Information Theoretic Capacity of Cellular Multiple Access Channel with Shadow Fading

Dimitrios Kaltakis, *Student Member, IEEE*, Muhammad A. Imran, *Member, IEEE*, and Costas Tzaras, *Member, IEEE*

I. INTRODUCTION

In the past, wireless systems were designed to accommodate a large number of voice and/or low data rate users. With the emergence and continuous growth of wireless data services, the value of a wireless network is not only defined by how many users it can support, but also by its ability to support high data capacity at localized spots.

Shannon developed a mathematical theory for the single-link channel capacity in [1] providing the framework for studying performance limits in communication. Shannon's work gave birth to the field of information theory. In information theoretic literature different approaches have been reported to determine maximum data rate and the means to achieve this under various assumptions and constraints. Despite the work in this field, the first important attempt to study the capacity of a cellular system was carried out in the previous decade by Wyner [2]. Wyner's model studies the uplink channel and although it considers a very crude approximation of path loss with no path loss variability across the cell, it manages to provide an insight into the cooperation of the base stations and the benefits that can be achieved through that process.

Fading was incorporated in Wyner's model by Somekh and Shamai in [3]. They maintained the assumption of a hyper-receiver with delay-less access to all cell-site receivers and assumed the same interference pattern as Wyner's. They used a "raster-scan" method to transform the two-dimensional system into an equivalent linear system in order to arrange the fading coefficients and the system's path gains in the channel matrix. Their results showed that for a certain range of relatively high inter-cell interference, the fading improves the system performance as compared to the case when there is no fading.

D. Kaltakis is with the Centre for Communications Systems Research, University of Surrey, Guildford, Surrey, GU2 7XH UK, e-mail: D.Kaltakis@surrey.ac.uk.

M. A. Imran and C. Tzaras are with the Centre for Communications Systems Research, University of Surrey.

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Letzepis in [4] modified the one-dimensional version of Wyner's model to account for the free-space path loss incurred between each user transmitter and cell-site receivers. The optimum capacity is estimated by using the Shannon-transform of the Marcenko-Pastur law. To keep the problem tractable, all users are assumed to be collocated at the cell-site receiver's position. The major contribution of the work is that it accounts for the interference caused not only from the two neighboring cells but from all cells in the system enabling the study of the effects of changing cell-density in the system. Another extension to Wyner's one-dimensional model was done in [5] where the users are modelled as located at the cell-boundaries with their signals received by only two cell-site receivers.

In [6] we find the optimum capacity of a Wyner-like two dimensional model assuming that each cell-site receiver is receiving signals from 5 tiers of users. Each tier is delineated based on the users' average distance from the center of the cell of interest. Each tier is considered to have a different path gain which corresponds to a specific power law path loss. The step approximation of the distance dependent path-loss provided a good insight of how the capacity is affected if one removes the crude assumption that all users in a cell experience the same attenuation.

One could get results even closer to reality if the users were considered distributed in the planar system and a different path attenuation coefficient was assigned to each one of them according to their distance from the BS of interest.

Up to now, there has not been reported any attempt to incorporate shadow fading in Wyner's model. There are numerous results for the capacity of conventional wireless systems with shadow fading [7],[8]. In [9] the authors derive the Shannon capacity of a single link channel under the effect of log-normal fading. The capacity is a function of the probability distribution function of the received SNR. Tse and Hanly in [10] derive the single-user delay limited capacity with shadow fading. There has also been attempts to characterize the ergodic capacity of composite marix channel models under the effect of large and small scale fading (see [11] and references within).

In this paper we provide a model which can be used to evaluate the uplink capacity of a cellular system under the notion of a hyper-receiver incorporating realistic system parameters. Using this model we were able to derive a closed form formula for the uplink capacity of a cellular system in which each transmitted signal experiences a distance dependent path loss, fast fading and shadow fading. Using this formula we study the effects that increasing cell radius and increasing standard deviation of the shadowing component have on the capacity of the system.

A. Outline

This paper is organized as follows: In section II we define the channel model. This definition includes the inputoutput equation and the distance dependent path loss along with the realistic system parameters that are incorporated in our model. We also define the small-scale and large scale fading models that are used for the derivation of the results. In section III we present the mathematical model and we derive the closed form formula for the capacity of the system. In section IV we present the definition of Rise over Thermal and the results obtained for the system's capacity are plotted. Finally in section V some conclusions are drawn.





B. Notation

Throughout this paper the following notations are used:

 ψ_v , ς_v refer to the linear amplitude value of the respective coefficients. ψ_p , ς_p refer to the linear power value, while $\overline{\psi}_p$, $\overline{\varsigma}_p$ refer to the dB power value of the respective coefficients.

II. CHANNEL MODEL

A. Input-Output Equation

We consider a Wyner-like [2] hexagonal cellular array model with N^2 cells and K uniformly distributed users in each cell. A BS at a cell will receive signals from all the users in the system.

It is more practical to obtain a representation of the cellular system in terms of a rectangular array as the representation in Figure 1 is less tractable. This can be done [2] by scaling and rotating the structure in Figure 1. The points of the rectangular array are indexed by (m, n) where m and n are the row and column numbers respectively.

According to the above, the received signal at the base stations' antenna referring to cell (m, n) is the sum of the transmitted signals each one multiplied by the corresponding path gain and fading coefficients. Hence, the received signal in a cell (m, n) is given by:

$$y_{m,n} = \sum_{k=1}^{K} \left[\psi_{m,n,k} \cdot g_{m,n,k} \cdot x_{m,n,k} \right] + \sum_{\acute{m},\acute{n}=-\infty}^{\infty} \sum_{k=1}^{K} \left[\psi_{\acute{m},\acute{n},k} \cdot g_{\acute{m},\acute{n},k} \cdot x_{\acute{m},\acute{n},k} \right] + z_{m,n} \quad (1)$$

where subscripts m, n and \acute{m}, \acute{n} (with $\acute{m}, \acute{n} \neq m, n$) identify the cell in which a user is located and k identifies the user index. Furthermore, ψ, g, x, y, z stand for path gain coefficients, fading coefficients, complex Gaussian inputs,

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complex Gaussian outputs and normalized (unit power) noise respectively. A power constraint is also considered for each input, $\mathbb{E}[(x_{m,n,k})^2] \leq P$.

B. Fading Model

In order to describe a fading coefficient we consider the following equation [12]:

$$g = \sqrt{\frac{\kappa}{\kappa+1}} e^{j\theta} + \sqrt{\frac{1}{\kappa+1}} \mathcal{CN}(0,1)$$
⁽²⁾

where κ is the ratio of the LoS and NLoS components and θ is the phase of the received signal. The fading coefficient is assumed normalized to unit power. In this model, $\kappa \to \infty$ corresponds to the case where all power is concentrated in the specular path. Whereas $\kappa \to 0$ corresponds to the most severe fading i.e. Rayleigh fading with no specular component.

Since a cellular system with distributed users is assumed, the transmitted signals cannot be synchronized with respect to the phase from the receiver's point of view. This, in fact, means that the signals are received with random, independent, normally distributed phase offsets.

For the capacity analysis that will be presented later on in this paper the expected value of the product of two fading coefficients will be needed. Since the received signals have random phase offsets the mean value of the product of two independent fading realisations is zero for Rayleigh ($\kappa \rightarrow 0$) as well as Rician ($\kappa \neq 0$) fading distributions. On the other hand the expectation of the product of a fading coefficient with the complex conjugate of itself is equal to its power, and thus equal to one (as the fading coefficients are assumed normalized to unit power).

C. Path Loss Model

We use a modified power law path loss model which can be expressed as:

$$\varsigma_v^2 = \frac{\frac{L_0}{r_0^{\eta}}}{\left(1 + \frac{r'}{r_0}\right)^{\eta}} \tag{3}$$

where L_0 is defined as the power received at the reference distance (r_0) when transmitted power is one unit and $r' + r_0$ is the actual distance from the BS.

It should be noted that the modified path loss model is not completely arbitrarily selected but it has a strong one-to-one correspondence to a practical system; A system where any transmitter can be placed in the system at a distance from the receive antenna which is constrained to the range $[r_0, \infty]$. This also implies that r' ranges in $[0, \infty]$, which creates an exclusion zone with a diameter of $2r_0$ around the BS of interest.

The modified path loss model attempts to capture two important phenomenon in the physical system: actual power (or envelope/voltage) attenuation in the physical system at any distance (w.r.t reference distance) and the rate at which this attenuation increases with the increasing distance from the reference point. The parameter L_0 captures the actual attenuation at the reference distance and the path loss exponent η captures the rate at which the



Fig. 2. Comparison of Empirical Models with Simplified Path Loss model using appropriate parameters for Simplified Path Loss Model

attenuation increases with the distance. In order to provide a one-to-one correspondence between the modified path loss model and the empirical models the values of these parameters need to be determined with the objective of providing a best-fit of the empirical data obtained in the field. We have selected two well-known empirical models. For a micro-cellular system the wideband PCS model is selected while for a macro-cellular system we selected the PCS extension to HATA model [13].

In order to see the relation of the modified path loss model and the empirical models we plot the results obtained by the two empirical models and the results for the modified model with varying η in Figure 2.

Based on the limitations of the two models (in terms of the parameters that can be used) we use the following parameters to approximate the path loss. For both models we use carrier frequency $f_c = 1.9$ GHz, transmit antenna height $H_{te} = 1.5m$ and power loss at reference distance $L_0 = 38$ dB. We use the minimum allowed receive antenna height for the macrocellular system model (30m) and the maximum allowed for the microcellular system (13.3m). We assume a line-of-sight dual slope environment for microcellular system and a small/medium sized city environment for the macrocellular system.

It can be observed that the microcellular model suggests a smaller value of $\eta = 2$ and the macrocellular model suggests a much larger value of $\eta = 3.5$. Even for the microcellular model, the path loss exponent after the Fresnel zone clearance is larger, i.e. $\eta \approx 3.5$. The macrocellular model also suggests a further loss of around 20 dB. Hence for practical results we use a value of $L_0 = -38$ dB and a path loss exponent of 2 for small cells and 3.5 for large cells.

In Table I we summarize the value/range of parameters that will be used in this paper to find the capacity of the Multiple Access Channel with shadow fading.

Parameter	Symbol	Value/Range (units)
Cell Radius	R	$0.1 - 10 \ km$
Reference Distance	r_0	1 m
Path Loss at r_0	L_0	38 dB
Path Loss Exponent	η	2-4 usual values {2, 3.5}
UTs per cell	K	20
UT Transmit Power	P	100-200 mW
Thermal Noise Density	N_0	-169 dBm/Hz
Channel Bandwidth	В	5 MHz

 TABLE I

 VALUE/RANGE OF PARAMETERS USED FOR FINDING THE CAPACITY OF THE PRACTICAL CELLULAR SYSTEMS

D. Shadow Fading Model

To incorporate shadowing in the system we need to find a model which can represent accurately the variation in received amplitude in the system due to the shadow fading [13], [14]. Goldsmith in [14] proposes the log-normal shadowing model which models accurately the above variation.

In the log-normal shadowing model the pdf of the random power gain (ratio of receive to transmit power, $\psi_p \triangleq \psi_v^2$) is given by:

$$p_{\Psi}(\psi_p) = \frac{\xi}{\sqrt{2\pi}\sigma_{\overline{\psi}_p}\psi_p} \exp\left[-\frac{(10\log_{10}\psi_p - \mu_{\overline{\psi}_p})^2}{2\sigma_{\overline{\psi}_p}^2}\right]$$
(4)

where $\xi = \frac{10}{\ln 10}$, $\mu_{\overline{\psi}_p}$ (average dB path gain due to distance dependent path loss[14]) is the mean of $\overline{\psi}_p = 10 \log_{10} \psi_p$ in dB, and $\sigma_{\overline{\psi}_p}$ is the standard deviation of ψ_p in dB. In our case $\mu_{\overline{\psi}_p}$ is a function of the path power loss for a distance r from the Base Stattion.

The joint pdf of the two random variables, the total power gain ψ_p and the distance dependent power loss ς_p ($\varsigma_p \triangleq \varsigma^2$) is given by:

$$p_{\Psi}(\psi_p|\overline{\varsigma}_p) = \frac{\xi}{\sqrt{2\pi}\sigma_{\overline{\psi}_p}\psi_p} \exp\left[-\frac{\left(10\log_{10}\psi_p - F(\overline{\varsigma}_p)\right)^2}{2\sigma_{\overline{\psi}_p}^2}\right]$$
(5)

where $F(\overline{\varsigma}_p)$ is a function of $\overline{\varsigma}_p$ which needs to be determined.

III. CAPACITY OF THE SYSTEM

A. General Capacity Formula

Following the classical Information Theoretic approach[15], the per-cell uplink capacity is given by:

$$C = \lim_{N \to \infty} \mathbb{E}\left(\frac{1}{N^2} \log_2\left(\det[\mathbf{I} + \gamma \cdot \mathbf{H}\mathbf{H}^{\dagger}]\right)\right)$$
(6)

where the matrix **H** is an $N^2 \times KN^2$ channel attenuation matrix for K users per cell and a planar array of $N \times N$ cells (or a linear array of N^2 cells) and $\gamma \triangleq P$. Note that **H** can be modelled as the overall system matrix which

is constituted of the amplitude/envelope attenuation (path-loss) factor and the fading coefficients, i.e. $\mathbf{H} = \Psi \odot \mathbf{G}$ where Ψ contains the path attenuation coefficients (ψ_v), \mathbf{G} contains the fading coefficients (g) and \odot denotes the Hadamard operator (element-wise multiplication of matrices).

We make a basic simplification here assuming that there are a sufficiently large number of users in each cell. When $K \gg 1$, the expectation in (6) can be moved inside the log-det[3] and then we have:

$$C = \lim_{N \to \infty} \left(\frac{1}{N^2} \log \left(\det \mathbb{E} \left[\mathbf{I} + \gamma \cdot \mathbf{H} \mathbf{H}^{\dagger} \right] \right) \right)$$
(7)

This gives an *upper bound* for the capacity which is *tight when* K *is large* (refer to Appendix A).

So, the problem reduces to characterize the mean value of $\Lambda \triangleq \mathbf{I} + \gamma \cdot \mathbf{H}\mathbf{H}^{\dagger}$ for the given statistical properties of fading and path gain coefficients. Path attenuation coefficients, are a function of UTs positions. As all these coefficients are dependent on distances between transmitter and receivers whose positions are fixed, Ψ is a deterministic matrix for a given snapshot of the system. Each received signal, experiences an independent fading coefficient, g. The statistical properties of these coefficients determine the capacity. To evaluate $\mathbb{E} [\Lambda]$, the following definition is used.

Definition: The expectation of the product between a complex fading coefficient g and the complex conjugate of another complex fading coefficient \hat{g} is defined as:

$$|m_g|^2 \triangleq \mathbb{E}\left[g \cdot \left(\hat{g}\right)^*\right]$$

This expectation is needed as by knowing it, every element of $\mathbb{E}[\Lambda]$ is known.

- If $g = \hat{g}$ then: $\mathbb{E}\left[g \cdot (\hat{g})^*\right] = \mathbb{E}\left[g^2\right]$. This is the case with the diagonal elements of $\mathbb{E}\left[\mathbf{\Lambda}\right]$, giving all diagonal entries in a closed form for the expectation of the covariance matrix.

- If $g \neq \hat{g}$ then one needs to find the value of $\mathbb{E}\left[g \cdot (\hat{g})^*\right] = |m_g|^2$ that will appear in some off-diagonal elements of $\mathbb{E}\left[\mathbf{\Lambda}\right]$.

As a result, according to the fading model presented in section II.B in this paper, all the off-diagonal entries of the expected covariance matrix are zero and all the diagonal entries are same (with $\mathbb{E}[gg^*] = 1$) and are given as:

$$\mathbb{E}[h[m] \cdot \hat{h}[m]^*] = \mathbb{E}\left[\psi_v^2\right] \tag{8}$$

$$\approx \lim_{K \to \infty} \frac{1}{KN^2} \sum_{k=1}^{KN^2} \psi_{v,k}^2 \tag{9}$$

Where $\psi_{v,k}$ is the attenuation factor between the base station of interest and the k^{th} user in the system. All the diagonal entries are same and each entry corresponds to a specific BS in the system. This simplifies (7) to the following form

$$C = \lim_{N \to \infty} \frac{1}{N^2} \log \left(\det \left[\mathbf{I} + \operatorname{diag} \left(\gamma \cdot K N^2 \mathbb{E}[\psi_v^2] \right) \right] \right)$$
(10)

But:

$$\mathbb{E}\left[\psi_{v}^{2}\right] = \mathbb{E}\left[\psi_{p}\right] = \int \psi_{p} \cdot p_{\Psi}(\psi_{p}) \mathbf{d}\psi_{p}$$
(11)

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Fig. 3. Reference area containing users with the same path loss.

with $p_{\Psi}(\psi_p)$ denoting the probability distribution function of the power attenuation factor ψ_p given by (5).

B. pdf of the Distance Dependent Path Loss

It is known that if the pdf of an independent random variable X is given by $p_X(x)$ then it is possible to calculate the probability density function of some variable Y which depends on x via y = e(x). Assuming y is a monotonic function the resulting density function is:

$$p_Y(y) = \left| \frac{1}{e'(e^{-1}(y))} \right| \cdot p_X(e^{-1}(y)),$$
(12)

where $x = e^{-1}(y)$ is the inverse function of y = e(x) and e' is the first derivative of that function.

Let's now consider the planar cellular system described in section II with \hat{K} users per unit area. The probability density function of the user position needs to be defined to use (12) and find the pdf of the distance dependent path loss in the system. To do this consider that r is the radial distance from the BS of interest. Lets also consider that all users in the disc formed by the two circles with radii $r + \frac{dr}{2}$ and $r - \frac{dr}{2}r$ (dr very small) have approximately the same path loss. The number of users with the same path loss can be found as (see Figure 3):

$$K_{pl} = \hat{K} \cdot \left[\pi \left(r + \frac{dr}{2} \right)^2 - \pi \left(r - \frac{dr}{2} \right)^2 \right] = 2\hat{K}\pi r dr$$
(13)

Assuming a maximum radial distance D, the probability of a user being in this area (thus having this path loss) is $\frac{2\pi \hat{K}}{\pi D^2 \hat{K}} r dr$. Thus, the pdf of the user position is given by:

$$p_R(r) = \frac{2}{D^2} r, \forall r \in [0, D]$$

$$\tag{14}$$

Lets consider the path loss model described in section II.C. The distance dependent path attenuation for each user in the system is given by:

$$\varsigma_v = e(r) = \frac{\sqrt{L_0}}{(1+r)^{\eta/2}} \tag{15}$$

Pdf of path loss. Maximum distance 5 Km, path loss factor=2



Fig. 4. pdf of the free space attenuation for the planar system. Coverage 5 Km, free space loss factor 2, $L_0 = 0$ dB

where η, L_0 are as defined above.

Now we can use (14),(15) in conjunction with (12) to find the probability density function of the distance dependent free space attenuation.

From (15) we have:

$$r = e^{-1}(\varsigma_v) = L_0^{1/\eta} \varsigma_v^{-2/\eta} - 1$$
(16)

and

$$e'(r) = -\frac{\sqrt{L_0}\eta}{2(1+r)^{\frac{\eta}{2}+1}}$$
(17)

Thus using (14),(16),(17) equation (12) yields:

$$p_{\Sigma}(\varsigma_{v}) = \frac{4}{D^{2}\eta} \left(L_{0}^{2/\eta} \varsigma_{v}^{-\frac{\eta+4}{\eta}} - L_{0}^{1/\eta} \varsigma_{v}^{-\frac{\eta+2}{\eta}} \right)$$
(18)

Equation (18) is the probability density function of the distance dependent attenuation in a planar system with the users uniformly distributed and the attenuation taking values between $\frac{\sqrt{L_0}}{(1+D)^{\eta/2}}$ and $\sqrt{L_0}$ (See Figure 4).

1) Capacity of a System without shadowing : Using equation (10) one can derive the per-cell capacity of a system where each user's signal experiences multipath fading and distance dependent path loss as:

$$C_{\text{no-shadowing}} = \log_2 \left(1 + KN^2 P \cdot \mathbb{E} \left[\varsigma_v^2 \right] \right)$$
(19)

With:

$$\mathbb{E}\left[\varsigma_{v}^{2}\right] = \int_{\frac{\sqrt{L_{0}}}{(1+D)^{\eta/2}}}^{\sqrt{L_{0}}} \varsigma_{v}^{2} \cdot p_{\Sigma}(\varsigma_{v}) \mathbf{d}\varsigma_{v}$$
(20)

Using (18) one can calculate (20) and thus provide a closed form solution for the per-cell capacity of the system:

$$C_{\text{no-shadowing}} = \begin{cases} \log_2 \left(1 + \frac{2KN^2 P L_0}{D^2} \left(\ln(1+D) + \frac{1}{1+D} - 1 \right) \right) & \eta = 2\\ \log_2 \left(1 + \frac{2KN^2 P L_0}{D^2} \cdot \frac{(1+D)^{2\eta-3} - (\eta-1)(1+D)^{\eta-1} + (\eta-2)(1+D)^{\eta-2}}{(\eta-2)(\eta-1)(1+D)^{2\eta-3}} \right) & \eta > 2 \end{cases}$$
(21)

C. Capacity with Shadow Fading

Let's consider the log-normal shadow fading model that was presented in section II.D of this paper. We find the capacity as follows. The joint pdf of the two random variables, the total power gain ψ_p and its mean ς_p (distance dependent power loss, $\varsigma_p \triangleq \varsigma^2$) is given by (5) but it is repeated here for convenience:

$$p_{\Psi}(\psi_p|\overline{\varsigma}_p) = \frac{\xi}{\sqrt{2\pi}\sigma_{\overline{\psi}_p}\psi_p} \exp\left[-\frac{\left(10\log_{10}\psi_p - F(\overline{\varsigma}_p)\right)^2}{2\sigma_{\overline{\psi}_p}^2}\right]$$
(22)

Since ς_v in (15) represents the distance dependent attenuation, equation (15) needs modification so as to represent the distance dependent power loss, at position r, as:

$$\varsigma_p = e_p(r) = \frac{L_0}{(1+r)^{\eta}}$$
(23)

Now following the same steps as in (16),(17),(18) equation (12) will yield the pdf of the distance dependent power loss:

$$p_{\Sigma}(\varsigma_p) = \frac{2}{\eta D^2} \left(L_0^{2/\eta} \varsigma_p^{-\frac{\eta+2}{\eta}} - L_0^{1/\eta} \varsigma_p^{-\frac{\eta+1}{\eta}} \right)$$
(24)

Since in (22) the value of the mean distance dependent power loss is in dB it is considered useful to transform eq. (24) so as to represent the pdf of the distance dependent power loss in dB. To do that equation (12) is once more used for the following transformation:

$$\overline{\varsigma}_p = 10 \log_{10} \varsigma_p,\tag{25}$$

with $p_{\Sigma}(\varsigma_p)$ given by (24).

Thus the pdf of the distance dependent power loss in dB:

$$p_{\overline{\Sigma}}(\overline{\varsigma}_p) = \frac{2}{D^2 \eta \xi} 10^{\frac{\overline{\varsigma}_p}{10}} \left(L_0^{2/\eta} 10^{-\frac{\overline{\varsigma}_p(\eta+2)}{10\eta}} - L_0^{1/\eta} 10^{-\frac{\overline{\varsigma}_p(\eta+1)}{10\eta}} \right)$$
(26)

The capacity can be calculated using (10) and by noticing that:

$$\mathbf{E}\left[\psi_{v}^{2}\right] = \mathbf{E}\left[\psi_{p}\right] = \int_{10\log_{10}\frac{L_{0}}{(1+D)^{\eta}}}^{10\log_{10}L_{0}} p_{\overline{\Sigma}}(\overline{\varsigma}_{p}) \int_{\psi>0}^{\infty} \psi_{p} p_{\Psi}(\psi_{p}|\overline{\varsigma}_{p}) \mathrm{d}\psi_{p} \mathrm{d}\overline{\varsigma}_{p} = \\ = \int_{10\log_{10}\frac{L_{0}}{(1+D)^{\eta}}}^{10\log_{10}L_{0}} p_{\overline{\Sigma}}(\overline{\varsigma}_{p}) \cdot \mathbf{E}\left[\psi_{p}|\overline{\varsigma}_{p}\right] \mathrm{d}\overline{\varsigma}_{p} \quad (27)$$

But:

$$\mathbf{E}\left[\psi_{p}|\overline{\varsigma}_{p}\right] = \int_{\psi>0}^{\infty} \psi_{p} \frac{\xi}{\sqrt{2\pi}\sigma_{\overline{\psi}_{p}}\psi_{p}} \exp\left[-\frac{(10\log_{10}\psi_{p} - F(\overline{\varsigma}_{p}))^{2}}{2\sigma_{\overline{\psi}_{p}}^{2}}\right] = \int_{\psi>0}^{\infty} \psi_{p} \frac{\xi}{\sqrt{2\pi}\sigma_{\overline{\psi}_{p}}\psi_{p}} \exp\left[-\frac{(\xi\ln\psi_{p} - F(\overline{\varsigma}_{p}))^{2}}{2\sigma_{\overline{\psi}_{p}}^{2}}\right] d\psi_{p} \quad (28)$$

Substituting $\psi_p = \exp u$ and thus $d\psi_p = \exp u du$:

$$(28) \Rightarrow \int_{-\infty}^{\infty} \frac{\xi \exp(2v)}{\sqrt{2\pi}\sigma_{\overline{\psi}_p}} \exp\left[-\frac{(\xi u - F(\overline{\zeta}_p))^2}{2\sigma_{\overline{\psi}_p}^2}\right] du = \\ = \int_{-\infty}^{\infty} \frac{\xi}{\sqrt{2\pi}\sigma_{\overline{\psi}_p}} \exp\left[-\frac{\xi^2 u^2 - 2\xi F(\overline{\zeta}_p) u + F(\overline{\zeta}_p)^2 - 2\sigma_{\overline{\psi}_p}^2 u}{2\sigma_{\overline{\psi}_p}^2}\right] = \\ = \int_{-\infty}^{\infty} \frac{\xi}{\sqrt{2\pi}\sigma_{\overline{\psi}_p}} \exp\left[-\frac{\left(u - \left(\frac{\sigma_{\overline{\psi}_p}^2 + F(\overline{\zeta}_p)}{\xi^2}\right)\right)^2 - \left(\frac{\sigma_{\overline{\psi}_p}^2 + F(\overline{\zeta}_p)}{\xi^2}\right)^2 + \frac{F(\overline{\zeta}_p)^2}{\xi^2}\right]}{\frac{2\sigma_{\overline{\psi}_p}^2}{\xi^2}}\right] = \\ = \exp\left(\frac{\sigma_{\overline{\psi}_p}^2 + F(\overline{\zeta}_p)}{2\xi^2}\right) \int_{-\infty}^{\infty} \frac{\xi}{\sqrt{2\pi}\sigma_{\overline{\psi}_p}} \exp\left[-\frac{\left(u - \left(\frac{\sigma_{\overline{\psi}_p}^2 + F(\overline{\zeta}_p)}{\xi^2}\right)\right)^2}{\frac{2\sigma_{\overline{\psi}_p}^2}{\xi^2}}\right] = \\ = \exp\left(\frac{\sigma_{\overline{\psi}_p}^2 + F(\overline{\zeta}_p)}{\xi^2}\right) \int_{-\infty}^{\infty} \frac{\xi}{\sqrt{2\pi}\sigma_{\overline{\psi}_p}} \exp\left[-\frac{\left(u - \left(\frac{\sigma_{\overline{\psi}_p}^2 + F(\overline{\zeta}_p)}{\xi^2}\right)\right)^2}{\frac{2\sigma_{\overline{\psi}_p}^2}{\xi^2}}\right] = \\ \exp\left(\frac{\sigma_{\overline{\psi}_p}^2 + F(\overline{\zeta}_p)}{\xi^2}\right) \int_{-\infty}^{\infty} \frac{\xi}{\sqrt{2\pi}\sigma_{\overline{\psi}_p}} \exp\left[-\frac{\left(u - \left(\frac{\sigma_{\overline{\psi}_p}^2 + F(\overline{\zeta}_p)}{\xi^2}\right)\right)^2}{\frac{2\sigma_{\overline{\psi}_p}^2}{\xi^2}}\right] = \\ \exp\left(\frac{\sigma_{\overline{\psi}_p}^2 + F(\overline{\zeta}_p)}{\xi^2}\right) \int_{-\infty}^{\infty} \frac{\xi}{\sqrt{2\pi}\sigma_{\overline{\psi}_p}} \exp\left[-\frac{\left(u - \left(\frac{\sigma_{\overline{\psi}_p}^2 + F(\overline{\zeta}_p)}{\xi^2}\right)\right)^2}{\frac{2\sigma_{\overline{\psi}_p}^2}{\xi^2}}\right] = \\ \exp\left(\frac{\sigma_{\overline{\psi}_p}^2 + F(\overline{\zeta}_p)}{\xi^2}\right) \int_{-\infty}^{\infty} \frac{\xi}{\sqrt{2\pi}\sigma_{\overline{\psi}_p}} \exp\left(-\frac{\sigma_{\overline{\psi}_p}^2 + F(\overline{\zeta}_p)}{\xi^2}\right)\right) \exp\left(-\frac{\sigma_{\overline{\psi}_p}^2 + F(\overline{\zeta}_p)}{\xi^2}\right) \exp\left(-\frac{\sigma_{\overline{\psi}_p}^2 + F(\overline{\zeta}_p)}{\xi^2}\right) \exp\left(-\frac{\sigma_{\overline{\psi}_p}^2 + F(\overline{\zeta}_p)}{\xi^2}\right) \exp\left(-\frac{\sigma_{\overline{\psi}_p}^2 + F(\overline{\zeta}_p)}{\xi^2}\right) \exp\left(-\frac{\sigma_{\overline{\psi}_p}^2 + F(\overline{\zeta}_p)}{\xi^2}\right)\right) \exp\left(-\frac{\sigma_{\overline{\psi}_p}^2 + F(\overline{\zeta}_p)}{\xi^2}\right) \exp\left(-\frac{\sigma_{\overline{\psi$$

Using equations (26), (29) it is derived from (27) that:

$$\mathbf{E}\left[\psi_{p}\right] = \frac{2}{D^{2}\eta\xi} \int_{10\log_{10}\frac{L_{0}}{(1+D)^{\eta}}}^{10\log_{10}\frac{L_{0}}{10}} 10^{\frac{\overline{\varsigma}_{p}}{10}} \left(L_{0}^{2/\eta}10^{-\frac{\overline{\varsigma}_{p}(\eta+2)}{10\eta}} - L_{0}^{1/\eta}10^{-\frac{\overline{\varsigma}_{p}(\eta+1)}{10\eta}}\right) \exp\left(\frac{\sigma_{\overline{\psi}_{p}}^{2}}{2\xi^{2}} + \frac{F(\overline{\varsigma}_{p})}{\xi}\right) d\overline{\varsigma}_{p}$$
(30)

In (30) $F(\overline{\varsigma}_p)$ is the distance dependent mean of the shadowing coefficient which depends on $\overline{\varsigma}_p$ (the distance dependent power loss when no shadowing is present). We need to find a relation between these in order to be able to derive the capacity of the system. One observation that can be made here is that by incorporating shadowing in a system, the mean value of the path gain, when compared to the no-shadowing situation, should be decreased. This means that a negative offset in the mean of (30) needs to be found. This negative offset shall be a function of the standard deviation of the shadowing component as the higher the standard deviation the further away the points of the path gain lie from their mean value (See Figure 5). As the normal distribution has infinite tail, one possible way to quantify this negative offset is to use the confidence intervals of the log-normal distribution so as to claim that a certain percentage (e.g. 99.99%) of the shadowing coefficient points lie below the mean value of the no-shadowing path gain. In Figure 6 we plot the pdf of the shadowing coefficient (5) if we set its mean to a specific value. It is



Fig. 5. Simple explanation for the negative offset that needs to be found so as to be able to compare the capacity with and without shadowing



Fig. 6. pdf of the shadowing coefficients for different values of the standard deviation multiplier and respective constant path loss value. The areas under the curve values denote the probability there will be points with value higher than that of the path loss in the abscence of shadowing.

obvious that if no negative offset is introduced then there is a high probability (50%) that the shadowing coefficient value will be higher than the path loss value in the abscense of shadowing, which is not possible.

Thus we define:

$$F(\overline{\varsigma}_p) \triangleq \overline{\varsigma}_p - c\sigma_{\overline{\psi}_p} \tag{31}$$

where c is the standard deviation multiplier determining the confidence interval of the normal distribution. A c value of 3 means that we are operating at 99.85% confidence, while a c value of 4 means a 99.995% confidence. In Figure 6 we have calculated the probability there will be shadowing coefficients with greater value than the path

loss coefficients in the abscense of shadowing, for two different shifts of the mean value.

Using (30),(31) equation (10) yields the per-cell capacity of the system with shadow fading which can be expressed in a closed form formula as:

$$C = \log_{2} \left(1 + \frac{2KN^{2}P}{D^{2}\eta\xi} \cdot \int_{10\log_{10}\frac{L_{0}}{(1+D)^{\eta}}}^{10\log_{10}L_{0}} 10^{\frac{\overline{\varsigma}_{p}}{10}} \left(L_{0}^{2/\eta}10^{-\frac{\overline{\varsigma}_{p}(\eta+2)}{10\eta}} - L_{0}^{1/\eta}10^{-\frac{\overline{\varsigma}_{p}(\eta+1)}{10\eta}} \right) \cdot \exp\left(\frac{\sigma_{\overline{\psi}_{p}}^{2}}{2\xi^{2}} + \frac{\overline{\varsigma}_{p} - c\sigma_{\overline{\psi}_{p}}}{\xi}\right) d\overline{\varsigma}_{p} \right)$$
(32)

IV. RESULTS

A. Definition of RoT

In the practical engineering design of cellular systems, the main figure of merit that determines the capacity (maximum reliable transmission rate with vanishingly small error rate) of a UT, is the SINR at the BS receiver, given as

$$SINR = \frac{P_R}{I + N_0}$$
(33)

where P_R is the received power at the BS of interest, N_0 is the thermal AWGN at the receiving BS and I is the inter-cell and intra-cell interference received from other UTs of the system. However, in the information-theoretic analysis of hyper-receiver cellular systems, the main figure of merit that determines the per-cell capacity (at any BT) is:

$$\operatorname{RoT} = \frac{\sum_{i} \alpha_i P_T}{N_0},\tag{34}$$

assuming that all UTs in the system transmit at their maximum allowable power constraint, P_T . The factor α_i denotes the relative attenuation experienced by the transmitted signal of each UT until it reaches the receiver. The numerator term $\sum_i \alpha_i P_T$ is the total received signal power (desired signal power for the base station in consideration and also the power of the signals intended for the other base stations in the system). Splitting the numerator into desired , P_R , and (conventionally termed) undesired signal, I, we can express RoT as:

$$\operatorname{RoT} = \frac{P_R + I}{N_0},\tag{35}$$

Which shows that the information theoretic approach of using a joint decoder has the potential of converting the conventionally harmful interference into a factor that increases the figure of merit RoT by moving the interference term from the denominator to the numerator.

Note that the numerator of (34) is a direct function of the transmit power of the UT. Hence we can define the ratio $\gamma \triangleq \frac{P_T}{N_0}$ and can also use this as the figure of merit. With this definition incorporated, the RoT is given as:

$$\operatorname{RoT} = \sum_{i} \alpha_{i} \gamma, \tag{36}$$

The main reasons that SINR does not constitute an appropriate figure of merit for information-theoretic analysis are:

- Inter-cell and intra-cell interference is not harmful and thus the term *I* cannot be used in the denominator when a joint decoder is considered.
- Since there is no harmful interference, there is no need for power control and thus the UTs constantly transmit with the maximum available power P_T . In this context, the transmit power P_T remains fixed for all the UTs, whereas the received power at each BS differs for each UT. In addition, since the objective function is the per-cell capacity, the power variable affecting the value of this function should have a constant value throughout the whole cell. In this direction, the per-cell capacity can be calculated as a function of P_T , which is a fixed system parameter, common for all the UTs of a cell.

It shall be noted that the problem of finding the (per-cell) capacity of a cellular system can be greatly simplified by focusing on the single BS receiver and its RoT. Due to the symmetry of the problem (ignoring the edge effects) all BS receivers are identical and system capacity is simply the per-cell capacity times the number of cells.

From (10) it follows:

$$C = \lim_{N \to \infty} \frac{1}{N^2} \log \left(\prod_{j=1}^{N^2} \left[1 + \gamma \cdot K N^2 \mathbb{E}[\psi_v^2] \right] \right)$$
$$= \lim_{N \to \infty} \log \left(1 + \gamma \cdot K N^2 \mathbb{E}[\psi_v^2] \right)$$
$$= \lim_{N \to \infty} \log \left(1 + \text{RoT} \right)$$
(37)

where

$$\operatorname{RoT} \triangleq \gamma \cdot \sum_{i=1}^{KN^2} \psi_{v,i}^2 = \gamma K N^2 \mathbb{E}[\psi_v^2],$$
(38)

as discussed before. This result suggests that the capacity formulations of the cellular system all fall on the same graph and the different system parameters define the range of operation on this graph by controlling the RoT in the system.

B. Capacity Results

In this section we present the capacity results obtained by plotting (21) and (32) versus various system parameters. Following the definition of RoT above, in Figure 7 we illustrate where our system operates on the capacity (bps/Hz/cell) v.s. RoT curve. The capacity is plotted for three different cases, no shadowing, mild shadowing conditions (4 dB standard deviation) and more severe shadow fading (8 dB standard deviation). The capacity region is obtained by varying the UT transmit power. The standard deviation multiplier that was selected for plotting the results was the one that provides 99.995% confidence. As the RoT curve illustrates the optimum capacity of the Multiple-access Channel under the notion of a hyper-receiver, Figure 7 provides a very useful insight on the operating region of systems under real-world assumptions. In Figures 8,9 we plot the per-cell capacity of the system v.s. the cell radius and the per-user transmit power respectively. The capacity difference between the three different cases illustrated here is almost constant with respect to cell radius (4.7bps/Hz capacity loss when we compare the



Fig. 7. Per cell capacity of the system with and without shadowing versus Rise over Thermal in dB. K=20 users per-cell, reference distance power loss -38 dB, path attenuation factor = 2. Cell radius 1000m. User power ranging from 100 mW to 200 mW. The RoT curve is also plotted.



Fig. 8. Per cell capacity of the system with and without shadowing. K=20 users per-cell, user power 200 mW, reference distance power loss -38 dB, path attenuation factor = 2. Cell radius ranging from 100m to 3Km.

no-shadowing and 4dB shadowing cases and 3.47 bps/Hz capacity loss when the severity of the shadowing increases from 4dB to 8dB).



Fig. 9. Per cell capacity of the system with and without shadowing. K=20 users per-cell, reference distance power loss -38 dB, path attenuation factor = 2. Cell radius 100m. User power ranging from 100 mW to 200 mW

V. CONCLUSIONS

In this paper we proposed a model which can be used to evaluate the uplink capacity of a cellular system under the notion of a hyper-receiver incorporating realistic system parameters. We derived a closed form formula for the uplink capacity of a cellular system in which each transmitted signal experiences a distance dependent path loss, fast fading and shadow fading. The results illustrate the uplink capacity range, in bps/Hz per cell, that a system employing full receiver co-operation is expected to operate. By incorporating some realistic assumptions about the system parameters we tried to make the results as close to reality as possible. The increasing standard deviation of the shadowing component, as well as the increasing cell radius, have, as expected, a negative impact on the capacity. Due to the fact that we didn't assume distance dependent standard deviation of the shadowing component the capacity gap between different shadowing conditions is constant with respect to the cell radius but it decreases as the severity of the fading increases.

APPENDIX A

LAW OF LARGE NUMBERS IN THE CAPACITY FORMULA

Consider the $N^2 \times KN^2$ matrix **H** with its elements to be random variables following some distribution. Consider also the following multiplication:

$$\mathbf{\Omega} = \mathbf{H}\mathbf{H}^{\dagger} \tag{39}$$

where \mathbf{H}^{\dagger} is the $KN^2 \times N^2$ Hermitian transpose matrix of \mathbf{H} . Each element of the matrix $\mathbf{\Omega}$ is the result of the multiplication of a row of matrix \mathbf{H} (which is a KN^2 vector) with a column of matrix \mathbf{H}^{\dagger} (again a KN^2 vector). Hence, each element of matrix $\mathbf{\Omega}$ is the KN^2 sum of random variables multiplied with the conjugate transpose of other random variables with all of them following the same distribution:

$$\varrho = \sum_{i=1}^{KN^2} \left[h_i \cdot \dot{h}_i^* \right] \tag{40}$$

From the above one can say that for a large number of $K \to \infty$ for every fixed N, the law of large numbers applies to each element of \mathbf{HH}^{\dagger} and thus ρ reaches its expected value, i.e. $\rho \simeq KN^2 \mathbb{E} \left[h \cdot \hat{h}^* \right]$.

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