see Figs. 2(b) and 2(c)). One can observe from Fig. 2 that the demand and transmission losses are supplied by the generators at every instant of time. The optimal cost is plotted in Fig. 3, and is evaluated to be \$11093.
- We verify the convergence time in these plots by evaluating the settling time T_s in (40). For the given values of σ, δ and $\mathcal{B}-$coefficients, the matrix $\mathcal{S}$ in Lemma 4 is obtained as (considering each entry with four significant decimal places):

$$\mathcal{S} = \begin{bmatrix} 0.1646 & 0.0000 & 0.0001 & 0.0000 \\ 0.0000 & 0.1645 & 0.0001 & 0.0002 \\ 0.0001 & 0.0001 & 0.1648 & 0.0002 \\ 0.0000 & 0.0002 & 0.0002 & 0.1646 \end{bmatrix}.$$

The minimum eigenvalues of $\mathcal{B}$ and $\mathcal{S}$ are $b_1 = -0.0161 \times 10^{-3}$ and $\tau_1 = 0.1644$, respectively. Further, $\rho = \min_i\{\mathcal{B}_{i0}\} = 1.0 \times 10^{-3}$ and the value of $\phi_2 = 0.5858$. It can be easily verified that $(1+\rho)\sigma + b_1\delta = (1 + 1.0 \times 10^{-3}) \times 0.164 + (-0.0161 \times 10^{-3}) \times 1.21 = 0.1641 > 0$, satisfying the Assumption (A2). Using the above values, the settling time is obtained as $T_s = 154.47$ sec, which supports our simulation results in Figs. 2 and 3.

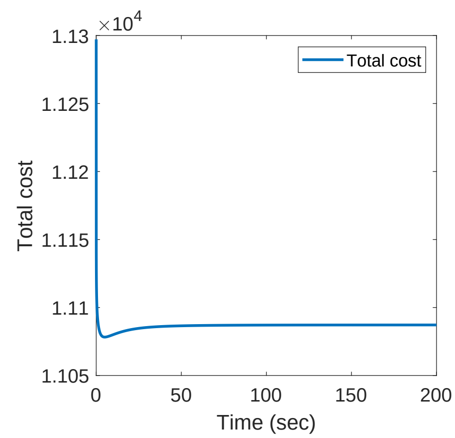


Fig. 3. Evolution of $\sum_{i=1}^{N} C_i(P_i)$ during 0–200 s.

- Furthermore, we have observed through simulations that the dynamics $\dot{z}_i$ is robust to the additive bounded disturbances with zero mean. That is, for any uniformly bounded zero-mean signal $w_i : \mathbb{R}_+ \to \mathbb{R}$ for each i, $\dot{z}_i = -k_1 \,\mathrm{sig}[\sum_{j\in N_i} a_{ij}(H_j\lambda_j - H_i\lambda_i)]^{\mu} - k_2 \,\mathrm{sig}[\sum_{j\in N_i} a_{ij}(H_j\lambda_j - H_i\lambda_i)]^{\nu} + w_i$ has no effect on the solution of algorithm (6).

V. Conclusion

In this paper, we investigated the EDP with Kron's modeled power transmission losses, under a few assumptions on the $\mathcal{B}-$loss coefficients, network topology, and the convexity of the cost functions associated with each generator. The time-varying power transmission losses are incorporated in the equality constraints of considered EDP. It is shown that the proposed consensus-based (partially distributed) algorithm solves the EDP in a finite time, which is upper bounded by a term relying on the eigenvalues of the matrix $\mathcal{B}$, local Laplacian, and the constants describing the convexity of the cost functions.

Although for the approximated power transmission losses, the proposed algorithm can be implemented in a fully distributed manner (see Remark 1). However, it remains a challenging problem to come up which such an algorithm accounting for Kron's modeled power transmission losses without an approximation. Besides, there are several possibilities for future work such as a) consideration of directed communication topology with time-delay in information sharing among generators b) incorporation of fluctuation in load demand.

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APPENDIX

A. Proof of Lemma 4

Proof. The proof is provided sequentially for each step.

(R1) By definition, $\mathcal{F} \in \mathbb{R}^N$ is a column vector (please notice the use of Hadamard product $\odot$) with i^{th} entry $\mathcal{F}_i = (\partial P_L/\partial P_i)(\partial C/\partial P_i)$, where,

- $\partial P_L/\partial P_i$ depends on both P_i and P_j, according to Lemma 3,
- $\partial C/\partial P_i = (\partial C/\partial C_i)(\partial C_i/\partial P_i) = \partial C_i/\partial P_i$ depends only on P_i (as $\partial C/\partial C_i = 1$, according to (3a)).

Therefore, $\nabla\mathcal{F}$ is an $N \times N$ matrix, whose diagonal entries can be obtained using chain rule as

$$[\nabla\mathcal{F}_{ii}] = \frac{\partial}{\partial P_i}\left[\frac{\partial P_L}{\partial P_i}\frac{\partial C_i}{\partial P_i}\right] = \frac{\partial}{\partial P_i}\left[\frac{\partial P_L}{\partial P_i}\right]\frac{\partial C_i}{\partial P_i} + \frac{\partial P_L}{\partial P_i}\frac{\partial^2 C_i}{\partial P_i^2},$$

which using Lemma 3 can be written as

$$\begin{aligned}
[\nabla\mathcal{F}_{ii}] &= 2\mathcal{B}_{ii}\left(\frac{\partial C_i(P_i)}{\partial P_i} + P_i\frac{\partial^2 C_i(P_i)}{\partial P_i^2}\right) \\
&\quad + \frac{\partial^2 C_i(P_i)}{\partial P_i^2}\left[\sum_{j=1,j\neq i}^{N} 2\mathcal{B}_{ij}P_j\right] + \mathcal{B}_{i0}\frac{\partial^2 C_i(P_i)}{\partial P_i^2} \\
&= 2\mathcal{B}_{ii}\frac{\partial C_i(P_i)}{\partial P_i} + \frac{\partial^2 C_i(P_i)}{\partial P_i^2}\left(\mathcal{B}_{i0} + 2\mathcal{B}_{i0}P_i + \sum_{j=1,j\neq i}^{N} 2\mathcal{B}_{ij}P_j\right) \\
&\geq 2\mathcal{B}_{ii}\frac{\partial C_i(P_i)}{\partial P_i} + \mathcal{B}_{i0}\frac{\partial^2 C_i(P_i)}{\partial P_i^2} \\
&\geq 2\mathcal{B}_{ii}\delta + \mathcal{B}_{i0}\sigma,
\end{aligned}$$

under Assumptions 2 and 3 for $P_i, P_j \geq 0, \forall i, j$. Similarly, the off-diagonal entries are given by

$$\begin{aligned}
[\nabla\mathcal{F}_{ij}] &= \frac{\partial}{\partial P_j}\left[\frac{\partial P_L}{\partial P_i}\frac{\partial C_i}{\partial P_i}\right] = \frac{\partial C_i}{\partial P_i}\frac{\partial}{\partial P_j}\left[\frac{\partial P_L}{\partial P_i}\right] \\
&= 2\mathcal{B}_{ij}\frac{\partial C_i(P_i)}{\partial P_i} \geq 2\mathcal{B}_{ij}\delta.
\end{aligned}$$

(R2) Clearly, $\mathcal{M} \in \mathbb{R}^N$ is a column vector with i^{th} entry $\mathcal{M}_i = [\mathcal{B}_{ii}P_i + (\mathcal{B}_{i0}/2)](\partial C_i(P_i)/\partial P_i)$, which depends only on P_i for each i. As a result, $\nabla\mathcal{M}$ is an $N \times N$ diagonal matrix with diagonal entries

$$\begin{aligned}
[\nabla\mathcal{M}_{ii}] &= \mathcal{B}_{ii}\frac{\partial C_i(P_i)}{\partial P_i} + \left[\mathcal{B}_{ii}P_i + \frac{\mathcal{B}_{i0}}{2}\right]\frac{\partial^2 C_i(P_i)}{\partial P_i^2} \\
&\geq \left[\mathcal{B}_{ii}\delta + \frac{\mathcal{B}_{i0}}{2}\sigma\right],
\end{aligned}$$

for $P_i \geq 0, \forall i$ and following Assumptions 2 and 3.

(R3) The poof of this statement is straightforward and follows the similar steps as above.

(R4) From Assumption 3, it is obvious that $\mathcal{S}$ is a symmetric matrix. Further, using (R3) it trivially holds that $\mathcal{S} \preccurlyeq \mathcal{Q}$.

(R5) Since $\mathcal{S}$ is a symmetric matrix, its eigenvalues are real and can be arranged as $\tau_1 \leq \tau_2 \cdots \leq \tau_N$. By construction, $\mathcal{S}$ can be written as the summation of two symmetric matrices $\sigma\text{diag}\{(1+\mathcal{B}_{i0})\}$ and $\delta\mathcal{B}$, where the constants σ and δ are defined in Assumption 2. Notice that the eigenvalues of $\sigma\text{diag}\{(1+\mathcal{B}_{i0})\}$ are $[\sigma(1+\mathcal{B}_{i0})]_{i=1}^N$, while for $\delta\mathcal{B}$ are $b_1\delta \leq b_2\delta \leq \cdots \leq b_N\delta$, if $\delta > 0$; and $b_N\delta \leq b_{N-1}\delta \leq \cdots \leq b_1\delta$, if $\delta < 0$. Now, the results immediately follows by applying the Weyl's theorem [[31], Chapter 4, pg. 239].

□