

Concentric Permutation Source Codes

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Abstract

Permutation codes are a class of structured vector quantizers with a computationally-simple encoding procedure based on sorting the scalar components. Using a codebook comprising several permutation codes as subcodes preserves the simplicity of encoding while increasing the number of rate-distortion operating points, improving the convex hull of operating points, and increasing design complexity. We show that when the subcodes are designed with the same composition, optimization of the codebook reduces to a lower-dimensional vector quantizer design within a single cone. Heuristics for reducing design complexity are presented, including an optimization of the rate allocation in a shape-gain vector quantizer with gain-dependent wrapped spherical shape codebook.

Index Terms

Gaussian source, group codes, integer partitions, order statistics, permutation codes, rate allocation, source coding, spherical codes, vector quantization

I. INTRODUCTION

A permutation source code [1], [2] places all codewords on a single sphere by using the permutations of an *initial codeword*. The size of the codebook is determined by multiplicities of repeated entries in the initial codeword, and the complexity of optimal encoding is low. In the limit of large vector dimension, an optimal permutation code for a memoryless source performs as well as entropy-constrained *scalar* quantization [3]. This could be deemed a disappointment because the constraint of placing all codewords on a single sphere does not preclude performance approaching the rate-distortion bound when coding a memoryless Gaussian source [4]. An advantage that remains is that the fixed-rate output of the permutation source code avoids the possibility of buffer overflow associated with entropy coding highly nonequiprobable outputs of a quantizer [5].

The performance gap between permutation codes and optimal spherical codes, along with the knowledge that the performance of permutation codes does not improve monotonically with increasing vector dimension [6], motivates the present paper. We consider generalizing permutation source codes to have more than one initial codeword. While adding very little to the encoding complexity, this makes the codebook of the vector quantizer (VQ) lie in the union of concentric spheres rather than in a single sphere. Our use of multiple spheres is similar to the wrapped spherical shape-gain vector quantization of Hamkins and Zeger [7]; one of our results, which may be of independent of interest, is an optimal rate allocation for that technique. Our use of permutations could be replaced by the action of other groups to obtain further generalizations [8].

Design of a permutation source code includes selection of the multiplicities in the initial codeword; these multiplicities form a *composition* of the vector dimension [9, Ch. 5]. The generalization makes the design problem more difficult because there is a composition associated with each initial codeword. Our primary focus is on methods for reducing the design complexity. We demonstrate the effectiveness of these methods and improvements over ordinary permutation source codes through simulations.

The use of multiple initial codewords was introduced as “composite permutation coding” by Lu *et al.* [10], [11] and applied to speech/audio coding by Abe *et al.* [12]. These previous works restrict the constituent permutation source codes to have the same number of codewords, neglect the design of compositions, and use an iterative VQ design algorithm at the full vector dimension. In contrast, we allow the compositions to be identical or different, thus allowing the sizes of subcodes to differ. In the case of a single, common composition, we show that a reduced-dimension VQ design problem arises. For the general case, we provide a rate allocation across subcodes.

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The generalization that we study maintains the low $O(n \log n)$ encoding complexity for vectors of dimension n that permutation source codes achieve. Vector permutation codes are a different generalization with improved performance [13]. Their encoding procedure, however, requires solving the assignment problem in combinatorial optimization [14] and has complexity $O(n^2 \sqrt{n} \log n)$.

The paper is organized as follows: We review the attainment of the rate–distortion bound by spherical source codes and the basic formulation of permutation coding in Section II. Section III introduces concentric permutation codes and discusses the difficulty of their optimization. One simplification that reduces the design complexity—the use of a single common composition for all initial codewords—is discussed in Section IV. The use of a common composition obviates the issue of allocating rate amongst concentric spheres of codewords. Section V returns to the general case, with compositions that are not necessarily identical. We develop fixed- and variable-rate generalizations of wrapped spherical shape–gain vector quantization for the purpose of guiding the rate allocation problem. Concluding comments appear in Section VI.

II. BACKGROUND

Let $X \in \mathbb{R}^n$ be a random vector with independent $\mathcal{N}(0, \sigma^2)$ components. We wish to approximate X with a codeword \hat{X} drawn from a finite codebook \mathcal{C} . We want small per-component mean-squared error (MSE) distortion $D = n^{-1} E[\|X - \hat{X}\|^2]$ when the approximation \hat{X} is represented with nR bits. In the absence of entropy coding, this means the codebook has size 2^{nR} . For a given codebook, the distortion is minimized when \hat{X} is the codeword closest to X .

A. Spherical Codes

In a *spherical (source) code*, all codewords lie on a single sphere in \mathbb{R}^n . Nearest-neighbor encoding with such a codebook partitions \mathbb{R}^n into 2^{nR} cells that are (unbounded) convex cones with apexes at the origin. In other words, the representations of X and αX are the same for any scalar $\alpha > 0$. Thus a spherical code essentially ignores $\|X\|$, placing all codewords at radius

$$E[\|X\|] = \frac{\sqrt{2\pi\sigma^2}}{\beta(n/2, 1/2)} \approx \sigma \sqrt{n-1/2},$$

where $\beta(\cdot, \cdot)$ is the beta function, while representing $X/\|X\|$ with nR bits.

Sakrison [4] first analyzed the performance of spherical codes for memoryless Gaussian sources. Following [4], [7], the distortion can be decomposed as

$$D = \frac{1}{n} E \left[\left\| \frac{E[\|X\|]}{\|X\|} X - \hat{X} \right\|^2 \right] + \frac{1}{n} \text{var}(\|X\|). \quad (1)$$

The first term is the distortion between the projection of X to the code sphere and its representation on the sphere, and the second term is the distortion incurred from the projection. The second term vanishes as n increases even though no bits are spent to convey the norm of X . Placing codewords uniformly at random on the sphere controls the first term sufficiently for achieving the rate–distortion bound as $n \rightarrow \infty$.

B. Permutation Codes

1) *Definition and Encoding*: A *permutation code* (PC) is a special spherical code in which all the codewords are related by permutation. Permutation channel codes were introduced by Slepian [15] and modified through the duality between source encoding and channel decoding by Dunn [1]. They were then developed by Berger *et al.* [2], [3], [16].

There are two variants of permutation codes:

Variant I: Let $\mu_1 > \mu_2 > \dots > \mu_K$ be real numbers, and let n_1, n_2, \dots, n_K be positive integers with sum equal to n (an *ordered* composition of n). The *initial codeword* of the codebook \mathcal{C} has the form

$$\hat{x}_{\text{init}} = (\underbrace{\mu_1, \dots, \mu_1}_{\leftarrow n_1 \rightarrow}, \underbrace{\mu_2, \dots, \mu_2}_{\leftarrow n_2 \rightarrow}, \dots, \underbrace{\mu_K, \dots, \mu_K}_{\leftarrow n_K \rightarrow}), \quad (2)$$

where each μ_i appears n_i times. The codebook is the set of all distinct permutations of \hat{x}_{init} . The number of codewords in \mathcal{C} is thus given by the multinomial coefficient

$$M = \frac{n!}{n_1! n_2! \cdots n_K!}. \quad (3)$$

The permutation structure of the codebook enables low-complexity nearest-neighbor encoding [2]: map X to the codeword \hat{X} whose components have the same order as X ; in other words, replace the n_1 largest components of X with μ_1 , the n_2 next-largest components of X with μ_2 , and so on.

Variant II: The initial codeword \hat{x}_{init} still has the form (2), but now all its entries are nonnegative; i.e., $\mu_1 > \mu_2 > \cdots > \mu_K \geq 0$. The codebook now consists of all possible permutations of \hat{x}_{init} in which each nonzero component is possibly negated. The number of codewords is thus given by

$$M = 2^h \cdot \frac{n!}{n_1! n_2! \cdots n_K!}, \quad (4)$$

where h is the number of positive components in \hat{x}_{init} . Optimal encoding is again simple [2]: map X to the codeword \hat{X} whose components have the same order in absolute value and match the signs of corresponding components of X .

Since the complexity of sorting is $O(n \log n)$ operations, the encoding complexity is much lower than with an unstructured VQ and only $O(\log n)$ times higher than scalar quantization.

2) *Performance and Optimization:* For i.i.d. sources, each codeword is chosen with equal probability. Consequently, there is no improvement from entropy coding and the per-letter rate is simply $R = n^{-1} \log M$.

Let $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$ denote the order statistics of random vector $X = (X_1, \dots, X_n)$, and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$ denote the order statistics of random vector $|X| \triangleq (|X_1|, \dots, |X_n|)$.¹ With these notations and an initial codeword given by (2), the per-letter distortion of optimally-encoded Variant I and Variant II codes can be deduced simply by realizing which order statistics are mapped to each element of \hat{x}_{init} :

$$D_{\text{I}} = n^{-1} E \left[\sum_{i=1}^K \sum_{\ell \in I_i} (\xi_\ell - \mu_i)^2 \right] \quad \text{and} \quad (5)$$

$$D_{\text{II}} = n^{-1} E \left[\sum_{i=1}^K \sum_{\ell \in I_i} (\eta_\ell - \mu_i)^2 \right], \quad (6)$$

where I_i s are the groups of indices generated by the composition, i.e.,

$$I_1 = \{1, 2, \dots, n_1\}, \quad I_i = \left\{ \left(\sum_{m=1}^{i-1} n_m \right) + 1, \dots, \left(\sum_{m=1}^i n_m \right) \right\}, \quad i \geq 2.$$

Given a composition (n_1, n_2, \dots, n_K) , minimization of D_{I} or D_{II} can be done separately for each μ_i , yielding optimal values

$$\mu_i = n_i^{-1} \sum_{\ell \in I_i} E[\xi_\ell], \quad \text{for Variant I, and} \quad (7)$$

$$\mu_i = n_i^{-1} \sum_{\ell \in I_i} E[\eta_\ell], \quad \text{for Variant II.} \quad (8)$$

Overall minimization of D_{I} or D_{II} over the choice of K , $\{n_i\}_{i=1}^K$, and $\{\mu_i\}_{i=1}^K$ subject to a rate constraint is difficult because of the integer constraint of the composition.

The analysis of [3] shows that as n grows large, the composition can be designed to give performance equal to optimal entropy-constrained scalar quantization (ECSQ) of X . Heuristically, it seems that for large block lengths, PCs suffer because there are too many permutations ($n^{-1} \log_2 n!$ grows) and the vanishing fraction that are chosen to meet a rate constraint do not form a good code. The technique we study in this paper is for moderate values of n , for which the second term of (1) is not negligible; thus, it is not adequate to place all codewords on a single sphere.

III. PERMUTATION CODES WITH MULTIPLE INITIAL CODEWORDS

In this paper, we generalize ordinary PCs by allowing multiple initial codewords. The resulting codebook is contained in a set of concentric spheres.

¹Because of the convention $\mu_i > \mu_{i+1}$ established by Berger *et al.* [2], it is natural to index the order statistics in descending order as shown, which is opposite to the ascending convention in the order statistics literature [17].

A. Basic Construction

Let J be a positive integer. We will define a *concentric permutation (source) code* (CPC) with J initial codewords. This is equivalent to having a codebook that is the union of J PCs. Each notation from Section II-B is extended with a superscript or subscript $j \in \{1, 2, \dots, J\}$ that indexes the constituent PC. Thus, \mathcal{C}_j is the subcodebook of full codebook $\mathcal{C} = \cup_{j=1}^J \mathcal{C}_j$ consisting of all M_j distinct permutations of initial vector

$$\hat{x}_{\text{init}}^j = \left(\mu_1^j, \dots, \mu_1^j, \dots, \mu_{K_j}^j, \dots, \mu_{K_j}^j \right), \quad (9)$$

where each μ_i^j appears n_i^j times, $\mu_1^j > \mu_2^j > \dots > \mu_{K_j}^j$ (all of which are nonnegative for Variant II), and $\sum_{i=1}^{K_j} n_i^j = n$. Also, $\{I_i^j\}_{i=1}^{K_j}$ are sets of indices generated by the j th composition.

Proposition 1: Nearest-neighbor encoding of X with codebook \mathcal{C} can be accomplished with the following procedure:

- 1) For each j , find $\hat{X}_j \in \mathcal{C}_j$ whose components have the same order as X .
- 2) Encode X with \hat{X} , the nearest codeword amongst $\{\hat{X}_j\}_{j=1}^J$.

Proof: Suppose $X' \in \mathcal{C}$ is an arbitrary codeword. Since $\mathcal{C} = \cup_{j=1}^J \mathcal{C}_j$, there must exist $j_0 \in \{1, 2, \dots, J\}$ such that $X' \in \mathcal{C}_{j_0}$. We have

$$\|X - \hat{X}\| \stackrel{(a)}{\leq} \|X - \hat{X}_{j_0}\| \stackrel{(b)}{\leq} \|X - X'\|,$$

where (a) follows from the second step of the algorithm, and (b) follows from the first step and the optimality of the encoding for ordinary PCs. ■

The first step of the algorithm requires $O(n \log n) + O(Jn)$ operations (sorting components of X and reordering each \hat{x}_{init}^j according to the index matrix obtained from the sorting); the second step requires $O(Jn)$ operations. The total complexity of encoding is therefore $O(n \log n)$, provided that we keep $J = O(\log n)$. In fact, in this rough accounting, the encoding with $J = O(\log n)$ is as cheap as the encoding for ordinary PCs.

For i.i.d. sources, codewords within a subcodebook are approximately equally likely to be chosen, but codewords in different subcodebooks may have very different probabilities. Using entropy coding yields

$$R \approx n^{-1} \left[H(\{p_j\}_{j=1}^J) + \sum_{j=1}^J p_j \log M_j \right], \quad (10)$$

where $H(\cdot)$ denotes the entropy of a distribution, p_j is the probability of choosing subcodebook \mathcal{C}_j , and M_j is the number of codewords in \mathcal{C}_j . Note that (10) is suggestive of a two-stage encoding scheme with a variable-rate code for the index of the chosen subcodebook and a fixed-rate code for the index of the chosen codeword within the subcodebook. Without entropy coding, the rate is

$$R = n^{-1} \log \left(\sum_{j=1}^J M_j \right). \quad (11)$$

The per-letter distortion for Variant I codes is now given by

$$D = n^{-1} E \left[\min_{1 \leq j \leq J} \|X - \hat{X}_j\|^2 \right] = n^{-1} E \left[\min_{1 \leq j \leq J} \sum_{i=1}^{K_j} \sum_{\ell \in I_i^j} \left(\xi_\ell - \mu_i^j \right)^2 \right], \quad (12)$$

where (12) is obtained by rearranging the components of X and \hat{X}_j in descending order. The distortion for Variant II codes has the same form as (12) with $\{\xi_\ell\}$ replaced by $\{\eta_\ell\}$.

B. Optimization

In general, finding the best ordinary PC requires an exhaustive search over all compositions of n . (Assuming a precomputation of all the order statistic means, the computation of the distortion for a given composition through either (5) or (6) is simple [2].) The search space can be reduced for certain distributions of X using [2, Thm. 3], but seeking the optimal code still quickly becomes intractable as n increases.

Our generalization makes the design problem considerably more difficult. Not only do we need J compositions, but the distortion for a given composition is not as easy to compute. Because of the minimization over j in (12), we lack a simple expression for μ_i^j 's in terms of the composition and the order statistic means as given in (7). The

relevant means are of conditional order statistics, conditioned on which subcodebook is selected; this depends on all J compositions.

In the remainder of the paper, we consider two ways to reduce the design complexity. In Section IV, we fix all subcodebooks to have a common composition. Along with reducing the design space, this restriction induces a structure in the full codebook that enables the joint design of $\{\mu_i^j\}_{j=1}^J$ for any i . In Section V, we take a brief detour into the optimal rate allocations in a wrapped spherical shape–gain vector quantizer with gain-dependent shape codebook. We use these rate allocations to pick the sizes of subcodebooks $\{C_j\}_{j=1}^J$.

The simplifications presented here still leave high design complexity for large n . Thus, some simulations use complexity-reducing heuristics including our conjecture that an analogue to [2, Thm. 3] holds. Since our numerical designs are not provably optimal, the improvements from allowing multiple initial codewords could be somewhat larger than we demonstrate.

IV. DESIGN WITH COMMON COMPOSITION

In this section, assume that the J compositions are identical, i.e., the n_i^j s have no dependence on j . The subcodebook sizes are also equal, and dropping unnecessary sub- and superscripts we write the common composition as $\{n_i\}_{i=1}^K$ and the size of a single subcodebook as M .

A. Common Compositions Give Common Conic Partitions

The Voronoi regions of the code now have a special geometric structure. Recall that any spherical code partitions \mathbb{R}^n into (unbounded) convex cones. Having a common composition implies that each subcodebook induces the same conic Voronoi structure on \mathbb{R}^n . The full code divides each of the M cones into J Voronoi regions.

The following theorem precisely maps the encoding of a CPC to a vector quantization problem. For compositions other than $(1, 1, \dots, 1)$, the VQ design problem is in a dimension strictly lower than n .

Theorem 1: For fixed common composition (n_1, n_2, \dots, n_K) , the initial codewords $\{(\mu_1^j, \dots, \mu_1^j, \dots, \mu_K^j, \dots, \mu_K^j)\}_{j=1}^J$ of a Variant I CPC are optimal if and only if $\{\mu^1, \dots, \mu^J\}$ are representation points of the optimal J -point vector quantization of $\bar{\xi} \in \mathbb{R}^K$, where

$$\mu^j = \left(\sqrt{n_1} \mu_1^j, \sqrt{n_2} \mu_2^j, \dots, \sqrt{n_K} \mu_K^j \right), \quad 1 \leq j \leq J,$$

$$\bar{\xi} = \left(\frac{1}{\sqrt{n_1}} \sum_{\ell \in I_1} \xi_\ell, \frac{1}{\sqrt{n_2}} \sum_{\ell \in I_2} \xi_\ell, \dots, \frac{1}{\sqrt{n_K}} \sum_{\ell \in I_K} \xi_\ell \right).$$

Proof: Rewrite the distortion as follows:

$$\begin{aligned} nD &= E \left[\min_{1 \leq j \leq J} \sum_{i=1}^K \sum_{\ell \in I_i} (\xi_\ell - \mu_i^j)^2 \right] = E \left[\min_{1 \leq j \leq J} \sum_{i=1}^K \left(\sum_{\ell \in I_i} (\xi_\ell)^2 - 2\mu_i^j \sum_{\ell \in I_i} \xi_\ell + n_i (\mu_i^j)^2 \right) \right] \\ &= E \left[\min_{1 \leq j \leq J} \sum_{i=1}^K \left(\frac{1}{\sqrt{n_i}} \sum_{\ell \in I_i} \xi_\ell - \sqrt{n_i} \mu_i^j \right)^2 \right] + E \left[\sum_{i=1}^K \sum_{\ell \in I_i} (\xi_\ell)^2 \right] - E \left[\sum_{i=1}^K \left(\frac{1}{\sqrt{n_i}} \sum_{\ell \in I_i} \xi_\ell \right)^2 \right] \\ &= E \left[\min_{1 \leq j \leq J} \|\bar{\xi} - \mu^j\|^2 \right] + E [\|X\|^2] - E \left[\sum_{i=1}^K \left(\frac{1}{\sqrt{n_i}} \sum_{\ell \in I_i} \xi_\ell \right)^2 \right]. \end{aligned} \quad (13)$$

Since the second and third terms of (13) do not depend on $\{\hat{x}_{\text{init}}^j\}_{j=1}^J$, minimizing D is equivalent to minimizing the first term of (13). By definition of a K -dimensional VQ, that term is minimized if and only if $\{\mu^1, \dots, \mu^J\}$ are optimal representation points of the J -point VQ of random vector $\bar{\xi}$, completing the proof. ■

For any fixed composition, one can implement the J -point VQ design inspired by Theorem 1, using the Lloyd-Max algorithm [18], [19], to obtain $\{\mu^1, \dots, \mu^J\} \subset \mathbb{R}^K$ and then apply the mapping stated in the theorem to obtain the J desired initial codewords in \mathbb{R}^n . Theorem 1 can be trivially extended for Variant II codes by simply replacing $\{\xi_\ell\}$ with $\{\eta_\ell\}$.

Figure 1 compares the performance of an ordinary Variant I PC ($J = 1$) with variable-rate CPCs with $J = 3$ initial vectors. For a given composition, the distortion of the optimal ordinary PC is computed using (7) and

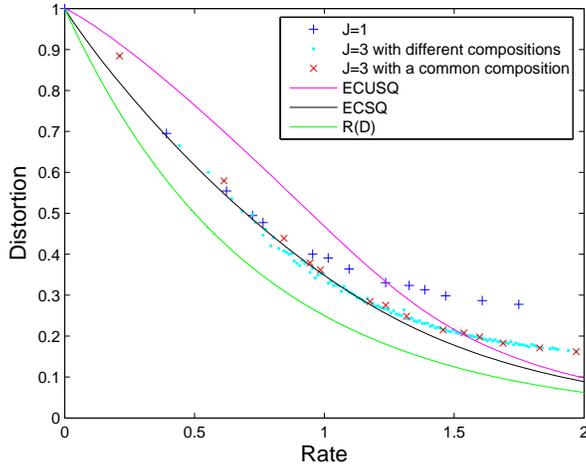


Fig. 1. Rate–distortion performance for variable-rate coding of i.i.d. $\mathcal{N}(0,1)$ source with block length $n = 7$. Ordinary Variant I PCs ($J = 1$) are compared with CPCs with $J = 3$. Codes with common compositions are designed according to Theorem 1. Codes with different compositions are designed with heuristic selection of compositions guided by Conjecture 2 and Algorithm 1. For clarity, amongst approximately-equal rates, only operational points with the lowest distortion are plotted.

variances of the order statistics (see [2, Eq. (13)]), whereas that of the optimal CPC is estimated empirically from 500 000 samples generated according to the $\mathcal{N}(0,1)$ distribution. Figure 1 and several subsequent figures include for comparison the rate–distortion bound and the performances of two types of entropy-constrained scalar quantization: uniform thresholds with uniform codewords (labeled ECUSQ) and uniform thresholds with optimal codewords (labeled ECSQ). At all rates, the latter is a very close approximation to optimal ECSQ; in particular, it has optimal rate–distortion slope at rate zero [20].

B. Optimization of Composition

Although the optimization of compositions is not easy even for ordinary PCs, for a certain class of distributions, there is a useful necessary condition for the optimal composition [2, Thm. 3]. The following conjecture is an analogue of that condition.

Conjecture 1: Suppose that $J > 1$ and that $E[\eta_\ell]$ is a convex function of ℓ , i.e.

$$E[\eta_{\ell+2}] - 2E[\eta_{\ell+1}] + E[\eta_\ell] \geq 0, \quad 1 \leq \ell \leq n-2. \quad (14)$$

Then the optimum n_i for Variant II CPCs increases monotonically with i .

The convexity of $E[\eta_\ell]$ holds for a large class of source distributions (see [2, Thm. 4]), including Gaussian ones. Conjecture 1 greatly reduces the search space for optimal compositions for such sources.

The conjecture is proven if one can show that the distortion associated with the composition $(n_1, \dots, n_m, n_{m+1}, \dots, n_K)$, where $n_m > n_{m+1}$, can be decreased by reversing the roles of n_m and n_{m+1} . As a plausibility argument for the conjecture, we will show that the reversing has the desired property when an additional constraint is imposed on the codewords. With the composition fixed, let

$$\zeta \triangleq \frac{1}{r} \sum_{L+1}^{L+r} \eta_\ell - \frac{2}{q-r} \sum_{L+r+1}^{L+q} \eta_\ell + \frac{1}{r} \sum_{L+q+1}^{L+q+r} \eta_\ell, \quad (15)$$

where $L = n_1 + n_2 + \dots + n_{m-1}$. The convexity of $E[\eta_\ell]$ implies the nonnegativity of $E[\zeta]$ (see [2, Thm. 2]). Using the total expectation theorem, $E[\zeta]$ can be written as the difference of two nonnegative terms,

$$\bar{\zeta}_+ \triangleq \Pr(\zeta \geq 0)E[\zeta | \zeta \geq 0] \quad \text{and} \quad \bar{\zeta}_- \triangleq -\Pr(\zeta < 0)E[\zeta | \zeta < 0].$$

Since $E[\zeta] \geq 0$ and probabilities are nonnegative, it is clear that $\bar{\zeta}_+ \geq \bar{\zeta}_-$. Therefore, the following set is non-empty:

$$\Omega_m = \left\{ \left\{ \mu_i^j \right\}_{i,j} \text{ s.t. } \frac{\min_j (\mu_m^j - \mu_{m+1}^j)}{\max_j (\mu_m^j - \mu_{m+1}^j)} \geq \frac{\bar{\zeta}_-}{\bar{\zeta}_+} \right\}. \quad (16)$$

With the notations above, we are now ready to state the proposition. If the restriction of the codewords were known to not preclude optimality, then Conjecture 1 would be proven.

Proposition 2: Suppose that $J > 1$ and $E[\eta_\ell]$ is a convex function of ℓ . If $n_m > n_{m+1}$ for some m , and the constraint Ω_m given in (16) is imposed on the codewords, then the distortion associated with the composition $(n_1, \dots, n_m, n_{m+1}, \dots, n_K)$ can be decreased by reversing the roles of n_m and n_{m+1} .

Proof: See Appendix A. ■

A straightforward extension of Conjecture 1 for Variant I codes is the following:

Conjecture 2: Suppose that $J > 1$, and that $E[\xi_\ell]$ is convex over $\mathcal{S}_1 \triangleq \{1, 2, \dots, \lfloor K/2 \rfloor\}$ and concave over $\mathcal{S}_2 \triangleq \{\lfloor K/2 \rfloor + 1, \lfloor K/2 \rfloor + 2, \dots, K\}$. Then the optimum n_i for Variant I CPCs increases monotonically with $i \in \mathcal{S}_1$ and decreases monotonically with $i \in \mathcal{S}_2$.

The convexity of $E[\xi_\ell]$ holds for a large class of source distributions (see [2, Thm. 5]). We will later restrict the compositions, while doing simulations for Variant I codes and Gaussian sources, to satisfy Conjecture 2.

V. DESIGN WITH DIFFERENT COMPOSITIONS

Suppose now that the compositions of subcodebooks can be different. The Voronoi partitioning of \mathbb{R}^n is much more complicated, lacking the separability discussed in the previous section.² Furthermore, the apparent design complexity for the compositions is increased greatly to equal the number of compositions raised to the J th power, namely $2^{J(n-1)}$.

In this section we first outline an algorithm for local optimization of initial vectors with all the compositions fixed. Then we address a portion of the composition design problem which is the sizing of the subcodebooks. For this, we extend the high-resolution analysis of [7]. For brevity, we limit our discussion to Variant I CPCs; Variant II could be generalized similarly.

A. Local Optimization of Initial Vectors

Let $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ denote the ordered vector of X . Given J initial codewords $\{\hat{x}_{\text{init}}^j\}_{j=1}^J$, for each j , let $R_j \subset \mathbb{R}^n$ denote the quantization region of ξ corresponding to codeword \hat{x}_{init}^j , and let $E_j[\cdot]$ denote the expectation conditioned on $\xi \in R_j$. If R_j is fixed, consider the distortion conditioned on $\xi \in R_j$

$$D_j = n^{-1} E \left[\sum_{i=1}^{K_j} \sum_{\ell \in I_i^j} (\xi_\ell - \mu_i^j)^2 \mid \xi \in R_j \right]. \quad (17)$$

By extension of an argument in [2], D_j is minimized with

$$\mu_i^j = \frac{1}{n_i^j} \sum_{\ell \in I_i^j} E_j[\xi_\ell], \quad 1 \leq i \leq K_j. \quad (18)$$

For a given set $\{R_j\}_{j=1}^J$, since the total distortion is determined by

$$D = \sum_{j=1}^J \Pr(\xi \in R_j) D_j,$$

it will decrease if μ_i^j 's are set to the new values given by (18) for all $1 \leq j \leq J$ and for all $1 \leq i \leq K_j$.

From the above analysis, a Lloyd algorithm can be developed to design initial codewords as given in Algorithm 1. This algorithm is similar to the algorithm in [10], but here the compositions can be arbitrary. Algorithm 1 was used to produce the operating points shown in Figure 1 for CPCs with different compositions in which the distortion of a locally-optimal code was computed empirically from 500 000 samples generated according to $\mathcal{N}(0, 1)$ distribution. We can see through the figure that common compositions can produce almost the same distortion as possibly-different compositions for the same rate. However, allowing the compositions to be different yields many more rates. The number of rates is explored in Appendix B.

²For a related two-dimensional visualization, compare [21, Fig. 3] against [21, Figs. 7–13].

Algorithm 1 Lloyd Algorithm for Initial Codeword Optimization from Given Composition

- 1) Order vector X to get ξ
 - 2) Choose an arbitrary initial set of J representation vectors $\hat{x}_{\text{init}}^1, \hat{x}_{\text{init}}^2, \dots, \hat{x}_{\text{init}}^J$.
 - 3) For each j , determine the corresponding quantization region R_j of ξ .
 - 4) For each j , set \hat{x}_{init}^j to the new value given by (18).
 - 5) Repeat steps 3 and 4 until further improvement in MSE is negligible.
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B. Wrapped Spherical Shape–Gain Vector Quantization

Hamkins and Zeger [7] introduced a type of spherical code for \mathbb{R}^n where a lattice in \mathbb{R}^{n-1} is “wrapped” around the code sphere. They applied the wrapped spherical code (WSC) to the shape component in a shape–gain vector quantizer.

We generalize this construction to allow the size of the shape codebook to depend on the gain. Along this line of thinking, Hamkins [22, pp. 102–104] provided an algorithm to optimize the number of codewords on each sphere. However, neither analytic nor experimental improvement was demonstrated. In contrast, our approach based on high-resolution optimization gives an explicit expression for the improvement in signal-to-noise ratio (SNR). While our results may be of independent interest, our present purpose is to guide the selection of $\{M_j\}_{j=1}^J$ in CPCs.

A *shape–gain* vector quantizer (VQ) decomposes a source vector X into a *gain* $g = \|X\|$ and a *shape* $S = X/g$, which are quantized to \hat{g} and \hat{S} , respectively, and the approximation is $\hat{X} = \hat{g} \cdot \hat{S}$. We optimize here a wrapped spherical VQ with gain-dependent shape codebook. The gain codebook, $\{\hat{g}_1, \hat{g}_2, \dots, \hat{g}_J\}$, is optimized for the gain pdf, e.g., using the scalar Lloyd–Max algorithm [18], [19]. For each gain codeword \hat{g}_j , a shape subcodebook is generated by wrapping the sphere packing $\Lambda \subset \mathbb{R}^{n-1}$ on to the unit sphere in \mathbb{R}^n . The same Λ is used for each j , but the density (or scaling) of the packing may vary with j . Thus the normalized second moment $G(\Lambda)$ applies for each j while minimum distance d_Λ^j depends on the quantized gain \hat{g}_j . We denote such a sphere packing as (Λ, d_Λ^j) .

The per-letter MSE distortion will be

$$\begin{aligned}
 D &= n^{-1} E \left[\|X - \hat{g} \hat{S}\|^2 \right] \\
 &= n^{-1} E \left[\|X - \hat{g} S\|^2 \right] + 2n^{-1} E \left[(X - \hat{g} S)^T (\hat{g} S - \hat{g} \hat{S}) \right] + n^{-1} E \left[\|\hat{g} S - \hat{g} \hat{S}\|^2 \right] \\
 &= \underbrace{n^{-1} E \left[\|X - \hat{g} S\|^2 \right]}_{D_g} + \underbrace{n^{-1} E \left[\|\hat{g} S - \hat{g} \hat{S}\|^2 \right]}_{D_s},
 \end{aligned}$$

where the omitted cross term is zero due to the independence of g and \hat{g} from S [7]. The gain distortion, D_g , is given by

$$D_g = \frac{1}{n} \int_0^\infty (r - \hat{g}(r))^2 f_g(r) dr,$$

where $\hat{g}(\cdot)$ is the quantized gain and $f_g(\cdot)$ is the pdf of g .

Conditioned on the gain codeword \hat{g}_j chosen, the shape S is distributed uniformly on the unit sphere in \mathbb{R}^n , which has surface area $S_n = 2\pi^{n/2}/\Gamma(n/2)$. Thus, as shown in [7], for asymptotically high shape rate R_s , the conditional distortion $E[\|S - \hat{S}\|^2 | \hat{g}_j]$ is equal to the distortion of the lattice quantizer with codebook (Λ, d_Λ^j) for a uniform source in \mathbb{R}^{n-1} . Thus,

$$E \left[\|S - \hat{S}\|^2 | \hat{g}_j \right] = (n-1) G(\Lambda) V_j(\Lambda)^{2/(n-1)}, \quad (19)$$

where $V_j(\Lambda)$ is the volume of a Voronoi region of the $(n-1)$ -dimensional lattice (Λ, d_Λ^j) . Therefore, for a given

gain codebook $\{\hat{g}_1, \hat{g}_2, \dots, \hat{g}_J\}$, the shape distortion D_s can be approximated by

$$\begin{aligned}
D_s &= \frac{1}{n} E \left[\|\hat{g} S - \hat{g} \hat{S}\|^2 \right] = \frac{1}{n} \sum_{j=1}^J p_j \hat{g}_j^2 E \left[\|S - \hat{S}\|^2 \mid \hat{g} = \hat{g}_j \right] \\
&\stackrel{(a)}{\approx} \frac{1}{n} \sum_{j=1}^J p_j \hat{g}_j^2 (n-1) G(\Lambda) V_j(\Lambda)^{2/(n-1)} \\
&\stackrel{(b)}{\approx} \frac{1}{n} \sum_{j=1}^J p_j \hat{g}_j^2 (n-1) G(\Lambda) (S_n/M_j)^{2/(n-1)} \\
&= \frac{n-1}{n} G(\Lambda) S_n^{2/(n-1)} \sum_{j=1}^J p_j \hat{g}_j^2 M_j^{-2/(n-1)} = C \cdot \sum_{j=1}^J p_j \hat{g}_j^2 M_j^{\frac{-2}{n-1}},
\end{aligned}$$

where p_j is the probability of \hat{g}_j being chosen; (a) follows from (19); (b) follows from the high-rate assumption and neglecting the overlapping regions, with M_j representing the number of codewords in the shape subcodebook associated with \hat{g}_j ; and

$$C \triangleq \frac{n-1}{n} G(\Lambda) \left(2\pi^{n/2} / \Gamma(n/2) \right)^{2/(n-1)}. \quad (20)$$

C. Rate Allocations

The optimal rate allocation for high-resolution approximation to WSC given below will be used as the rate allocation across subcodebooks in our CPCs.

1) *Variable-Rate Coding*: Before stating the theorem, we need the following lemma.

Lemma 1: If there exist constants C_s and C_g such that

$$\lim_{R_s \rightarrow \infty} D_s \cdot 2^{2(n/(n-1))R_s} = C_s \quad \text{and} \quad \lim_{R_g \rightarrow \infty} D_g \cdot 2^{2nR_g} = C_g,$$

then the minimum of $D = D_s + D_g$ subject to the constraint $R = R_s + R_g$ satisfies

$$\lim_{R \rightarrow \infty} D 2^{2R} = \frac{n}{(n-1)^{1-1/n}} \cdot C_g^{1/n} C_s^{1-1/n}$$

and is achieved by $R_s = R_s^*$ and $R_g = R_g^*$, where

$$R_s^* = \left(\frac{n-1}{n} \right) \left[R + \frac{1}{2n} \log \left(\frac{C_s}{C_g} \cdot \frac{1}{n-1} \right) \right], \quad (21)$$

$$R_g^* = \left(\frac{1}{n} \right) \left[R - \frac{n-1}{2n} \log \left(\frac{C_s}{C_g} \cdot \frac{1}{n-1} \right) \right]. \quad (22)$$

Proof: See [7, Thm. 1]. ■

Theorem 2: Let $X \in \mathbb{R}^n$ be an i.i.d. $\mathcal{N}(0, \sigma^2)$ vector, and let Λ be a lattice in \mathbb{R}^{n-1} with normalized second moment $G(\Lambda)$. Suppose X is quantized by an n -dimensional shape-gain VQ at rate $R = R_g + R_s$ with gain-dependent shape codebook constructed from Λ with different minimum distances. Also, assume that a variable-rate coding follows the quantization. Then, the asymptotic decay of the minimum mean-squared error D is given by

$$\lim_{R \rightarrow \infty} D 2^{2R} = \frac{n}{(n-1)^{1-1/n}} \cdot C_g^{1/n} C_s^{1-1/n} \quad (23)$$

and is achieved by $R_s = R_s^*$ and $R_g = R_g^*$, where R_s^* and $R_g = R_g^*$ are given in (21) and (22),

$$C_s = \frac{n-1}{n} G(\Lambda) \left(2\pi^{n/2} / \Gamma(n/2) \right)^{2/(n-1)} \cdot 2\sigma^2 e^{\psi(n/2)}, \quad C_g = \sigma^2 \cdot \frac{3^{n/2} \Gamma^3(\frac{n+2}{6})}{8n\Gamma(n/2)},$$

and $\psi(\cdot)$ is the digamma function.

Proof: We first minimize D_s for a given gain codebook $\{\hat{g}_j\}_{j=1}^J$. From (20), ignoring the constant C , we must perform the minimization

$$\min_{M_1, \dots, M_J} \sum_{j=1}^J p_j \hat{g}_j^2 M_j^{2/(1-n)} \quad \text{subject to} \quad \sum_{j=1}^J p_j \log M_j = nR_s. \quad (24)$$

Using a Lagrange multiplier to get an unconstrained problem, we obtain the objective function

$$f = \sum_{j=1}^J p_j \hat{g}_j^2 M_j^{2/(1-n)} - \lambda \sum_{j=1}^J p_j \log M_j.$$

Neglecting the integer constraint, we can take the partial derivatives

$$\frac{\partial f}{\partial M_j} = \frac{2}{1-n} p_j \hat{g}_j^2 M_j^{(n+1)/(1-n)} - \lambda p_j M_j^{-1}, \quad 1 \leq j \leq J.$$

Setting $\frac{\partial f}{\partial M_j} = 0$, $1 \leq j \leq J$, yields

$$M_j = [\lambda(1-n)/(2\hat{g}_j^2)]^{(1-n)/2}. \quad (25)$$

Substituting into the constraint (24), we get

$$\sum_{j=1}^J p_j \log \left[\lambda(1-n)/(2\hat{g}_j^2) \right]^{(1-n)/2} = nR_s.$$

Thus,

$$[\lambda(1-n)/2]^{(1-n)/2} = 2^{nR_s - (n-1)\sum_{k=1}^J p_k \log \hat{g}_k} = 2^{nR_s - (n-1)E[\log \hat{g}]}$$

Therefore, it follows from (25) that the optimal size for the j th shape subcodebook for a given gain codebook is

$$M_j = \hat{g}_j^{n-1} \cdot 2^{nR_s^* - (n-1)E[\log \hat{g}]}, \quad 1 \leq j \leq J. \quad (26)$$

The resulting shape distortion is

$$D_s \approx C \cdot \sum_{j=1}^J p_j \hat{g}_j^2 \left(\hat{g}_j^{n-1} 2^{nR_s^* - (n-1)E[\log \hat{g}]} \right)^{2/(1-n)} = C \cdot 2^{2E[\log \hat{g}]} \cdot 2^{-2(n/(n-1))R_s^*},$$

where C is the same constant as specified in (20). Hence,

$$\lim_{R \rightarrow \infty} D_s \cdot 2^{2(n/(n-1))R_s^*} = C \cdot \lim_{R_s^* \rightarrow \infty} 2^{2E[\log \hat{g}]} \stackrel{(a)}{=} C \cdot 2^{2E[\log g]} \stackrel{(b)}{=} C \cdot 2\sigma^2 e^{\psi(n/2)} = C_s, \quad (27)$$

where (a) follows from the high-rate assumption; and (b) follows from computing the expectation $E[\log g]$. On the other hand, it is shown in [7, Thm. 1] that

$$\lim_{R \rightarrow \infty} D_g \cdot 2^{2n(R-R_s^*)} = \lim_{R \rightarrow \infty} D_g \cdot 2^{2nR_g^*} = C_g. \quad (28)$$

The limits (27) and (28) now allow us to apply Lemma 1 to obtain the desired result. \blacksquare

Through this theorem we can verify the rate–distortion improvement as compared to independent shape–gain encoding by comparing C_g and C_s in the distortion formula to the analogous quantities in [7, Thm. 1]. C_g remains the same whereas C_s , which plays a more significant role in the distortion formula, is scaled by a factor of $2e^{\psi(n/2)}/n < 1$. In particular, the improvement in signal-to-quantization noise ratio achieved by the WSC with gain-dependent shape codebook is given by

$$\Delta_{\text{SNR}} \text{ (in dB)} = -10(1 - 1/n) \log_{10}(2e^{\psi(n/2)}/n). \quad (29)$$

From the theory of the gamma function [23, Eq. 29], we know that, for $s \in \mathbb{C}$,

$$\lim_{|s| \rightarrow \infty} [\psi(s) - \ln(s)] = 0.$$

It follows that $[\psi(n/2) - \ln(n/2)] \rightarrow 0$, and thus $\Delta_{\text{SNR}}(n) \rightarrow 0$, as $n \rightarrow \infty$; this is not surprising because of the ‘‘sphere hardening’’ effect. This improvement is plotted in Figure 2 as a function of block length n in the range between 5 and 50.

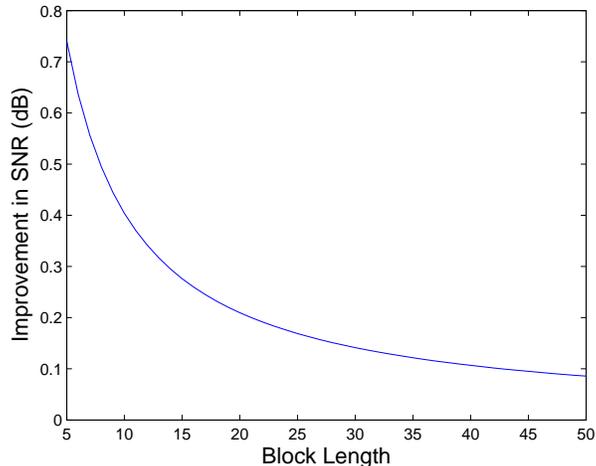


Fig. 2. Improvement in signal-to-quantization noise ratio of WSC with gain-dependent shape quantizer specified in (29), as compared to the asymptotic rate-distortion performance given in [7, Thm. 1]

2) *Fixed-Rate Coding*: A similar optimal rate allocation is possible for fixed-rate coding.

Theorem 3: Let $X \in \mathbb{R}^n$ be an i.i.d. $\mathcal{N}(0, \sigma^2)$ vector, and let Λ be a lattice in \mathbb{R}^{n-1} with normalized second moment $G(\Lambda)$. Suppose X is quantized by an n -dimensional shape-gain VQ at rate R with gain-dependent shape codebook constructed from Λ with different minimum distances. Also, assume that J gain codewords are used and that a fixed-rate coding follows the quantization. Then, the optimal number of codewords in each subcodebook is

$$M_j = 2^{nR} \cdot \frac{(p_j \hat{g}_j^2)^{(n-1)/(n+1)}}{\sum_{k=1}^J (p_k \hat{g}_k^2)^{(n-1)/(n+1)}}, \quad 1 \leq j \leq J, \quad (30)$$

where $\{\hat{g}_1, \hat{g}_2, \dots, \hat{g}_J\}$ is the optimal gain codebook. The resulting asymptotic decay of the shape distortion D_s is given by

$$\lim_{R \rightarrow \infty} D_s 2^{2(n/(n-1))R} = C \cdot \left[\sum_{j=1}^J (p_j \hat{g}_j^2)^{\frac{n-1}{n+1}} \right]^{\frac{n+1}{n-1}}, \quad (31)$$

where p_j is probability of \hat{g}_j being chosen and C is the same constant as given in (20).

Proof: For a given gain codebook $\{\hat{g}_j\}_{j=1}^J$, the optimal subcodebook sizes are given by the optimization

$$\min_{M_1, \dots, M_J} \sum_{j=1}^J p_j \hat{g}_j^2 M_j^{2/(1-n)} \quad \text{subject to} \quad \sum_{j=1}^J M_j = 2^{nR}. \quad (32)$$

Similarly to the variable-rate case, we can use a Lagrange multiplier to obtain an unconstrained optimization with the objective function

$$h = \sum_{j=1}^J p_j \hat{g}_j^2 M_j^{2/(1-n)} - \lambda \sum_{j=1}^J M_j.$$

Again, assuming high rate, we can ignore the integer constraints on M_j to take partial derivatives. Setting them equal to zero, one can obtain

$$M_j = [\lambda(1-n)/(2p_j \hat{g}_j^2)]^{(1-n)/(n+1)}. \quad (33)$$

Substituting into the constraint (32) yields

$$\sum_{j=1}^J \left[\lambda(1-n)/(2p_j \hat{g}_j^2) \right]^{(1-n)/(n+1)} = 2^{nR}.$$

Hence,

$$\lambda^{(n-1)/(n+1)} = 2^{-nR} \sum_{k=1}^J \left(\frac{1-n}{2p_k \hat{g}_k^2} \right)^{(1-n)/(n+1)}. \quad (34)$$

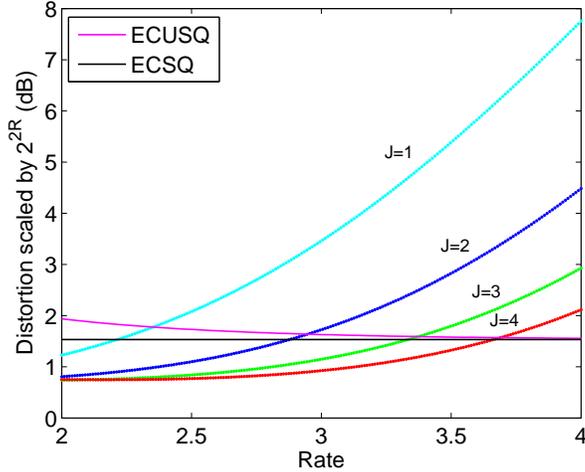


Fig. 3. High-resolution approximation of the rate–distortion performance of WSC with gain-dependent shape codebooks and fixed-rate coding for an i.i.d. $\mathcal{N}(0, 1)$ source with block length $n = 25$.

Combining (34) and (33) give us

$$M_j = \lambda^{(1-n)/(n+1)} \left(\frac{1-n}{2p_j \hat{g}_j^2} \right)^{(1-n)/(n+1)} = 2^{nR} \frac{(p_j \hat{g}_j^2)^{(n-1)/(n+1)}}{\sum_{k=1}^J (p_k \hat{g}_k^2)^{(n-1)/(n+1)}}, \quad 1 \leq j \leq J.$$

With the high-rate assumption, the resulting shape distortion will be

$$\begin{aligned} D_s &= C \sum_{j=1}^J p_j \hat{g}_j M_j^{2/(1-n)} = C \sum_{j=1}^J p_j \hat{g}_j \left[\frac{2^{nR} (p_j \hat{g}_j^2)^{(n-1)/(n+1)}}{\sum_{k=1}^J (p_k \hat{g}_k^2)^{(n-1)/(n+1)}} \right]^{2/(1-n)} \\ &= C \cdot 2^{-2(n/(n-1))R} \left[\sum_{j=1}^J (p_j \hat{g}_j^2)^{(n-1)/(n+1)} \right]^{\frac{n+1}{n-1}} \end{aligned} \quad (35)$$

where $C = \frac{n-1}{n} G(\Lambda) (2\pi^{n/2} / \Gamma(n/2))^{2/(n-1)}$, completing the proof. \blacksquare

Figure 3 illustrates the resulting performance as a function of the rate for several values of J . As expected, for a fixed block size n , higher rates require higher values of J (more concentric spheres) to attain good performance, and the best performance is improved by increasing the maximum value for J .

D. Using WSC Rate Allocation for Permutation Codes

In this section we use the optimal rate allocations for WSC to guide the design of CPCs at a given rate. The rate allocations are used to set target sizes for each subcodebook. Then for each subcodebook \mathcal{C}_j , a composition meeting the constraint on M_j is selected (using heuristics inspired by Conjecture 2). Algorithm 1 of Section V-A is then used for those compositions to compute the actual rate and distortion.

For the variable-rate case, Theorem 2 provides the key rate allocation step in the design procedure given in Algorithm 2. Similarly, Theorem 3 leads to the design procedure for the fixed-rate case given in Algorithm 3. Each case requires as input not only the rate R but also the number of initial codewords J .

Results for the fixed-rate case are plotted in Figure 4. This demonstrates that using the rate allocation of WSC with gain-dependent shape codebook actually yields good CPCs for most of the rates. Figure 5 demonstrates the improvement that comes with allowing more initial codewords. The distortion is again computed empirically from Gaussian samples. It has a qualitative similarity with Figure 3.

Algorithm 2 Design Algorithm for Variable-Rate Case

- 1) Compute R_s^* and R_g^* from (21) and (22), respectively.
 - 2) For $1 \leq j \leq J$, compute M_j from (26).
 - 3) For $1 \leq j \leq J$, search through all possible compositions of n that satisfy Conjecture 2, choosing the one that produces the number of codewords closest to M_j .
 - 4) Run Algorithm 1 for the J compositions chosen in step 4 to generate the initial codewords and to compute the actual rate and distortion.
-

Algorithm 3 Design Algorithm for Fixed-Rate Case

- 1) Use the scalar Lloyd-Max algorithm to optimize J gain codewords.
 - 2) For $1 \leq j \leq J$, compute M_j from (30)
 - 3) Repeat steps 3 and 4 of Algorithm 2.
-

VI. CONCLUSIONS

We have studied a generalization of permutation codes in which more than one initial codeword is allowed. This improves rate–distortion performance while adding very little to encoding complexity. However, the design complexity is increased considerably. To reduce the design complexity, we explore a method introduced by Lu *et al.* of restricting the subcodebooks to share a common composition; and we introduce a method of allocating rates across subcodebooks using high-resolution analysis of wrapped spherical codes. Simulations suggest that these heuristics are effective, but obtaining theoretical guarantees remains an open problem.

ACKNOWLEDGMENT

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APPENDIX A PROOF OF PROPOSITION 2

Consider a new composition $\{n'_1, n'_2, \dots, n'_K\}$ obtained by swapping n_m and n_{m+1} , i.e.,

$$n'_i = \begin{cases} n_i, & i \neq m \text{ or } m + 1; \\ n_{m+1}, & i = m; \\ n_m, & i = m + 1. \end{cases}$$

Let $\{I'_i\}$ denote groups of indices generated by composition $\{n'_i\}$. Suppose that D is the optimal distortion associated with $\{n_i\}$,

$$D = n^{-1} E \left[\min_{1 \leq j \leq J} \sum_{i=1}^K \sum_{\ell \in I_i} \left(\eta_\ell - \mu_i^j \right)^2 \right],$$

where $\{\mu_i^j\}$ is the optimum of the minimization of the right side over Ω_m . Consider a suboptimal distortion D' associated with $\{n'_i\}$,

$$D' = n^{-1} E \left[\min_{1 \leq j \leq J} \sum_{i=1}^K \sum_{\ell \in I'_i} \left(\eta_\ell - \tilde{\mu}_i^j \right)^2 \right],$$

where $\{\tilde{\mu}_i^j\}$ is constructed from $\{\mu_i^j\}$ as follows: for each j ,

$$\tilde{\mu}_i^j = \begin{cases} \mu_i^j, & i \neq m \text{ or } m + 1; \\ \frac{2n_m \mu_m^j + (n_{m+1} - n_m) \mu_{m+1}^j}{n_m + n_{m+1}}, & i = m; \\ \frac{(n_m - n_{m+1}) \mu_m^j + 2n_{m+1} \mu_{m+1}^j}{n_m + n_{m+1}}, & i = m + 1. \end{cases} \quad (36)$$

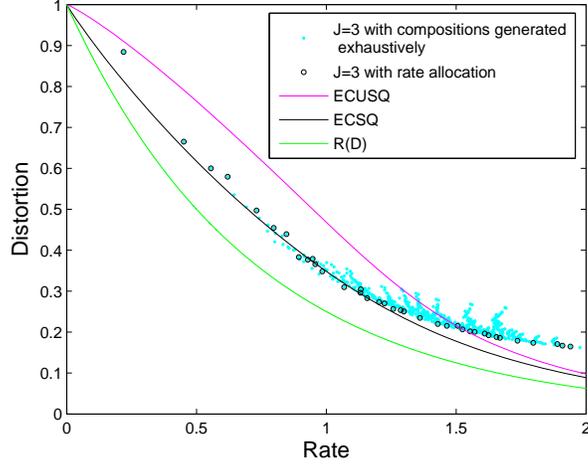


Fig. 4. Rate–distortion performance of fixed-rate CPCs designed with compositions guided by the WSC rate allocation and Algorithm 1, in comparison with codes designed with exhaustive search over a heuristic subset of compositions guided by Conjecture 2. Computation uses i.i.d. $\mathcal{N}(0, 1)$ source, $n = 7$, and $J = 3$.

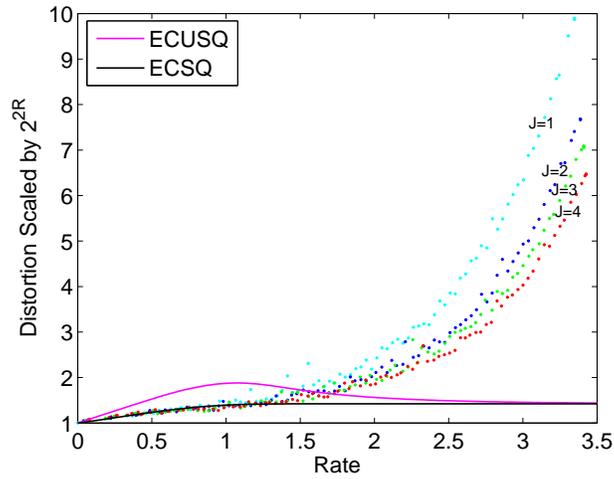


Fig. 5. Rate–distortion performance for variable-rate coding of i.i.d. $\mathcal{N}(0, 1)$ source with block length $n = 25$. CPCs with different compositions are designed using rate allocations from Theorem 3 and initial codewords locally optimized by Algorithm 1. The rate allocation computation assumes $G(\Lambda_{24}) \approx 0.065771$ [24, p. 61].

Note that, for the above construction, we have $\tilde{\mu}_m^j - \tilde{\mu}_{m+1}^j = \mu_m^j - \mu_{m+1}^j$, for all $j \in \{1, 2, \dots, J\}$. Therefore $\{\tilde{\mu}_i^j\}$ also satisfies Ω_m , and so forms a valid codebook corresponding to composition $\{n_i^j\}$. Thus, it will be sufficient if we can show $D > D'$. On the other hand, it is easy to verify that, for all j ,

$$n_{m+1}(\tilde{\mu}_m^j)^2 + n_m(\tilde{\mu}_{m+1}^j)^2 = n_m(\mu_m^j)^2 + n_{m+1}(\mu_{m+1}^j)^2.$$

Hence,

$$\sum_{i=1}^K n_i^j(\tilde{\mu}_i^j)^2 = \sum_{i=1}^K n_i(\mu_i^j)^2, \quad \text{for all } j. \quad (37)$$

Now consider the difference between D and D' :

$$\begin{aligned} \Delta &= n(D - D') = E \left[\min_j \sum_{i=1}^K \sum_{\ell \in I_i} (\eta_\ell - \mu_i^j)^2 - \min_j \sum_{i=1}^K \sum_{\ell \in I'_i} (\eta_\ell - \tilde{\mu}_i^j)^2 \right] \\ &\stackrel{(a)}{\geq} E \left[\min_j \left\{ \sum_{i=1}^K \left(n_i (\mu_i^j)^2 - 2\mu_i^j \sum_{\ell \in I_i} \eta_\ell \right) - \sum_{i=1}^K \left(n'_i (\tilde{\mu}_i^j)^2 - 2\tilde{\mu}_i^j \sum_{\ell \in I'_i} \eta_\ell \right) \right\} \right] \\ &\stackrel{(b)}{=} 2E \left[\min_j \left\{ \tilde{\mu}_m^j \sum_{\ell=L+1}^{L+r} \eta_\ell + \tilde{\mu}_{m+1}^j \sum_{\ell=L+r+1}^{L+r+q} \eta_\ell - \mu_m^j \sum_{\ell=L+1}^{L+q} \eta_\ell - \mu_{m+1}^j \sum_{\ell=L+q+1}^{L+q+r} \eta_\ell \right\} \right], \end{aligned}$$

where (a) uses the fact that $\min f - \min g \geq \min\{f - g\}$, for arbitrary functions f, g ; and (b) follows from (37) in which $q = n_m$, $r = n_{m+1}$, and $L = n_1 + n_2 + \dots + n_{m-1}$. Now using the formulae of $\tilde{\mu}_m^j$ and $\tilde{\mu}_{m+1}^j$ in (36), we obtain

$$\begin{aligned} \Delta &\geq 2E \left[\min_j \left\{ \frac{(q-r)(\mu_m^j - \mu_{m+1}^j)}{q+r} \sum_{\ell=L+1}^{L+r} \eta_\ell - \frac{2r(\mu_m^j - \mu_{m+1}^j)}{q+r} \sum_{\ell=L+r+1}^{L+q} \eta_\ell \right. \right. \\ &\quad \left. \left. + \frac{(q-r)(\mu_m^j - \mu_{m+1}^j)}{q+r} \sum_{\ell=L+q+1}^{L+q+r} \eta_\ell \right\} \right] \\ &= \frac{2r(q-r)}{q+r} E \left[\min_j \left\{ (\mu_m^j - \mu_{m+1}^j) \zeta \right\} \right] \\ &\stackrel{(a)}{=} \frac{2r(q-r)}{q+r} \left[\bar{\zeta}_+ \cdot \min_j \left\{ \mu_m^j - \mu_{m+1}^j \right\} - \bar{\zeta}_- \cdot \max_j \left\{ \mu_m^j - \mu_{m+1}^j \right\} \right] \\ &\stackrel{(b)}{\geq} 0, \end{aligned}$$

where ζ is the random variable specified in (15); (a) follows from the total expectation theorem; and (b) follows from constraint Ω_m and that $q > r$. The nonnegativity of Δ has proved the proposition.

APPENDIX B THE NUMBER AND DENSITY OF DISTINCT RATES

In this appendix, we discuss the distinct rate points at which fixed-rate ordinary PCs, CPCs with common compositionss, and CPCs with possibly-different compositionss may operate. For brevity, we restrict attention to Variant I codes.

The number of codewords (and therefore the rate) for an ordinary PC is determined by the multinomial coefficient (3). The multinomial coefficient is invariant to the order of the n_i s, and so we are interested in the number of unordered compositionss (or *integer partitions*) of n , $P(n)$. Hardy and Ramanujan [25] gave the asymptotic formula: $P(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}$. One might think that the number of possible distinct rate points is $P(n)$, but different sets of $\{n_i\}$ can yield the same multinomial coefficient. For example, at $n = 7$ both $(3, 2, 2)$ and $(4, 1, 1, 1)$ yield $M = 210$. Thus we are instead interested in the number of distinct multinomial coefficients, $N_{\text{mult}}(n)$ [26], [27, A070289]. Clearly $P(n) \geq N_{\text{mult}}(n)$. A lower bound to $N_{\text{mult}}(n)$ is the number of unordered partitions of n into parts that are prime, $P_{\mathbb{P}}(n)$, with asymptotic formula: $P_{\mathbb{P}}(n) \sim \exp\left\{\frac{2\pi\sqrt{n}}{\sqrt{3}\log n}\right\}$. Thus the number of distinct rate points for ordinary PCs grows exponentially with block length.

It follows easily that the average density of distinct rate points on the interval of possible rates grows without bound. Denote this average density by $\delta(n)$. The interval of possible rates is $[0, \log n!/n]$, so applying the upper and lower bounds gives the asymptotic expression

$$\frac{n \exp\left\{\frac{2\pi}{\sqrt{3}}\sqrt{\frac{n}{\log n}}\right\}}{\log n!} \lesssim \delta(n) \lesssim \frac{e^{\pi\sqrt{2n/3}}}{4\sqrt{3}\log n!}.$$

TABLE I
NUMBER OF RATE POINTS

n	$J = 1$	$J = 2$	$J = 3$	$J = 4$
2	2	3	4	5
3	3	6	10	15
4	5	15	33	56
5	7	27	68	132
6	11	60	207	517
7	14	97	415	1202
8	20	186	1038	3888
9	27	335	2440	11911

Taking the limits of the bounds then yields $\lim_{n \rightarrow \infty} \delta(n) = +\infty$.

The following proposition addresses the maximum gap between rate points, giving a result stronger than the statement on average density.

Proposition 3: The maximum spacing between any pair of rate points goes to 0 as $n \rightarrow \infty$.

Proof: First note that there are rate points at $0, \frac{\log[n]}{n}, \frac{\log[(n)(n-1)]}{n}, \frac{\log[(n)(n-1)(n-2)]}{n}, \dots$ induced by integer partitions $(n), (n-1, 1), (n-2, 1, 1), (n-3, 1, 1, 1), \dots$. The lengths of the intervals between these rate points is $\frac{\log(n)}{n}, \frac{\log(n-1)}{n}, \frac{\log(n-2)}{n}, \dots$ which decreases as one moves to larger rate points, so the first one is the largest. If there are other achievable rates between $\frac{\log[n]}{n}$ and $\frac{\log[(n)(n-1)]}{n}$ and so on, they only act to decrease the size of the intervals between successive rate points. So the interval between 0 and $\frac{\log[n]}{n}$ is the largest.

Taking $n \rightarrow \infty$ for the largest interval gives $\lim_{n \rightarrow \infty} \frac{\log n}{n} = 0$, so the maximum distance between any rate points goes to zero. ■

$N_{\text{mult}}(n)$ is the number of distinct rate points for ordinary PCs. If fixed-rate CPCs are restricted to have common compositions, then they too have the same number of distinct rate points. If different compositions are allowed, the number of distinct rate points may increase dramatically.

Recall the rate expression (11), and notice that distinct values of $\sum M_j$ will yield distinct rate points. Somewhat similarly to the distinct subset sum problem [28, pp. 174–175], we want to see how many distinct sums are obtainable from subsets of size J selected with replacement from the possible multinomial coefficients of a given block length n . This set is denoted $\mathcal{M}(n)$ and satisfies $|\mathcal{M}(n)| = N_{\text{mult}}(n)$; for example, $\mathcal{M}(4) = \{1, 4, 6, 12, 24\}$.

For a general set of integers of size $N_{\text{mult}}(n)$, the number of distinct subset sums is upper-bounded by $\binom{N_{\text{mult}}(n)+J-1}{J}$. This is achieved, for example, by the set $\{1^1, 2^2, \dots, N_{\text{mult}}(n)^{N_{\text{mult}}(n)}\}$. The number of distinct subset sums, however, can be much smaller. For example, for the set $\{1, 2, \dots, N_{\text{mult}}(n)\}$, this number is $J N_{\text{mult}}(n) - J + 1$. We have been unable to obtain a general expression for the set $\mathcal{M}(n)$; this seems to be a difficult number theoretic problem. It can be noted, however, that this number may be much larger than $N_{\text{mult}}(n)$.

Exact computations for the number of distinct rate points at small values of n and J are provided in Table I.

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