On High-Rate Full-Diversity 2×2 Space-Time Codes with Low-Complexity Optimum Detection

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***Abstract*—The 2×2 MIMO profiles included in Mobile WiMAX specifications are Alamouti's space-time code (STC) for transmit diversity and spatial multiplexing (SM). The former has full diversity and the latter has full rate, but neither of them has both of these desired features. An alternative 2×2 STC, which is both full rate and full diversity, is the Golden code. It is the best known 2×2 STC, but it has a high decoding complexity. Recently, the attention was turned to the decoder complexity, this issue was included in the STC design criteria, and different STCs were proposed. In this paper, we first present a full-rate full-diversity 2×2 STC design leading to substantially lower complexity of the optimum detector compared to the Golden code with only a slight performance loss. We provide the general optimized form of this STC and show that this scheme achieves the diversity-multiplexing frontier for square QAM signal constellations. Then, we present a variant of the proposed STC, which provides a further decrease in the detection complexity with a rate reduction of 25% and show that this provides an interesting trade-off between the Alamouti scheme and SM.**

***Index Terms*—ML detection, multiple-input multiple-output (MIMO), space-time codes (STCs).**

I. Introduction

MULTIPLE-input multiple-output (MIMO) techniques based on using multiple antennas at transmitter and receiver can provide spatial diversity, multiplexing gain, interference suppression, and make various tradeoffs between them. These techniques have been incorporated in all of the recently developed wireless communications system specifications including the IEEE 802.16e-2005 standard [1] for mobile broadband wireless access systems. From the MIMO schemes included in the IEEE 802.16e specifications, the WiMAX Forum has specified two mandatory profiles for use on the downlink. One of them is based on the space–time code (STC) proposed by Alamouti for transmit diversity [2]. This code achieves a diversity order that is equal to twice the number of antennas at the receiver, but it is only half-rate. (In this paper, the rate is defined as the number of transmitted symbols per antenna use.) The other profile is spatial multiplexing (SM), which uses two transmit antennas to transmit two independent data streams. This scheme is full-rate, but it does not benefit from any diversity gain at the transmitter.

Although it can be anticipated that these STCs will be two basic profiles of most future standards, there is a need to include a new code combining their respective advantages while avoiding their drawbacks. Such a code actually exists in [1]. This code is a variant of the Golden code [3], which is known to be one of the best 2×2 STCs achieving the diversity-multiplexing frontier [4]. But the problem of this code is its detection complexity, which grows as the fourth-power of the signal constellation size, and this makes it impractical for low-cost wireless user terminals. Recently, motivated by the orthogonality of the Alamouti scheme, new full-rate full-diversity (FR-FD) 2×2 STCs were proposed independently in [5], [8]–[10]. These codes achieve the diversity-multiplexing frontier with reduced detection complexity. In this paper, we describe the STC originally proposed in [5], discuss its basic properties, and compare it with the best known STC to date. We provide the general optimized form of this STC whose optimum detection complexity (using exhaustive search) grows at most quadratically with the size of the signal constellation and show that this scheme achieves the diversity-multiplexing frontier for square QAM signal constellations. We also present a rate-$3/4$ variant of this STC which provides an interesting trade-off between the Alamouti scheme and SM.

The rest of the paper is organized as follows. First, in Section II, we briefly discuss the general design criteria for STCs. Sections III and IV are devoted to the proposed 2×2 STCs and the relevant comparisons. Specifically, we first describe the proposed scheme and compare its features with the best known alternatives. Then, we describe the corresponding maximum likelihood (ML) detector including both exhaustive search and sphere decoder (SD), and analyze the optimized form of the proposed STC. In Section IV, we present the rate-$3/4$ STC with a further reduction in receiver complexity. Finally, we present some numerical comparisons in Section V, and we give our conclusions in Section VI.

Notation: Matrices (resp., column vectors) are set in boldface capital (resp., lowercase) letters. a_{kl} denotes the entry of matrix $\mathbf{A}$ at its kth row and lth column, and b_k denotes the kth element of the column vector $\mathbf{b}$. The operators $(\cdot)^*$, $(\cdot)^T$, and $(\cdot)^H$ stand for complex conjugate, transpose, and conjugate transpose, respectively. $||\cdot||^2$ denotes Frobenius norm of the enclosed vector. $\Re\{\cdot\}$ (resp., $\Im\{\cdot\}$) denotes the real (resp., imaginary) part of the enclosed term.

II. STC Design Criteria

A. Pairwise Error Probability Analysis

We will start with a brief discussion on the most common design criteria for STCs. We consider that the transmitter does not have any channel state information while the receiver

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knows the channel perfectly. For 2×2 MIMO transmission, we write

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{Z}, \tag{1}$$

where $\mathbf{H}$ is the 2×2 channel matrix with the entries h_{kl} of complex channel gains, $\mathbf{X}$ is the 2×2 codeword matrix

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \tag{2}$$

whose elements take values from the codebook $\mathcal{X}$, $\mathbf{Y}$ includes the received signals and $\mathbf{Z}$ denotes the matrix of additive circularly symmetric complex Gaussian noise samples with spectral density N_0, respectively. Recently proposed STC schemes mainly rely on analysis of the pairwise error probability (PEP) $P(\mathbf{X} \rightarrow \hat{\mathbf{X}})$ which is the probability that $\hat{\mathbf{X}}$ is detected while $\mathbf{X}$ is transmitted. At high signal-to-noise ratio (SNR) values, Chernoff bound analysis of the PEP leads to the well-known *rank criterion* [6] and *determinant criterion* [7]. If the difference matrix $(\mathbf{X} - \hat{\mathbf{X}})$ is full rank for all codeword pairs, then the code is said to have *full diversity*. For high SNR values, the most important parameter is the diversity gain, which dominates the steepness of the bit-error rate (BER) curve. After ensuring full-diversity, we need to maximize the coding gain which can be defined for a 2×2 STC as

$$\delta(\mathcal{X}) = \min_{\substack{\mathbf{X}, \hat{\mathbf{X}} \in \mathcal{X} \\ \mathbf{X} \neq \hat{\mathbf{X}}}} |\det(\mathbf{X} - \hat{\mathbf{X}})|^2. \tag{3}$$

The STCs presented in the sequel are examples of such schemes which have a large coding gain. Other design criteria can also be added. Among them, we mention below the requirement that the constellation has cubic shaping. Here, we will optimize the proposed code such that the coding gain is maximized and does not depend on the constellation size. The last property ensures that the optimized code will have full diversity.

B. Detection Complexity

In the design of STCs another important criterion is the decoding complexity. This is highly crucial especially for mobile applications. The Golden code is the best known full-rate 2×2 STC which satisfies the rank criterion with a high coding gain. However, optimum detection has a high computational complexity. Therefore, other FR-FD STCs should be introduced as alternatives to the Golden code which have lower optimum decoding complexity. The results available in the literature suggest that there is an intrinsic tradeoff between error performance and detection complexity [5], [8]–[11]. However, theoretical tradeoff limits have not been exhibited yet.

The Golden code, which we denote by $\mathbf{X}_g$ in the sequel, provides FR-FD with a coding gain of $16/5$ and achieves substantially better performance than SM whose diversity order is limited to the number of receive antennas. But, as explained above, this code has an inherent detection complexity problem.

For full-rate 2×2 STCs, the optimum receiver evaluates the ML function expressed as:

$$D\left(s_1, s_2, s_3, s_4\right) = \left\|\mathbf{Y} - \mathbf{H}\mathbf{X}\right\|^2, \tag{4}$$

for all symbol quadruplets (s_1, s_2, s_3, s_4) and selects the one which minimizes this norm. The norm given in (4) is actually the squared Euclidean distance between the received noisy signal and the noiseless signal corresponding to that quadruplet. For a signal constellation with M points, this receiver involves the computation of M^4 Euclidean distances and selects the symbol quadruplet minimizing this distance. This complexity is of course prohibitive in practical applications with the 16-QAM and 64-QAM signal constellations and current state of technology. Therefore, one has to resort to suboptimum receivers which may degrade the performance severely. One possible solution is to use SD whose performance and complexity are upper bounded by those of ML detection based on exhaustive search. However, even the use of SD would require a high number of computations for satisfactory detection performance. This motivates the use of new STCs which have close performance to that of $\mathbf{X}_g$ with lower detection complexity.

III. Proposed STC and Comparison with the Existing FR-FD 2 × 2 STCs

Now, we turn our attention to the recently proposed FR-FD 2×2 STC schemes. They attempt to maximize both the diversity gain and the coding gain, while leading to an optimum detection of reduced complexity. More specifically, these schemes are FR-FD 2×2 STCs whose optimum receiver has a complexity that is *only* proportional to M^2 (see [5], [8]–[11] for more detail). Comparing their complexity to that associated to $\mathbf{X}_g$, it becomes clear that these codes make the implementation of FR-FD 2×2 STCs with an optimum receiver more realistic.

Such an STC first appears in [10], but its low decoding complexity property was only realized in [8], independent of our work in [5]. The STC presented in [8] is a combination of the original Alamouti scheme and a precoded scheme having also an Alamouti structure. In contrast, our STC directly combines two Alamouti schemes and evenly distributes the average transmitted energy for each symbol per channel use. Since the transmitted signal is a combination of two symbols only, it has a lower peak-to-average power ratio (PAPR) than the STCs presented in [8][10] whose components are sums of more than two signals. Table I shows such a comparison of constellation PAPR, which is defined as the ratio of the peak power to the average power transmitted per antenna. It gives evidence that the proposed scheme provides more than 1.2 dB PAPR gain. This is an obvious upside that one may be willing to take in exchange of the minor SNR loss and the lack of cubic shaping which is explained below. In this code, a group of 4 symbols (s_1, s_2, s_3, s_4) is transmitted as follows:

$$\mathbf{X}_{new} = \begin{bmatrix} as_1 + bs_3 & -cs_2^* - ds_4^* \\ as_2 + bs_4 & cs_1^* + ds_3^* \end{bmatrix}. \tag{5}$$

A careful look clearly shows that (5) is nothing but a simple linear combination of two Alamouti schemes. Here, a, b, c, and d are complex-valued design parameters. They are chosen such that the resulting STC attains FR-FD transmission in a quasi-static Rayleigh fading channel. The constraints we introduce

$$\mathbf{G}=\begin{bmatrix} \Re\{a\} & -\Im\{a\} & 0 & 0 & \Re\{b\} & -\Im\{b\} & 0 & 0 \\ \Im\{a\} & \Re\{a\} & 0 & 0 & \Im\{b\} & \Re\{b\} & 0 & 0 \\ 0 & 0 & \Re\{a\} & -\Im\{a\} & 0 & 0 & \Re\{b\} & -\Im\{b\} \\ 0 & 0 & \Im\{a\} & \Re\{a\} & 0 & 0 & \Im\{b\} & \Re\{b\} \\ 0 & 0 & -\Re\{c\} & -\Im\{c\} & 0 & 0 & -\Re\{d\} & -\Im\{d\} \\ 0 & 0 & -\Im\{c\} & \Re\{c\} & 0 & 0 & -\Im\{d\} & \Re\{d\} \\ \Re\{c\} & \Im\{c\} & 0 & 0 & \Re\{d\} & \Im\{d\} & 0 & 0 \\ \Im\{c\} & -\Re\{c\} & 0 & 0 & \Im\{d\} & -\Re\{d\} & 0 & 0 \end{bmatrix}$$

TABLE I
COMPARISON OF CONSTELLATION PAPR (DB) PER TRANSMIT ANTENNA.

	QPSK	16-QAM	64-QAM
Proposed STC	2.8136	5.3663	6.4934
STC in [8]	4.0866	6.6393	7.7663

in (5) are the average transmit power constraints, i.e.,

$$|a|^2+|b|^2=|c|^2+|d|^2=1 \text{ and } |a|^2+|c|^2=|b|^2+|d|^2=1. \tag{6}$$

Here, the first condition ensures the transmission of equal average power at each symbol time, while the second condition ensures that equal average total power is transmitted for each symbol. As is shown in the sequel, these constraints allow us to simplify the optimization procedure. Before giving the details related to the optimization of the design parameters we will give a brief comparison among the existing STCs and explain the reduced complexity detection capability of the presented STC.

A. Comparison with the Existing STCs

Similar to the other STCs mentioned above, the proposed STC $\mathbf{X}_{new}$ falls into the class of linear dispersion codes [12] which can be written in the form

$$\mathbf{X}=\sum_{k=1}^{4}(s_{k,R}\mathbf{A}_k+js_{k,I}\mathbf{B}_k), \tag{7}$$

where $s_{k,R}$ and $s_{k,I}$ denote the real and imaginary parts of the symbol s_k, respectively, and $\mathbf{A}_k$, $\mathbf{B}_k$, $k=1,\ldots,4$, are 2×2 complex-valued weight matrices of $\mathbf{X}$. Generally speaking, the matrices $\mathbf{A}_k$, $\mathbf{B}_k$, $k=1,\ldots,4$ have to be designed such that

$$\sum_{k=1}^{4}\operatorname{tr}(\mathbf{A}_k^H\mathbf{A}_k+\mathbf{B}_k^H\mathbf{B}_k)=8 \tag{8}$$

in order to conserve the total average transmitted power, where $\operatorname{tr}(\cdot)$ denotes the trace of the enclosed matrix. With the constraint of equal average energy transmission for each symbol, (8) turns to

$$\operatorname{tr}(\mathbf{A}_k^H\mathbf{A}_k+\mathbf{B}_k^H\mathbf{B}_k)=2 \text{ for all } k=1,\ldots,4. \tag{9}$$

It can be easily shown that the aforementioned STCs satisfy (9). Indeed, (9) is equivalent to the transmit power constraint (6) introduced for the design of $\mathbf{X}_{new}$. Furthermore, as shown below, the magnitudes of all the parameters appear to be equal in $\mathbf{X}_{new}$ and this allows the transmitter to transmit all the symbols with the same average power at each channel use. This property is unique to $\mathbf{X}_{new}$.

Now, in order to make a more detailed comparison, we use vector representation and introduce the following notation. First, define the column vectors $\bar{\mathbf{x}}=[x_{11},x_{21},x_{12},x_{22}]^T$, $\bar{\mathbf{y}}=[y_{11},y_{21},y_{12},y_{22}]^T$ and $\bar{\mathbf{z}}=[z_{11},z_{21},z_{12},z_{22}]^T$, which are obtained by stacking the columns of the matrices $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$, respectively, one after the other. Next, we define the corresponding real-valued column vector as

$$\begin{aligned}\bar{\mathbf{x}}_R &= [\Re\{x_{11}\},\Im\{x_{11}\},\Re\{x_{21}\},\Im\{x_{21}\},\\ &\quad \ldots,\Re\{x_{22}\},\Im\{x_{22}\}]^T.\end{aligned} \tag{10}$$

It is known that any linear dispersion code in the form of (7) can be expressed as $\bar{\mathbf{x}}_R=\mathbf{G}\bar{\mathbf{s}}_R$, where $\bar{\mathbf{s}}_R$ collects the real and imaginary parts of the symbols from the symbol vector $\mathbf{s}=[s_1,s_2,s_3,s_4]^T$ as in (10). Here, $\mathbf{G}$ is the real generator matrix of the STC, which can be written for the proposed STC as shown at the top of the page.

The generator matrix $\mathbf{G}$ of the Golden code $(\mathbf{X}_g)$ and the STC presented in [8] has the property

$$\mathbf{G}\mathbf{G}^T=\mathbf{G}^T\mathbf{G}=\mathbf{I}_8, \tag{11}$$

where $\mathbf{I}_N$ denotes the $N\times N$ identity matrix. Therefore, the properties of the input signal $\mathbf{s}$ are not changed and the resulting STC is said to have cubic shaping [3]. This also implies that the average power of the input symbol vector $\mathbf{s}$ remains unchanged whatever the properties of the signal. On the other hand, the property (11) is not satisfied with $\mathbf{X}_{new}$, i.e., we have

$$\begin{aligned}\mathbf{G}\mathbf{G}^T=\mathbf{G}^T\mathbf{G}=&\begin{bmatrix} 1 & 0 & 0 & 0 & \phi & -\psi & 0 & 0 \\ 0 & 1 & 0 & 0 & \psi & \phi & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \phi & -\psi \\ 0 & 0 & 0 & 1 & 0 & 0 & \psi & \phi \\ \phi & \psi & 0 & 0 & 1 & 0 & 0 & 0 \\ -\psi & \phi & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \phi & \psi & 0 & 0 & 1 & 0 \\ 0 & 0 & -\psi & \phi & 0 & 0 & 0 & 1 \end{bmatrix}\\ &\neq \mathbf{I}_8,\end{aligned}$$

where $\phi=\Re\{ab^*\}+\Re\{cd^*\}$ and $\psi=-\Im\{ab^*\}+\Im\{cd^*\}$. Hence, this code does not have cubic shaping. In order to make a fair comparison between $\mathbf{X}_g$ and $\mathbf{X}_{new}$, we need to know the statistics of the input symbols. In fact, if the input symbols s_1,s_2,s_3,s_4 are either non-zero mean independent symbols or correlated symbols, then, the performance of $\mathbf{X}_{new}$ will deviate from that of $\mathbf{X}_g$. However, since quite powerful interleavers are used in all current system specifications, it is reasonable to assume that the data symbols at the input of the space-time encoder will be uncorrelated. Hence, the average performance of the system will not be affected by the absence of property (11). Indeed, the average power is conserved when the input symbols have no correlation, i.e., we have $E[||\mathbf{G}\bar{\mathbf{s}}_R||^2] = E[\bar{\mathbf{s}}_R^T\mathbf{G}^T\mathbf{G}\bar{\mathbf{s}}_R] = E[||\bar{\mathbf{s}}_R||^2] = E[||\mathbf{s}||^2]$, for both $\mathbf{X}_g$ and $\mathbf{X}_{new}$.

B. Reduced-Complexity ML Detection

Before describing the reduced-complexity ML detection, we provide the following proposition. The idea of constructing such an STC is based upon the following proposition.

Proposition 1: Any 2×2 matrix in the form of (5) with either $|a| = |c|$ or $|b| = |d|$ is ML detectable with exhaustive search complexity $\mathcal{O}(M^2)$.

We provide the proof of this proposition considering the exhaustive search and then explain the corresponding reduced complexity SD.

1) Exhaustive Search: For the sake of simplicity, we first explain the interesting features of the code $\mathbf{X}_{new}$ given in (5) considering the exhaustive ML procedure. The exhaustive ML detector makes a search over all possible values of the transmitted symbols and decides in favor of (s_1, s_2, s_3, s_4) which minimizes the Euclidean distance $D(s_1, s_2, s_3, s_4)$ written as

$$D(s_1, s_2, s_3, s_4) = \sum_{k=1}^{2} |y_{k1} - h_{k1}(as_1 + bs_3) - h_{k2}(as_2 + bs_4)|^2 + \sum_{l=1}^{2} |y_{l2} + h_{l1}(cs_2^* + ds_4^*) - h_{l2}(cs_1^* + ds_3^*)|^2. \quad (12)$$

As explained above, an exhaustive search clearly involves the computation of M^4 metrics and $M^4 - 1$ comparisons, which is excessive for the 16-QAM and 64-QAM signal constellations. But the proposed STC lends itself to a low-complexity implementation of the ML detector. In order to see this complexity reduction more clearly, we expand $D(s_1, s_2, s_3, s_4)$ given in (12). It is straightforward to show that the ML metric in (12) can be written as

$$\begin{aligned} D(s_1, s_2, s_3, s_4) &= C + g_1(s_1, s_3) + g_2(s_2, s_4) \\ &+ \sum_{k=1}^{2} 2\Re\Big\{h_{k1}h_{k2}^*\Big(|a|^2 s_1 s_2^* + ab^* s_1 s_4^* \\ &+ a^* b s_2^* s_3 + |b|^2 s_3 s_4^*\Big)\Big\} \\ &- \sum_{l=1}^{2} 2\Re\Big\{h_{l1}h_{l2}^*\Big(|c|^2 s_1 s_2^* + c^* d s_1 s_4^* \\ &+ cd^* s_2^* s_3 + |d|^2 s_3 s_4^*\Big)\Big\} \end{aligned} \quad (13)$$

where C is a constant independent of the symbols, and $g_1(s_1, s_3)$ and $g_2(s_2, s_4)$ are functions of the symbol pairs (s_1, s_3) and (s_2, s_4), respectively. It is clear from equality (13) that when $|a| = |c|$ the ML metric will reduce to a form $D(s_1, s_2, s_3, s_4) = C + f_1(s_1, s_3, s_4) + f_2(s_2, s_3, s_4)$ where $f_1(s_1, s_3, s_4)$ (resp. $f_2(s_2, s_3, s_4)$) is a function that has no terms involving s_2 (resp. s_1). Therefore, for some given values of the symbol pair (s_3, s_4), $f_1(s_1, s_3, s_4)$ and $f_2(s_2, s_3, s_4)$ can be minimized separately and we can get the ML estimate of s_1 and s_2 independently using a simple decision circuit (2-D threshold detector). As a consequence, the elimination of the terms involving both s_1 and s_2, and, thereby, decreasing the complexity of the detector without losing the optimality is possible if and only if (iff) the coefficients a and c have the same magnitude.

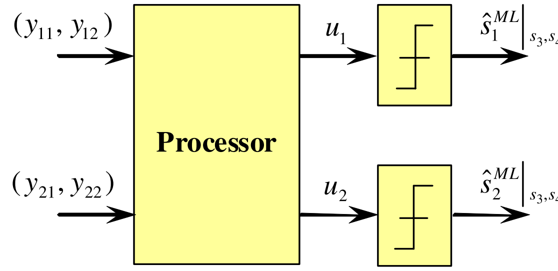

Fig. 1. Processing of the received signals to determine the ML estimate of symbols s_1 and s_2 conditional on a particular combination of symbols s_3 and s_4.

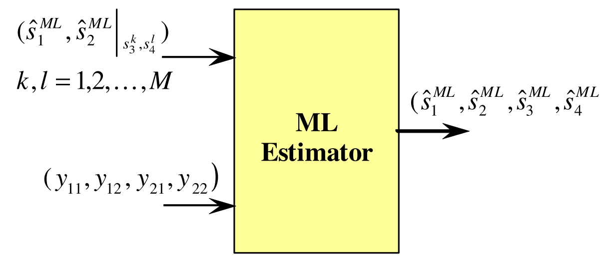

Fig. 2. Second stage of the estimator.

ML estimation of s_1 and s_2 conditional on (s_3, s_4) is illustrated in Fig. 1. In this way, for a given symbol pair (s_3, s_4), we get the ML estimate of (s_1, s_2), which we denote $(\hat{s}_1^{ML}, \hat{s}_2^{ML}|_{s_3,s_4})$. Now, instead of computing the metric $D(s_1, s_2, s_3, s_4)$ for all (s_1, s_2, s_3, s_4) values, we only need to compute it for $((\hat{s}_1^{ML}, \hat{s}_2^{ML}|_{s_3,s_4}), s_3, s_4)$, with s_3 and s_4 spanning the signal constellation. Specifically, let (s_3^k, s_4^l) indicate that symbol s_3 takes the kth point of the signal constellation and symbol s_4 takes the lth point of the signal constellation. The optimum receiver computes the metric $D(s_1, s_2, s_3, s_4)$ for $((\hat{s}_1^{ML}, \hat{s}_2^{ML}|_{s_3^k,s_4^l}), s_3^k, s_4^l)$, where $k, l = 1, \ldots, M$. This procedure, which is illustrated in Fig. 2, reduces the ML receiver complexity from M^4 to M^2, where M is the size of the signal constellation.

Note that the special structure of (5) allows the ML detector also to work the other way round: Instead of deriving the ML estimate of the symbol pair (s_1, s_2) conditional on (s_3^k, s_4^l) and then computing the metric $D(s_1, s_2, s_3, s_4)$ for $((\hat{s}_1^{ML}, \hat{s}_2^{ML}|_{s_3^k,s_4^l}), s_3^k, s_4^l)$, we can first estimate the symbol pair (s_3, s_4) conditional on (s_1^k, s_2^l) and then compute the metric $D(s_1, s_2, s_3, s_4)$ for $(s_1^k, s_2^l, (\hat{s}_3^{ML}, \hat{s}_4^{ML}|_{s_1^k,s_2^l}))$, $k, l = 1, \ldots, M$, and select the quadruplet (s_1, s_2, s_3, s_4) minimizing the metric. In this case, the necessary condition for the elimination of the terms involving both s_1 and s_2 becomes $|b| = |d|$ and the ML metric reduces to the form $D(s_1, s_2, s_3, s_4) = C + f_3(s_3, s_1, s_2) + f_4(s_4, s_1, s_2)$ where $f_3(s_3, s_1, s_2)$ (resp. $f_4(s_4, s_1, s_2)$) is a function that has no terms involving s_4 (resp. s_3).

Another way of demonstrating the optimality condition involves showing that the corresponding columns of the equivalent channel matrix (combining the channel matrix and the generator matrix $\mathbf{G}$ of the STC) are orthogonal iff $|a| = |c|$ for the forward detection and $|b| = |d|$ for the reverse detection.

2) Sphere Decoding: Now, we will describe the SD and present a reduced-complexity detector employing this algorithm. Utilizing the definition (10) for the column vectors $\bar{\mathbf{y}}$ and $\bar{\mathbf{z}}$, we can express the system in (1) as [13]

$$\bar{\mathbf{y}}_R = \breve{\mathbf{H}}\mathbf{G}\bar{\mathbf{s}}_R + \bar{\mathbf{z}}_R. \tag{14}$$

Here, $\breve{\mathbf{H}}$ is obtained from the channel matrix $\mathbf{H}$ as $\breve{\mathbf{H}} = (1/2)\mathbf{I}_2 \otimes (\mathbf{H} \otimes \mathbf{E} + \mathbf{H}^* \otimes \mathbf{E}^*)$ where $\otimes$ stands for Kronecker product and $\mathbf{E} = \begin{bmatrix} 1 & j \\ -j & 1 \end{bmatrix}$. Then, the ML metric (4) can be rewritten as

$$D\left(s_1, s_2, s_3, s_4\right) = \left\|\bar{\mathbf{y}}_R - \breve{\mathbf{H}}\mathbf{G}\bar{\mathbf{s}}_R\right\|^2. \tag{15}$$

Minimization of (15) can be implemented using the SD algorithm [14]. To this end, the matrix $\breve{\mathbf{H}}\mathbf{G}$ is first decomposed using $\mathbf{QR}$ decomposition as $\breve{\mathbf{H}}\mathbf{G} = \mathbf{QR}$, where $\mathbf{Q}$ is an 8×8 unitary matrix and $\mathbf{R}$ is an 8×8 upper triangular matrix. Multiplying (14) from the left-hand side with $\mathbf{Q}^H$, we can write

$$\tilde{\bar{\mathbf{y}}}_R = \mathbf{Q}^H \bar{\mathbf{y}}_R = \mathbf{R}\bar{\mathbf{s}}_R + \mathbf{Q}^H \bar{\mathbf{z}}_R. \tag{16}$$

Then, the SD finds

$$\hat{\bar{\mathbf{s}}}_R = \arg\min_{\bar{\mathbf{s}}_R} \left\|\tilde{\bar{\mathbf{y}}}_R - \mathbf{R}\bar{\mathbf{s}}_R\right\|^2. \tag{17}$$

The search procedure of this standard real SD should be performed by using a tree search with 8 levels. Now, using the special structures of its real generator matrix $\mathbf{G}$ and the upper triangular matrix $\mathbf{R}$, we will show that $\mathbf{X}_{new}$ lends itself to a reduced-complexity implementation of the SD.

Using the fact that the QR decomposition coincides with the Gram-Schmidt orthogonalization procedure applied to the columns of the matrix $\breve{\mathbf{H}}\mathbf{G}$ (see [11] for a more detailed discussion), it can be shown that the upper-triangular matrix $\mathbf{R}$ is

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{0}_4 & \mathbf{R}_{22} \end{bmatrix}, \tag{18}$$

where $\mathbf{R}_{11}$ and $\mathbf{R}_{22}$ are 4×4 *identity* matrices scaled with constant factors, and $\mathbf{0}_4$ is a 4×4 zero matrix. The interesting property of the matrices $\mathbf{R}_{11}$ and $\mathbf{R}_{22}$ comes from the special structure of the real generator matrix $\mathbf{G}$. Indeed, the real generator matrix $\mathbf{G}$ of (5) can be decomposed as $\mathbf{G} = [\mathbf{G}_1|\mathbf{G}_2]$, where each $\mathbf{G}_i$ is equivalent to the real generator matrix of the Alamouti scheme. This can also be deduced from the structure of the proposed STC since it combines two Alamouti type STCs. This allows us to decouple the estimation of symbol pairs and simplify the receiver architecture. More formally, this allows the SD to be performed only for 4 levels and the SD finds

$$\hat{\bar{\mathbf{s}}}_R^{(5,8)} = \arg\min_{\bar{\mathbf{s}}_R^{(5,8)}} \left\|\tilde{\bar{\mathbf{y}}}_R^{(5,8)} - \mathbf{R}_{22}\bar{\mathbf{s}}_R^{(5,8)}\right\|^2. \tag{19}$$

Here, we used the notation $\mathbf{x}^{(k,l)} = [x_k, \ldots, x_l]^T$ in which the symbols are collected from the vector $\mathbf{x}$ either in increasing or decreasing order of indices from k to l. Once the symbol vector $\bar{\mathbf{s}}_R^{(5,8)}$ is obtained using the reduced SD, the remaining symbols in $\bar{\mathbf{s}}_R^{(1,4)}$ are decoded simply as in Alamouti scheme after the cancellation of the interference of $\bar{\mathbf{s}}_R^{(5,8)}$ from $\bar{\mathbf{s}}_R^{(1,4)}$ using $\mathbf{R}_{12}$. Moreover, since the matrix $\mathbf{G}$ is a combination of two 'Alamouti-type' real generator matrices, the decoding can be performed the other way round with the same complexity: By decoding in the reverse direction, the SD finds

$$\hat{\bar{\mathbf{s}}}_R^{(4,1)} = \arg\min_{\bar{\mathbf{s}}_R^{(4,1)}} \left\|\tilde{\bar{\mathbf{y}}}_R^{(4,1)} - \mathbf{R}_{11}\bar{\mathbf{s}}_R^{(4,1)}\right\|^2, \tag{20}$$

and the remaining symbols in $\bar{\mathbf{s}}_R^{(8,5)}$ are obtained using symbol-by-symbol 'Alamouti' decoding. This simply allows evaluating soft information in the form of log-likelihood ratios for all symbol bits with the same receiver architecture. This is particularly important if we need soft data for further decoding stages – which is the case in real system architectures.

It is also worth noting that since the matrix $\mathbf{R}_{22}$ (resp., $\mathbf{R}_{11}$, for the reverse detection order) is an identity matrix scaled with a constant factor, the number of computations will be reduced further in the SD process in (19) (resp. in (20)) compared to the standard SD computations with 4 level tree search.

C. Parameter Optimization

Although the direct optimization of the design parameters a, b, c, d in the code matrix is infeasible especially for higher constellation sizes, the average transmit power constraints given in (6) allow a decrease in the number of parameters to be optimized. These equalities together with the constraint $|a| = |c|$ for optimal delectability lead immediately to the fact that all the design parameters should have the same magnitude, i.e., $|a| = |b| = |c| = |d| = 1/\sqrt{2}$. Now, without any loss of generality, we may set $a = c = 1/\sqrt{2}$. This decreases the number of unknown parameters without affecting the coding gain. Then, the remaining parameter pair (b, c) can be optimized numerically leading to a full-diversity scheme with large coding gain. Note that the values of a and c affect the shape of the resulting lattice structure. Hence, depending on the constellation size, they can be optimized such that the number of nearest points (the so-called kissing number [15]) is minimized. Here, our interest is on the maximization of the coding gain, and the optimization is carried out considering only the above mentioned design criteria (cf. Section II).

In order to set the values of the remaining parameters b and d, one may perform an exhaustive search so as to maximize the coding gain (and, thus, to ensure the full diversity) for QPSK signaling. This optimization leads to a set of parameter values which result in a coding gain of 2. From this set we take the parameter pair as $b = [(1-\sqrt{7}) + i(1+\sqrt{7})]/(4\sqrt{2})$ and $d = -ib$, and introduce the following proposition. The proof of this proposition is provided in the Appendix.

Proposition 2: The proposed full-rate STC (5) with the parameter values given above has a constant coding gain across square M-QAM constellations, independent of M.

Note that a constant coding gain which does not depend on the constellation size implies that the code $\mathbf{X}_{new}$ achieves the diversity-multiplexing frontier [16].

IV. Rate-3/4 2×2 STC

The STC given in (5) can be modified for a further reduction in the optimum detector complexity. More specifically, by setting $s_4 = s_3$ in (5) and scaling the energy of this symbol, we obtain the following 2×2 code with rate 3/4:

$$\mathbf{X}_{new}^{3/4} = \begin{bmatrix} as_1 + bs_3/\sqrt{2} & -cs_2^* - ds_3^*/\sqrt{2} \\ as_2 + bs_3/\sqrt{2} & cs_1^* + ds_3^*/\sqrt{2} \end{bmatrix}, \qquad (21)$$

where the notation $\mathbf{X}_{new}^{3/4}$ is used to distinguish the proposed code $\mathbf{X}_{new}$ (5) from its reduced-rate version.

In order to detect the symbols transmitted using (21), the full ML detector makes an exhaustive search over all possible values of the transmitted symbols and decides in favor of the triplet (s_1, s_2, s_3) which minimizes the Euclidean distance that we denote by $D(s_1, s_2, s_3)$. Specifically, this exhaustive search involves the computation of M^3 metrics and $M^3 - 1$ comparisons, which is also excessive for the 16-QAM and 64-QAM signal constellations. Now, dropping the symbol s_4 lends itself to a lower-complexity implementation of the ML detector at the price of transmission rate reduction.

More precisely, following the same procedure as that presented for the full-rate case in Section III, it can be seen that the signals u_k, $k = 1, 2$, will have only terms involving the respective symbol s_k and the estimation of symbols s_k, $k = 1, 2$, will benefit from full fourth-order spatial diversity. By sending the signals u_1 and u_2 to a threshold detector, we get the ML estimate of symbol s_1 and s_2 conditional only on the symbol s_3. Note that, as a natural consequence of similarity to the full-rate case, the elimination of the terms involving s_2 can be possible iff the coefficients a and c have the same magnitude. In this way, for a given value of symbol s_3, we get the ML estimate of (s_1, s_2), which we denote $(\hat{s}_1^{ML}, \hat{s}_2^{ML}|_{s_3})$. Now, instead of computing the metric $D(s_1, s_2, s_3)$ for all (s_1, s_2, s_3) values, we only need to compute it for $((\hat{s}_1^{ML}, \hat{s}_2^{ML}|_{s_3}), s_3)$. In other words, the optimum receiver computes the metric $D(s_1, s_2, s_3)$ for $((\hat{s}_1^{ML}, \hat{s}_2^{ML}|_{s_3^l}), s_3^l)$, $l = 1, \ldots, M$. This procedure evidently reduces the ML receiver complexity from M^3 to M.

Optimization of the parameters in the reduced-rate case can be performed similarly to the full-rate case. The parameters a and c can be set to $1/\sqrt{2}$ without any loss of generality. In terms of the average transmitted power, the desired conditions can be expressed as $|a|^2 + |b|^2/2 = |c|^2 + |d|^2/2$ and $|a|^2 + |c|^2 = |b|^2 + |d|^2$. Using these constraints, we can easily obtain $|b| = |d| = 1/\sqrt{2}$. Then, an exhaustive search maximizing (3) gives a set of parameter pairs one of them being $b = d = (1 + i\sqrt{7})/4$. It is also interesting to note that, by using the optimized values of the full-rate case, one can obtain the optimum values of the rate-3/4 case without any need for exhaustive search. Indeed, further analysis shows that the non-vanishing coding gain is achieved with a value having its square equal to the product of the optimized values b and d corresponding to the full rate case. Now, we provide the following proposition. The proof follows the same steps as those of Proposition 2.

Proposition 3: The proposed rate-3/4 2×2 STC (21) with $b = d = (1 + i\sqrt{7})/4$ has a constant coding gain for square M-QAM constellations independent of M.

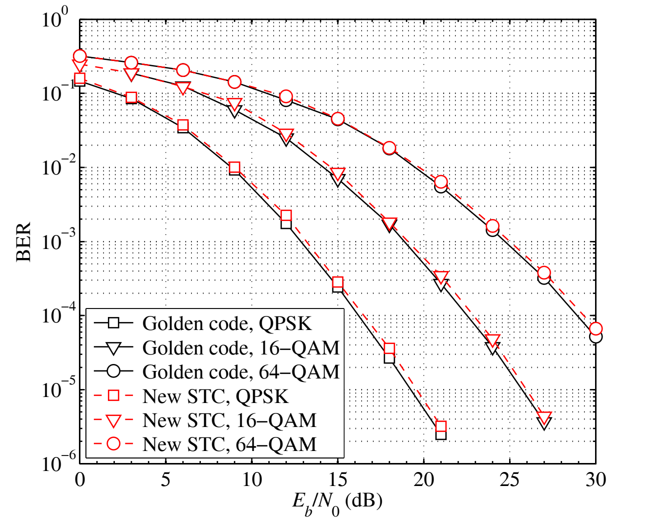

Fig. 3. Performance comparison between $\mathbf{X}_g$ with full SD and $\mathbf{X}_{new}$ with reduced SD.

V. Results

In this section, we present some comparisons between the aforementioned new STCs and the existing alternatives. The simulations were carried out for the QPSK, 16-QAM and 64-QAM signal constellations, and the results are obtained for an uncorrelated Rayleigh fading channel with $E[|h_{kl}|^2] = 1$ for all k, l. Two receive antennas were used in all cases.

A. Performance Comparison in the Full-Rate Case

We first give performance comparisons between the best-known full-rate 2×2 STCs and the proposed full-rate code (5). Fig. 3 shows the BER performance as a function of E_b/N_0, where E_b denotes the average signal energy per bit, and provides comparisons between $\mathbf{X}_{new}$, namely, the new STC, and $\mathbf{X}_g$ (the Golden code). We use full SD for $\mathbf{X}_g$ and reduced-complexity SD (cf. Section III.B) for $\mathbf{X}_{new}$. It can be seen that $\mathbf{X}_{new}$ achieves the same diversity gain and gives essentially the same results as $\mathbf{X}_g$ at substantially lower complexity. The performance curves for the STC proposed in [8] were not included in this figure, but we observed that they are quite indistinguishable from those of $\mathbf{X}_{new}$. Such comparisons also exist in [11] and coincide with our observations. Indeed, their conclusion is that the performance of $\mathbf{X}_{new}$ is marginally inferior to that of [8] and very close to that of $\mathbf{X}_g$.

The complexity reduction can be observed from Fig. 4, where the number of visited nodes [18] are plotted as a function of SNR E_b/N_0. As seen in Fig. 4, $\mathbf{X}_{new}$ results in a considerable reduction in the number of computations. Since the number of visited nodes has a large impact on the required chip area per throughput [18], these results indicate that $\mathbf{X}_{new}$ enables to reduce the hardware complexity without any significant performance degradation.

We may also consider same suboptimum detectors in order to see the performance difference with similar receiver complexity. For such a comparison between $\mathbf{X}_g$ and $\mathbf{X}_{new}$, we initially use SDs with tree search levels of 2, 4 and 6, and

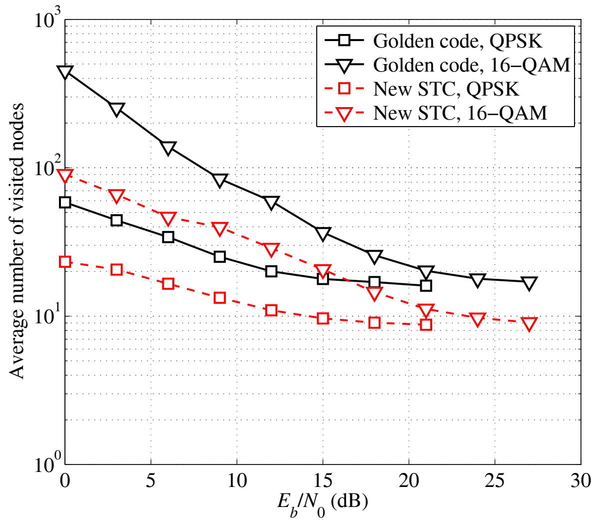

Fig. 4. Average number of visited nodes in full SD used to decode $\mathbf{X}_g$ and reduced SD used to decode $\mathbf{X}_{new}$.

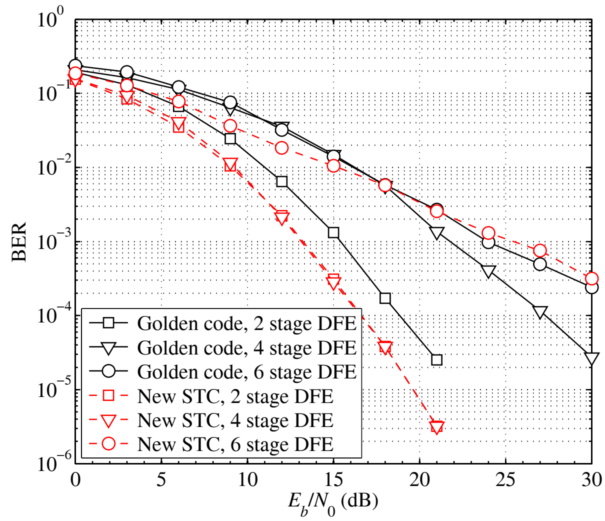

Fig. 5. Performance comparison between $\mathbf{X}_g$ and $\mathbf{X}_{new}$ with the same detector complexity (QPSK).

then employ zero-forcing decision-feedback equalization (ZF-DFE) for the rest of the symbols. In Fig. 5, we depict the BER curves for QPSK where we employed ZF-DFE for detecting 2, 4 and 6 real symbols, respectively. For similar detection complexities, $\mathbf{X}_{new}$ outperforms $\mathbf{X}_g$ by about 2.4 dB at the BER of 10^{-3} when a 2-stage ZF-DFE is used, and this gain increases to about 8.1 dB when a 4-stage ZF-DFE is used. On the other hand, when we use SD only for the first two real symbols and detect the rest using ZF-DFE, neither of the two STCs benefits from the available diversity. For low SNR, $\mathbf{X}_{new}$ provides better performance than $\mathbf{X}_g$, while for high SNR $\mathbf{X}_g$ slightly outperforms $\mathbf{X}_{new}$. In Fig. 6, we compare the number of visited points in the SD in each case. As expected, they have comparable results. Moreover, since the SD with 2- and 4-level ZF-DFE already provides optimum results for $\mathbf{X}_{new}$, the detector converges more quickly because of the more reliable results. Similar conclusions can be drawn for 16-QAM as shown in Fig. 7.

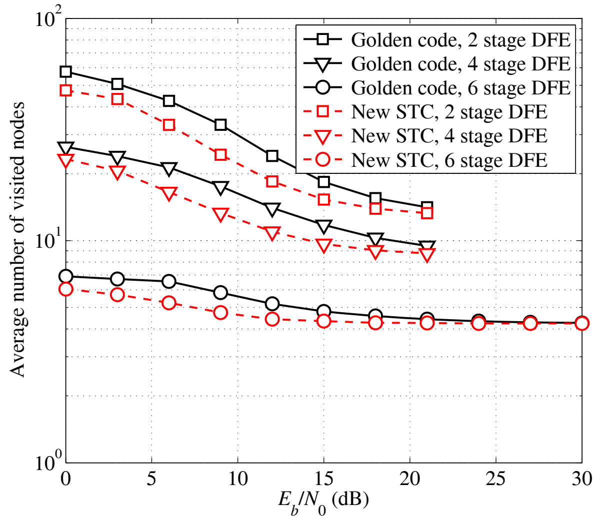

Fig. 6. Average number of visited nodes in the SD part of the suboptimum receiver used to decode $\mathbf{X}_g$ and $\mathbf{X}_{new}$ (QPSK).

B. Performance Comparison in the Rate-$3/4$ Case

We now provide a performance comparison between $\mathbf{X}_{new}^{3/4}$, namely, the proposed rate-$3/4$ STC, and the two MIMO schemes in current mobile WiMAX system specifications (Alamouti's STC and SM). Fig. 8 shows the BER performance as a function of E_s/N_0, E_s denoting the average transmitted signal energy per antenna use. With the optimized values, the proposed STC maximizes the diversity gain and, therefore, it achieves the same BER curve slope as Alamouti's STC with a constant coding gain independent of the constellation size. This is a crucial property as in the full-rate case, since we do not want vanishing determinants.

The results of Fig. 8 indicate that the Alamouti scheme achieves a BER of 10^{-3} with an SNR of 10 dB for QPSK, 16.6 dB for 16-QAM, and 22.4 dB for 64-QAM. Next, we can observe that SM achieves this BER with an SNR of 18.6 dB for QPSK and 26.6 dB for 16-QAM. With 64-QAM, this MIMO scheme requires an SNR well in excess of 30 dB to reach this BER performance level. Finally, our new STC achieves a BER of 10^{-3} with an SNR of 13.8 dB for QPSK, 21.7 dB for 16-QAM, and 28.6 dB for 64-QAM. Clearly, the Alamouti scheme has the best BER performance, but also the lowest bit rate on a given channel bandwidth. The SM scheme doubles the bit rate, but it involves a strong SNR loss, which increases at lower BER values. As evidenced from these results, the proposed rate-$3/4$ scheme is an interesting alternative to these two MIMO schemes, as it substantially improves BER performance compared to SM at the price of a 25% decrease in bit rate, and it increases the bit rate by 50% compared to the Alamouti scheme at the price of some SNR loss.

A closer examination of the results shows that at the spectral efficiency of 3 bits per antenna use, the proposed technique outperforms Alamouti's STC. Indeed, the new STC with 16-QAM and Alamouti's STC with 64-QAM have a spectral

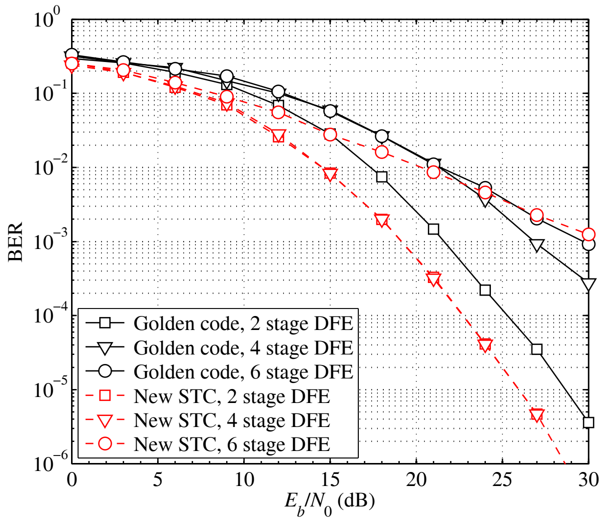

Fig. 7. Performance comparison between $\mathbf{X}_g$ and $\mathbf{X}_{new}$ with the same detector complexity (16-QAM).

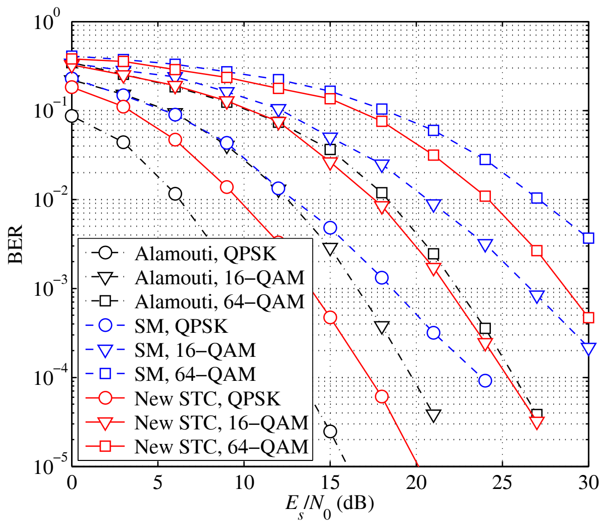

Fig. 8. Performance comparison of the proposed rate-3/4 STC with Alamouti's STC and SM.

efficiency of 3 bits per antenna use, and the results indicate that at the BER of 10^{-3}, the former outperforms the latter by 0.7 dB.

VI. Conclusions

In this paper, we have presented two new full-diversity 2×2 STC designs with an inherent low-complexity optimum decoder. First, we have analyzed the proposed full-rate STC and proved that it has full diversity with a non-vanishing coding gain. We have compared its performance and detection complexity to those of the Golden code, and the results indicated that the proposed scheme achieves the same performance while reducing the decoder complexity by orders of magnitude depending on the signal constellation. Furthermore, it was also observed that when used at a similar decoder complexity, the new STC may bring a considerable performance gain compared to the Golden code. Second, we have presented a full-diversity rate-$3/4$ 2×2 STC whose optimum decoder complexity grows only linearly with the number of constellation points. We have compared its performance to the two MIMO schemes included in the IEEE 802.16e-2005 specifications, and the results indicated that it stands as an interesting alternative providing further tradeoffs between performance and spectral efficiency.

Appendix

Proof of Proposition 2

Here, we prove that the coding gain of $\mathbf{X}_{new}$ is constant across square QAM constellations, and takes the value of 2. We want to compute

$$\min_{\mathbf{X}_{new}\neq\hat{\mathbf{X}}_{new}} \left|\det(\mathbf{X}_{new} - \hat{\mathbf{X}}_{new})\right|^2. \tag{22}$$

Assume that the signal s_k belong to the square lattice $2\mathbb{Z}(i)+1$, which has elements $m+in$, m and n odd integers. Thus, the difference $(\mathbf{X}_{new} - \hat{\mathbf{X}}_{new})$ has the same form as $\mathbf{X}_{new}$, with s_k changed into Δs_k, and Δs_k taking values in $2\mathbb{Z}(i)$ with elements $\mu + i\nu$, μ and ν *even* integers. With the above mentioned numerically optimized values for QPSK, the minimum value in (22) is achieved for example when $\Delta s_2 = \Delta s_3 = 2$ and $\Delta s_1 = \Delta s_4 = 0$, and is equal to 2. To prove that this minimum value is constant across constellations, observe first that the values $\Delta s_2 = \Delta s_3 = 2$ and $\Delta s_1 = \Delta s_4 = 0$ are compatible with any constellation size. Consequently, it suffices to prove that, with no constraint on the constellation size, we have $|\det(\mathbf{X}_{new} - \hat{\mathbf{X}}_{new})|^2 \geq 2$ for any pair $\mathbf{X}_{new} \neq \hat{\mathbf{X}}_{new}$.

Let us rewrite $\Delta s_k = 2m_k$, $k = 1, \ldots, 4$, $m_k \in \mathbb{Z}(i)$. With this notation, after some algebra, we obtain

$$\begin{aligned}\left|\det(\mathbf{X}_{new} - \hat{\mathbf{X}}_{new})\right|^2 &= 4(|m_1|^2 + |m_2|^2)^2 \\ &\quad + 4(|m_3|^2 + |m_4|^2)^2 \\ &\quad - 6(|m_1|^2 + |m_2|^2)(|m_3|^2 + |m_4|^2) \\ &\quad + 8\left[\Re\{(1-i)(m_1 m_3^* + m_2 m_4^*)\}\right]^2 \\ &\quad + 4(|m_1|^2 + |m_2|^2 + |m_3|^2 + |m_4|^2) \\ &\quad \times \Re\{(1-i)(m_1 m_3^* + m_2 m_4^*)\}\end{aligned} \tag{23}$$

where we use the equality $\Re\{\alpha\} + \Im\{\alpha\} = \Re\{(1-i)\alpha\}$. With the definition $u \stackrel{\Delta}{=} \Re\{(1-i)(m_1 m_3^* + m_2 m_4^*)\}$, the last two terms in (23) can be lower-bounded as

$$\begin{aligned}&8u^2 + 4(|m_1|^2 + |m_2|^2 + |m_3|^2 + |m_4|^2)u \\ &\quad = 8\left[u + (|m_1|^2 + |m_2|^2 + |m_3|^2 + |m_4|^2)/4\right]^2 \\ &\qquad -(|m_1|^2 + |m_2|^2 + |m_3|^2 + |m_4|^2)^2/2 \\ &\quad \geq -(|m_1|^2 + |m_2|^2 + |m_3|^2 + |m_4|^2)^2/2\end{aligned}$$

In conclusion, we obtain $|\det(\mathbf{X}_{new} - \hat{\mathbf{X}}_{new})|^2 \geq 3.5(|m_1|^2 + |m_2|^2 - |m_3|^2 - |m_4|^2)^2$. Now, if $|m_1|^2 + |m_2|^2 - |m_3|^2 - |m_4|^2 \neq 0$, since $m_k \in \mathbb{Z}(i)$, we must have $|m_1|^2 + |m_2|^2 - |m_3|^2 - |m_4|^2 \geq 1$, and hence $|\det(\mathbf{X}_{new} - \hat{\mathbf{X}}_{new})|^2 \geq 3.5 > 2$. Therefore, the existence of a minimum in (22) taking a value smaller than 2 requires the condition $|m_1|^2 + |m_2|^2 = |m_3|^2 + |m_4|^2$. In this case, we have

$$\left|\det(\mathbf{X}_{new} - \hat{\mathbf{X}}_{new})\right|^2 = 2\left[|m_1|^2 + |m_2|^2 + 2\Re\{(1-i)(m_1 m_3^* + m_2 m_4^*)\}\right]^2 . \quad (24)$$

In (24), the squared function in brackets takes on integer values, and, hence, if it is not 0, it must be greater than or equal to 1. Thus, under the condition that it is not 0, no value of $|\det(\mathbf{X}_{new} - \hat{\mathbf{X}}_{new})|^2$ can be smaller than 2.

The proof is complete if we can prove that $|\det(\mathbf{X}_{new} - \hat{\mathbf{X}}_{new})|^2 \neq 0$ when $\mathbf{X}_{new} \neq \hat{\mathbf{X}}_{new}$ and $|m_1|^2 + |m_2|^2 = |m_3|^2 + |m_4|^2$. To do this, note first that the term $(1 - i)$ in (24) can be omitted: in fact, its presence has the effect of transforming m_3 and m_4 into $(1 + i)m_3$ and $(1 + i)m_4$, respectively. Since $(1+i)m_3$ and $(1+i)m_4$ take on values in the same set as m_3 and m_4, we can conclude that the term in brackets in the right-hand side of (24) can be rewritten as

$$|m_1 + m_3|^2 + |m_2 + m_4|^2 - |m_3|^2 - |m_4|^2 \quad (25)$$

while the condition $|m_1|^2 + |m_2|^2 = |m_3|^2 + |m_4|^2$ becomes $2(|m_1|^2+|m_2|^2) = |m_3|^2+|m_4|^2$. Consequently, our problem reduces to showing that the system

$$\begin{aligned} 2(|m_1|^2 + |m_2|^2) &= |m_3|^2 + |m_4|^2 \text{ and} \\ |m_3|^2 + |m_4|^2 &= |m_1 + m_3|^2 + |m_2 + m_4|^2 \end{aligned} \quad (26)$$

admits no solution in $\mathbb{Z}(i)$, except the trivial one $m_1 = m_2 = m_3 = m_4 = 0$. To prove the latter statement, we invoke the theory of Lipschitz quaternions [19], which are Hamilton quaternions with integer components. They have the form $z = n_0 + n_1 i + n_2 j + n_3 k$ with $n_0, n_1, n_2, n_3 \in \mathbb{Z}$, $i^2 = j^2 = k^2 = -1$, $ij = -ji$, and $k = ij$. The *reduced norm* of the quaternion z, given by $\mathcal{N}(z) \triangleq n_0^2 + n_1^2 + n_2^2 + n_3^2$ satisfies the *Lagrange identity* $\mathcal{N}(z_1 z_2) = \mathcal{N}(z_2 z_1) = \mathcal{N}(z_1)\mathcal{N}(z_2)$ where z_1 and z_2 are two quaternions. In our problem, define the two Lipschitz quaternions $z_1 \triangleq m_1 + jm_2$ and $z_2 \triangleq m_3 + jm_4$. Then, (26) becomes

$$2\mathcal{N}(z_1) = \mathcal{N}(z_2) \text{ and } \mathcal{N}(z_2) = \mathcal{N}(z_1 + z_2). \quad (27)$$

Now, using Lagrange identity, we see that, to satisfy $\mathcal{N}(z_2)/\mathcal{N}(z_1) = 2$, we must have $z_2 = \beta z_1$ or $z_2 = z_1 \beta$, where β is a Lipschitz quaternion with norm 2, and is hence the product of any element in $\{1 + i, 1 + j, 1 + k\}$ by any element in $\{\pm 1, \pm i, \pm j, \pm k\}$ (the Lipschitz quaternions with norm 1). Finally, replacing z_2 in the second equality of (27) with βz_1, we obtain

$$\mathcal{N}(\beta)\mathcal{N}(z_1) = \mathcal{N}(1 + \beta)\mathcal{N}(z_1). \quad (28)$$

If $z_1 \neq 0$, and hence $\mathcal{N}(z_1) \neq 0$, we have $\mathcal{N}(\beta) = \mathcal{N}(1+\beta)$, which cannot be satisfied by any of the values of β listed above. Thus, we must have $z_1 = 0$, which also yields $z_2 = 0$. This simply implies the fact that (22) may be smaller than 2 iff $\Delta s_1 = \Delta s_2 = \Delta s_3 = \Delta s_4 = 0$. This completes the proof. □

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